MORE ON MORPHISMS

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Contents

1. Introduction 2
2. Thickennings 2
3. Morphisms of thickenings 5
4. Picard groups of thickenings 8
5. First order infinitesimal neighbourhood 9
6. Formally unramified morphisms 10
7. Universal first order thickenings 12
8. Formally étale morphisms 18
9. Infinitesimal deformations of maps 20
10. Infinitesimal deformations of schemes 24
11. Formally smooth morphisms 32
12. Smoothness over a Noetherian base 37
13. The naïve cotangent complex 39
14. Pushouts in the category of schemes, I 41
15. Openness of the flat locus 45
16. Critère de platitude par fibres 46
17. Normalization revisited 49
18. Normal morphisms 50
19. Regular morphisms 52
20. Cohen-Macaulay morphisms 53
21. Slicing Cohen-Macaulay morphisms 57
22. Generic fibres 60
23. Relative assassins 64
24. Reduced fibres 67
25. Irreducible components of fibres 69
26. Connected components of fibres 74
27. Connected components meeting a section 78
28. Dimension of fibres 80
29. Theorem of the cube 81
30. Limit arguments 87
31. Étale neighbourhoods 90
32. Étale neighbourhoods and Artin approximation 92
33. Étale neighbourhoods and branches 99
34. Slicing smooth morphisms 101
35. Finite free locally dominates étale 106
36. Étale localization of quasi-finite morphisms 107
37. Étale localization of integral morphisms 110
38. Zariski’s Main Theorem 111

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1. Introduction

In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [DG67].

2. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 2.1. Thickenings.

(1) We say a scheme $X'$ is a *thickening* of a scheme $X$ if $X$ is a closed subscheme of $X'$ and the underlying topological spaces are equal.
(2) We say a scheme $X'$ is a first order thickening of a scheme $X$ if $X$ is a closed subscheme of $X'$ and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining $X$ has square zero.

(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a morphism of thickenings is a morphism $f' : X' \to Y'$ such that $f''(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme $Y$. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.

(4) Let $S$ be a scheme. We similarly define thickenings over $S$, and morphisms of thickenings over $S$. This means that the schemes $X, X', Y, Y'$ above are schemes over $S$, and that the morphisms $X \to X', Y \to Y'$ and $f' : X' \to Y'$ are morphisms over $S$.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a finite order thickening if the ideal sheaf $\mathcal{I}$ is “globally” nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^n+1 = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 = \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \cdots \subset \mathcal{I} \subset \mathcal{O}_{X'},$$

and we see that $X'$ is filtered by closed subspaces

$$X = X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over $S$. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickenings are described as follows (see Modules, Lemma 26.11).

**Lemma 2.2.** Let $X$ be a scheme over a base $S$. Consider a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

1. $\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and
2. $\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over $S$. Moreover, any first order thickening over $S$ is of this form.

**Proof.** It is clear that $X'$ is a locally ringed space. Let $U = \text{Spec}(B)$ be an affine open of $X$. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 26.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring $A$. By Schemes, Lemma 6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \to \text{Spec}(A)$$
coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$
\begin{array}{ccc}
(U, \mathcal{O}_X|_U) & \longrightarrow & \text{Spec}(B) \\
\downarrow & & \downarrow \\
(U, \mathcal{A}|_U) & \longrightarrow & \text{Spec}(\mathcal{A})
\end{array}
$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $p \subset A$ we obtain a commutative diagram of short exact sequences

$$
\begin{array}{cccc}
0 & \longrightarrow & I_p & \longrightarrow & A_p & \longrightarrow & B_p & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I_u & \longrightarrow & A_u & \longrightarrow & \mathcal{O}_{X,u} & \longrightarrow & 0.
\end{array}
$$

The left and right vertical arrows are isomorphisms because $I$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and $X'$ is a scheme as desired. □

**Lemma 2.3.** Any thickening of an affine scheme is affine.

**Proof.** This is a special case of Limits, Proposition 11.2. □

**Proof for a finite order thickening.** Suppose that $X \subset X'$ is a finite order thickening with $X$ affine. Then we may use Serre’s criterion to prove $X'$ is affine. More precisely, we will use Cohomology of Schemes, Lemma 3.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. It suffices to show that $H^1(X', \mathcal{F}) = 0$. Denote $i : X \to X'$ the given closed immersion and denote $\mathcal{I} = \text{Ker}(i^\sharp : \mathcal{O}_{X'} \to i_* \mathcal{O}_X)$. By our discussion of finite order thickenings (following Definition 2.1) there exists an $n \geq 0$ and a filtration

$$
0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}
$$

by quasi-coherent submodules such that $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$. Namely, we can take $\mathcal{F}_a = i^\sharp \mathcal{F}$. Then $\mathcal{F}_a/\mathcal{F}_{a+1} = i_* \mathcal{G}_a$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}_a$, see Morphisms, Lemma 4.1. We obtain

$$
H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) = H^1(X', i_* \mathcal{G}_a) = H^1(X, \mathcal{G}_a) = 0
$$

The second equality comes from Cohomology of Schemes, Lemma 2.4 and the last equality from Cohomology of Schemes, Lemma 2.2. Thus $\mathcal{F}$ has a finite induction argument that $H^1(X', \mathcal{F}) = 0$. □

**Lemma 2.4.** Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.

**Proof.** Omitted. □

**Lemma 2.5.** If $S \subset S'$ and $S' \subset S''$ are thickenings, then so is $S \subset S''$.

**Proof.** Omitted. □
\textbf{Lemma 2.6.} The property of being a thickening is fpqc local. Similarly for first order thickenings.

Proof. The statement means the following: Let \( X \to X' \) be a morphism of schemes and let \( \{ g_i : X_i' \to X' \} \) be an fpqc covering such that the base change \( X_i \to X'_i \) is a thickening for all \( i \). Then \( X \to X' \) is a thickening. Since the morphisms \( g_i \) are jointly surjective we conclude that \( X \to X' \) is surjective. By Descent, Lemma 20.19 we conclude that \( X \to X' \) is a closed immersion. Thus \( X \to X' \) is a thickening. We omit the proof in the case of first order thickenings. \( \square \)

3. Morphisms of thickenings

\textbf{Lemma 3.1.} Let \( (f,f') : (X \subset X') \to (S \subset S') \) be a morphism of thickenings. Then

1. \( f \) is an affine morphism if and only if \( f' \) is an affine morphism,
2. \( f \) is a surjective morphism if and only if \( f' \) is a surjective morphism,
3. \( f \) is quasi-compact if and only if \( f' \) quasi-compact,
4. \( f \) is universally closed if and only if \( f' \) is universally closed,
5. \( f \) is integral if and only if \( f' \) is integral,
6. \( f \) is (quasi-)separated if and only if \( f' \) is (quasi-)separated,
7. \( f \) is universally injective if and only if \( f' \) is universally injective,
8. \( f \) is universally open if and only if \( f' \) is universally open,
9. \( f \) is quasi-affine if and only if \( f' \) is quasi-affine, and
10. add more here.

Proof. Observe that \( S \to S' \) and \( X \to X' \) are universal homeomorphisms (see for example Morphisms, Lemma 43.6). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 2.3 which tells us that there is a 1-to-1 correspondence between affine opens of \( S \) and \( S' \) and between affine opens of \( X \) and \( X' \). Part (9) follows from Limits, Lemma 11.3 and the remark just made about affine opens of \( S \) and \( S' \). Part (5) follows from (1) and (4) by Morphisms, Lemma 42.7 Finally, note that

\[ S \times_X S = S \times_{X'} S \to S \times_{X'} S' \to S' \times_X S' \]

is a thickening (the two arrows are thickenings by Lemma 2.4). Hence applying (3) and (4) to the morphism \( (S \subset S') \to (S \times_X S \to S' \times_{X'} S') \) we obtain (6). \( \square \)

\textbf{Lemma 3.2.} Let \( (f,f') : (X \subset X') \to (S \subset S') \) be a morphism of thickenings. Let \( \mathcal{L}' \) be an invertible sheaf on \( X' \) and denote \( \mathcal{L} \) the restriction to \( X \). Then \( \mathcal{L}' \) is \( f' \)-ample if and only if \( \mathcal{L} \) is \( f \)-ample.

Proof. Recall that being relatively ample is a condition for each affine open in the base, see Morphisms, Definition 35.1. By Lemma 2.3 there is a 1-to-1 correspondence between affine opens of \( S \) and \( S' \). Thus we may assume \( S \) and \( S' \) are affine and we reduce to proving that \( \mathcal{L}' \) is ample if and only if \( \mathcal{L} \) is ample. This is Limits, Lemma 11.4 \( \square \)

\textbf{Lemma 3.3.} Let \( (f,f') : (X \subset X') \to (S \subset S') \) be a morphism of thickenings such that \( X = S \times_{S'} X' \). If \( S \subset S' \) is a finite order thickening, then
(1) \( f \) is a closed immersion if and only if \( f' \) is a closed immersion,
(2) \( f \) is locally of finite type if and only if \( f' \) is locally of finite type,
(3) \( f \) is locally quasi-finite if and only if \( f' \) is locally quasi-finite,
(4) \( f \) is locally of finite type of relative dimension \( d \) if and only if \( f' \) is locally of finite type of relative dimension \( d \),
(5) \( \Omega_{X/S} = 0 \) if and only if \( \Omega_{X'/S'} = 0 \),
(6) \( f \) is unramified if and only if \( f' \) is unramified,
(7) \( f \) is proper if and only if \( f' \) is proper,
(8) \( f \) is finite if and only if \( f' \) is finite,
(9) \( f \) is a monomorphism if and only if \( f' \) is a monomorphism,
(10) \( f \) is an immersion if and only if \( f' \) is an immersion, and
(11) add more here.

**Proof.** The properties \( \mathcal{P} \) listed in the lemma are all stable under base change, hence if \( f' \) has property \( \mathcal{P} \), then so does \( f \). See Schemes, Lemmas 18.2 and 23.5 and Morphisms, Lemmas 14.4, 19.13, 28.2, 31.10, 33.5, 39.5, and 42.6.

The interesting direction in each case is therefore to assume that \( f \) has the property and deduce that \( f' \) has it too. By induction on the order of the thickening we may assume that \( S \subset S' \) is a first order thickening, see discussion immediately following Definition 2.1.

Most of the proofs will use a reduction to the affine case. Let \( U' \subset S' \) be an affine open and let \( V' \subset X' \) be an affine open lying over \( U' \). Let \( U' = \text{Spec}(A') \) and denote \( I \subset A' \) be the ideal defining the closed subscheme \( U' \setminus S \). Say \( V' = \text{Spec}(B') \). Then \( V' \setminus X = \text{Spec}(B'/IB') \). Setting \( A = A'/I \) and \( B = B'/IB' \) we get a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & IB' & \rightarrow & B' & \rightarrow & B & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & IA' & \rightarrow & A' & \rightarrow & A & \rightarrow & 0
\end{array}
\]

with exact rows and \( I^2 = 0 \).

The translation of (1) into algebra: If \( A \rightarrow B \) is surjective, then \( A' \rightarrow B' \) is surjective. This follows from Nakayama’s lemma (Algebra, Lemma 19.1).

The translation of (2) into algebra: If \( A \rightarrow B \) is a finite type ring map, then \( A' \rightarrow B' \) is a finite type ring map. This follows from Nakayama’s lemma (Algebra, Lemma 19.1) applied to a map \( A'[x_1, \ldots, x_n] \rightarrow B' \) such that \( A[x_1, \ldots, x_n] \rightarrow B \) is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma 19.6.

Proof of (4). Follows from (2) and that the additional property of “being of relative dimension \( d \)” can be checked on fibres (by definition, see Morphisms, Definition 28.1).

The translation of (5) into algebra: If \( \Omega_{B/A} = 0 \), then \( \Omega_{B'/A'} = 0 \). By Algebra, Lemma 130.12 we have \( 0 = \Omega_{B'/A'} = \Omega_{B'/A'}/f\Omega_{B'/A'} \). Hence \( \Omega_{B'/A'} = 0 \) by Nakayama’s lemma (Algebra, Lemma 19.1).

The translation of (6) into algebra: If \( A \rightarrow B \) is unramified map, then \( A' \rightarrow B' \) is unramified. Since \( A \rightarrow B \) is of finite type we see that \( A' \rightarrow B' \) is of finite type by
MORE ON MORPHISMS

(2) above. Since $A \to B$ is unramified we have $\Omega_{B/A} = 0$. By part (5) we have $\Omega_{B'/A'} = 0$. Thus $A' \to B'$ is unramified.

Proof of (7). Follows by combining (2) with results of Lemma 3.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 3.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 42.4). 

Proof of (9). As $f$ is a monomorphism we have $X = X \times_S X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_S X \subset X' \times_{S'} X')$. We conclude $X' \to X' \times_{S'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{S'} X'$ is contained in the pullback of the ideal $I \subset O_{S'}$ cutting out $S$ in $S'$. Indeed, $X = X \times_S X = (X' \times_{S'} X') \times_{S'} S$ is contained in $X'$. Hence by Morphisms, Lemma 31.7, it suffices to show that $\Omega_{X'/S'} = 0$ which follows from (5) and the corresponding statement for $X/S$.

Proof of (10). If $f : X \to S$ is an immersion, then it factors as $X \to U \to S$ where $U \to S$ is an open immersion and $X \to U$ is a closed immersion. Let $U' \subset S'$ be the open subscheme whose underlying topological space is the same as $U$. Then $X' \to S'$ factors through $U'$ and we conclude that $X' \to U'$ is a closed immersion by part (1). This finishes the proof.

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from $f$ to $f'$), we instead take as given that each of $f$ and $f'$ is locally of finite type.

**Lemma 3.4.** Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings. Assume $f$ and $f'$ are locally of finite type and $X = Y \times_Y Y'$. Then

1. $f$ is locally quasi-finite if and only if $f'$ is locally quasi-finite,
2. $f$ is finite if and only if $f'$ is finite,
3. $f$ is a closed immersion if and only if $f'$ is a closed immersion,
4. $\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,
5. $f$ is unramified if and only if $f'$ is unramified,
6. $f$ is a monomorphism if and only if $f'$ is a monomorphism,
7. $f$ is an immersion if and only if $f'$ is an immersion,
8. $f$ is proper if and only if $f'$ is proper, and
9. add more here.

**Proof.** The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if $f'$ has property $\mathcal{P}$, then so does $f$. See Schemes, Lemmas 18.2, 23.5 and Morphisms, Lemmas 19.13, 28.2, 31.10, 33.3, 39.5, 42.6. Hence in each case we need only to prove that if $f$ has the desired property, so does $f'$.

A morphism is locally quasi-finite if and only if it is locally of finite type and the scheme theoretic fibres are discrete spaces, see Morphisms, Lemma 19.8. Since the underlying topological space is unchanged by passing to a thickening, we see that $f'$ is locally quasi-finite if (and only if) $f$ is. This proves (1).
Case (2) follows from case (5) of Lemma 3.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms, Lemma 42.4).

Case (3). This follows immediately from Morphisms, Lemma 43.7.

Case (4) follows from the following algebra statement: Let \(A\) be a ring and let \(I \subset A\) be a locally nilpotent ideal. Let \(B\) be a finite type \(A\)-algebra. If \(\Omega(B/IB)/(A/I) = 0\), then \(\Omega_B/A = 0\). Namely, the assumption means that \(I \Omega_B/A = 0\), see Algebra, Lemma 130.12. On the other hand \(\Omega_B/A\) is a finite \(B\)-module, see Algebra, Lemma 130.16. Hence the vanishing of \(\Omega_B/A\) follows from Nakayama’s lemma (Algebra, Lemma 19.1) and the fact that \(IB\) is contained in the Jacobson radical of \(B\).

Case (5) follows immediately from (4) and Morphisms, Lemma 33.2.

Proof of (6). As \(f\) is a monomorphism we have \(X = X \times_Y X\). We may apply the results proved so far to the morphism of thickenings \((X \subset X') \to (X \times_Y X \subset X' \times_Y X')\). We conclude \(\Delta_{X'/Y'} : X' \to X' \times_Y X'\) is a closed immersion by (3). In fact \(\Delta_{X'/Y'}\) is a bijection on underlying sets, hence \(\Delta_{X'/Y'}\) is a thickening. On the other hand \(\Delta_{X'/Y'}\) is locally of finite presentation by Morphisms, Lemma 20.12. In other words, \(\Delta_{X'/Y'}(X')\) is cut out by a quasi-coherent sheaf of ideals \(J \subset \mathcal{O}_{X' \times_Y X'}\) of finite type. Since \(\Omega_{X'/Y'} = 0\) by (5) we see that the conormal sheaf of \(X' \to X' \times_Y X'\) is zero by Morphisms, Lemma 31.7. In other words, \(J/J^2 = 0\). This implies \(\Delta_{X'/Y'}\) is an isomorphism, for example by Algebra, Lemma 20.5.

Proof of (7). If \(f : X \to Y\) is an immersion, then it factors as \(X \to V \to Y\) where \(V \to Y\) is an open immersion and \(X \to V\) is a closed immersion. Let \(V' \subset Y'\) be the open subscheme whose underlying topological space is the same as \(V\). Then \(X' \to V'\) factors through \(V'\) and we conclude that \(X' \to V'\) is a closed immersion by part (3).

Case (8) follows from Lemma 3.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type.

4. Picard groups of thickenings

0C6Q Some material on Picard groups of thickenings.

0C6R Lemma 4.1. Let \(X \subset X'\) be a first order thickening with ideal sheaf \(\mathcal{I}\). Then there is a canonical exact sequence

\[
0 \rightarrow H^0(X, \mathcal{I}) \rightarrow H^0(X', \mathcal{O}_{X'}) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{I}) \rightarrow \text{Pic}(X') \rightarrow \text{Pic}(X) \rightarrow H^2(X, \mathcal{I}) \rightarrow \ldots \rightarrow \ldots
\]

of abelian groups.
Proof. This is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

where the first map sends a local section $f$ of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_X^*$. We also use the identification of the Picard group of a ringed space with the first cohomology group of the sheaf of invertible functions, see Cohomology, Lemma 6.1.

Lemma 4.2. Let $X \subset X'$ be a thickening. Let $n$ be an integer invertible in $\mathcal{O}_X$. Then the map $\text{Pic}(X')[n] \to \text{Pic}(X)[n]$ is bijective.

Proof for a finite order thickening. By the general principle explained following Definition 2.1 this reduces to the case of a first order thickening. Then may use Lemma 4.1 to see that it suffices to show that $H^1(X, \mathcal{I})[n]$, $H^1(X, \mathcal{I})/n$, and $H^2(X, \mathcal{I})[n]$ are zero. This follows as multiplication by $n$ on $\mathcal{I}$ is an isomorphism as it is an $\mathcal{O}_X$-module.

Proof in general. Let $\mathcal{I} \subset \mathcal{O}_X$, be the quasi-coherent ideal sheaf cutting out $X$. Then we have a short exact sequence of abelian groups

$$0 \to (1 + \mathcal{I})^* \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

We obtain a long exact cohomology sequence as in the statement of Lemma 4.1 with $H^i(X, \mathcal{I})$ replaced by $H^i(X, (1 + \mathcal{I})^*)$. Thus it suffices to show that raising to the $n$th power is an isomorphism $(1 + \mathcal{I})^* \to (1 + \mathcal{I})^*$. Taking sections over affine opens this follows from Algebra, Lemma 31.8.

5. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that $i$ identifies $Z$ with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining $Z$ in $U$. Then we can consider the closed subscheme $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 5.1. Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood of $Z$ in $X$ is the first order thickening $Z \subset Z'$ over $X$ described above.

This thickening has the following universal property (which will alluage any fears that the construction above depends on the choice of the open $U$).

Lemma 5.2. Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood $Z'$ of $Z$ in $X$ has the following universal property: Given any commutative diagram

$$
\begin{array}{ccc}
Z & \leftarrow & T \\
\uparrow & & \uparrow \\
X & \leftarrow & T'
\end{array}
$$

where $T \subset T'$ is a first order thickening over $X$, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over $X$. 

Proof. Let $U \subset X$ be the open used in the construction of $Z'$, i.e., an open such that $Z$ is identified with a closed subscheme of $U$ cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of $b$ as a morphism into $U$. Let $\mathcal{J} \subset \mathcal{O}_{Z'}$ be the ideal cutting out $T$. Since $b(T') \subset Z$ by the diagram above we see that $\mathcal{J}(b^{-1}(T^2)) = 0$. By Schemes, Lemma 4.4, this implies that $b$ factors through $Z'$. Denote $a' : T' \to Z'$ this factorization and everything is clear.

Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of $Z$ in $X$. Then the diagram

$$
\begin{array}{ccc}
Z & \to & Z' \\
\downarrow & & \downarrow \\
Z & \to & X
\end{array}
$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 30.3. This map is an isomorphism.

Proof. This is clear from the construction of $Z'$ above.

6. Formally unramified morphisms

Recall that a ring map $R \to A$ is called formally unramified (see Algebra, Definition 145.1) if for every commutative solid diagram

$$
\begin{array}{ccc}
A & \to & B/I \\
\downarrow & & \downarrow \\
R & \to & B
\end{array}
$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 6.1. Let $f : X \to S$ be a morphism of schemes. We say $f$ is formally unramified if given any solid commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & T \\
f \downarrow & \swarrow & \downarrow i \\
S & \leftarrow & T'
\end{array}
$$

where $T \subset T'$ is a first order thickening of affine schemes over $S$ there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

Lemma 6.2. If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & T \\
f \downarrow & \swarrow & \downarrow i \\
S & \leftarrow & T'
\end{array}
$$
where $T \subset T'$ is a first order thickening of schemes over $S$ there exists at most one dotted arrow making the diagram commute. In other words, in Definition 6.1 the condition that $T$ be affine may be dropped.

**Proof.** This is true because a morphism is determined by its restrictions to affine opens. □

**Lemma 6.3.** A composition of formally unramified morphisms is formally unramified.

**Proof.** This is formal. □

**Lemma 6.4.** A base change of a formally unramified morphism is formally unramified.

**Proof.** This is formal. □

**Lemma 6.5.** Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If $f$ is formally unramified, so is $f|_U : U \to V$.

**Proof.** Consider a solid diagram

$$
\begin{array}{c}
U \\
\downarrow{f|_U} \\
V
\end{array}
\quad
\begin{array}{c}
T \\
\downarrow{i} \\
T'
\end{array}
\quad
\begin{array}{c}
\leftarrow \quad a \\
\leftarrow \quad \leftarrow
\end{array}
$$

as in Definition 6.1. If $f$ is formally ramified, then there exists at most one $S$-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into $U$. □

**Lemma 6.6.** Let $f : X \to S$ be a morphism of schemes. Assume $X$ and $S$ are affine. Then $f$ is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.

**Proof.** This is immediate from the definitions (Definition 6.1 and Algebra, Definition 145.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. □

Here is a characterization in terms of the sheaf of differentials.

**Lemma 6.7.** Let $f : X \to S$ be a morphism of schemes. Then $f$ is formally unramified if and only if $\Omega_{X/S} = 0$.

**Proof.** We give two proofs.

First proof. It suffices to show that $\Omega_{X/S}$ is zero on the members of an affine open covering of $X$. Choose an affine open $U \subset X$ with $f(U) \subset V$ where $V \subset S$ is an affine open of $S$. By Lemma 6.5 the restriction $f_U : U \to V$ is formally unramified. By Morphisms, Lemma 31.5 we see that $\Omega_{X/S}|_U$ is the quasi-coherent sheaf associated to the $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$. By Lemma 6.6 we see that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is a formally unramified ring map. Hence by Algebra, Lemma 145.2 we conclude that $\Omega_{X/S}|_U = 0$ as desired.

Second proof. We recall some of the arguments of the proof of Morphisms, Lemma 31.5. Let $W \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into $W$. Let $\mathcal{J} \subset \mathcal{O}_W$ be the ideal sheaf of this closed immersion. Let
$X' \subset W$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that $p_1$ and $p_2$ agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_X' \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^*(f) - p_2^*(f)$ for $f$ a local section of $\mathcal{O}_X$.

**Lemma 6.8.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

1. The morphism $f$ is unramified (resp. $G$-unramified), and
2. the morphism $f$ is locally of finite type (resp. locally of finite presentation) and formally unramified.

**Proof.** Use Lemma 6.7 and Morphisms, Lemma 33.2. □

7. Universal first order thickenings

Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of $Z$ over $X$.

**Lemma 7.1.** Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of $Z$ over $X$. □
Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of $Z$ over $X$ if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over $X$ by gluing, starting with the affine case which is Algebra, Lemma 146.1. We begin with some general remarks.

If a universal first order thickening of $Z$ over $X$ exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of $Z$ over $X$. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of $V$ over $U$. Namely, suppose given any diagram

$$
\begin{array}{ccc}
V & \xleftarrow{a} & T \\
\downarrow{h} & & \downarrow{} \\
U & \xleftarrow{b} & T'
\end{array}
$$

where $T \subset T'$ is a first order thickening over $U$. By the universal property of $Z'$ we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of $(a, a')$ as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over $U$. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$ is a universal first order thickening over $U$, then $Z \subset Z'$ is a universal first order thickening over $X$.

Before we glue affine pieces let us show that the lemma holds if $Z$ and $X$ are affine. Say $X = \text{Spec}(R)$ and $Z = \text{Spec}(S)$. By Algebra, Lemma 146.1 there exists a first order thickening $Z \subset Z'$ over $X$ which has the universal property of the lemma for diagrams

$$
\begin{array}{ccc}
Z & \xleftarrow{a} & T \\
\downarrow{h} & & \downarrow{} \\
X & \xleftarrow{b} & T'
\end{array}
$$

where $T, T'$ are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over $X$ such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that $a'_i$ and $a'_j$ agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms $a'_i$ glue to a global morphism $a' : T' \to Z'$ over $X$ as desired. Thus the lemma holds if $X$ and $Z$ are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each $Z_i$ maps into an affine open $U_i$ of $X$. By Lemma 6.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over $U_i$. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of $Z_i$ over $X$. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$. By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over $X$. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 14 to get a
first order thickening $Z \subset Z'$ over $X$ which the property that the open subscheme $Z'_i \subset Z'$ with $Z_i = Z'_i \cap Z$ is a universal first order thickening of $Z_i$ over $X$.

It turns out that this implies formally that $Z'$ is a universal first order thickening of $Z$ over $X$. Namely, we have the universal property for any diagram

$$
\begin{array}{ccc}
Z & \xleftarrow{a} & T \\
\downarrow{h} & & \downarrow \\
X & \xleftarrow{b} & T'
\end{array}
$$

where $a(T)$ is contained in some $Z_i$. Given a general diagram we can choose an open covering $T' = \bigcup T'_i$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a'_i : T'_i \to Z'$ over $X$ such that $a'_i|_{T_i} = a|_{T_i}$. We see that $a'_i$ and $a'_j$ necessarily agree on $T'_i \cap T'_j$ since both $a'_i|_{T'_i \cap T'_j}$ and $a'_j|_{T'_i \cap T'_j}$ are solutions of the problem of mapping into the universal first order thickening $Z'_i \cap Z'_j$ of $Z_i \cap Z_j$ over $X$. Hence the morphisms $a'_i$ glue to a global morphism $a' : T' \to Z'$ over $X$ as desired. This finishes the proof. □

Definition 7.2. Let $h : Z \to X$ be a formally unramified morphism of schemes.

1. The universal first order thickening of $Z$ over $X$ is the thickening $Z \subset Z'$ constructed in Lemma 7.1.
2. The conormal sheaf of $Z$ over $X$ is the conormal sheaf of $Z$ in its universal first order thickening $Z'$ over $X$.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$
0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0
$$
on $Z$. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 7.3. Let $i : Z \to X$ be an immersion of schemes. Then

1. $i$ is formally unramified,
2. the universal first order thickening of $Z$ over $X$ is the first order infinitesimal neighbourhood of $Z$ in $X$ of Definition 5.1, and
3. the conormal sheaf of $i$ in the sense of Morphisms, Definition 30.1 agrees with the conormal sheaf of $i$ in the sense of Definition 7.2.

Proof. By Morphisms, Lemmas 33.7 and 33.8 an immersion is unramified, hence formally unramified by Lemma 6.8. The other assertions follow by combining Lemmas 5.2 and 5.3 and the definitions. □

Lemma 7.4. Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening $Z'$ is formally unramified over $X$.

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 146.5. Then Lemma 6.7 implies what we want. The second is a direct argument as follows.
Let $T \subset T'$ be a first order thickening. Let

\[
\begin{array}{ccc}
Z' & \xleftarrow{c} & T \\
\downarrow{a,b} & & \downarrow \\
X & \xleftarrow{\alpha} & T'
\end{array}
\]

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since $Z'$ is a first order thickening of $Z$, we see that $T'$ is a first order thickening of $T'_a$. Moreover, since $c = a|_T$ we see that $T_0 = a^{-1}(Z)(T_a)$ scheme theoretically). As $T'$ is a first order thickening of $T$ it follows that $T'_a$ is a first order thickening of $T_0$. Now $a|_{T_a}$ and $b|_{T_a}$ are morphisms of $T'_a$ into $Z'$ over $X$ which agree on $T_0$ as morphisms into $Z$. Hence by the universal property of $Z'$ we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus $a$ and $b$ are morphism from the first order thickening $T'_a$ of $T'_a$ whose restrictions to $T'_a$ agree as morphisms into $Z$. Thus using the universal property of $Z'$ once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired.

**Lemma 7.5.** Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

with $h$ and $h'$ formally unramified. Let $Z \subset Z'$ be the universal first order thickening of $Z$ over $X$. Let $W \subset W'$ be the universal first order thickening of $W$ over $Y$. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over $Y$ which fits into the following commutative diagram

\[
\begin{array}{ccc}
& & Z' \\
& \xleftarrow{e} & \\
Z & \xrightarrow{f} & X & \xrightarrow{g} & W' \\
\downarrow{f} & & \downarrow{h} & & \downarrow{h'} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

In particular the morphism $(f, f')$ of thickenings induces a morphism of conormal sheaves $f^*C_{W/Y} \to C_{Z/X}$.

**Proof.** The first assertion is clear from the universal property of $W'$. The induced map on conormal sheaves is the map of Morphisms, Lemma 30.3 applied to $(Z \subset Z') \to (W \subset W')$.

**Lemma 7.6.** Let

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

be a fibre product diagram in the category of schemes with $h'$ formally unramified. Then $h$ is formally unramified and if $W \subset W'$ is the universal first order thickening
of $W$ over $Y$, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of $Z$ over $X$. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 7.5 is surjective.

**Proof.** The morphism $h$ is formally unramified by Lemma 6.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because $W'$ has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 30.4 for why this implies that the map of conormal sheaves is surjective. \hfill $\square$

**Lemma 7.7.** Let
\[
\begin{array}{ccc}
Z & \to & X \\
\downarrow h & & \downarrow f \\
W & \to & Y \\
\downarrow g & & \\
W' & \to & \\
\end{array}
\]

be a fibre product diagram in the category of schemes with $h'$ formally unramified and $g$ flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

**Proof.** Flatness is preserved under base change, see Morphisms, Lemma 24.8. Hence the first statement follows from the description of $W'$ in Lemma 7.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because $W'$ has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 30.4 for why this implies that the map of conormal sheaves is an isomorphism. \hfill $\square$

**Lemma 7.8.** Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z, U \subset X$ be opens such that $h(V) \subset U$. Let $Z'$ be the universal first order thickening of $Z$ over $X$. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of $V$ over $U$ is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

**Proof.** The first statement is Lemma 6.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 7.1, or from Algebra, Lemma 146.4 or deduced from Lemma 7.7. \hfill $\square$

**Lemma 7.9.** Let $h : Z \to X$ be a formally unramified morphism of schemes over $S$. Let $Z \subset Z'$ be the universal first order thickening of $Z$ over $X$ with structure morphism $h' : Z' \to X$. The canonical map
\[ c_{h'} : (h')^*\Omega_{X/S} \to \Omega_{Z'/S} \]
induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

**Proof.** The map $c_{h'}$ is the map defined in Morphisms, Lemma 31.8. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 7.8 and Morphisms, Lemma 31.3. In this case the result is Algebra, Lemma 146.5. \hfill $\square$

**Lemma 7.10.** Let $h : Z \to X$ be a formally unramified morphism of schemes over $S$. There is a canonical exact sequence
\[ \mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0. \]
The first arrow is induced by $d_{Z'/S}$ where $Z'$ is the universal first order neighborhood of $Z$ over $X$.

**Proof.** We know that there is a canonical exact sequence

$$C_{Z/Z'} \rightarrow \Omega_{Z'/S} \otimes \mathcal{O}_Z \rightarrow \Omega_{Z/S} \rightarrow 0.$$  

see Morphisms, Lemma 31.15. Hence the result follows on applying Lemma 7.9. □

**Lemma 7.11.** Let

$$\xymatrix{ Z \ar[r]^i \ar[d]^j & X \ar[d] \cr Y & }$$

be a commutative diagram of schemes where $i$ and $j$ are formally unramified. Then there is a canonical exact sequence

$$C_{Z/Y} \rightarrow C_{Z/X} \rightarrow i^* \Omega_{X/Y} \rightarrow 0$$

where the first arrow comes from Lemma 7.5 and the second from Lemma 7.10.

**Proof.** Denote $Z \rightarrow Z'$ the universal first order thickening of $Z$ over $X$. Denote $Z \rightarrow Z''$ the universal first order thickening of $Z$ over $Y$. By Lemma 7.10 here is a canonical morphism $Z' \rightarrow Z''$ so that we have a commutative diagram

$$\xymatrix{ Z \ar[r]^i \ar[d]^{j'} & Z' \ar[r] \ar[d] & X \ar[d] \cr Y' \ar[r] & Y' \times_Y Z' \ar[r] & Y }$$

Apply Morphisms, Lemma 31.18 to the left triangle to get an exact sequence

$$C_{Z/Z''} \rightarrow C_{Z/Z'} \rightarrow (i')^* \Omega_{Z/Z''} \rightarrow 0$$

As $Z''$ is formally unramified over $Y$ (see Lemma 7.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 6.7 and Morphisms, Lemma 31.9). Then we have $(i')^* \Omega_{Z'/Y} = i^* \Omega_{X/Y}$ by Lemma 7.9. □

**Lemma 7.12.** Let $Z \rightarrow Y \rightarrow X$ be formally unramified morphisms of schemes.

1. If $Z \subset Z'$ is the universal first order thickening of $Z$ over $X$ and $Y \subset Y'$ is the universal first order thickening of $Y$ over $X$, then there is a morphism $Z' \rightarrow Y'$ and $Y \times_Y Z'$ is the universal first order thickening of $Z$ over $Y$.

2. There is a canonical exact sequence

$$i^* C_{Y/X} \rightarrow C_{Z/X} \rightarrow C_{Z/Y} \rightarrow 0$$

where the maps come from Lemma 7.5 and $i : Z \rightarrow Y$ is the first morphism.

**Proof.** The map $h : Z' \rightarrow Y'$ in (1) comes from Lemma 7.5. The assertion that $Y \times_Y Z'$ is the universal first order thickening of $Z$ over $Y$ is clear from the universal properties of $Z'$ and $Y'$. By Morphisms, Lemma 30.5 we have an exact sequence

$$(i')^* C_{Y \times_Y Z'/Z'} \rightarrow C_{Z/Z'} \rightarrow C_{Z/Y \times_Y Z'} \rightarrow 0$$

where $i' : Z \rightarrow Y \times_Y Z'$ is the given morphism. By Morphisms, Lemma 30.4 there exists a surjection $h^* C_{Y/Y'} \rightarrow C_{Y \times_Y Z'/Z'}$. Combined with the equalities $C_{Y/Y'} = C_{Y/X}, C_{Z/Z'} = C_{Z/X},$ and $C_{Z/Y \times_Y Z'} = C_{Z/Y}$ this proves the lemma. □
8. Formally étale morphisms

Recall that a ring map $R \to A$ is called \textit{formally étale} (see Algebra, Definition 147.1) if for every commutative solid diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B/I \\
\downarrow & & \downarrow \\
R & \longrightarrow & B
\end{array}
\]

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

\textbf{Definition 8.1.} Let $f : X \to S$ be a morphism of schemes. We say $f$ is \textit{formally étale} if given any solid commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{T} & T' \\
\downarrow \ f \downarrow & & \downarrow \ i \\
S & \xleftarrow{T'}
\end{array}
\]

where $T \subset T'$ is a first order thickening of affine schemes over $S$ there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 6.7.

\textbf{Lemma 8.2.} If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{T} & T' \\
\downarrow \ f \downarrow & & \downarrow \ i \\
S & \xleftarrow{T'}
\end{array}
\]

where $T \subset T'$ is a first order thickening of schemes over $S$ there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 8.1 the condition that $T$ be affine may be dropped.

\textbf{Proof.} Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that $a'_i$ and $a'_j$ agree on any affine open subscheme of $T'_i \cap T'_j$. Hence $a'_i$ and $a'_j$ agree on $T'_i \cap T'_j$. Thus we see that the morphisms $a'_i$ glue to a global morphism $a' : T' \to X$. The uniqueness of $a'$ we have seen in Lemma 6.2. \hfill \square

\textbf{Lemma 8.3.} A composition of formally étale morphisms is formally étale.

\textbf{Proof.} This is formal. \hfill \square

\textbf{Lemma 8.4.} A base change of a formally étale morphism is formally étale.

\textbf{Proof.} This is formal. \hfill \square

\textbf{Lemma 8.5.} Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If $f$ is formally étale, so is $f|_U : U \to V$. \hfill \square
Proof. Consider a solid diagram

\[
\begin{array}{c}
U \\
\downarrow f|_U \\
V \\
\end{array}
\begin{array}{c}
\mathbb{T} \\
\downarrow a \\
\mathbb{T}' \\
\end{array}
\begin{array}{c}
\mathbb{T} \\
\downarrow a' \\
\mathbb{T}' \\
\end{array}
\]

as in Definition 8.1. If \( f \) is formally ramified, then there exists exactly one \( S \)-morphism \( a' : \mathbb{T}' \to \mathbb{X} \) such that \( a'|_{\mathbb{T}} = a \). Since \( |\mathbb{T}'| = |\mathbb{T}| \) we conclude that \( a'(\mathbb{T}') \subset U \) which gives our unique morphism from \( \mathbb{T}' \) into \( U \).

\[\square\]

04FE Lemma 8.6. Let \( f : \mathbb{X} \to \mathbb{S} \) be a morphism of schemes. The following are equivalent:

1. \( f \) is formally étale,
2. \( f \) is formally unramified and the universal first order thickening of \( \mathbb{X} \) over \( \mathbb{S} \) is equal to \( \mathbb{X} \),
3. \( f \) is formally unramified and \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} = 0 \), and
4. \( \Omega_{\mathbb{X}/\mathbb{S}} = 0 \) and \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} = 0 \).

Proof. Actually, the last assertion only make sense because \( \Omega_{\mathbb{X}/\mathbb{S}} = 0 \) implies that \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} \) is defined via Lemma 6.7 and Definition 7.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that \( f \) is formally unramified. Hence we may assume \( f \) is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening \( \mathbb{X}' \) of \( \mathbb{X} \) over \( \mathbb{S} \) and the fact that \( \mathbb{X} = \mathbb{X}' \iff \mathcal{C}_{\mathbb{X}/\mathbb{S}} = 0 \) since after all by definition \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} = \mathcal{C}_{\mathbb{X}/\mathbb{X}'} \) is the ideal sheaf of \( \mathbb{X} \) in \( \mathbb{X}' \). \[\square\]

04FF Lemma 8.7. An unramified flat morphism is formally étale.

Proof. Say \( \mathbb{X} \to \mathbb{S} \) is unramified and flat. Then \( \Delta : \mathbb{X} \to \mathbb{X} \times_{\mathbb{S}} \mathbb{X} \) is an open immersion, see Morphisms, Lemma 33.13. We have to show that \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} \) is zero. Consider the two projections \( p,q : \mathbb{X} \times_{\mathbb{S}} \mathbb{X} \to \mathbb{X} \). As \( f \) is formally unramified (see Lemma 6.8), \( q \) is formally unramified (see Lemma 6.4). As \( f \) is flat, \( p \) is flat, see Morphisms, Lemma 24.8. Hence \( p^*\mathcal{C}_{\mathbb{X}/\mathbb{S}} = \mathcal{C}_q \) by Lemma 7.7 where \( \mathcal{C}_q \) denotes the conormal sheaf of the formally unramified morphism \( q : \mathbb{X} \times_{\mathbb{S}} \mathbb{X} \to \mathbb{X} \). But \( \Delta(\mathbb{X}) \subset \mathbb{X} \times_{\mathbb{S}} \mathbb{X} \) is an open subscheme which maps isomorphically to \( \mathbb{X} \) via \( q \). Hence by Lemma 7.8 we see that \( \mathcal{C}_{q|\Delta(\mathbb{X})} = \mathcal{C}_{\mathbb{X}/\mathbb{X}} = 0 \). In other words, the pullback of \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} \) to \( \mathbb{X} \) via the identity morphism is zero, i.e., \( \mathcal{C}_{\mathbb{X}/\mathbb{S}} = 0 \). \[\square\]

02HL Lemma 8.8. Let \( f : \mathbb{X} \to \mathbb{S} \) be a morphism of schemes. Assume \( \mathbb{X} \) and \( \mathbb{S} \) are affine. Then \( f \) is formally étale if and only if \( \mathcal{O}_{\mathbb{S}}(\mathbb{S}) \to \mathcal{O}_{\mathbb{X}}(\mathbb{X}) \) is a formally étale ring map.

Proof. This is immediate from the definitions (Definition 8.1 and Algebra, Definition 147.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. \[\square\]

02HM Lemma 8.9. Let \( f : \mathbb{X} \to \mathbb{S} \) be a morphism of schemes. The following are equivalent:

1. The morphism \( f \) is étale, and
2. the morphism \( f \) is locally of finite presentation and formally étale.
**Proof.** Assume $f$ is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 34. Hence $f$ is locally of finite presentation and formally étale, see Lemma 8.7.

Conversely, suppose that $f$ is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on $X$ and $S$, see Morphisms, Lemma 34.2. By Lemma 8.5 we can cover $X$ by affine opens $U$ which map into affine opens $V$ such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 20.2). By Lemma 8.8 we see that the ring maps $O(V) \to O(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 147.2. (We will give another proof of this implication when we discuss formally smooth morphisms.) □

9. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening $X'$ of $X$ can be seen as a sheaf on $X$.

**Lemma 9.1.** Let $S$ be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over $S$. Let $(a,a'), (b,b'): (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over $S$. Assume that

1. $a = b$, and
2. the two maps $a^*C_{Y/Y'} \to C_{X/X'}$ (Morphisms, Lemma 30.3) are equal.

Then the map $(a')^* - (b')^*$ factors as

\[ O_{Y'} \to O_Y \xrightarrow{D} a_* C_{X/X'} \to a_* O_X, \]

where $D$ is an $O_S$-derivation.

**Proof.** Instead of working on $Y$ we work on $X$. The advantage is that the pullback functor $a^{-1}$ is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & C_{X/X'} & \to & O_{X'} & \to & O_X & \to & 0 \\
& & \uparrow (a')^* & & \uparrow (b')^* & & \downarrow & & \\
0 & \to & a^{-1} C_{Y/Y'} & \to & a^{-1} O_{Y'} & \to & a^{-1} O_Y & \to & 0
\end{array}
\]

Now it is a general fact that in such a situation the difference of the $O_S$-algebra maps $(a')^*$ and $(b')^*$ is an $O_S$-derivation from $a^{-1} O_Y$ to $C_{X/X'}$. By adjointness of the functors $a^{-1}$ and $a_*$ this is the same thing as an $O_S$-derivation from $O_Y$ into $a_* C_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write $D$ as

\[ D = d_{Y/S} \circ \theta \]

where $\theta$ is an $O_Y$-linear map $\theta : \Omega_{Y/S} \to a_* C_{X/X'}$. Of course, then by adjunction again we may view $\theta$ as an $O_X$-linear map $\theta : a^* \Omega_{Y/S} \to C_{X/X'}$.

**Lemma 9.2.** Let $S$ be a scheme. Let $(a,a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over $S$. Let

\[ \theta : a^* \Omega_{Y/S} \to C_{X/X'} \]
be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 9.1 hold and the derivation $D$ and $\theta$ are related by Equation (9.1.1).

**Proof.** We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where $D$ is as in Equation (9.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 9.1 hold. Equation (9.1.1) holds by construction. □

**Remark 9.3.** Assumptions and notation as in Lemma 9.2. The action of a local section $\theta$ on $a'$ is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation $D$ which is related to $\theta$ by Equation (9.1.1).

**Lemma 9.4.** Let $S$ be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over $S$. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/S} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that

1. $a'$ is a morphism over $S$,
2. $a'|_U = a|_U$, and
3. the induced map $a^*\mathcal{C}_{Y/S}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of $A$ to $U$.

Here $U = X \cap U'$. Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on $X'$.

**Proof.** Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. □

In the following lemma we identify sheaves on $X$ and any thickening of $X$.

**Lemma 9.5.** Same notation and assumptions as in Lemma 9.4. There is an action of the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf $\{\text{9.4.1}\}$. Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf $\{\text{9.4.1}\}$ has a section.

**Proof.** This is a combination of Lemmas 9.1, 9.2, and 9.4 □

**Remark 9.6.** A special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where $Y = Y'$. In this case the map $A$ is always zero. The sheaf of Lemma 9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$. 
Another special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where \( S \) itself is a thickening \( Z \subset Z' = S \) and \( Y = Z \times_Z Y' \). Picture

\[
\begin{array}{c}
\xymatrix{(X \subset X') \ar[rr]^-{(a',?)} & & (Y \subset Y') \\
(Z \subset Z') \ar[rr]^-{(g,g')} & & (h,h')
\end{array}
\]

In this case the map \( A: a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'} \) is determined by \( a \): the map \( h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'} \) is surjective (because we assumed \( Y = Z \times_Z Y' \)), hence the pullback \( g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \) is surjective, and the composition \( g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'} \) has to be the canonical map induced by \( g' \). Thus the sheaf of Lemma 9.4 is just given by the rule

\[
U' \mapsto \{ a': U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U \}
\]

and we act on this by the sheaf \( \text{Hom}_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}) \).

Let \( S \) be a scheme. Let \( X \subset X' \) be a first order thickening over \( S \). Let \( Y \) be a scheme over \( S \). Let \( a', b': X' \to Y \) be two morphisms over \( S \) with \( a = a'|_X = b'|_X \). This gives rise to a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a} & X' \\
\downarrow & & \downarrow \quad (b',a') \\
Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y
\end{array}
\]

Since the horizontal arrows are immersions with conormal sheaves \( \mathcal{C}_{X/X'} \) and \( \Omega_{Y/S} \), by Morphisms, Lemma 30.3, we obtain a map \( \theta: a^*\Omega_{Y/Z} \to \mathcal{C}_{X/X'} \). Then this \( \theta \) and the derivation \( D \) of Lemma 9.1 are related by Equation (9.1.1).

**Proof.** Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If \( f \) is a local section of \( \mathcal{O}_Y \), then \( 1 \otimes f - f \otimes 1 \) is a local section of \( \mathcal{C}_{Y/(Y \times_S Y)} \) corresponding to \( d_{Y/S}(f) \). It is mapped to the local section \( (a')^!(f) - (b')^!(f) = D(f) \) of \( \mathcal{C}_{X/X'} \). In other words, \( \theta(d_{Y/S}(f)) = D(f) \).

For later purposes we need a result that roughly states that the construction of Lemma 9.2 is compatible with étale localization.

Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow & & \downarrow \\
S_1 & \xleftarrow{g} & S_2
\end{array}
\]

be a commutative diagram of schemes with \( X_2 \to X_1 \) and \( S_2 \to S_1 \) étale. Then the map \( f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2} \) of Morphisms, Lemma 31.8 is an isomorphism.

**Proof.** We recall that an étale morphism \( U \to V \) is a smooth morphism with \( \Omega_{U/V} = 0 \). Using this we see that Morphisms, Lemma 31.9 implies \( \Omega_{X_2/S_2} = \Omega_{X_2/S_1} \) and Morphisms, Lemma 32.16 implies that the map \( f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1} \) (for the morphism \( f \) seen as a morphism over \( S_1 \)) is an isomorphism. Hence the lemma follows.
04BY Lemma 9.10. Consider a commutative diagram of first order thickenings

\[
\begin{array}{ccc}
(T_2 \subset T'_2) & \xrightarrow{(a_2,a'_2)} & (X_2 \subset X'_2) \\
(h,h') & & (f,f') \quad \text{and a commutative} \\
(T_1 \subset T'_1) & \xrightarrow{(a_1,a'_1)} & (X_1 \subset X'_1) \\
\end{array}
\]

with \( X_2 \to X_1 \) and \( S_2 \to S_1 \) étale. For any \( \mathcal{O}_{T_1} \)-linear map \( \theta_1 : a_1^* \Omega_{X_1/S_1} \to C_{T_1/T'_1} \) let \( \theta_2 \) be the composition

\[
a_2^* \Omega_{X_2/S_2} \xrightarrow{h^*a_1^* \Omega_{X_1/S_1}} \xrightarrow{h^*\theta_1} h^*C_{T_1/T'_1} \xrightarrow{C_{T_2/T'_2}}
\]

(= equality sign is explained in the proof). Then the diagram

\[
\begin{array}{ccc}
T'_2 & \xrightarrow{\theta_2 \cdot a'_2} & X'_2 \\
\downarrow & & \downarrow \\
T'_1 & \xrightarrow{\theta_1 \cdot a'_1} & X'_1
\end{array}
\]

commutes where the actions \( \theta_2 \cdot a'_2 \) and \( \theta_1 \cdot a'_1 \) are as in Remark 9.3.

**Proof.** The equality sign comes from the identification \( f^* \Omega_{X_1/S_1} = \Omega_{X_2/S_2} \) of Lemma 9.9. Namely, using this we have

\[
a_2^* \Omega_{X_2/S_2} = a_2^* f^* \Omega_{X_1/S_1} = h^* a_1^* \Omega_{X_1/S_1}
\]

because \( f \circ a_2 = a_1 \circ h \). Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain commutative diagrams

\[
\begin{array}{ccc}
(B'_2, I_2) & \xleftarrow{a'_2} & (A'_2, J_2) \\
\uparrow_{h'} & & \uparrow_{f'} \\
(B'_1, I_1) & \xleftarrow{a'_1} & (A'_1, J_1)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A'_2 & \xleftarrow{R_2} & A'_1 \\
\uparrow & & \uparrow \\
R_1
\end{array}
\]

The notation signifies that \( I_1, I_2, J_1, J_2 \) are ideals of square zero and maps of pairs are ring maps sending ideals into ideals. Set \( A_1 = A'_1/J_1, A_2 = A'_2/J_2, B_1 = B'_1/I_1, \) and \( B_2 = B'_2/I_2 \). We are given that

\[
A_2 \otimes_{A_1} \Omega_{A_1/R_1} \xrightarrow{\theta_2} \Omega_{A_2/R_2}
\]

is an isomorphism. Then \( \theta_1 : B_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1 \) is \( B_1 \)-linear. This gives an \( R_1 \)-derivation \( D_1 = \theta_1 \circ d_{A_1/R_1} : A_1 \to I_1 \). In a similar way we see that \( \theta_2 : B_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2 \) gives rise to a \( R_2 \)-derivation \( D_2 = \theta_2 \circ d_{A_2/R_2} : A_2 \to I_2 \). The construction of \( \theta_2 \) implies the following compatibility between \( \theta_1 \) and \( \theta_2 \): for every \( x \in A_1 \) we have

\[
h'(D_1(x)) = D_2(f'(x))
\]

as elements of \( I_2 \). We may view \( D_1 \) as a map \( A'_1 \to B'_1 \) using \( A'_1 \to A_1 \xrightarrow{D_1} I_1 \to B_1 \) similarly we may view \( D_2 \) as a map \( A'_2 \to B'_2 \). Then the displayed equality holds for \( x \in A'_1 \). By the construction of the action in Lemma 9.2 and Remark 9.3 we know that \( \theta_1 \cdot a'_1 \) corresponds to the ring map \( a'_1 + D_1 : A'_1 \to B'_1 \) and \( \theta_2 \cdot a'_2 \) corresponds to the ring map \( a'_2 + D_2 : A'_2 \to B'_2 \). By the displayed equality we obtain that

\[
h'(a'_1 + D_1) = (a'_2 + D_2) \circ f'
\]

as desired. \( \square \)
Remark 9.11. Lemma [9.10] can be improved in the following way. Suppose that we have commutative diagrams as in Lemma [9.10] but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to C_{T_1/T_1}$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to C_{T_2/T_2}$ such that $f^*O_{X_2} \to f^*D_2 \to f^*a_2^*C_{T_2/T_2}$ induced by $(k')^t$ is commutative where $D_1$ corresponds to $\theta_i$ as in Equation (9.1.1). Then we have the conclusion of Lemma [9.10]. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a $\theta_2$ from $\theta_1$.

10. Infinitesimal deformations of schemes

Let $(f,f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that $f$ is flat. Then the following are equivalent

1. $f'$ is flat and $X = S \times_{S'} X'$, and
2. the canonical map $f^*C_{S/S'} \to C_{X/X'}$ is an isomorphism.

Proof. As the problem is local on $X'$ we may assume that $X, X', S, S'$ are affine schemes. Say $S' = \text{Spec}(A')$, $X' = \text{Spec}(B')$, $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$
\begin{array}{cccc}
0 & \to & J & \to & B' & \to & B & \to & 0 \\
0 & \to & I & \to & A' & \to & A & \to & 0
\end{array}
$$

with exact rows. The canonical map of the lemma is the map

$I \otimes_A B = I \otimes_{A'} B' \to J$.

The assumption that $f$ is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\text{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma [74.8]). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark [74.9]). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\text{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence $B'$ is flat over $A'$ by Algebra, Lemma [98.8].

The following lemma is the “nilpotent” version of the “critère de platitude par fibres”, see Section 16.
Lemma 10.2. Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
& \searrow & \downarrow \\
& (S \subset S') &
\end{array}
\]

of thickenings. Assume

1. \(X'\) is flat over \(S'\),
2. \(f\) is flat,
3. \(S \subset S'\) is a finite order thickening, and
4. \(X = S \times_{S'} X'\) and \(Y = S \times_{S'} Y'\).

Then \(f'\) is flat and \(Y'\) is flat over \(S'\) at all points in the image of \(f'\).

Proof. Immediate consequence of Algebra, Lemma 100.8.

Many properties of morphisms of schemes are preserved under flat deformations.

Lemma 10.3. Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
& \searrow & \downarrow \\
& (S \subset S') &
\end{array}
\]

of thickenings. Assume \(S \subset S'\) is a finite order thickening, \(X'\) flat over \(S'\), \(X = S \times_{S'} X'\), and \(Y = S \times_{S'} Y'\). Then

1. \(f\) is flat if and only if \(f'\) is flat,
2. \(f\) is an isomorphism if and only if \(f'\) is an isomorphism,
3. \(f\) is an open immersion if and only if \(f'\) is an open immersion,
4. \(f\) is quasi-compact if and only if \(f'\) is quasi-compact,
5. \(f\) is universally closed if and only if \(f'\) is universally closed,
6. \(f\) is (quasi-)separated if and only if \(f'\) is (quasi-)separated,
7. \(f\) is a monomorphism if and only if \(f'\) is a monomorphism,
8. \(f\) is surjective if and only if \(f'\) is surjective,
9. \(f\) is universally injective if and only if \(f'\) is universally injective,
10. \(f\) is affine if and only if \(f'\) is affine,
11. \(f\) is locally of finite type if and only if \(f'\) is locally of finite type,
12. \(f\) is locally quasi-finite if and only if \(f'\) is locally quasi-finite,
13. \(f\) is locally of finite presentation if and only if \(f'\) is locally of finite presentation,
14. \(f\) is locally of finite type of relative dimension \(d\) if and only if \(f'\) is locally of finite type of relative dimension \(d\),
15. \(f\) is universally open if and only if \(f'\) is universally open,
16. \(f\) is syntomic if and only if \(f'\) is syntomic,
17. \(f\) is smooth if and only if \(f'\) is smooth,
18. \(f\) is unramified if and only if \(f'\) is unramified,
19. \(f\) is étale if and only if \(f'\) is étale,
20. \(f\) is proper if and only if \(f'\) is proper,
f is integral if and only if \( f' \) is integral,

(22) \( f \) is finite if and only if \( f' \) is finite,

(23) \( f \) is finite locally free (of rank \( d \)) if and only if \( f' \) is finite locally free (of rank \( d \)), and

(24) add more here.

**Proof.** The assumptions on \( X \) and \( Y \) mean that \( f \) is the base change of \( f' \) by \( X \to X' \). The properties \( P \) listed in (1) – (23) above are all stable under base change, hence if \( f' \) has property \( P \), then so does \( f \). See Schemes, Lemmas 18.2.5, 21.12 and 23.5 and Morphisms, Lemmas 9.4, 10.4, 14.4, 19.13, 20.4, 28.2, 29.4, 32.5, 33.5, 34.4, 39.5, 42.6, and 46.4.

The interesting direction in each case is therefore to assume that \( f \) has the property and deduce that \( f' \) has it too. By induction on the order of the thickening we may assume that \( S \subset S' \) is a first order thickening, see discussion immediately following Definition 2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let \( W' \subset S' \) be an affine open and let \( U' \subset X' \) and \( V' \subset Y' \) be affine opens lying over \( W' \) with \( f'(U') \subset V' \). Let \( W' = \text{Spec}(R') \) and denote \( I \subset R' \) be the ideal defining the closed subscheme \( W' \cap S' \). Say \( U' = \text{Spec}(B') \) and \( V' = \text{Spec}(A') \). Then we get a commutative diagram

\[
\begin{array}{ccc}
0 & \to & IB' & \to & B' & \to & B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & IA' & \to & A' & \to & A & \to & 0
\end{array}
\]

with exact rows. Moreover \( IB' \cong I \otimes_R B \), see proof of Lemma 10.1. (II) The morphisms \( X \to X' \) and \( Y \to Y' \) are universal homeomorphisms. Hence the topology of the maps \( f \) and \( f' \) (after any base change) is identical. (III) If \( f \) is flat, then \( f' \) is flat and \( Y' \to S' \) is flat at every point in the image of \( f' \), see Lemma 10.2.

Ad (1). This is general remark (III).

Ad (2). Assume \( f \) is an isomorphism. By (III) we see that \( Y' \to S' \) is flat. Choose an affine open \( V' \subset Y' \) and set \( U' = (f')^{-1}(V') \). Then \( V = Y \cap V' \) is affine which implies that \( V \cong f^{-1}(V) = U = Y \times_{Y'} U' \) is affine. By Lemma 2.3, we see that \( U' \) is affine. Thus we have a diagram as in the general remark (I) and moreover \( IA \cong I \otimes_R A \) because \( R' \to A' \) is flat. Then \( IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA' \) and \( A \cong B \). By the exactness of the rows in the diagram above we see that \( A' \cong B' \), i.e., \( U' \cong V' \). Thus \( f' \) is an isomorphism.

Ad (3). Assume \( f \) is an open immersion. Then \( f \) is an isomorphism of \( X \) with an open subscheme \( V \subset Y \). Let \( V' \subset Y' \) be the open subscheme whose underlying topological space is \( V \). Then \( f' \) is a map from \( X' \) to \( V' \) which is an isomorphism by (2). Hence \( f' \) is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 3.1 for a more general statement.
Ad (6). Note that $X \times_Y X = Y \times_Y (X' \times_Y X')$ so that $X' \times_Y X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 3.1 for a more general statement.

Ad (7). Assume $f$ is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_Y X'$. The base change of $\Delta_{X/Y}$ by $S \to S'$ is $\Delta_{X'/Y'}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (10). Assume $f$ is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_Y U'$ is affine. By Lemma 2.3, we see that $U'$ is affine. Hence $f'$ is affine. See also Lemma 3.1 for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in $B$ generate $B$ as an $A$-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma 3.3 for a more general statement.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 19.6. See also Lemma 3.3 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal $K'$ by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As $B'$ is flat over $R'$ we see that $K' \otimes_R R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since $I$ is nilpotent we see that $f'_1, \ldots, f'_m$ generate $K'$ by Nakayama’s lemma, see Algebra, Lemma 19.1. See also Lemma 3.3 for a more general statement.

Ad (14). Follows from (11) and general remark (II). See also Lemma 3.3 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (16). Assume $f$ is syntomic. By (13) $f'$ is locally of finite presentation, by general remark (III) $f'$ is flat and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is syntomic by Morphisms, Lemma 29.11.

Ad (17). Assume $f$ is smooth. By (13) $f'$ is locally of finite presentation, by general remark (III) $f'$ is flat and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is smooth by Morphisms, Lemma 32.3.

Ad (18). Assume $f$ unramified. By (11) $f'$ is locally of finite type and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is unramified by Morphisms, Lemma 33.12. See also Lemma 3.3 for a more general statement.
Ad (19). Assume \( f \) étale. By (13) \( f' \) is locally of finite presentation, by general remark (III) \( f' \) is flat, and the fibres of \( f' \) are the fibres of \( f \). Hence \( f' \) is étale by Morphisms, Lemma 34.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 3.3 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 42.7. See also Lemma 3.1 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 42.4. See also Lemma 3.3 for a more general statement.

Ad (23). Assume \( f \) finite locally free. By (22) we see that \( f' \) is finite, by general remark (III) \( f' \) is flat, and by (13) \( f' \) is locally of finite presentation. Hence \( f' \) is finite locally free by Morphisms, Lemma 46.2.

The following lemma is the “locally nilpotent” version of the “critère de platitude par fibres”, see Section 16.

**Lemma 10.4.** Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \stackrel{(f,f')}{\longrightarrow} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & & 
\end{array}
\]

of thickenings. Assume

1. \( Y' \to S' \) is locally of finite type,
2. \( X' \to S' \) is flat and locally of finite presentation,
3. \( f \) is flat, and
4. \( X = S \times_{S'} X' \) and \( Y = S \times_{S'} Y' \).

Then \( f' \) is flat and for all \( y' \in Y' \) in the image of \( f' \) the local ring \( O_{Y',y'} \) is flat and essentially of finite presentation over \( O_{S',s'} \).

**Proof.** Immediate consequence of Algebra, Lemma 127.10

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

**Lemma 10.5.** Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \stackrel{(f,f')}{\longrightarrow} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & & 
\end{array}
\]

of thickenings. Assume \( Y' \to S' \) locally of finite type, \( X' \to S' \) flat and locally of finite presentation, \( X = S \times_{S'} X' \), and \( Y = S \times_{S'} Y' \). Then

- (1) \( f \) is flat if and only if \( f' \) is flat,
- (2) \( f \) is an isomorphism if and only if \( f' \) is an isomorphism,
- (3) \( f \) is an open immersion if and only if \( f' \) is an open immersion,
- (4) \( f \) is quasi-compact if and only if \( f' \) is quasi-compact,
- (5) \( f \) is universally closed if and only if \( f' \) is universally closed,
f is (quasi-)separated if and only if $f'$ is (quasi-)separated,
(7) $f$ is a monomorphism if and only if $f'$ is a monomorphism,
(8) $f$ is surjective if and only if $f'$ is surjective,
(9) $f$ is universally injective if and only if $f'$ is universally injective,
(10) $f$ is affine if and only if $f'$ is affine,
(11) $f$ is locally quasi-finite if and only if $f'$ is locally quasi-finite,
(12) $f$ is locally of finite type of relative dimension $d$ if and only if $f'$ is locally of finite type of relative dimension $d$,
(13) $f$ is universally open if and only if $f'$ is universally open,
(14) $f$ is syntomic if and only if $f'$ is syntomic,
(15) $f$ is smooth if and only if $f'$ is smooth,
(16) $f$ is unramified if and only if $f'$ is unramified,
(17) $f$ is étale if and only if $f'$ is étale,
(18) $f$ is proper if and only if $f'$ is proper,
(19) $f$ is finite if and only if $f'$ is finite,
(20) $f$ is finite locally free (of rank $d$) if and only if $f'$ is finite locally free (of rank $d$), and
(21) add more here.

Proof. The assumptions on $X$ and $Y$ mean that $f$ is the base change of $f'$ by $X \to X'$. The properties $P$ listed in (1) – (20) above are all stable under base change, hence if $f'$ has property $P$, then so does $f$. See Schemes, Lemmas 18.2, 19.3, 21.12 and 23.5 and Morphisms, Lemmas 9.4, 10.4, 11.8, 19.13, 28.2, 29.4, 32.5, 33.5, 34.4, 39.5, 42.6, and 46.4.

The interesting direction in each case is therefore to assume that $f$ has the property and deduce that $f'$ has it too. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over $W'$ with $f'(U') \subset V'$. Let $W' = \text{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \text{Spec}(B')$ and $V' = \text{Spec}(A')$. Then we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

with exact rows. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps $f$ and $f'$ (after any base change) is identical. (III) If $f$ is flat, then $f'$ is flat and $Y' \to S'$ is flat at every point in the image of $f'$, see Lemma 10.2.

Ad (1). This is general remark (III).

Ad (2). Assume $f$ is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 2.3 we see that $U'$ is affine. Thus we have a diagram as in the general remark (I). By Algebra, Lemma 125.10 we see that $A' \to B'$ is an isomorphism, i.e., $U' \cong V'$. Thus $f'$ is an isomorphism.
Ad (3). Assume \( f \) is an open immersion. Then \( f \) is an isomorphism of \( X \) with an open subscheme \( V \subset Y \). Let \( V' \subset Y' \) be the open subscheme whose underlying topological space is \( V \). Then \( f' \) is a map from \( X' \) to \( V' \) which is an isomorphism by (2). Hence \( f' \) is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (6). Note that \( X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X') \) so that \( X' \times_{Y'} X' \) is a thickening of \( X \times_Y X \). Hence the topology of the maps \( \Delta_{X/Y} \) and \( \Delta_{X'/Y'} \) matches and we win. See also Lemma 3.1 for a more general statement.

Ad (7). Assume \( f \) is a monomorphism. Consider the diagonal morphism \( \Delta_{X'/Y'} : X' \to X' \times_{Y'} X' \). Observe that \( X' \times_{Y'} X' \to S' \) is locally of finite type. The base change of \( \Delta_{X'/Y'} \) by \( S \to S' \) is \( \Delta_{X/Y} \) which is an isomorphism by assumption. By (2) we conclude that \( \Delta_{X'/Y'} \) is an isomorphism.

Ad (8). This is clear. See also Lemma 3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (10). Assume \( f \) is affine. Choose an affine open \( V' \subset Y' \) and set \( U' = (f')^{-1}(V') \). Then \( V = Y \cap V' \) is affine which implies that \( U = Y \times_Y U' \) is affine. By Lemma 2.3 we see that \( U' \) is affine. Hence \( f' \) is affine. See also Lemma 3.1 for a more general statement.

Ad (11). Follows from the fact that \( f' \) is locally of finite type (by Morphisms, Lemma 14.8) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 19.6.

Ad (12). Follows from general remark (II) and the fact that \( f' \) is locally of finite type (Morphisms, Lemma 14.8).

Ad (13). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (14). Assume \( f \) is syntomic. By Morphisms, Lemma 20.11 \( f' \) is locally of finite presentation. By general remark (III) \( f' \) is flat. The fibres of \( f' \) are the fibres of \( f \). Hence \( f' \) is syntomic by Morphisms, Lemma 29.11

Ad (15). Assume \( f \) is smooth. By Morphisms, Lemma 20.11 \( f' \) is locally of finite presentation. By general remark (III) \( f' \) is flat. The fibres of \( f' \) are the fibres of \( f \). Hence \( f' \) is smooth by Morphisms, Lemma 32.3

Ad (16). Assume \( f \) unramified. By Morphisms, Lemma 14.8 \( f' \) is locally of finite type. The fibres of \( f' \) are the fibres of \( f \). Hence \( f' \) is unramified by Morphisms, Lemma 33.12

Ad (17). Assume \( f \) étale. By Morphisms, Lemma 20.11 \( f' \) is locally of finite presentation. By general remark (III) \( f' \) is flat. The fibres of \( f' \) are the fibres of \( f \). Hence \( f' \) is étale by Morphisms, Lemma 34.8

Ad (18). This follows from a combination of (6), the fact that \( f \) is locally of finite type (Morphisms, Lemma 14.8), (4), and (5).
Ad (19). Assume $f$ finite locally free. By (19) we see that $f'$ is finite. By general remark (III) $f'$ is flat. By Morphisms, Lemma 20.11 $f'$ is locally of finite presentation. Hence $f'$ is finite locally free by Morphisms, Lemma 46.2.

0D4F \textbf{Lemma 10.6 (Deformations of projective schemes).} Let $f : X \to S$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $\mathcal{L}$ be $f$-ample. Assume $S$ is quasi-compact. There exists a $d_0 \geq 0$ such that for every cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{i} & S'
\end{array}
$$

where $S \subset S'$ is a thickening and $f'$ is proper, flat, of finite presentation we have

1. $R^p(f')_* (\mathcal{L}')^\otimes d = 0$ for all $p > 0$ and $d \geq d_0$.
2. $A'_d = (f')_* (\mathcal{L}')^\otimes d$ is finite locally free for $d \geq d_0$.
3. $A' = \mathcal{O}_S \oplus \bigoplus_{d \geq d_0} A'_d$ is a quasi-coherent $\mathcal{O}_S$-algebra of finite presentation,
4. there is a canonical isomorphism $r' : X' \to \text{Proj}_{X'}(A')$, and
5. there is a canonical isomorphism $\theta' : (r')^* \mathcal{O}_{\text{Proj}_{X'}(A')}(1) \to \mathcal{L}'$.

The construction of $A'$, $r'$, $\theta'$ is functorial in the data $(X', S', i', i', f', \mathcal{L}')$.

\textbf{Proof.} We first describe the maps $r'$ and $\theta'$. Observe that $\mathcal{L}'$ is $f'$-ample. There is a canonical map of quasi-coherent graded $\mathcal{O}_S$-algebras $A' : \bigoplus_{d \geq 0} (f')_* (\mathcal{L}')^\otimes d$ which is an isomorphism in degrees $\geq d_0$. Hence this induces an isomorphism on relative $\text{Proj}$ compatible with the Serre twists of the structure sheaf, see Constructions, Lemma 18.4. Hence we get the morphism $r'$ by Morphisms, Lemma 35.4 (which in turn appeals to the construction given in Constructions, Lemma 19.1) and it is an isomorphism by Morphisms, Lemma 41.17. We get the map $\theta'$ from Constructions, Lemma 19.1. By Properties, Lemma 28.2 we find that $\theta'$ is an isomorphism (this also uses that the morphism $r'$ over affine opens of $S'$ is the same as the morphism from Properties, Lemma 26.9 as is explained in the proof of Morphisms, Lemma 41.17).

Assuming the vanishing and local freeness stated in parts (1) and (2), the functoriality of the construction can be seen as follows. Suppose that $h : T \to S'$ is a morphism of schemes, denote $f_T : X'_T \to T$ the base change of $f'$ and $\mathcal{L}_T$ the pullback of $\mathcal{L}$ to $X'_T$. By cohomology and base change (as formulated in Derived Categories of Schemes, Lemma 21.3 for example) we have the corresponding vanishing over $T$ and moreover $h^* A'_d = f_T^* \mathcal{L}_T^{\otimes d}$ (and thus the local freeness of pushforwards as well as the finite generation of the corresponding graded $\mathcal{O}_T$-algebra $A_T$). Hence the morphism $r_T : X_T \to \text{Proj}_T(\bigoplus f_T^* \mathcal{L}_T^{\otimes d})$ is simply the base change of $r'$ to $T$ and the pullback of $\theta'$ is the map $\theta_T$.

Having said all of the above, we see that it suffices to prove (1), (2), and (3). Pick $d_0$ such that $R^p f_* \mathcal{L}^{\otimes d} = 0$ for all $d \geq d_0$ and $p > 0$, see Cohomology of Schemes, Lemma 16.1. We claim that $d_0$ works.

By cohomology and base change (Derived Categories of Schemes, Lemma 27.4) we see that $E'_d = Rf'_* (\mathcal{L}')^\otimes d$ is a perfect object of $D(\mathcal{O}_{S'})$ and its formation commutes
with arbitrary base change. In particular, \( E_d = \text{Li}^* E'_d = R f_* L^{\otimes d} \). By Derived Categories of Schemes, Lemma 29.3 we see that for \( d \geq d_0 \) the complex \( E_d \) is isomorphic to the finite locally free \( S \)-module \( f_* L^{\otimes d} \) placed in cohomological degree 0. Then by Derived Categories of Schemes, Lemma 28.3 we conclude that \( E'_d \) is isomorphic to a finite locally free module placed in cohomological degree 0. Of course this means that \( E'_d = \mathcal{A}'_d[0] \), that \( R^p f'_*(L')^{\otimes d} = 0 \) for \( p > 0 \), and that \( \mathcal{A}'_d \) is finite locally free. This proves (1) and (2).

The last thing we have to show is finite presentation of \( \mathcal{A}' \) as a sheaf of \( \mathcal{O}_S \)-algebras (this notion was introduced in Properties, Section 22). Let \( U' = \text{Spec}(R') \subset S' \) be an affine open. Then \( \mathcal{A}' = \mathcal{A}'(U') \) is a graded \( R' \)-algebra whose graded parts are finite projective \( R' \)-modules. We have to show that \( \mathcal{A}' \) is a finitely presented \( R' \)-algebra. We will prove this by reduction to the Noetherian case. Namely, we can find a finite type \( \mathbf{Z} \)-subalgebra \( R'_0 \subset R' \) and a pair \( (X'_0, L'_0) \) over \( R'_0 \) whose base change is \( (X'_{1'}, L'_{1'}) \), see Lemmas 10.3, 13.1, 8.7 and 4.15. Cohomology of Schemes, Lemma 16.1 implies \( \mathcal{A}'_0 = \bigoplus_{d \geq 0} H^0(X'_0, (L'_0)^{\otimes d}) \) is a finitely generated graded \( R'_0 \)-algebra and implies there exists a \( d'_0 \) such that \( H^p(X'_0, (L'_0)^{\otimes d}) = 0 \), \( p > 0 \) for \( d \geq d'_0 \). By the arguments given above applied to \( X'_0 \to \text{Spec}(R'_0) \) and \( L'_0 \) we see that \( \mathcal{A}'_d \) is a finite projective \( R'_0 \)-module and that

\[
\mathcal{A}'_d = \mathcal{A}'_d(U') = H^0(X'_{1'}, (L'_{1'})^{\otimes d}|X'_{1'}) = H^0(X'_0, (L'_0)^{\otimes d}) \otimes_{R'_0} R' = (\mathcal{A}'_0)_d \otimes_{R'_0} R'
\]

for \( d \geq d'_0 \). Now a small twist in the argument is that we don’t know that we can choose \( d'_0 \) equal to \( d'_0^\square \). To get around this we use the following sequence of arguments to finish the proof:

(a) The algebra \( B = R'_0 \oplus \bigoplus_{d \geq \max(d_0, d'_0)} (\mathcal{A}'_0)_d \) is an \( R'_0 \)-algebra of finite type:

- apply the Artin-Tate lemma to \( B \subset A'_0 \), see Algebra, Lemma 50.7.
(b) As \( R'_0 \) is Noetherian we see that \( B \) is an \( R'_0 \)-algebra of finite presentation.
(c) By right exactness of tensor product we see that \( B \otimes_{R'_0} R' \) is an \( R' \)-algebra of finite presentation.
(d) By the displayed equalities this exactly says that \( C = R' \oplus \bigoplus_{d \geq \max(d_0, d'_0)} A'_d \) is an \( R' \)-algebra of finite presentation.
(e) The quotient \( A'/C \) is the direct sum of the finite projective \( R' \)-modules \( A'_d \), \( d_0 \leq d \leq \max(d_0, d'_0) \), hence finitely presented as \( R' \)-module.
(f) The quotient \( A'/C \) is finitely presented as a \( C \)-module by Algebra, Lemma 6.4.
(g) Thus \( A' \) is finitely presented as a \( C \)-module by Algebra, Lemma 5.3.
(h) By Algebra, Lemma 5.4 this implies \( A' \) is finitely presented as a \( C \)-algebra.
(i) Finally, by Algebra, Lemma 6.2 applied to \( R' \to C \to A' \) this implies \( A' \) is finitely presented as an \( R' \)-algebra.

This finishes the proof.

\[\square\]

### 11. Formally smooth morphisms

Michael Artin’s position on differential criteria of smoothness (e.g., Morphisms, Lemma 32.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism \( X \to S \). Such a morphism

\[1\text{With the same properties as those enjoyed by } X' \to S' \text{ and } L', \text{ i.e., } X'_0 \to \text{Spec}(R'_0) \text{ is flat and proper and } L'_0 \text{ is ample.}\]

\[2\text{Actually, one can reduce to this case by doing more limit arguments.}\]
is characterized by the property that $T$-valued points of $X$ lift to infinitesimal thickenings of $T$ provided $T$ is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 11.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called \textit{formally smooth} (see Algebra, Definition \ref{alg-def-form-smooth}) if for every commutative solid diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B/I \\
\downarrow & & \downarrow \\
R & \longrightarrow & B
\end{array}
\]

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

**Definition 11.1.** Let $f : X \to S$ be a morphism of schemes. We say $f$ is \textit{formally smooth} if given any solid commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & T \\
\uparrow & & \uparrow \\
f & \nearrow & i \\
S & \leftarrow & T'
\end{array}
\]

where $T \subset T'$ is a first order thickening of affine schemes over $S$ there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that $T'$ be affine could be dropped, see Lemmas 6.2 and 8.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on $T'$. In fact, analyzing the proof of Lemma 11.10 shows that this would be equivalent to the definition as it currently stands. In particular, the being formally smooth is Zariski local on the source (and in fact it is smooth local on the soure, insert future reference here).

**Lemma 11.2.** A composition of formally smooth morphisms is formally smooth.

\textbf{Proof.} Omitted.

**Lemma 11.3.** A base change of a formally smooth morphism is formally smooth.

\textbf{Proof.} Omitted, but see Algebra, Lemma\ref{alg-lem-base-change-form-smooth} for the algebraic version.

**Lemma 11.4.** Let $f : X \to S$ be a morphism of schemes. Then $f$ is formally étale if and only if $f$ is formally smooth and formally unramified.

\textbf{Proof.} Omitted.

**Lemma 11.5.** Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If $f$ is formally smooth, so is $f|_U : U \to V$. 
Proof. Consider a solid diagram

\[
\begin{array}{c}
U \\
\downarrow f|_U \\
\downarrow V \\
\end{array}
\begin{array}{ccc}
& a & \\
\downarrow & \downarrow i & \\
T & \downarrow & T' \\
\end{array}
\]

as in Definition 11.1. If \( f \) is formally smooth, then there exists an \( S \)-morphism \( a' : T' \rightarrow X \) such that \( a'|_T = a \). Since the underlying sets of \( T \) and \( T' \) are the same we see that \( a' \) is a morphism into \( U \) (see Schemes, Section 3). And it clearly is a \( V \)-morphism as well. Hence the dotted arrow above as desired. □

Lemma 11.6. Let \( f : X \rightarrow S \) be a morphism of schemes. Assume \( X \) and \( S \) are affine. Then \( f \) is formally smooth if and only if \( \mathcal{O}_S(S) \rightarrow \mathcal{O}_X(X) \) is a formally smooth ring map.

Proof. This is immediate from the definitions (Definition 11.1 and Algebra, Definition 137.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. □

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 6.1) that we can recognize whether a morphism \( f : X \rightarrow S \) is smooth in terms of “simple” properties of the functor \( h_X : \text{Sch}/S \rightarrow \text{Sets} \).

Lemma 11.7 (Infinitesimal lifting criterion). Let \( f : X \rightarrow S \) be a morphism of schemes. The following are equivalent:

1. The morphism \( f \) is smooth, and
2. the morphism \( f \) is locally of finite presentation and formally smooth.

Proof. Assume \( f : X \rightarrow S \) is locally of finite presentation and formally smooth. Consider a pair of affine opens \( \text{Spec}(A) = U \subset X \) and \( \text{Spec}(R) = V \subset S \) such that \( f(U) \subset V \). By Lemma 11.5 we see that \( U \rightarrow V \) is formally smooth. By Lemma 11.6 we see that \( R \rightarrow A \) is formally smooth. By Morphisms, Lemma 20.2 we see that \( R \rightarrow A \) is of finite presentation. By Algebra, Proposition 137.13 we see that \( R \rightarrow A \) is smooth. Hence by the definition of a smooth morphism we see that \( X \rightarrow S \) is smooth.

Conversely, assume that \( f : X \rightarrow S \) is smooth. Consider a solid commutative diagram

\[
\begin{array}{c}
X \\
\downarrow f \\
S \\
\end{array}
\begin{array}{ccc}
& T & \\
\downarrow & \downarrow i & \\
T' & \downarrow & \\
\end{array}
\]

as in Definition 11.1. We will show the dotted arrow exists thereby proving that \( f \) is formally smooth.

Let \( \mathcal{F} \) be the sheaf of sets on \( T' \) of Lemma 9.4 in the special case discussed in Remark 9.6. Let

\[
\mathcal{H} = \text{Hom}_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})
\]

be the sheaf of \( \mathcal{O}_T \)-modules with action \( a^*\Omega_{X/S}, \mathcal{C}_{T/T'} \) as in Lemma 9.5. Our goal is simply to show that \( \mathcal{F}(T) \neq \emptyset \). In other words we are trying to show that \( \mathcal{F} \) is a trivial \( \mathcal{H} \)-torsor on \( T \) (see Cohomology, Section 4). There are two steps: (I) To
show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 4.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 4.3) we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 13.3.

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of $t$ such that $a(U)$ is contained in an affine open $\text{Spec}(A) = W \subset X$ which maps to an affine open $\text{Spec}(R) = V \subset S$. By Morphisms, Lemma 32.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 137.13 the ring map $R \to A$ is formally smooth. Lemma 11.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a $V$-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 31.13 we see that $\Omega_X/S$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 32.12). Hence $\mathcal{F}$ is quasi-coherent by the discussion in Schemes, Section 21. Thus by Cohomology of Schemes, Lemma 2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired.

Locally projective quasi-coherent modules are defined in Properties, Section 21.

**Lemma 11.8.** Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on $X$.

**Proof.** Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 11.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 11.6 Hence by Algebra, Lemma 137.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega(U, \mathcal{O}_U)|_{\Gamma(U, \mathcal{O}_U)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 21.

**Lemma 11.9.** Let $T$ be an affine scheme. Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules. Consider $\mathcal{H} = \text{Hom}_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ is locally projective, then $H^1(T, \mathcal{H}) = 0$.

**Proof.** By the definition of a locally projective sheaf on a scheme (see Properties, Definition 21.1) we see that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology, Lemma 11.12 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Cohomology of Schemes, Lemma 2.2.

**Lemma 11.10.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. $f$ is formally smooth,
2. for every $x \in X$ there exist opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth,
3. for every pair of affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is formally smooth, and
4. there exists an affine open covering $Y = \bigcup V_j$ and for each $j$ an affine open covering $f^{-1}(V_j) = \bigcup U_{{ij}}$ such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a formally smooth ring map for all $j$ and $i$.

**Proof.** The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), and (2) $\Rightarrow$ (4) follow from Lemma 11.5. The implication (3) $\Rightarrow$ (4) is immediate.
Assume (4). The proof that $f$ is formally smooth is the same as the second part of the proof of Lemma 11.7. Consider a solid commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{a} & T \\
\downarrow{f} & & \downarrow{i} \\
Y & \xleftarrow{i} & T'
\end{array}
$$

as in Definition 11.1. We will show the dotted arrow exists thereby proving that $f$ is formally smooth. Let $F$ be the sheaf of sets on $T'$ of Lemma 9.4 as in the special case discussed in Remark 9.6. Let

$\mathcal{H} = \text{Hom}_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$

be the sheaf of $\mathcal{O}_T$-modules on $T$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 9.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $F$ into a pseudo $\mathcal{H}$-torsor, see Cohomology, Definition 4.1. Our goal is to show that $F$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $F$ is a torsor we have to show that $F$ locally has a section. (II) To show that $F$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$, see Cohomology, Lemma 4.3.

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $W \subset T$ neighbourhood of $t$ such that $a(W)$ is contained in $U_{ji}$ for some $i, j$. Let $W' \subset T'$ be the corresponding open subscheme. By assumption (4) we can lift $a|_W : W \to U_{ji}$ to a $V_j$-morphism $a' : W' \to U_{ji}$ showing that $F(W')$ is nonempty.

Finally we prove (II). By Lemma 11.8 we see that $\Omega_{U_{ji}/V_j}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.2. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.3. Hence

$H^1(T, \mathcal{H}) = H^1(T, \text{Hom}_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})) = 0$

by Lemma 11.9 as desired. □

06B6 **Lemma 11.11.** Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume $f$ is formally smooth. Then

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 31.9) is short exact.

**Proof.** The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 130.7 is exact. This is Algebra, Lemma 137.9. □

06B7 **Lemma 11.12.** Let $h : Z \to X$ be a formally unramified morphism of schemes over $S$. Assume that $Z$ is formally smooth over $S$. Then the canonical exact sequence

$$0 \to C_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 7.10 is short exact.

**Proof.** Let $Z \to Z'$ be the universal first order thickening of $Z$ over $X$. From the proof of Lemma 7.10 we see that our sequence is identified with the sequence

$$C_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$
Since $Z \to S$ is formally smooth we can locally on $Z'$ find a left inverse $Z' \to Z$ over $S$ to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 31.16. □

Lemma 11.13. Let

\[
\begin{array}{ccc}
Z & \to & X \\
\downarrow^i & & \downarrow^f \\
\downarrow_j & & \\
Z' & \to & Y
\end{array}
\]

be a commutative diagram of schemes where $i$ and $j$ are formally unramified and $f$ is formally smooth. Then the canonical exact sequence

\[0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0\]

of Lemma 7.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of $Z$ over $X$. Denote $Z \to Z''$ the universal first order thickening of $Z$ over $Y$. By Lemma 7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

\[
\begin{array}{ccc}
Z & \to & Z' \\
\downarrow^i & & \downarrow^a \\
\downarrow^j & & \downarrow^f \\
Z'' & \to & Y
\end{array}
\]

In the proof of Lemma 7.11 we identified the sequence above with the sequence

\[\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0\]

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As $f$ is formally smooth there exists a morphism $h : U'' \to X$ which agrees with $i$ on $U$ and such that $f \circ h$ equals $b_{|U''}$. Since $Z'$ is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of $Z''$ implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence $g$ is a left inverse to $k$. Picture

\[
\begin{array}{ccc}
U & \to & Z' \\
\downarrow^g & & \downarrow^k \\
U'' & \to & Z''
\end{array}
\]

Thus $g$ induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over $U$. □

12. Smoothness over a Noetherian base

It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

Lemma 12.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that $S$ is locally Noetherian and $f$ locally of finite type. The following are equivalent:

1) $f$ is smooth at $x$,..
(2) for every solid commutative diagram

\[
\begin{array}{c}
X \\ f
\end{array} \xleftarrow{\alpha} \begin{array}{c}
\text{Spec}(B) \\ i
\end{array} \\
\begin{array}{c}
S \\ \beta
\end{array} \xleftarrow{} \begin{array}{c}
\text{Spec}(B')
\end{array}
\]

where \( B' \to B \) is a surjection of local rings with \( \text{Ker}(B' \to B) \) of square zero, and \( \alpha \) mapping the closed point of \( \text{Spec}(B) \) to \( x \) there exists a dotted arrow making the diagram commute,

(3) same as in (2) but with \( B' \to B \) ranging over small extensions (see Algebra, Definition [14.0.1]), and

(4) same as in (2) but with \( B' \to B \) ranging over small extensions such that \( \alpha \) induces an isomorphism \( \kappa(x) \to \kappa(m) \) where \( m \subset B \) is the maximal ideal.

**Proof.** Choose an affine neighbourhood \( V \subset S \) of \( f(x) \) and choose an affine neighbourhood \( U \subset X \) of \( x \) such that \( f(U) \subset V \). For any “test” diagram as in (2) the morphism \( \alpha \) will map \( \text{Spec}(B) \) into \( U \) and the morphism \( \beta \) will map \( \text{Spec}(B') \) into \( V \) (see Schemes, Section [13]). Hence the lemma reduces to the morphism \( f|_U : U \to V \) of affines. (Indeed, \( V \) is Noetherian and \( f|_U \) is of finite type, see Properties, Lemma [5.2] and Morphisms, Lemma [14.2]) In this affine case the lemma is identical to Algebra, Lemma [140.2]. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions “centered” at points of finite type (see Morphisms, Section [15]).

**02HY Lemma 12.2.** Let \( f : X \to S \) be a morphism of schemes. Assume that \( S \) is locally Noetherian and \( f \) locally of finite type. The following are equivalent:

(1) \( f \) is smooth,

(2) for every solid commutative diagram

\[
\begin{array}{c}
X \\ f
\end{array} \xleftarrow{\alpha} \begin{array}{c}
\text{Spec}(B) \\ i
\end{array} \\
\begin{array}{c}
S \\ \beta
\end{array} \xleftarrow{} \begin{array}{c}
\text{Spec}(B')
\end{array}
\]

where \( B' \to B \) is a small extension of Artinian local rings and \( \beta \) of finite type (!) there exists a dotted arrow making the diagram commute.

**Proof.** If \( f \) is smooth, then the infinitesimal lifting criterion (Lemma [11.7]) says \( f \) is formally smooth and (2) holds.

Assume (2). The set of points \( x \in X \) where \( f \) is not smooth forms a closed subset \( T \) of \( X \). By the discussion in Morphisms, Section [15] if \( T \neq \emptyset \) there exists a point \( x \in T \subset X \) such that the morphism

\[ \text{Spec}(\kappa(x)) \to X \to S \]

is of finite type (namely, pick any point \( x \) of \( T \) which is closed in an affine open of \( X \)). By Morphisms, Lemma [15.2] given any local Artinian ring \( B' \) with residue field \( \kappa(x) \) then any morphism \( \beta : \text{Spec}(B') \to S \) is of finite type. Thus we see that all the diagrams used in Lemma [12.1] (4) correspond to diagrams as in the current lemma (2). Whence \( X \to S \) is smooth a \( x \) a contradiction. □
Here is a useful application.

**Lemma 12.3.** Let \( f : X \to S \) be a finite type morphism of locally Noetherian schemes. Let \( Z \subset S \) be a closed subscheme with \( n \)th infinitesimal neighbourhood \( Z_n \subset S \). Set \( X_n = Z_n \times_S X \).

1. If \( X_n \to Z_n \) is smooth for all \( n \), then \( f \) is smooth at every point of \( f^{-1}(Z) \).
2. If \( X_n \to Z_n \) is étale for all \( n \), then \( f \) is étale at every point of \( f^{-1}(Z) \).

**Proof.** Assume \( X_n \to Z_n \) is smooth for all \( n \). Let \( x \in X \) be a point lying over a point of \( Z \). Given a small extension \( B' \to B \) and morphisms \( \alpha, \beta \) as in Lemma 12.1 part (3) the maximal ideal of \( B' \) is nilpotent (as \( B' \) is Artinian) and hence the morphism \( \beta \) factors through \( Z_n \) and \( \alpha \) factors through \( X_n \) for a suitable \( n \). Thus the lifting property for \( X_n \to Z_n \) kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. \( \square \)

**Lemma 12.4.** Let \( f : X \to S \) be a morphism of locally Noetherian schemes. Let \( Z \subset S \) be a closed subscheme with \( n \)th infinitesimal neighbourhood \( Z_n \subset S \). Set \( X_n = Z_n \times_S X \). If \( X_n \to Z_n \) is flat for all \( n \), then \( f \) is flat at every point of \( f^{-1}(Z) \).

**Proof.** This is a translation of Algebra, Lemma 98.11 into the language of schemes. \( \square \)

### 13. The naive cotangent complex

This section is the continuation of Modules, Section 28 which in turn continues the discussion in Algebra, Section 133.

**Definition 13.1.** Let \( f : X \to Y \) be a morphism of schemes. The naive cotangent complex of \( f \) is the complex defined in Modules, Definition 28.6. Notation: \( NL_f \) or \( NL_{X/Y} \).

**Lemma 13.2.** Let \( f : X \to Y \) be a morphism of schemes. Let \( \text{Spec}(A) = U \subset X \) and \( \text{Spec}(R) = V \subset S \) be affine opens with \( f(U) \subset V \). There is a canonical map

\[
\tilde{NL}_{A/R} \to NL_{X/Y}|_U
\]

of complexes which is an isomorphism in \( D(\mathcal{O}_U) \).

**Proof.** From the construction of \( NL_{X/Y} \) in Modules, Section 28 we see there is a canonical map of complexes \( NL_{\mathcal{O}_X(U)} / f^{-1}\mathcal{O}_Y(U) \to NL_{X/Y}(U) \) of \( A = \mathcal{O}_X(U) \)-modules, which is compatible with further restrictions. Using the canonical map \( R \to f^{-1}\mathcal{O}_Y(U) \) we obtain a canonical map \( NL_{A/R} \to NL_{\mathcal{O}_X(U)} / f^{-1}\mathcal{O}_Y(U) \) of complexes of \( A \)-modules. Using the universal property of the \( ^- \) functor (see Schemes, Lemma 7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows from Algebra, Lemma 133.11 and Modules, Lemma 28.4 (and the description of the stalks of \( \mathcal{O}_X \) and \( f^{-1}\mathcal{O}_Y \) at a point \( p \in \text{Spec}(A) \) as \( A_p \) and \( R_q \) where \( q = R \cap p \); references used are Schemes, Lemma 5.4 and Sheaves, Lemma 21.5). \( \square \)

**Lemma 13.3.** Let \( f : X \to Y \) be a morphism of schemes. The cohomology sheaves of the complex \( NL_{X/Y} \) are quasi-coherent, zero outside degrees \(-1, 0\) and equal to \( \Omega_{X/Y} \) in degree 0.
\textbf{Proof.} By construction of the naive cotangent complex in Modules, Section 28 we have that $NL_{X/Y}$ is a complex sitting in degrees $-1, 0$ and that its cohomology in degree 0 is $\Omega_{X/Y}$. The sheaf of differentials is quasi-coherent (by Morphisms, Lemma 31.7). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking over affines using Lemma 13.2. □

\textbf{Lemma 13.4.} Let $f : X \to Y$ be a morphism of schemes. If $f$ is locally of finite presentation, then $NL_{X/Y}$ is locally on $X$ quasi-isomorphic to a complex

$$\ldots \to 0 \to F^{-1} \to F^0 \to 0 \to \ldots$$

of quasi-coherent $O_X$-modules with $F^0$ of finite presentation and $F^{-1}$ of finite type.

\textbf{Proof.} By Lemma 13.2 it suffices to show that $NL_{A/R}$ has this shape if $R \to A$ is a finitely presented ring map. Write $A = R[x_1, \ldots, x_n]/I$ with $I$ finitely generated. Then $I/I^2$ is a finite $A$-module and $NL_{A/R}$ is quasi-isomorphic to

$$\ldots \to 0 \to I/I^2 \to \bigoplus_{i=1}^n \text{Ad}x_i \to 0 \to \ldots$$

by Algebra, Section 133 and in particular Algebra, Lemma 133.2. □

\textbf{Lemma 13.5.} Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) $f$ is formally smooth,
(2) $H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.

\textbf{Proof.} This follows from Algebra, Proposition 137.8 and Lemma 11.10. □

\textbf{Lemma 13.6.} Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) $f$ is formally étale,
(2) $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.

\textbf{Proof.} A formally étale morphism is formally smooth and hence we have $H^{-1}(NL_{X/Y}) = 0$ by Lemma 13.5. On the other hand, we have $\Omega_{X/Y} = 0$ by Lemma 8.6. Conversely, if (2) holds, then $f$ is formally smooth by Lemma 13.5 and formally unramified by Lemma 6.7 and hence formally étale by Lemmas 11.4. □

\textbf{Lemma 13.7.} Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) $f$ is smooth, and
(2) $f$ is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.

\textbf{Proof.} This follows from the definition of a smooth ring homomorphism (Algebra, Definition 136.1), Lemma 13.2 and the definition of a smooth morphism of schemes (Morphisms, Definition 32.1). We also use that finite locally free is the same as finite projective for modules over rings (Algebra, Lemma 77.2). □

\textbf{Lemma 13.8.} Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a canonical six term exact sequence

$$H^{-1}(f^*NL_{Y/Z}) \to H^{-1}(NL_{X/Z}) \to H^{-1}(NL_{X/Y}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of cohomology sheaves.
Proof. Special case of Modules, Lemma \[28.5\]

\[\square\]

**Lemma 13.9.** Consider a cartesian diagram of schemes

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow^{g'} & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

If \(Y' \to Y\) is flat, then the canonical map \((g')^* \mathcal{NL}_{X/Y} \to \mathcal{NL}_{X'/Y'}\) is a quasi-isomorphism.

**Proof.** By Lemma 13.2 this follows from Algebra, Lemma 133.8

\[\square\]

**Lemma 13.10.** Consider a cartesian diagram of schemes

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow^{g'} & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

If \(X \to Y\) is flat, then the canonical map \((g')^* \mathcal{NL}_{X/Y} \to \mathcal{NL}_{X'/Y'}\) is a quasi-isomorphism. If in addition \(\mathcal{NL}_{X/Y}\) has tor-amplitude in \([-1,0]\) then \(L(g')^* \mathcal{NL}_{X/Y} \to \mathcal{NL}_{X'/Y'}\) is a quasi-isomorphism too.

**Proof.** Translated into algebra this is Divided Power Algebra, Lemma 11.5. To do the translation use Lemma 13.2 and Derived Categories of Schemes, Lemmas 3.5 and 9.4

\[\square\]

14. Pushouts in the category of schemes, I

In this section we construct pushouts of \(Y \leftarrow X \to X'\) where \(X \to Y\) is affine and \(X \to X'\) is a thickening. This will actually be an important case for us, hence a detailed discussion is merited. In Section 59 we discuss a more interesting and more difficult case. See Categories, Section 9 for a general discussion of pushouts in any category.

**Lemma 14.1.** Let \(A' \to A\) be a surjection of rings and let \(B \to A\) be a ring map. Let \(B' = B \times_A A'\) be the fibre product of rings. Set \(S = \text{Spec}(A)\), \(S' = \text{Spec}(A')\), \(T = \text{Spec}(B)\), and \(T' = \text{Spec}(B')\). Then

\[
\begin{array}{ccc}
S & \longrightarrow & S' \\
\downarrow^{i} & & \downarrow^{i'} \\
T & \longrightarrow & T'
\end{array}
\]

is a pushout of schemes.

**Proof.** By More on Algebra, Lemma 6.2 we have \(T' = T \amalg_S S'\) as topological spaces, i.e., the diagram is a pushout in the category of topological spaces. Next, consider the map

\[
((i')^*(f')^*) : \mathcal{O}_{T'} \longrightarrow i'_* \mathcal{O}_T \times g_* \mathcal{O}_S f'_* \mathcal{O}_{S'}
\]

where \(g = i' \circ f = f' \circ i\). We claim this map is an isomorphism of sheaves of rings. Namely, we can view both sides as quasi-coherent \(\mathcal{O}_T\)-modules (use Schemes, Lemmas 24.1 for the right hand side) and the map is \(\mathcal{O}_{T'}\)-linear. Thus it suffices to
show the map is an isomorphism on the level of global sections (Schemes, Lemma 7.5). On global sections we recover the identification $B' \to B \times_A A'$ from statement of the lemma (this is how we chose $B'$).

Let $X$ be a scheme. Suppose we are given morphisms of schemes $m' : S' \to X$ and $n : T \to X$ such that $m' \circ i = n \circ f$ (call this $m$). We get a unique map of topological spaces $n' : T' \to X$ compatible with $m'$ and $n$ as $T' = T \amalg_S S'$ (see above). By the description of $\mathcal{O}_{T'}$ in the previous paragraph we obtain a unique homomorphism of sheaves of rings

$$(n')^\sharp : \mathcal{O}_X \longrightarrow (n')_*\mathcal{O}_{T'} = m'_*\mathcal{O}_T \times_{m_*\mathcal{O}_S} n_*\mathcal{O}_S$$

given by $(m')^\circ$ and $n^\circ$. Thus $(n', (n')^\sharp)$ is the unique morphism of ringed spaces $T' \to X$ compatible with $m'$ and $n$. To finish the proof it suffices to show that $n'$ is a morphism of schemes, i.e., a morphism of locally ringed spaces.

Let $t' \in T'$ with image $x \in X$. We have to show that $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ is local. If $t' \notin T$, then $t'$ is the image of a unique point $s' \in S'$ and $\mathcal{O}_{T',t'} = \mathcal{O}_{S',s'}$. Namely, $S' \setminus S \to T' \setminus T$ is an isomorphism of schemes as $B' \to A'$ induces an isomorphism $\text{Ker}(B' \to B) = \text{Ker}(A' \to A)$. If $t'$ is the image of $t \in T$, then we know that the composition $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'} \to \mathcal{O}_{T,t}$ is local and we conclude also.

**Lemma 14.2.** Consider a commutative diagram of schemes

$$\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow j & & \downarrow a \\
Y & \xrightarrow{b} & W
\end{array}$$

and set $c = a \circ i = b \circ j$. If there exists an fpqc covering $\{W_i \to W\}$ such that for all $i$ and $i'$ the diagrams

$$\begin{array}{ccc}
Z \times_{c,W} W_i & \xrightarrow{j} & X \times_{a,W} W_i \\
\downarrow & & \downarrow \\
Y \times_{b,W} W_i & \xrightarrow{} & W_i
\end{array}$$

$$\begin{array}{ccc}
Z \times_{c,W} (W_i \times_W W_{i'}) & \xrightarrow{j} & X \times_{a,W} (W_i \times_W W_{i'}) \\
\downarrow & & \downarrow \\
Y \times_{b,W} (W_i \times_W W_{i'}) & \xrightarrow{} & (W_i \times_W W_{i'})
\end{array}$$

are cocartesian, then so is the original diagram.

**Proof.** Namely, for a scheme $T$ a morphism $W \to T$ is the same thing as a collection of morphism $W_i \to T$ which agree on overlaps, see Descent, Lemma 10.7.

**Lemma 14.3.** Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Then there exists a pushout

$$\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{} & Y'
\end{array}$$

in the category of schemes. Moreover, $Y \subset Y'$ is a thickening, $X = Y \times_{Y'} X'$, and $\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$ as sheaves on $|Y| = |Y'|$.  

MORE ON MORPHISMS 42
Proof. We first construct $Y'$ as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals $f$. As structure sheaf $\mathcal{O}_{Y'}$, we take the right hand side of the equation of the lemma. To see that $Y'$ is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume $Y$ is affine. Then $X$ is affine (as $f$ is affine) and $X'$ is affine as well (see Lemma 2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_A A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\text{Spec}(B') = \text{Spec}(B)$ (as topological spaces). We claim that $Y' = \text{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_g$. Namely, by More on Algebra, Lemma 5.3 we see that

$$(B')_g = B_g \times_{A_h} A'_h,$$

where $h \in A, h' \in A'$ are the images of $g'$. Since $B_g$, resp. $A_h$, resp. $A'_h$ is equal to $\mathcal{O}_Y(D(g))$, resp. $f_*\mathcal{O}_X(D(g))$, resp. $f'_*\mathcal{O}_{Y'}(D(g))$ the claim follows.

It remains to show that $Y'$ is the pushout. The discussion above shows the scheme $Y'$ has an affine open covering $Y' = \bigcup W'_i$ such that the corresponding opens $U'_i \subset X'$, $W_i \subset Y$, and $U_i \subset X$ are affine open. Moreover, if $A'_i, B_i, A_i$ are the rings corresponding to $U'_i, W_i, U_i$, then $W'_i$ corresponds to $B_i \times_{A_i} A'_i$. Thus we can apply Lemmas 14.1 and 14.2 to conclude our construction is a pushout in the category of schemes. 

\[\square\]

In the following lemma we use the fibre product of categories as defined in Categories, Example 30.3.

**Lemma 14.4.** Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_{X} X'$ be the pushout (see Lemma 14.3). Base change gives a functor

$$F : (\text{Sch}/Y') \to (\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$$

given by $V' \mapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which has a left adjoint

$$G : (\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X') \to (\text{Sch}/Y')$$

which sends the triple $(V, U', \varphi)$ to the pushout $V \amalg_{(V \times_{Y} X)} U'$. Finally, $F \circ G$ is isomorphic to the identity functor.

**Proof.** Let $(V, U', \varphi)$ be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_{Y'} X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies $U$ with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 11.8. Hence we can apply Lemma 14.3 to get a pushout $V' = V \amalg_{U} U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that $V'$ is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on $U$ as morphisms into $Y'$. Setting $G(V, U', \varphi) = V'$ gives the functor $G$.

Let us prove that $G$ is a left adjoint to $F$. Let $Z$ be a scheme over $Y'$. We have to show that

$$\text{Mor}(V', Z) = \text{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}$ the composition of $g'$ with the morphism $V \to V'$,
Let $g : V \to Z \times_Y Y$, resp. $f' : U' \to Z \times_Y X'$. Then $(g,f')$ is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g,f') : (V,U',\varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of $g$, resp. $f$ by $Z \times_Y X' \to Z$, resp. $Z \times_Y Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from $U$ to $Z$. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that $F \circ G$ is isomorphic to the identity we have to show that the adjunction mapping $(V,U',\varphi) \to F(G(V,U',\varphi))$ is an isomorphism. To do this we may work affine locally. Say $X = \text{Spec}(A)$, $X' = \text{Spec}(A')$, and $Y = \text{Spec}(B)$. Then $A' \to A$ and $B \to A$ are ring maps as in More on Algebra, Lemma 6.4 and $Y' = \text{Spec}(B')$ with $B' = B \times_A A'$. Next, suppose that $V = \text{Spec}(D)$, $U' = \text{Spec}(C')$ and $\varphi$ is given by an $A$-algebra isomorphism $D \otimes_B A \to C' \otimes_A A = C'/IC'$. Set $D' = D \times_{C'/IC'} C''$. In this case the statement we have to prove is that $D' \otimes_B A \cong D$ and $D' \otimes_B A' \cong C'$. This is a special case of More on Algebra, Lemma 6.4. 

**Lemma 14.5.** Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 14.3). Let $V' \to Y'$ be a morphism of schemes. Set $V = Y \times_Y V'$, $U' = X' \times_Y V'$, and $U = X \times_Y V'$. There is an equivalence of categories between

1. quasi-coherent $O_{V'}$-modules flat over $Y'$, and
2. the category of triples $(G,F',\varphi)$ where
   a. $G$ is a quasi-coherent $O_V$-module flat over $Y$,
   b. $F'$ is a quasi-coherent $O_{U'}$-module flat over $X'$, and
   c. $\varphi : (U \to V)^*G \to (U \to U')^*F'$ is an isomorphism of $O_U$-modules.

The equivalence maps $G'$ to $((V \to V')^*G', (U' \to V')^*G'$, can). Suppose $G'$ corresponds to the triple $(G,F',\varphi)$. Then

1. $G'$ is a finite type $O_{V'}$-module if and only if $G$ and $F'$ are finite type $O_V$ and $O_{U'}$-modules.
2. If $V' \to Y'$ is locally of finite presentation, then $G'$ is an $O_{V'}$-module of finite presentation if and only if $G$ and $F'$ are $O_V$ and $O_{U'}$-modules of finite presentation.

**Proof.** A quasi-inverse functor assigns to the triple $(G,F',\varphi)$ the fibre product $(V \to V'), (U \to U'), \varphi$ $V' \times_{(U \to U') F'}$. This works, because on affines we recover the equivalence of More on Algebra, Lemma 7.5. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 7.4 and 7.6. 

**Lemma 14.6.** In the situation of Lemma 14.4. If $V' = G(V,U',\varphi)$ for some triple $(V,U',\varphi)$, then

1. $V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,
2. $V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,
3. $V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,
4. $V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,
(5) $V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and
(6) add more here as needed.

If $W'$ is flat over $Y'$, then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence $F$ and $G$ define mutually quasi-inverse functors between the category of schemes flat over $Y'$ and the category of triples $(V, U', \varphi)$ with $V \to Y$ and $U' \to X'$ flat.

**Proof.** Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7.

15. Openness of the flat locus

This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

**Theorem 15.1.** Let $S$ be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $O_X$-module which is locally of finite presentation. Then

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in $X$.

**Proof.** We may test for openness locally on $X$ hence we may assume that $f$ is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 128.4.

**Lemma 15.2.** Let $S$ be a scheme. Let

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $O_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = g'(x')$.

(1) If $\mathcal{F}$ is flat over $S$ at $x$, then $(g')^* \mathcal{F}$ is flat over $S'$ at $x'$.

(2) If $g$ is flat at $s'$ and $(g')^* \mathcal{F}$ is flat over $S'$ at $x'$, then $\mathcal{F}$ is flat over $S$ at $x$.

In particular, if $g$ is flat, $f$ is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 15.1 commutes with base change.

**Proof.** Consider the commutative diagram of local rings

$$\begin{array}{ccc}
O_{X', x'} & \hookleftarrow & O_{X, x} \\
\uparrow & & \uparrow \\
O_{S', s'} & \hookleftarrow & O_{S, s}
\end{array}$$

Note that $O_{X', x'}$ is a localization of $O_{X, x} \otimes_{O_{S, x}} O_{S', s'}$, and that $((g')^* \mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes O_{X, x} O_{X', x'}$. Hence the lemma follows from Algebra, Lemma 99.1.
16. Critère de platitude par fibres

Consider a commutative diagram of schemes (left hand diagram)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{S} & & \downarrow{Y} \\
& \downarrow{Y_s} & \\
& \text{Spec}(\kappa(s)) &
\end{array}
\]

and a quasi-coherent \(O_X\)-module \(F\). Given a point \(x \in X\) lying over \(s \in S\) with image \(y = f(x)\) we consider the question: Is \(F\) flat over \(Y\) at \(x\)? If \(F\) is flat over \(S\) at \(x\), then the theorem states this question is intimately related to the question of whether the restriction of \(F\) to the fibre

\[F_s = (X_s \to X)^* F\]

is flat over \(Y_s\) at \(x\). Below you will find a “Noetherian” version, a “finitely presented” version, and earlier we treated a “nilpotent” version, see Lemma 10.2.

**Theorem 16.1.** Let \(S\) be a scheme. Let \(f : X \to Y\) be a morphism of schemes over \(S\). Let \(F\) be a quasi-coherent \(O_X\)-module. Let \(x \in X\). Set \(y = f(x)\) and \(s \in S\) the image of \(x\) in \(S\). Assume \(S, X, Y\) locally Noetherian, \(F\) coherent, and \(F_x \neq 0\).

Then the following are equivalent:

1. \(F\) is flat over \(S\) at \(x\), and \(F_s\) is flat over \(Y_s\) at \(x\), and
2. \(Y\) is flat over \(S\) at \(y\) and \(F\) is flat over \(Y\) at \(x\).

**Proof.** Consider the ring maps

\[O_{S,s} \to O_{Y,y} \to O_{X,x}\]

and the module \(F_x\). The stalk of \(F_s\) at \(x\) is the module \(F_x / m_s F_x\) and the local ring of \(Y_s\) at \(y\) is \(O_{Y,y} / m_s O_{Y,y}\). Thus the implication (1) \(\Rightarrow\) (2) is Algebra, Lemma 98.15. If (2) holds, then the first ring map is faithfully flat and \(F_x\) is flat over \(O_{Y,y}\) so by Algebra, Lemma 38.4 we see that \(F_x\) is flat over \(O_{S,s}\). Moreover, \(F_x / m_s F_x\) is the base change of the flat module \(F_x\) by \(O_{Y,y} \to O_{Y,y} / m_s O_{Y,y}\), hence flat by Algebra, Lemma 38.7. □

Here is the non-Noetherian version.

**Theorem 16.2.** Let \(S\) be a scheme. Let \(f : X \to Y\) be a morphism of schemes over \(S\). Let \(F\) be a quasi-coherent \(O_X\)-module. Assume

1. \(X\) is locally of finite presentation over \(S\),
2. \(F\) an \(O_X\)-module of finite presentation, and
3. \(Y\) is locally of finite type over \(S\).

Let \(x \in X\). Set \(y = f(x)\) and let \(s \in S\) be the image of \(x\) in \(S\). If \(F_x \neq 0\), then the following are equivalent:

1. \(F\) is flat over \(S\) at \(x\), and \(F_s\) is flat over \(Y_s\) at \(x\), and
2. \(Y\) is flat over \(S\) at \(y\) and \(F\) is flat over \(Y\) at \(x\).

Moreover, the set of points \(x\) where (1) and (2) hold is open in \(\text{Supp}(F)\).

**Proof.** Consider the ring maps

\[O_{S,s} \to O_{Y,y} \to O_{X,x}\]

and the module \(F_x\). The stalk of \(F_s\) at \(x\) is the module \(F_x / m_s F_x\) and the local ring of \(Y_s\) at \(y\) is \(O_{Y,y} / m_s O_{Y,y}\). Thus the implication (1) \(\Rightarrow\) (2) is Algebra, Lemma
If (2) holds, then the first ring map is faithfully flat and \( F_x \) is flat over \( \mathcal{O}_{Y,y} \), so by Algebra, Lemma 38.4 we see that \( F_x \) is flat over \( \mathcal{O}_{S,s} \). Moreover, \( F_x/m_s F_x \) is the base change of the flat module \( F_x \) by \( \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_s \mathcal{O}_{Y,y} \), hence flat by Algebra, Lemma 38.7.

By Morphisms, Lemma 20.11 the morphism \( f \) is locally of finite presentation. Consider the set

\[
U = \{ x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S \}.
\]

This set is open in \( X \) by Theorem 15.1. Note that if \( x \in U \), then \( F_x \) is flat at \( x \) over \( Y_s \) as a base change of a flat module under the morphism \( Y_s \to Y \), see Morphisms, Lemma 24.7. Hence at every point of \( U \cap \text{Supp}(\mathcal{F}) \) condition (1) is satisfied. On the other hand, it is clear that if \( x \in \text{Supp}(\mathcal{F}) \) satisfies (1) and (2), then \( x \in U \). Thus the open set we are looking for is \( U \cap \text{Supp}(\mathcal{F}) \).

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

**Lemma 16.3.** Let \( S \) be a scheme. Let \( f : X \to Y \) be a morphism of schemes over \( S \). Assume

1. \( S, X, Y \) are locally Noetherian,
2. \( X \) is flat over \( S \),
3. for every \( s \in S \) the morphism \( f_s : X_s \to Y_s \) is flat.

Then \( f \) is flat. If \( f \) is also surjective, then \( Y \) is flat over \( S \).

**Proof.** This is a special case of Theorem 16.1.

**Lemma 16.4.** Let \( S \) be a scheme. Let \( f : X \to Y \) be a morphism of schemes over \( S \). Assume

1. \( X \) is locally of finite presentation over \( S \),
2. \( X \) is flat over \( S \),
3. for every \( s \in S \) the morphism \( f_s : X_s \to Y_s \) is flat, and
4. \( Y \) is locally of finite type over \( S \).

Then \( f \) is flat. If \( f \) is also surjective, then \( Y \) is flat over \( S \).

**Proof.** This is a special case of Theorem 16.2.

**Lemma 16.5.** Let \( S \) be a scheme. Let \( f : X \to Y \) be a morphism of schemes over \( S \). Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Assume

1. \( X \) is locally of finite presentation over \( S \),
2. \( \mathcal{F} \) an \( \mathcal{O}_X \)-module of finite presentation,
3. \( \mathcal{F} \) is flat over \( S \), and
4. \( Y \) is locally of finite type over \( S \).

Then the set

\[
U = \{ x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y \}
\]

is open in \( X \) and its formation commutes with arbitrary base change: If \( S' \to S \) is a morphism of schemes, and \( U' \) is the set of points of \( X' = X \times_S S' \) where \( \mathcal{F}' = \mathcal{F} \times_S S' \) is flat over \( Y' = Y \times_S S' \), then \( U' = U \times_S S' \).
Proof. By Morphisms, Lemma \ref{lem-flat-morphism} the morphism \( f \) is locally of finite presentation. Hence \( U \) is open by Theorem \ref{thm-flat-morphism}. Because we have assumed that \( \mathcal{F} \) is flat over \( S \) we see that Theorem \ref{thm-flat-morphism} implies
\[
U = \{ x \in X | \mathcal{F}_s \text{ flat at } x \text{ over } Y_s \}.
\]
where \( s \) always denotes the image of \( x \) in \( S \). (This description also works trivially when \( \mathcal{F}_x = 0 \).) Moreover, the assumptions of the lemma remain in force for the morphism \( f' : X' \to Y' \) and the sheaf \( \mathcal{F}' \). Hence \( U' \) has a similar description. In other words, it suffices to prove that given \( s' \in S' \) mapping to \( s \in S \) we have
\[
\{ x' \in X'_{s'} | \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_s \}
\]
is the inverse image of the corresponding locus in \( X_s \). This is true by Lemma \ref{lem-flat-morphism} because in the cartesian diagram
\[
\begin{array}{ll}
X'_{s'} & \to X_s \\
\downarrow & \downarrow \\
Y'_{s'} & \to Y_s
\end{array}
\]
the horizontal morphisms are flat as they are base changes by the flat morphism \( \text{Spec}(k(s')) \to \text{Spec}(k(s)) \).

05VK Lemma \ref{lem-flat-morphism}. Let \( S \) be a scheme. Let \( f : X \to Y \) be a morphism of schemes over \( S \). Assume
\begin{itemize}
  \item[(1)] \( X \) is locally of finite presentation over \( S \),
  \item[(2)] \( X \) is flat over \( S \), and
  \item[(3)] \( Y \) is locally of finite type over \( S \).
\end{itemize}
Then the set
\[
U = \{ x \in X | X \text{ flat at } x \text{ over } Y \}
\]
is open in \( X \) and its formation commutes with arbitrary base change.

Proof. This is a special case of Lemma \ref{lem-flat-morphism}.

The following lemma is a variant of Algebra, Lemma \ref{lem-flat-module}. Note that the hypothesis that \( (\mathcal{F}_s)_x \) is a flat \( \mathcal{O}_{X_s,x} \)-module means that \( (\mathcal{F}_s)_x \) is a free \( \mathcal{O}_{X_s,x} \)-module which is always the case if \( x \in X_s \) is a generic point of an irreducible component of \( X_s \) and \( X_s \) is reduced (namely, in this case \( \mathcal{O}_{X_s,x} \) is a field, see Algebra, Lemma \ref{lem-field}).

080Q Lemma \ref{lem-flat-free}. Let \( f : X \to S \) be a morphism of schemes of finite presentation. Let \( \mathcal{F} \) be a finitely presented \( \mathcal{O}_X \)-module. Let \( x \in X \) with image \( s \in S \). If \( \mathcal{F} \) is flat at \( x \) over \( S \) and \( (\mathcal{F}_s)_x \) is a flat \( \mathcal{O}_{X_s,x} \)-module, then \( \mathcal{F} \) is finite free in a neighbourhood of \( x \).

Proof. If \( \mathcal{F}_x \otimes k(x) \) is zero, then \( \mathcal{F}_x = 0 \) by Nakayama’s lemma (Algebra, Lemma \ref{lem-nakayama}) and hence \( \mathcal{F} \) is zero in a neighbourhood of \( x \) (Modules, Lemma \ref{lem-flat-zero}) and the lemma holds. Thus we may assume \( \mathcal{F}_x \otimes k(x) \) is not zero and we see that Theorem \ref{thm-flat-morphism} applies with \( f = \text{id} : X \to X \). We conclude that \( \mathcal{F}_x \) is flat over \( \mathcal{O}_{X,x} \). Hence \( \mathcal{F}_x \) is free, see Algebra, Lemma \ref{lem-flat-free} for example. Choose an open neighbourhood \( x \in U \subset X \) and sections \( s_1, \ldots, s_r \in \mathcal{F}(U) \) which map to a basis in \( \mathcal{F}_x \). The corresponding map \( \psi : \mathcal{O}_U^r \to \mathcal{F}|_U \) is surjective after shrinking \( U \) (Modules, Lemma \ref{lem-flat-free}). Then \( \text{Ker}(\psi) \) is of finite type (see Modules, Lemma \ref{lem-flat-free}) and \( \text{Ker}(\psi)_x = 0 \). Whence after shrinking \( U \) once more \( \psi \) is an isomorphism.
0CZR **Lemma 16.8.** Let \( f : X \to S \) be a morphism of schemes which is locally of finite presentation. Let \( \mathcal{F} \) be a finitely presented \( \mathcal{O}_X \)-module flat over \( S \). Then the set
\[
\{ x \in X : \mathcal{F} \text{ free in a neighbourhood of } x \}
\]
is open in \( X \) and its formation commutes with arbitrary base change \( S' \to S \).

**Proof.** Openness holds trivially. Let \( x \in X \) mapping to \( s \in S \). By Lemma 16.7 we see that \( x \) is in our set if and only if \( \mathcal{F}|_{X_s} \) is flat at \( x \) over \( X_s \). Clearly this is also equivalent to \( \mathcal{F} \) being flat at \( x \) over \( X \) (because this statement is implied by freeness of \( \mathcal{F}_x \) and implies flatness of \( \mathcal{F}|_{X_s} \) at \( x \) over \( X_s \)). Thus the base change statement follows from Lemma 16.5 applied to \( \text{id} : X \to X \) over \( S \). \( \square \)

17. Normalization revisited

081J Normalization commutes with smooth base change.

081K **Lemma 17.1.** Let \( f : Y \to X \) be a smooth morphism of schemes. Let \( A \) be a quasi-coherent sheaf of \( \mathcal{O}_X \)-algebras. The integral closure of \( \mathcal{O}_Y \) in \( f^*A \) is equal to \( f^*A' \) where \( A' \subset A \) is the integral closure of \( \mathcal{O}_X \) in \( A \).

**Proof.** This is a translation of Algebra, Lemma 144.4 into the language of schemes. Details omitted. \( \square \)

03GV **Lemma 17.2** (Normalization commutes with smooth base change). Let \[
\begin{array}{ccc}
Y_2 & \longrightarrow & Y_1 \\
\downarrow f_2 & & \downarrow f_1 \\
X_2 & \varphi \longrightarrow & X_1
\end{array}
\]
be a fibre square in the category of schemes. Assume \( f_1 \) is quasi-compact and quasi-separated, and \( \varphi \) is smooth. Let \( Y_i \to X'_i \to X_i \) be the normalization of \( X_i \) in \( Y_i \). Then \( X'_2 \cong X_2 \times_{X_1} X'_1 \).

**Proof.** The base change of the factorization \( Y_1 \to X'_1 \to X_1 \) to \( X_2 \) is a factorization \( Y_2 \to X_2 \times_{X_1} X'_1 \to X_2 \) and \( X_2 \times_{X_1} X'_1 \to X_2 \) is integral (Morphisms, Lemma 42.6). Hence we get a morphism \( h : X'_2 \to X_2 \times_{X_1} X'_1 \) by the universal property of Morphisms, Lemma 51.4. Observe that \( X'_2 \) is the relative spectrum of the integral closure of \( \mathcal{O}_{X_2} \) in \( f_{2,*}\mathcal{O}_{Y_2} \). If \( A' \subset f_{1,*}\mathcal{O}_{Y_1} \) denotes the integral closure of \( \mathcal{O}_{X_2} \), then \( X_2 \times_{X_1} X'_1 \) is the relative spectrum of \( \varphi^*A' \), see Constructions, Lemma 4.6. By Cohomology of Schemes, Lemma 5.2 we know that \( f_{2,*}\mathcal{O}_{Y_2} = \varphi^*f_{1,*}\mathcal{O}_{Y_1} \). Hence the result follows from Lemma 17.1. \( \square \)

07TD **Lemma 17.3** (Normalization and smooth morphisms). Let \( X \to Y \) be a smooth morphism of schemes. Assume every quasi-compact open of \( Y \) has finitely many irreducible components. Then the same is true for \( X \) and there is a canonical isomorphism \( X'' = X \times_Y Y'' \).

**Proof.** By Descent, Lemma 13.3 every quasi-compact open of \( X \) has finitely many irreducible components. Note that \( X_{\text{red}} = X \times_Y Y_{\text{red}} \) as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 15.1. Hence we may assume that \( X \) and \( Y \) are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that \( X' = X \times_Y Y'' \) is a normal scheme by Descent, Lemma 15.2. The morphism \( X' \to Y'' \) is smooth (hence flat) thus the
generic points of irreducible components of $X'$ lie over generic points of irreducible components of $Y'$. Since $Y' \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y'$ induces an isomorphism on fibres over generic points of $Y$). We conclude that there exists a factorization $X' \to X' \to X$, see Morphisms, Lemma 52.5 which is an isomorphism as $X'$ is normal and integral over $X$. □

**Lemma 17.4 (Normalization and henselization).** Let $X$ be a locally Noetherian scheme. Let $\nu : X' \to X$ be the normalization morphism. Then for any point $x \in X$ the base change

$$X' \times_X \text{Spec}(\mathcal{O}_X^h) \to \text{Spec}(\mathcal{O}_X^h), \text{ resp. } X' \times_X \text{Spec}(\mathcal{O}_X^{sh}) \to \text{Spec}(\mathcal{O}_X^{sh})$$

is the normalization of $\text{Spec}(\mathcal{O}_X^h)$, resp. $\text{Spec}(\mathcal{O}_X^{sh})$.

**Proof.** Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of $X$ passing through $x$. The base change of the normalization to $\text{Spec}(\mathcal{O}_X^h)$ is the spectrum of the integral closure of $\mathcal{O}_{X,x}$ in $\prod \kappa(\eta_i)$. This follows from our construction of the normalization of $X$ in Morphisms, Definition 52.1 and Morphisms, Lemma 51.1; you can also use the description of the normalization in Morphisms, Lemma 52.3.

Thus we reduce to the following algebra problem. Let $A$ be a Noetherian local ring; recall that this implies the henselization $A^h$ and strict henselization $A^{sh}$ are Noetherian too (More on Algebra, Lemma 44.3). Let $p_1, \ldots, p_r$ be its minimal primes. Let $A'$ be the integral closure of $A$ in $\prod \kappa(p_i)$. Problem: show that $A' \otimes_A A^h$, resp. $A' \otimes_A A^{sh}$ is constructed from the Noetherian local ring $A^h$, resp. $A^{sh}$ in the same manner.

Since $A^h$, resp. $A^{sh}$ are colimits of étale $A$-algebras, we see that the minimal primes of $A$ and $A^h$ are exactly the primes of $A^h$, resp. $A^{sh}$ lying over the minimal primes of $A$ (by going down, see Algebra, Lemmas 38.19 and 29.7). Thus More on Algebra, Lemma 44.13 tells us that $A^h \otimes_A \prod \kappa(p_i)$, resp. $A^{sh} \otimes_A \prod \kappa(p_i)$ is the product of the residue fields at the minimal primes of $A^h$, resp. $A^{sh}$. We know that taking the integral closure in an overring commutes with étale base change, see Algebra, Lemma 144.2. Writing $A^h$ and $A^{sh}$ as a limit of étale $A$-algebras we see that the same thing is true for the base change to $A^h$ and $A^{sh}$ (you can also use the more general Algebra, Lemma 144.5). □

### 18. Normal morphisms

In the article [DM69] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

**Definition 18.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres $X_y$ are locally Noetherian schemes.

1. Let $x \in X$, and $y = f(x)$. We say that $f$ is normal at $x$ if $f$ is flat at $x$, and the scheme $X_y$ is geometrically normal at $x$ over $\kappa(y)$ (see Varieties, Definition 10.1).

2. We say $f$ is a normal morphism if $f$ is normal at every point of $X$.

The other types are coprof $\leq k$, Cohen-Macaulay, $(S_k)$, regular, $(R_k)$, and reduced. See [DG67] IV Definition 6.8.1]. Gorenstein morphisms will be defined in Duality for Schemes, Section 24.
So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

**Lemma 18.2.** Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of $f$ are locally Noetherian. The following are equivalent

1. $f$ is normal, and
2. $f$ is flat and its fibres are geometrically normal schemes.

**Proof.** This follows directly from the definitions. □

**Lemma 18.3.** A smooth morphism is normal.

**Proof.** Let $f : X \to Y$ be a smooth morphism. As $f$ is locally of finite presentation, see Morphisms, Lemma 32.8, the fibres $X_y$ are locally of finite type over a field, hence locally Noetherian. Moreover, $f$ is flat, see Morphisms, Lemma 32.9. Finally, the fibres $X_y$ are smooth over a field (by Morphisms, Lemma 32.5) and hence geometrically normal by Varieties, Lemma 25.4. Thus $f$ is normal by Lemma 18.2. □

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

**Lemma 18.4.** The property $P(f) =$ “the fibres of $f$ are locally Noetherian” is local in the fppf topology on the source and the target.

**Proof.** Let $f : X \to Y$ be a morphism of schemes. Let $I = \{ \varphi_i : Y_i \to Y \}_{i \in I}$ be an fppf covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$  

Moreover, as $\varphi_i$ is of finite presentation the field extension $\kappa(y) \subset \kappa(y_i)$ is finitely generated. Hence in this situation we have that $X_y$ is locally Noetherian if and only if $X_{i,y_i}$ is locally Noetherian, see Varieties, Lemma 11.1. This fact implies locality on the target.

Let $\{ X_i \to X \}$ be an fppf covering of $X$. Let $y \in Y$. In this case $\{ X_{i,y} \to X_y \}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 13.1. □

**Lemma 18.5.** The property $P(f) =$ “the fibres of $f$ are locally Noetherian and $f$ is normal” is local in the fppf topology on the target and local in the smooth topology on the source.

**Proof.** We have $P(f) = P_1(f) \wedge P_2(f) \wedge P_3(f)$ where $P_1(f) =$ “the fibres of $f$ are locally Noetherian”, $P_2(f) =$ “$f$ is flat”, and $P_3(f) =$ “the fibres of $f$ are geometrically normal”. We have already seen that $P_1$ and $P_2$ are local in the fppf topology on the source and the target, see Lemma 18.4 and Descent, Lemmas 20.13 and 24.1. Thus we have to deal with $P_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $I = \{ \varphi_i : Y_i \to Y \}_{i \in I}$ be an fpqc covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$
Hence in this situation we have that $X_y$ is geometrically normal if and only if $X_{i,y_i}$ is geometrically normal, see Varieties, Lemma \[10.4\] This fact implies $P_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of $X$. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $P_3$ for the smooth topology on the source follows from Descent, Lemma \[15.2\] Combining the above the lemma follows. \hfill \Box

## 19. Regular morphisms

07R6 Compare with Section \[18\] The algebraic version of this notion is discussed in More on Algebra, Section \[40\].

07R7 **Definition 19.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres $X_y$ are locally Noetherian schemes.

1. Let $x \in X$, and $y = f(x)$. We say that $f$ is regular at $x$ if $f$ is flat at $x$, and the scheme $X_y$ is geometrically regular at $x$ over $\kappa(y)$ (see Varieties, Definition \[12.1\]).

2. We say $f$ is a regular morphism if $f$ is regular at every point of $X$.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

07R8 **Lemma 19.2.** Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of $f$ are locally Noetherian. The following are equivalent

1. $f$ is regular,
2. $f$ is flat and its fibres are geometrically regular schemes,
3. for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $O(V) \to O(U)$ is regular,
4. there exists an open covering $Y = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and
5. there exists an affine open covering $Y = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring maps $O(V_j) \to O(U_i)$ are regular.

**Proof.** The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition $f$ is flat at $x$ if and only if $O_{Y,y} \to O_{X,x}$ is a flat ring map, and $X_y$ is geometrically regular at $x$ over $\kappa(y)$ if and only if $O_{X_{i,y}} = O_{X,x}/m_xO_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence whether or not $f$ is regular at $x$ depends only on the local homomorphism of local rings $O_{Y,y} \to O_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition \[40.1\]) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(p)$ are Noetherian and geometrically regular for all primes $p \subset A$. By Varieties, Lemma \[12.3\] this is equivalent to Spec($B \otimes_A \kappa(p)$) being a geometrically regular scheme over $\kappa(p)$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that $f$ is flat (see Morphisms, Lemma \[24.3\]). Moreover, if $y \in Y$, then $y \in V_j$ for some $j$ and we see that $X_y = \bigcup_{i \in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma \[12.3\]. Another application of Varieties, Lemma \[12.3\] shows that $X_y$ is geometrically regular. Hence (2) holds and the proof of the lemma is finished. \hfill \Box
Lemma 19.3. A smooth morphism is regular.

Proof. Let $f : X \to Y$ be a smooth morphism. As $f$ is locally of finite presentation, see Morphisms, Lemma 32.8, the fibres $X_y$ are locally of finite type over a field, hence locally Noetherian. Moreover, $f$ is flat, see Morphisms, Lemma 32.9. Finally, the fibres $X_y$ are smooth over a field (by Morphisms, Lemma 32.5) and hence geometrically regular by Varieties, Lemma 25.4. Thus $f$ is regular by Lemma 19.2.

Lemma 19.4. The property $\mathcal{P}(f) = \text{"the fibres of } f \text{ are locally Noetherian and } f \text{ is regular"}$ is local in the fppf topology on the target and local in the smooth topology on the source.

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \land \mathcal{P}_2(f) \land \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) = \text{"the fibres of } f \text{ are locally Noetherian"}$, $\mathcal{P}_2(f) = \text{"}f\text{ is flat"}$, and $\mathcal{P}_3(f) = \text{"the fibres of } f \text{ are geometrically regular"}$. We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 18.4 and Descent, Lemmas 20.15 and 24.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that $X_y$ is geometrically regular if and only if $X_{i,y_i}$ is geometrically regular, see Varieties, Lemma 12.4. This fact implies $\mathcal{P}_3$ is fppf local on the target.

Let $\{X_i \to X\}$ be a smooth covering of $X$. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 15.4. Combining the above the lemma follows.

20. Cohen-Macaulay morphisms

Compare with Section 18. Note that, as pointed out in Algebra, Section 162 and Varieties, Section 13 “geometrically Cohen-Macaulay” is the same as plain Cohen-Macaulay.

Definition 20.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres $X_y$ are locally Noetherian schemes.

1. Let $x \in X$, and $y = f(x)$. We say that $f$ is Cohen-Macaulay at $x$ if $f$ is flat at $x$, and the local ring of the scheme $X_y$ at $x$ is Cohen-Macaulay.
2. We say $f$ is a Cohen-Macaulay morphism if $f$ is Cohen-Macaulay at every point of $X$.

Here is a translation.

Lemma 20.2. Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of $f$ are locally Noetherian. The following are equivalent

1. $f$ is Cohen-Macaulay, and
2. $f$ is flat and its fibres are Cohen-Macaulay schemes.

Proof. This follows directly from the definitions.
Lemma 20.3. Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is locally of finite type and Cohen-Macaulay. For every point $x$ in $X$ with image $y$ in $Y$,

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where $X_y$ denotes the fiber over $y$.

Proof. After replacing $X$ by an open neighborhood of $x$, there is a natural number $d$ such that all fibers of $X \to Y$ have dimension $d$ at every point, see Morphisms, Lemma 28.4. Then $f$ is flat, locally of finite type and of relative dimension $d$. Hence the result follows from Morphisms, Lemma 28.6. □

Lemma 20.4. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the fibres of $f$, $g$, and $g \circ f$ are locally Noetherian. Let $x \in X$ with images $y \in Y$ and $z \in Z$.

(1) If $f$ is Cohen-Macaulay at $x$ and $g$ is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at $x$.

(2) If $f$ and $g$ are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.

(3) If $g \circ f$ is Cohen-Macaulay at $x$ and $f$ is flat at $x$, then $f$ is Cohen-Macaulay at $x$ and $g$ is Cohen-Macaulay at $f(x)$.

(4) If $f \circ g$ is Cohen-Macaulay and $f$ is flat, then $f$ is Cohen-Macaulay and $g$ is Cohen-Macaulay at every point in the image of $f$.

Proof. Consider the map of Noetherian local rings

$$\mathcal{O}_{Y,z,y} \to \mathcal{O}_{X,x}$$

and observe that its fibre is

$$\mathcal{O}_{X,x}/m_{Y,z,y}\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$$

Thus the lemma this follows from Algebra, Lemma [158.3]. □

Lemma 20.5. Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If $X$ is Cohen-Macaulay, then $f$ is Cohen-Macaulay and $\mathcal{O}_{Y,f(x)}$ is Cohen-Macaulay for all $x \in X$.

Proof. After translating into algebra this follows from Algebra, Lemma [158.3]. □

Lemma 20.6. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres $X_y$ are locally Noetherian schemes. Let $f' : X' \to Y'$ be the base change of $f$. Let $x' \in X'$ be a point with image $x \in X$.

(1) If $f$ is Cohen-Macaulay at $x$, then $f' : X' \to Y'$ is Cohen-Macaulay at $x'$.

(2) If $f$ is flat at $x$ and $f'$ is Cohen-Macaulay at $x'$, then $f$ is Cohen-Macaulay at $x$.

(3) If $Y' \to Y$ is flat at $f'(x')$ and $f'$ is Cohen-Macaulay at $x'$, then $f$ is Cohen-Macaulay at $x$.

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y) \subset \kappa(y')$ is finitely generated. Hence also all the fibres $X_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma [11.1]. Thus the
lemma makes sense. Set \( y' = f'(x') \) and \( y = f(x) \). Hence we get the following commutative diagram of local rings

\[
\begin{array}{ccc}
\mathcal{O}_{X',x'} & \xleftarrow{\mathcal{O}_{X,x}} & \\
\uparrow & & \uparrow \\
\mathcal{O}_{Y',y'} & \xleftarrow{\mathcal{O}_{Y,y}} & \\
\end{array}
\]

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume \( f \) is Cohen-Macaulay at \( x \). The flatness of \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) implies the flatness of \( \mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'} \), see Algebra, Lemma \( \text{[DG67]} \). The fact that \( \mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x} \) is Cohen-Macaulay implies that \( \mathcal{O}_{X',x'}/m_y \mathcal{O}_{X',x'} \) is Cohen-Macaulay, see Varieties, Lemma \( \text{[13.1]} \). Hence we see that \( f' \) is Cohen-Macaulay at \( x' \).

Assume \( f \) is flat at \( x \) and \( f' \) is Cohen-Macaulay at \( x' \). The fact that \( \mathcal{O}_{X',x'}/m_y \mathcal{O}_{X',x'}/m_y \mathcal{O}_{X,x} \) is Cohen-Macaulay implies that \( \mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x} \) is Cohen-Macaulay, see Varieties, Lemma \( \text{[13.1]} \). Hence we see that \( f' \) is Cohen-Macaulay at \( x' \).

Assume \( Y' \to Y \) is flat at \( y' \) and \( f' \) is Cohen-Macaulay at \( x' \). The flatness of \( \mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'} \) and \( \mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'} \) implies the flatness of \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \), see Algebra, Lemma \( \text{[DG67]} \). The fact that \( \mathcal{O}_{X',x'}/m_y \mathcal{O}_{X,x} \) is Cohen-Macaulay implies that \( \mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x} \) is Cohen-Macaulay, see Varieties, Lemma \( \text{[13.1]} \). Hence we see that \( f' \) is Cohen-Macaulay at \( x ' \).

\[045U \textbf{Lemma 20.7.} \] Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let

\[ W = \{ x \in X \mid f \text{ is Cohen-Macaulay at } x \} \]

Then

1. \( W = \{ x \in X \mid \mathcal{O}_{X(x),x} \text{ is Cohen-Macaulay} \} \),
2. \( W \) is open in \( X \),
3. \( W \) dense in every fibre of \( X \to S \),
4. the formation of \( W \) commutes with arbitrary base change of \( f \): For any morphism \( g : S' \to S \), consider the base change \( f' : X' \to S' \) of \( f \) and the projection \( g' : X' \to X \). Then the corresponding set \( W' \) for the morphism \( f' \) is equal to \( W' = (g')^{-1}(W) \).

\[ \textbf{Proof.} \] As \( f \) is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma \( \text{[129.5]} \). Part (4) follows either from Algebra, Lemma \( \text{[129.7]} \) or Varieties, Lemma \( \text{[13.1]} \). \( \square \)

\[0BUU \textbf{Lemma 20.8.} \] Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let \( x \in X \) with image \( s \in S \). Set \( d = \dim_x(X_s) \). The following are equivalent

1. \( f \) is Cohen-Macaulay at \( x \),
2. there exists an open neighbourhood \( U \subset X \) of \( x \) and a locally quasi-finite morphism \( U \to \mathbb{A}^d_S \) over \( S \) which is flat at \( x \),
3. there exists an open neighbourhood \( U \subset X \) of \( x \) and a locally quasi-finite flat morphism \( U \to \mathbb{A}^d_S \) over \( S \),
4. for any \( S \)-morphism \( g : U \to \mathbb{A}^d_S \) of an open neighbourhood \( U \subset X \) of \( x \) we have: \( g \) is quasi-finite at \( x \) \( \Rightarrow \) \( g \) is flat at \( x \).
Proof. Openness of flatness shows (2) and (3) are equivalent, see Theorem \[15.1\] Choose affine open \( U = \text{Spec}(A) \subseteq X \) with \( x \in U \) and \( V = \text{Spec}(R) \subseteq S \) with \( f(U) \subseteq V \). Then \( R \to A \) is a flat ring map of finite presentation. Let \( p \subseteq A \) be the prime ideal corresponding to \( x \). After replacing \( A \) by a principal localization we may assume there exists a quasi-finite map \( R[x_1, \ldots, x_d] \to A \), see Algebra, Lemma \[124.2\] Thus there exists at least one pair \((U, g)\) consisting of an open neighbourhood \( U \subseteq X \) of \( x \) and a locally4 quasi-finite morphism \( g : U \to A^d \).

Claim: Given \( R \to A \) flat and of finite presentation, a prime \( p \subseteq A \) and \( \varphi : R[x_1, \ldots, x_d] \to A \) quasi-finite at \( p \) we have: \( \text{Spec}(\varphi) \) is flat at \( p \) if and only if \( \text{Spec}(A) \to \text{Spec}(R) \) is Cohen-Macaulay at \( p \). Namely, by Theorem \[16.2\] flatness may be checked on fibres. The same is true for being Cohen-Macaulay (as \( A \) is already assumed flat over \( R \)). Thus the claim follows from Algebra, Lemma \[129.1\] The claim shows that (1) is equivalent to (4) and combined with the fact that we have constructed a suitable \((U, g)\) in the second paragraph, the claim also shows that (1) is equivalent to (2).

\[ \square \]

\[ \text{Lemma 20.9.} \] Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. For \( d \geq 0 \) there exist opens \( U_d \subseteq X \) with the following properties

1. \( W = \bigcup_{d \geq 0} U_d \) is dense in every fibre of \( f \), and
2. \( U_d \to S \) is of relative dimension \( d \) (see Morphisms, Definition \[28.4\]).

Proof. This follows by combining Lemma \[20.7\] with Morphisms, Lemma \[28.4\]

\[ \square \]

\[ \text{Lemma 20.10.} \] Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Suppose \( x' \sim x \) is a specialization of points of \( X \) with image \( s' \sim s \) in \( S \). If \( x \) is a generic point of an irreducible component of \( X_s \) then

\[ \dim_x(X_{s'}) = \dim_x(X_s). \]

Proof. The point \( x \) is contained in \( U_d \) for some \( d \), where \( U_d \) as in Lemma \[20.9\]

\[ \square \]

\[ \text{Lemma 20.11.} \] The property \( \mathcal{P}(f) = \text{“the fibres of } f \text{ are locally Noetherian and } f \text{ is Cohen-Macaulay”} \) is local in the fppf topology on the target and local in the syntomic topology on the source.

Proof. We have \( \mathcal{P}(f) = \mathcal{P}_1(f) \land \mathcal{P}_2(f) \) where \( \mathcal{P}_1(f) = \text{“} f \text{ is flat”} \), and \( \mathcal{P}_2(f) = \text{“the fibres of } f \text{ are locally Noetherian and Cohen-Macaulay”} \). We know that \( \mathcal{P}_1 \) is local in the fppf topology on the source and the target, see Descent, Lemmas \[20.15\] and \[24.1\] Thus we have to deal with \( \mathcal{P}_2 \).

Let \( f : X \to Y \) be a morphism of schemes. Let \( \{ \varphi_i : Y_i \to Y \}_{i \in I} \) be an fppf covering of \( Y \). Denote \( f_i : X_i \to Y_i \) the base change of \( f \) by \( \varphi_i \). Let \( i \in I \) and let \( y_i \in Y_i \) be a point. Set \( y = \varphi_i(y_i) \). Note that

\[ X_{i/y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y, \]

and that \( \kappa(y) \subseteq \kappa(y_i) \) is a finitely generated field extension. Hence if \( X_y \) is locally Noetherian, then \( X_{i/y_i} \) is locally Noetherian, see Varieties, Lemma \[11.1\] And if in addition \( X_y \) is Cohen-Macaulay, then \( X_{i/y_i} \) is Cohen-Macaulay, see Varieties, Lemma \[13.1\] Thus \( \mathcal{P}_2 \) is fppf local on the target.

4If \( S \) is quasi-separated, then \( g \) will be quasi-finite.
Let $\{X_i \to X\}$ be a syntomic covering of $X$. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $P_2$ for the syntomic topology on the source follows from Descent, Lemma [14.2]. Combining the above the lemma follows.

### 21. Slicing Cohen-Macaulay morphisms

The results in this section eventually lead to the assertion that the fppf topology is the same as the “finitely presented, flat, quasi-finite” topology. The following lemma is very closely related to Divisors, Lemma [18.9].

#### Lemma 21.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in m_x \subset O_{X,x}$. Assume

1. $f$ is locally of finite presentation,
2. $f$ is flat at $x$, and
3. the image $\overline{h}$ of $h$ in $O_{X,s,x} = O_{X,x}/m_sO_{X,x}$ is a nonzerodivisor.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ such that $h$ comes from $h \in \Gamma(U, O_U)$ and such that $D = V(h)$ is an effective Cartier divisor in $U$ with $x \in D$ and $D \to S$ flat and locally of finite presentation.

**Proof.** We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem [15.1]) we may assume, after replacing $X$ by an open neighbourhood of $x$, that $X \to S$ is flat. We may also assume that $X$ and $S$ are affine. After possible shrinking $X$ a bit we may assume that there exists an $h \in \Gamma(X, O_X)$ which maps to our given $h$.

We may write $S = \text{Spec}(A)$ and we may write $A = \text{colim}_i A_i$ as a directed colimit of finite type $\mathbb{Z}$ algebras. Then by Algebra, Lemma [163.1] or Limits, Lemmas [10.1] [8.2] and [10.1] we can find a cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X_0 \\
\downarrow f & & \downarrow f_0 \\
S & \longrightarrow & S_0
\end{array}
$$

with $f_0$ flat and of finite presentation, $X_0$ affine, and $S_0$ affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of $x$, resp. $s$. We may also assume there exists an element $h_0 \in \Gamma(X_0, O_{X_0})$ which restricts to $h$ on $X$. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma [10.1] by thinking of $h$ as a morphism $X \to A^{1}_{\mathbb{K}}$.) Note that $O_{X_s,x}$ is a localization of $O_{(X_0)_{s_0},x_0} \otimes_{\kappa(s)} \kappa(s)$, so that $O_{(X_0)_{s_0},x_0} \to O_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in O_{(X_0)_{s_0},x_0}$ is contained in $m_{(X_0)_{s_0},x_0}$ and is a nonzerodivisor.

We claim that after replacing $X_0$ by a principal open neighbourhood of $x_0$ the element $h_0$ is a nonzerodivisor in $B_0 = \Gamma(X_0, O_{X_0})$ such that $B_0/h_0B_0$ is flat over $A_0 = \Gamma(S_0, O_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat $A_0$-modules. Hence this remains exact on tensoring with $A$ (by Algebra, Lemma [38.12]) and the lemma follows.
It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $q \subset B$ be a prime lying over $p \subset A$. Assume $h \in q$ maps to a nonzerodivisor in $B_q/pB_q$. Goal: show that after possible replacing $B$ by $B_q$ for some $g \in B$, $g \notin q$ the element $h$ becomes a nonzerodivisor and $B/hB$ becomes flat over $A$. By Algebra, Lemma 98.2 we see that $h$ is a nonzerodivisor in $B_q$ and that $B_q/hB_q$ is flat over $A$. By openness of flatness, see Algebra, Theorem 128.4 or Theorem 15.1 we see that $B/hB$ is flat over $A$ after replacing $B$ by $B_q$ for some $g \in B$, $g \notin q$. Finally, let $I = \{ b \in B \mid hb = 0 \}$ be the annihilator of $h$. Then $IB_q = 0$ as $h$ is a nonzerodivisor in $B_q$. Also $I$ is finitely generated as $B$ is Noetherian. Hence there exists a $g \in B$, $g \notin q$ such that $IB_g = 0$. After replacing $B$ by $B_g$ we see that $h$ is a nonzerodivisor.

06LI Lemma 21.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume

(1) $f$ is locally of finite presentation,
(2) $f$ is flat at $x$, and
(3) the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X,x} = \mathcal{O}_{X,x}/m_x \mathcal{O}_{X,x}$ form a regular sequence.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme $S'$ over $S$.

Proof. (Our conventions on regular sequences imply that $h_i \in m_x$ for each $i$.) The case $r = 1$ follows from Lemma 21.1 combined with Divisors, Lemma 18.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case $r > 1$ follows from a straightforward induction on $r$ (applying the result for $r = 1$ exactly $r$ times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 22. Namely, first by openness of flatness (see Theorem 15.1) we may assume, after replacing $X$ by an open neighbourhood of $x$, that $X \to S$ is flat. We may also assume that $X$ and $S$ are affine. After possible shrinking $X$ a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that $Z_s = Z$ is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 20) and that the topology on $Z_s$ is induced from the topology on $X$ (see Schemes, Lemma 18.5). Hence after shrinking $X$ a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the $r$ elements $h_i|_{X_s}$, see Divisors, Lemma 20.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

\[
\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(x)}(\kappa(x)),
\]

\[
\dim_x(Z_s) = \dim(\mathcal{O}_{Z_s,x}) + \text{trdeg}_{\kappa(x)}(\kappa(x)),
\]

\[
\dim(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{Z_s,x}) + r
\]

the first two equalities by Algebra, Lemma 115.3 and the second by $r$ times applying Algebra, Lemma 99.12. Hence Divisors, Lemma 22.7 part (3) applies to show that (after Zariski shrinking $X$) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 22.4 applies (which gives the flatness and the statement on base change). □
Lemma 21.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is locally of finite presentation,
2. $f$ is flat at $x$, and
3. $\mathcal{O}_{X, x}$ has depth $\geq 1$.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ and an effective Cartier divisor $D \subset U$ containing $x$ such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X, x}$ which maps to a nonzerodivisor in $\mathcal{O}_{X, x}$ and apply Lemma 21.1. □

Lemma 21.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is locally of finite presentation,
2. $f$ is Cohen-Macaulay at $x$, and
3. $x$ is a closed point of $X_s$.

Then there exists a regular immersion $Z \to X$ containing $x$ such that

(a) $Z \to S$ is flat and locally of finite presentation,
(b) $Z \to S$ is locally quasi-finite, and
(c) $Z_s = \{x\}$ set theoretically.

Proof. We may and do replace $S$ by an affine open neighbourhood of $s$. We will prove the lemma for affine $S$ by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take $Z$ to be an open neighbourhood of $x$. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at $x$, see Morphisms, Lemma [28.5]. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma [53.2]. Thus after replacing $X$ by $U$ we see that the fibre $X_s$ is a finite discrete set. Hence after replacing $X$ by a further affine open neighbourhood of $X$ we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on $X_s$ is induced from the topology on $X$, see Schemes, Lemma [18.5]). This proves the lemma in this case.

Next, assume $d > 0$. Note that because $x$ is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma [19.3]). Thus we see

$$\text{depth}(\mathcal{O}_{X, x}) = \dim(\mathcal{O}_{X, x}) = d > 0$$

the first equality as $\mathcal{O}_{X, x}$ is Cohen-Macaulay and the second by Morphisms, Lemma [27.1]. Thus we may apply Lemma 21.3 to find a diagram

$$
\begin{array}{ccc}
D & \to & U \\
\downarrow & & \downarrow \\
& & X \\
& \searrow & \\
& & S
\end{array}
$$

with $x \in D$. Note that $\mathcal{O}_{D, x} = \mathcal{O}_{X, x}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma [18.1]. Hence $\mathcal{O}_{D, x}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X, x}$, see Algebra, Lemma [103.2] for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at $x$ with $\dim_x(D_x) = \dim_x(X_x) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular
immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 21.7. This finishes the proof.

**Lemma 21.5.** Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $s \in S$ be a point in the image of $f$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
S & \downarrow & \\
\end{array}
\]

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

**Proof.** The fibre $X_s$ is not empty by assumption. Hence there exists a closed point $x \in X_s$ where $f$ is Cohen-Macaulay, see Lemma 20.7. Apply Lemma 21.4 and set $S' = S$.

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the “quasi-finite, flat, finite presentation” topology.

**Lemma 21.6.** Let $S$ be a scheme. Let $U = \{S_i \to S\}_{i \in I}$ be an fppf covering of $S$, see Topologies, Definition 7.1. Then there exists an fppf covering $V = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 8.1) $U$ such that each $T_j \to S$ is locally quasi-finite.

**Proof.** For every $s \in S$ there exists an $i \in I$ such that $s$ is in the image of $S_i \to S$. By Lemma 21.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that $g_s$ factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of $S$ that refines $U$.

## 22. Generic fibres

Some results on the relationship between generic fibres and nearby fibres.

**Lemma 22.1.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.

**Proof.** Follows immediately from the more general Morphisms, Lemma 8.4.

**Lemma 22.2.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.

**Proof.** This follows, upon taking affine opens, from Algebra, Lemma 29.2. (Of course it also follows from generic flatness.)

**Lemma 22.3.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $Z \subset X$ is a closed subset with $Z_\eta$ nowhere dense in $X_\eta$, then there exists a nonempty open $V \subset Y$ such that $Z_V$ is nowhere dense in $X_V$ for all $y \in V$. 
Proof. Let $Y' \subset Y$ be the reduction of $Y$. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 43.6 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that $Y$ is integral, see Properties, Lemma 3.4. By Morphisms, Proposition 26.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to Z$ are flat and of finite presentation. We claim that $V$ works. Pick $y \in V$. If $Z_y$ has a nonempty interior, then $Z_y$ contains a generic point $\xi$ of an irreducible component of $X_y$. Note that $\eta \leadsto f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \leadsto \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 24.9 By Lemma 20.10 we see that
\[
\dim_\xi(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_\xi(X_\eta).
\]
Hence some irreducible component of $Z_\eta$ passing through $\xi'$ has dimension $\dim_\xi(X_\eta)$ which contradicts the assumption that $Z_\eta$ is nowhere dense in $X_\eta$ and we win. \qed

Lemma 22.4. Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. Let $U \subset X$ be an open subscheme such that $U_\eta$ is scheme theoretically dense in $X_\eta$. Then there exists a nonempty open $V \subset Y$ such that $U_\eta$ is scheme theoretically dense in $X_\eta$ for all $y \in V$.

Proof. Let $Y' \subset Y$ be the reduction of $Y$. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace $Y$ by $Y'$ and $X$ by $X'$. Thus we may assume that $Y$ is integral, see Properties, Lemma 3.4. We may also replace $Y$ by a nonempty affine open. In other words we may assume that $Y = \Spec(A)$ where $A$ is a domain with fraction field $K$.

As $f$ is of finite type we see that $X$ is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens $X_i$. By Morphisms, Definition 7.3 we see that $U_i = X_i \cap U$ is an open subscheme of $X_i$ such that $U_i \cap U$ is scheme theoretically dense in $X_i \cap U$. Thus it suffices to prove the result for the pairs $(X_i, U_i)$, in other words we may assume that $X$ is affine.

Write $X = \Spec(B)$. Note that $B_K$ is Noetherian as it is a finite type $K$-algebra. Hence $U_\eta$ is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that $U_\eta$ is scheme theoretically dense in $X_\eta$ means that $B_K \to \bigoplus_{j=1}^m B_{g_j}$ is injective, see Morphisms, Example 7.4. By Algebra, Lemma 23.3 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1}^m B_{K_{g_j}}, b \mapsto (g_1 b, \ldots, g_m b)$. Let $M$ be the cokernel of this map over $A$, i.e., such that we have an exact sequence
\[
0 \to I \to B \xrightarrow{(g_1, \ldots, g_m)} \bigoplus_{j=1}^m B \to M \to 0.
\]

After replacing $A$ by $A_h$ for some nonzero $h$ we may assume that $B$ is a flat, finitely presented $A$-algebra, and that $M$ is flat over $A$, see Algebra, Lemma 117.3. The flatness of $B$ over $A$ implies that $B$ is torsion free as an $A$-module, see More on Algebra, Lemma 22.9. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of $I_K$). Hence now we have a short exact sequence
\[
0 \to B \xrightarrow{(g_1, \ldots, g_m)} \bigoplus_{j=1}^m B \to M \to 0.
\]
with \( M \) flat over \( A \). Hence for every homomorphism \( A \to \kappa \) where \( \kappa \) is a field, we obtain a short exact sequence

\[
0 \to B \otimes_A \kappa \xrightarrow{(g_1 \otimes 1, \ldots, g_m \otimes 1)} \bigoplus_{j=1}^{1, \ldots, m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0
\]

see Algebra, Lemma 38.12. Reversing the arguments above this means that \( \bigcup D(g_j \otimes 1) \) is scheme theoretically dense in \( \text{Spec}(B \otimes_A \kappa) \). As \( \bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa \) we obtain that \( U_\kappa \) is scheme theoretically dense in \( X_\kappa \) which is what we wanted to prove.

\[\square\]

Suppose given a morphism of schemes \( f : X \to Y \) and a point \( y \in Y \). Recall that the fibre \( X_y \) is isomorphic to the subset \( f^{-1} \{ \{ y \} \} \) of \( X \) with induced topology, see Schemes, Lemma 18.5. Suppose given a closed subset \( T \subset X_y \). Let \( T \) be the closure of \( T(y) \) in \( X \). Endow \( T \) with the induced reduced scheme structure. Then \( T \) is a closed subscheme of \( X \) with the property that \( T_y = T(y) \) set-theoretically. In fact \( T \) is the smallest closed subscheme of \( X \) with this property. Thus it is “harmless” to denote a closed subset of \( X_y \) by \( T \) if we so desire. In the following lemma we apply this to the generic fibre of \( f \).

**Lemma 22.5.** Let \( f : X \to Y \) be a finite type morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \). Let \( X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta} \) be a covering of the generic fibre by closed subsets of \( X_\eta \). Let \( Z_i \) be the closure of \( Z_{i,\eta} \) in \( X \) (see discussion above). Then there exists a nonempty open \( V \subset Y \) such that \( X_y = Z_{1,y} \cup \ldots \cup Z_{n,y} \) for all \( y \in V \).

**Proof.** If \( Y \) is Noetherian then \( U = X \setminus (Z_1 \cup \ldots \cup Z_n) \) is of finite type over \( Y \) and we can directly apply Lemma 22.1 to get that \( U_Y = \emptyset \) for a nonempty open \( V \subset Y \). In general we argue as follows. As the question is topological we may replace \( Y \) by its reduction. Thus \( Y \) is integral, see Properties, Lemma 3.4. After shrinking \( Y \) we may assume that \( X \to Y \) is flat, see Morphisms, Proposition 26.1. In this case every point \( x \) in \( X_y \) is a specialization of a point \( x' \in X_y \) by Morphisms, Lemma 24.9. As the \( Z_i \) are closed in \( X \) and cover the generic fibre this implies that \( X_y = \bigcup Z_{i,y} \) for \( y \in Y \) as desired.

\[\square\]

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

**Lemma 22.6.** Let \( f : X \to Y \) be a morphism of schemes. Let \( \eta \in Y \) be a generic point of an irreducible component of \( Y \). Then \( (X_\eta)_{\text{red}} = (X_{\text{red}})_\eta \).

**Proof.** Choose an affine neighbourhood \( \text{Spec}(A) \subset Y \) of \( \eta \). Choose an affine open \( \text{Spec}(B) \subset X \) mapping into \( \text{Spec}(A) \) via the morphism \( f \). Let \( \mathfrak{p} \subset A \) be the minimal prime corresponding to \( \eta \). Let \( B_{\text{red}} \) be the quotient of \( B \) by the nilradical \( \sqrt{(0)} \). The algebraic content of the lemma is that \( C = B_{\text{red}} \otimes_A \kappa(\mathfrak{p}) \) is reduced. Denote \( I \subset A \) the nilradical so that \( A_{\text{red}} = A/I \). Denote \( \mathfrak{p}_{\text{red}} = \mathfrak{p}/I \) which is a minimal prime of \( A_{\text{red}} \) with \( \kappa(\mathfrak{p}) = \kappa(\mathfrak{p}_{\text{red}}) \). Since \( A \to B_{\text{red}} \) and \( A \to \kappa(\mathfrak{p}) \) both factor through \( A \to A_{\text{red}} \) we have \( C = B_{\text{red}} \otimes_{A_{\text{red}}} \kappa(\mathfrak{p}_{\text{red}}) \). Now \( \kappa(\mathfrak{p}_{\text{red}}) = (A_{\text{red}})_{\mathfrak{p}_{\text{red}}} \) is a localization by Algebra, Lemma 24.1. Hence \( C \) is a localization of \( B_{\text{red}} \) (Algebra, Lemma 11.15) and hence reduced.

\[\square\]
**Lemma 22.7.** Let \( f : X \to Y \) be a morphism of schemes. Assume that \( Y \) is irreducible and \( f \) is of finite type. There exists a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X_V \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & V \\
\end{array}
\]

where

1. \( V \) is a nonempty open of \( Y \),
2. \( X_V = V \times_Y X \),
3. \( g : Y' \to V \) is a finite universal homeomorphism,
4. \( X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red} \),
5. \( g' \) is a finite universal homeomorphism,
6. \( Y' \) is an integral affine scheme,
7. \( f' \) is flat and of finite presentation, and
8. the generic fibre of \( f' \) is geometrically reduced.

**Proof.** Let \( V = \text{Spec}(A) \) be a nonempty affine open of \( Y \). By assumption the Jacobson radical of \( A \) is a prime ideal \( p \). Let \( K = \kappa(p) \). Let \( p \) be the characteristic of \( K \) if positive and 1 if the characteristic is zero. By Varieties, Lemma [6.11] there exists a finite purely inseparable field extension \( K \subset K' \) such that \( X_{K'} \) is geometrically reduced over \( K' \). Choose elements \( x_1, \ldots, x_n \in K' \) which generate \( K' \) over \( K \). Let \( A' \subset K' \) be the finite \( A \)-subalgebra of \( K' \) generated by \( x_1, \ldots, x_n \). Note that \( A' \) is a domain with fraction field \( K' \). By Algebra, Lemma [45.7] we see that \( A \to A' \) is a universal homeomorphism. Set \( Y' = \text{Spec}(A') \). Set \( X' = (Y' \times_Y X)_{red} \). The generic fibre of \( X' \to Y' \) is \((X_K)_{red}\) by Lemma [22.6] which is geometrically reduced by construction. Note that \( X' \to X_V \) is a finite universal homeomorphism as the composition of the reduction morphism \( X' \to Y' \times_Y X \) (see Morphisms, Lemma [43.6]) and the base change of \( g \). At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking \( Y' \) and hence \( V \), see Morphisms, Proposition [26.1].

**Lemma 22.8.** Let \( f : X \to Y \) be a morphism of schemes. Assume that \( Y \) is irreducible and \( f \) is of finite type. There exists a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X_V \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & V \\
\end{array}
\]

where

1. \( V \) is a nonempty open of \( Y \),
2. \( X_V = V \times_Y X \),
3. \( g : Y' \to V \) is surjective finite étale,
4. \( X' = Y' \times_Y X = Y' \times_V X_V \),
5. \( g' \) is surjective finite étale,
6. \( Y' \) is an irreducible affine scheme, and
7. all irreducible components of the generic fibre of \( f' \) are geometrically irreducible.
Proof. Let $V = \text{Spec}(A)$ be a nonempty affine open of $Y$. By assumption the Jacobson radical of $A$ is a prime ideal $p$. Let $K = \kappa(p)$. By Varieties, Lemma \ref{varieties-lemma} there exists a finite separable field extension $K \subset K'$ such that all irreducible components of $X_{K'}$ are geometrically irreducible over $K'$. Choose an element $\alpha \in K'$ which generates $K'$ over $K$, see Fields, Lemma \ref{fields-lemma}. Let $P(T) \in K[T]$ be the minimal polynomial for $\alpha$ over $K$. After replacing $\alpha$ by $f\alpha$ for some $f \in A$, $f \not\in p$ we may assume that there exists a monic polynomial $T^d + a_1T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $q$ lying over $p$, such that $K = \kappa(p) \subset \kappa(q) = K'$ is finite separable, and such that $pA'_q$ is the maximal ideal of $A'_q$. Hence $g : Y' = \text{Spec}(A') \to V = \text{Spec}(A)$ is étale at $q$, see Algebra, Lemma \ref{algebra-lemma}. This means that there exists an open $W \subset \text{Spec}(A')$ such that $g|_W : W \to \text{Spec}(A)$ is étale. Since $g$ is finite and since $q$ is the only point lying over $p$ we see that $Z = g(Y' \setminus W)$ is a closed subset of $V$ not containing $p$. Hence after replacing $V$ by a principal affine open of $V$ which does not meet $Z$ we obtain that $g$ is finite étale. \hfill \qed

23. Relative assassins

\begin{lemma}
Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in \text{Ass}_{X/S}(\mathcal{F})$ and set $Z = \{\xi\} \subset X$. If $f$ is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of $V_s$ are elements of $\text{Ass}_{X/S}(\mathcal{F})$.
\end{lemma}

\begin{proof}
We may replace $S$ by an affine open neighbourhood of $f(\xi)$ and $X$ by an affine open neighbourhood of $\xi$. Hence we may assume $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ and that $f$ is given by the finite type ring map $A \to B$, see Morphisms, Lemma \ref{morphisms-lemma}. Moreover, we may write $\mathcal{F} = \mathcal{M}$ for some finite $B$-module $M$, see Properties, Lemma \ref{properties-lemma}. Let $q \subset B$ be the prime corresponding to $\xi$ and let $p \subset A$ be the corresponding prime of $A$. By assumption $q \in \text{Ass}_B(M \otimes_A \kappa(p))$, see Algebra, Remark \ref{algebra-remark} and Divisors, Lemma \ref{divisors-lemma}. With this notation $Z = V(q) \subset \text{Spec}(B)$. In particular $f(Z) \subset V(p)$. Hence clearly it suffices to prove the lemma after replacing $A$, $B$, and $M$ by $A/pA$, $B/pB$, and $M/pM$. In other words we may assume that $A$ is a domain with fraction field $K$ and $q \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma \ref{algebra-lemma} there exists a nonzero $g \in A$ such that $M_g$ is flat as an $A_q$-module. After replacing $A$ by $A_g$ we may assume that $M$ is flat as an $A$-module.

In this case, by Algebra, Lemma \ref{algebra-lemma} we see that $q$ is also an associated prime of $M$. Hence we obtain an injective $B$-module map $B/q \to M$. Let $Q$ be the cokernel so that we obtain a short exact sequence

$$0 \to B/q \to M \to Q \to 0$$

of finite $B$-modules. After applying generic flatness Algebra, Lemma \ref{algebra-lemma} once more, this time to the $B$-module $Q$, we may assume that $Q$ is a flat $A$-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma \ref{algebra-lemma}. In this situation $(B/q) \otimes_A \kappa(p') \subset M \otimes_A \kappa(p')$ for any prime
Let \( p' \) of \( A \). The lemma follows as a minimal prime \( q' \) of the support of \( (B/q) \otimes_A \kappa(p') \) is an associated prime of \( (B/q) \otimes_A \kappa(p') \) by Divisors, Lemma 2.9

**Lemma 23.2.** Let \( f : X \to Y \) be a morphism of schemes. Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Let \( U \subset X \) be an open subscheme. Assume

1. \( f \) is of finite type,
2. \( \mathcal{F} \) is of finite type,
3. \( Y \) is irreducible with generic point \( \eta \), and
4. \( \text{Ass}_X(f) \cap U \) is not contained in \( U_\eta \).

Then there exists a nonempty open subscheme \( V \subset Y \) such that for all \( y \in V \) the set \( \text{Ass}_Y(f_y) \) is not contained in \( U_y \).

**Proof.** Let \( Z \subset X \) be the scheme theoretic support of \( \mathcal{F} \), see Morphisms, Definition 5.5. Then \( Z_\eta \) is the scheme theoretic support of \( \mathcal{F}_\eta \) (Morphisms, Lemma 24.14). Hence the generic points of irreducible components of \( Z_\eta \) are contained in \( \text{Ass}_X(f) \cap U_\eta \) by Divisors, Lemma 2.9. Hence we see that \( Z_\eta \cap U_\eta = \emptyset \). Thus \( T = Z \setminus U \) is a closed subset of \( Z \) with \( T_\eta = \emptyset \). If we endow \( T \) with the induced reduced scheme structure then \( T \to Y \) is a morphism of finite type. By Lemma 22.1 there is a nonempty open \( V \subset Y \) with \( T_Y = \emptyset \). Then \( V \) works. □

**Lemma 23.3.** Let \( f : X \to Y \) be a morphism of schemes. Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Let \( U \subset X \) be an open subscheme. Assume

1. \( f \) is of finite type,
2. \( \mathcal{F} \) is of finite type,
3. \( Y \) is irreducible with generic point \( \eta \), and
4. \( \text{Ass}_X(f) \cap U_\eta \).

Then there exists a nonempty open subscheme \( V \subset Y \) such that for all \( y \in V \) we have \( \text{Ass}_Y(f_y) \cap U_y \).

**Proof.** (This proof is the same as the proof of Lemma 22.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace \( Y \) by its reduction. Hence we may assume that \( Y \) is integral, see Properties, Lemma 3.4. We may also assume that \( Y = \text{Spec}(A) \) is affine. Then \( A \) is a domain with fraction field \( K \).

As \( f \) is of finite type we see that \( X \) is quasi-compact. Write \( X = X_1 \cup \ldots \cup X_n \) for some affine opens \( X_i \) and set \( f_i = f|_{X_i} \). By assumption the generic fibre of \( U_i = X_i \cap U \) contains \( \text{Ass}_{X_i}(f_{i,\eta}) \). Thus it suffices to prove the result for the triples \( (X_i, f_i, U_i) \), in other words we may assume that \( X \) is affine.

Write \( X = \text{Spec}(B) \). Let \( N \) be a finite \( B \)-module such that \( \mathcal{F} = \tilde{N} \). Note that \( B_K \) is Noetherian as it is a finite type \( K \)-algebra. Hence \( U_\eta \) is quasi-compact. Thus we can find finitely many \( g_1, \ldots, g_m \in B \) such that \( D(g_j) \subset U \) and such that \( U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta \). Since \( \text{Ass}_{X_i}(f_{i,\eta}) \subset U_\eta \) we see that \( N_K \to \bigoplus_j (N_K)_{g_j} \) is injective. By Algebra, Lemma 23.4 this is equivalent to the injectivity of \( N_K \to \bigoplus_{j=1,\ldots,m} N_K, n \mapsto (g_1n, \ldots, g_mn) \). Let \( I \) and \( M \) be the kernel and cokernel of this map over \( A \), i.e., such that we have an exact sequence

\[
0 \to I \to N \xrightarrow{(g_1, \ldots, g_m)} \bigoplus_{j=1,\ldots,m} N \to M \to 0
\]

After replacing \( A \) by \( A_h \) for some nonzero \( h \) we may assume that \( B \) is a flat, finitely presented \( A \)-algebra and that both \( M \) and \( N \) are flat over \( A \), see Algebra, Lemma...
The flatness of $N$ over $A$ implies that $N$ is torsion free as an $A$-module, see More on Algebra, Lemma 22.9. Hence $N \subseteq N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subseteq N \subseteq N_K$ is a subset of $I_K$). Hence now we have a short exact sequence

$$0 \to N' \to \bigoplus_{j = 1, \ldots, m} N \to M \to 0$$

with $M$ flat over $A$. Hence for every homomorphism $A \to \kappa$ where $\kappa$ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \to \bigoplus_{j = 1, \ldots, m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 38.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\text{Ass}_{B \otimes_A \kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that $U_\kappa$ contains $\text{Ass}_{X \otimes \kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove.

**Lemma 23.4.** Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of $f$, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^* \mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case

$$\text{Ass}_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow \text{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

**Proof.** This follows immediately from Divisors, Lemma 7.3. See also Divisors, Remark 7.3.

**Lemma 23.5.** Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set

$$E = \{ y \in Y \mid \text{Ass}_{X_y}(\mathcal{F}_y) \subset U_y \}$$

is locally constructible in $Y$.

**Proof.** Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim} A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.11 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. After possibly increasing $i$ we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to $X$ is isomorphic to $\mathcal{F}$, see Limits, Lemma 10.2. After possibly increasing $i$ one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in $X$ is $U$, see Limits, Lemma 4.11. By Lemma 23.4 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in $Z$. This follows from Lemmas 23.2 and 23.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over $Z$. □
24. Reduced fibres

**Lemma 24.1.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) is nonreduced, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) is nonreduced.

**Proof.** Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' \to Y' \) be the base change of \( f \). Note that \( Y' \to Y \) induces a bijection on points and that \( X' \to X \) identifies fibres. Hence we may assume that \( Y' \) is reduced, i.e., integral, see Properties, Lemma \[8.4 \]. We may also replace \( Y \) by an affine open. Hence we may assume that \( Y = \text{Spec}(A) \) with \( A \) a domain. Denote \( K \) the fraction field of \( A \). Pick an affine open Spec\( (B) = U \subset X \) and a section \( h_\eta \in \Gamma(U, \mathcal{O}_U) = B_K \) which is nonzero and nilpotent. After shrinking \( Y \) we may assume that \( h \) comes from \( h \in \Gamma(U, \mathcal{O}_U) = B \). After shrinking \( Y \) a bit more we may assume that \( h \) is nilpotent. Let \( I = \{ b \in B \mid hb = 0 \} \) be the annihilator of \( h \). Then \( C = B/I \) is a finite type \( A \)-algebra whose generic fiber \( (B/I)_K \) is nonzero (as \( h_\eta \neq 0 \)). We apply generic flatness to \( A \to C \) and \( A \to B/hB \), see Algebra, Lemma \[117.3 \], and we obtain a \( g \in A, g \neq 0 \) such that \( C_g \) is free as an \( A_g \)-module and \( (B/hB)_g \) is flat as an \( A_g \)-module. Replace \( Y \) by \( D(g) \subset Y \). Now we have the short exact sequence

\[
0 \to C \to B \to B/hB \to 0.
\]

with \( B/hB \) flat over \( A \) and with \( C \) nonzero free as an \( A \)-module. It follows that for any homomorphism \( A \to \kappa \) to a field the ring \( C \otimes_A \kappa \) is nonzero and the sequence

\[
0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0
\]

is exact, see Algebra, Lemma \[38.12 \]. Note that \( B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa) \) by right exactness of tensor product. Thus we conclude that multiplication by \( h \) is not zero on \( B \otimes_A \kappa \). This clearly means that for any point \( y \in Y \) the element \( h \) restricts to a nonzero element of \( U_y \), whence \( X_y \) is nonreduced. \( \square \)

**Lemma 24.2.** Let \( f : X \to Y \) be a morphism of schemes. Let \( g : Y' \to Y \) be any morphism, and denote \( f' : X' \to Y' \) the base change of \( f \). Then

\[
\{ y' \in Y' \mid X_{y'} \text{ is geometrically reduced} \} = g^{-1}(\{ y \in Y \mid X_y \text{ is geometrically reduced} \}).
\]

**Proof.** This comes down to the statement that for \( y' \in Y' \) with image \( y \in Y \) the fibre \( X_{y'} = X_y \times_y y' \) is geometrically reduced over \( \kappa(y') \) if and only if \( X_y \) is geometrically reduced over \( \kappa(y) \). This follows from Varieties, Lemma \[6.6 \]. \( \square \)

**Lemma 24.3.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) is not geometrically reduced, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) is not geometrically reduced.

**Proof.** Apply Lemma \[22.7 \] to get

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X' \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & V \rightarrow Y
\end{array}
\]
with all the properties mentioned in that lemma. Let \( \eta' \) be the generic point of \( Y' \). Consider the morphism \( X' \to Y_{\eta'} \) (which is the reduction morphism) and the resulting morphism of generic fibres \( X_{\eta'} \to X_{\eta} \). Since \( X_{\eta'} \) is geometrically reduced, and \( X_{\eta} \) is not this cannot be an isomorphism, see Varieties, Lemma 26.1. Hence \( X_{\eta'} \) is nonreduced. Hence by Lemma 24.1 the fibres of \( X_{\eta'} \to Y' \) are nonreduced at all points \( y' \in V' \) of a nonempty open \( V' \subset Y' \). Since \( g : Y' \to V \) is a homeomorphism Lemma 24.2 proves that \( g(V') \) is the open we are looking for.

\[\text{Lemma 24.4.} \quad \text{Let } f : X \to Y \text{ be a morphism of schemes. Assume}
\]

(1) \( Y \) is irreducible with generic point \( \eta \),
(2) \( X_{\eta} \) is geometrically reduced, and
(3) \( f \) is of finite type.

Then there exists a nonempty open subscheme \( V \subset Y \) such that \( X_V \to V \) has geometrically reduced fibres.

**Proof.** Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' \to Y' \) be the base change of \( f \). Note that \( Y' \to Y \) induces a bijection on points and that \( X' \to X \) identifies fibres. Hence we may assume that \( Y' \) is reduced, i.e., integral, see Properties, Lemma 3.4. We may also replace \( Y \) by an affine open. Hence we may assume that \( Y = \text{Spec}(A) \) with \( A \) a domain. Denote \( K \) the fraction field of \( A \). After shrinking \( Y \) a bit we may also assume that \( X \to Y \) is flat and of finite presentation, see Morphisms, Proposition 26.1.

As \( X_{\eta} \) is geometrically reduced there exists an open dense subset \( V \subset X_{\eta} \) such that \( V \to \text{Spec}(K) \) is smooth, see Varieties, Lemma 25.7. Let \( U \subset X \) be the set of points where \( f \) is smooth. By Morphisms, Lemma 32.15 we see that \( V \subset U_{\eta} \). Thus the generic fibre of \( U \) is dense in the generic fibre of \( X \). Since \( X_{\eta} \) is reduced, it follows that \( U_{\eta} \) is scheme theoretically dense in \( X_{\eta} \), see Morphisms, Lemma 7.8. We note that as \( U \to Y \) is smooth all the fibres of \( U \to Y' \) are geometrically reduced. Thus it suffices to show that, after shrinking \( Y \), for all \( y \in Y \) the scheme \( U_{\eta} \) is scheme theoretically dense in \( X_{\eta} \), see Morphisms, Lemma 7.9. This follows from Lemma 22.3.

\[\text{Lemma 24.5.} \quad \text{Let } f : X \to Y \text{ be a morphism which is quasi-compact and locally of finite presentation. Then the set}
\]

\[ E = \{ y \in Y \mid X_{\eta} \text{ is geometrically reduced} \} \]

is locally constructible in \( Y \).

**Proof.** Let \( y \in Y \). We have to show that there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( E \cap V \) is constructible in \( V \). Thus we may assume that \( Y \) is affine. Then \( X \) is quasi-compact. Choose a finite affine open covering \( X = U_1 \cup \ldots \cup U_n \). Then the fibres of \( U_i \to Y \) at \( y \) form an affine open covering of the fibre of \( X \to Y \) at \( y \). Hence we may assume \( X \) is affine as well. Write \( Y = \text{Spec}(A) \). Write \( A = \text{colim} A_i \) as a directed limit of finite type \( \mathbf{Z} \)-algebras. By Limits, Lemma 10.1 we can find an \( i \) and a morphism \( f_i : X_i \to \text{Spec}(A_i) \) of finite presentation whose base change to \( Y \) recovers \( f \). By Lemma 24.2 it suffices to prove the lemma for \( f_i \). Thus we reduce to the case where \( Y \) is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that \( E \) is constructible in case \( Y \) is a Noetherian scheme. To see this let \( Z \subset Y \) be an irreducible closed subscheme. We have to show that \( E \cap Z \) either contains a nonempty open subset
or is not dense in \( Z \). If \( X_z \) is geometrically reduced, then Lemma 24.4 (applied to the morphism \( X_Z \to Z \)) implies that all fibres \( X_y \) are geometrically reduced for a nonempty open \( V \subset Z \). If \( X_z \) is not geometrically reduced, then Lemma 24.3 (applied to the morphism \( X_Z \to Z \)) implies that all fibres \( X_y \) are geometrically reduced for a nonempty open \( V \subset Z \). Thus we win.

**Lemma 24.6.** Let \( X \to \text{Spec}(R) \) be a proper flat morphism where \( R \) is a discrete valuation ring. If the special fibre is reduced, then both \( X \) and the generic fibre \( X_\eta \) are reduced.

**Proof.** Let \( x \in X \) be a point in the generic fibre \( X_\eta \) such that \( \mathcal{O}_{X, x} \) is nonreduced. Then \( \mathcal{O}_{X, x} \) is nonreduced. Let \( x \leadsto x' \) be a specialization with \( x' \) in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring \( A = \mathcal{O}_{X, x'} \). Let \( \pi \in R \) be a uniformizer. If \( a \in A \) then there exists an \( n \geq 0 \) and an element \( a' \in A \) such that \( a = \pi^n a' \) and \( a' \notin \pi A \). This follows from Krull intersection theorem (Algebra, Lemma 50.4). If \( a \) is nilpotent, so is \( a' \), because \( \pi \) is a nonzerodivisor by flatness of \( A \) over \( R \). But \( a' \) maps to a nonzero element of the reduced ring \( A/\pi A = \mathcal{O}_{X, x'} \). This is a contradiction unless \( A \) is reduced, which is what we wanted to show. \( \square \)

**Lemma 24.7.** Let \( f : X \to Y \) be a flat proper morphism of finite presentation. Then the set \( \{ y \in Y \mid X_y \text{ is geometrically reduced} \} \) is open in \( Y \).

**Proof.** We may assume \( Y \) is affine. Then \( Y \) is a cofiltered limit of affine schemes of finite type over \( \mathbf{Z} \). Hence we can assume \( X \to Y \) is the base change of \( X_0 \to Y_0 \) where \( Y_0 \) is the spectrum of a finite type \( \mathbf{Z} \)-algebra and \( X_0 \to Y_0 \) is flat and proper. See Limits, Lemma 10.1, 8.7, and 13.1. Since the formation of the set of points where the fibres are geometrically reduced commutes with base change (Lemma 24.2), we may assume the base is Noetherian.

Assume \( Y \) is Noetherian. The set is constructible by Lemma 24.5. Hence it suffices to show the set is stable under generalization (Topology, Lemma 19.10). By Properties, Lemma 5.10 we reduce to the case where \( Y = \text{Spec}(R) \), \( R \) is a discrete valuation ring, and the closed fibre \( X_y \) is geometrically reduced. To show: the generic fibre \( X_\eta \) is geometrically reduced.

If not then there exists a finite extension \( L \) of the fraction field of \( R \) such that \( X_L \) is not reduced, see Varieties, Lemma 6.4. There exists a discrete valuation ring \( R' \subset L \) with fraction field \( L \) dominating \( R \), see Algebra, Lemma 119.18. After replacing \( R \) by \( R' \) we reduce to Lemma 24.6. \( \square \)

## 25. Irreducible components of fibres

**Lemma 25.1.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) has \( n \) irreducible components, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) has at least \( n \) irreducible components.

**Proof.** As the question is purely topological we may replace \( X \) and \( Y \) by their reductions. In particular this implies that \( Y \) is integral, see Properties, Lemma 3.4. Let \( X_\eta = X_{1, \eta} \cup \ldots \cup X_{n, \eta} \) be the decomposition of \( X_\eta \) into irreducible components. Let \( X_i \subset X \) be the reduced closed subscheme whose generic fibre is \( X_{i, \eta} \). Note that
\[ Z_{i,j} = X_i \cap X_j \] is a closed subset of \( X_i \) whose generic fibre \( Z_{i,j,y} \) is nowhere dense in \( X_{i,y} \). Hence after shrinking \( Y \) we may assume that \( Z_{i,j,y} \) is nowhere dense in \( X_{i,y} \) for every \( y \in Y \), see Lemma \text{22.3}. After shrinking \( Y \) some more we may assume that \( X_y = \bigcup X_{i,y} \) for \( y \in Y \), see Lemma \text{22.5}. Moreover, after shrinking \( Y \) we may assume that each \( X_i \to Y \) is flat and of finite presentation, see Morphisms, Proposition \text{26.1}. The morphisms \( X_i \to Y \) are open, see Morphisms, Lemma \text{24.10}. Thus there exists an open neighbourhood \( V \) of \( \eta \) which is contained in \( f(X_i) \) for each \( i \). For each \( y \in V \) the schemes \( X_{i,y} \) are nonempty closed subsets of \( X_y \), we have \( X_y = \bigcup X_{i,y} \) and the intersections \( Z_{i,j,y} = X_{i,y} \cap X_{j,y} \) are not dense in \( X_{i,y} \). Clearly this implies that \( X_y \) has at least \( n \) irreducible components. 

**Lemma 25.2.** Let \( f : X \to Y \) be a morphism of schemes. Let \( g : Y' \to Y \) be any morphism, and denote \( f' : X' \to Y' \) the base change of \( f \). Then
\[
\{ y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible} \} = g^{-1}(\{ y \in Y \mid X_y \text{ is geometrically irreducible} \}).
\]

**Proof.** This comes down to the statement that for \( y' \in Y' \) with image \( y \in Y \) the fibre \( X'_{y'} = X_y \times_y y' \) is geometrically irreducible over \( \kappa(y') \) if and only if \( X_y \) is geometrically irreducible over \( \kappa(y) \). This follows from Varieties, Lemma \text{8.2}. 

**Lemma 25.3.** Let \( f : X \to Y \) be a morphism of schemes. Let \( n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\} \) be the function which associates to \( y \in Y \) the number of irreducible components of \( (X_y)_K \) where \( K \) is a separably closed extension of \( \kappa(y) \). This is well defined and if \( g : Y' \to Y \) is a morphism then
\[
n_{X'/Y'} = n_{X/Y} \circ g
\]
where \( X' \to Y' \) is the base change of \( f \).

**Proof.** Suppose that \( y' \in Y' \) has image \( y \in Y \). Suppose \( K \supset \kappa(y) \) and \( K' \supset \kappa(y') \) are separably closed extensions. Then we may choose a commutative diagram
\[
\begin{array}{ccc}
K & \longrightarrow & K' \\
\downarrow & & \downarrow \\
\kappa(y) & \longrightarrow & \kappa(y')
\end{array}
\]
of fields. The result follows as the morphisms of schemes
\[
(X'_{y'})_{K'} \leftrightarrow (X'_{y'})_{K''} = (X_y)_{K''} \leftrightarrow (X_y)_K
\]
induce bijections between irreducible components, see Varieties, Lemma \text{8.7}. 

**Lemma 25.4.** Let \( A \) be a domain with fraction field \( K \). Let \( P \in A[x_1, \ldots, x_n] \). Denote \( K \) the algebraic closure of \( K \). Assume \( P \) is irreducible in \( K[x_1, \ldots, x_n] \). Then there exists a \( f \in A \) such that \( P^f \in \kappa[x_1, \ldots, x_n] \) is irreducible for all automorphisms \( \varphi : A_f \to \kappa \) into fields.

**Proof.** There exists an automorphism \( \Psi \) of \( A[x_1, \ldots, x_n] \) over \( A \) such that \( \Psi(P) = ax_n^d + \text{lower order terms in } x_n \) with \( a \neq 0 \), see Algebra, Lemma \text{114.2}. We may replace \( P \) by \( \Psi(P) \) and we may replace \( A \) by \( A_a \). Thus we may assume that \( P \) is monic in \( x_n \) of degree \( d > 0 \). For \( i = 1, \ldots, n-1 \) let \( d_i \) be the degree of \( P \) in \( x_i \).
Note that this implies that $P^\varphi$ is monic of degree $d$ in $x_n$ and has degree $\leq d_i$ in $x_i$ for every homomorphism $\varphi : A \to \kappa$ where $\kappa$ is a field. Thus if $P^\varphi$ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with $Q_1, Q_2$ monic of degree $e_1, e_2 \geq 0$ in $x_n$ with $e_1 + e_2 = d$ and having degree $\leq d_i$ in $x_i$ for $i = 1, \ldots, n - 1$. In other words we can write

$$Q_j = x_n^{e_j} + \sum_{0 \leq l < e_j} \left( \sum_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l$$

where the sum is over the set $\mathcal{L}$ of multi-indices $L$ of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the $A$-algebra

$$B_{e_1, e_2} = A[\{a_{1,l,L} \} \cap_{l \leq l_1, L \in \mathcal{L}}, \{a_{2,l,L} \} \cap_{l \leq l_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial $P - Q_1 Q_2 \in A[\{a_{1,l,L} \} \cap_{l \leq l_1, L \in \mathcal{L}}, \{a_{2,l,L} \} \cap_{l \leq l_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$. By the Hilbert Nullstellensatz, see Algebra, Theorem 33.1 this means that $B_{e_1, e_2} \otimes_A K = 0$. As $B_{e_1, e_2}$ is a finitely generated $A$-algebra this signifies that we can find an $f_{e_1, e_2} \in A$ such that $(B_{e_1, e_2})_{f_{e_1, e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1, e_2}} \to \kappa$ is a homomorphism to a field, then $P^\varphi$ does not have a factorization $P^\varphi = Q_1 Q_2$ with $Q_1$ of degree $e_1$ in $x_n$ and $Q_2$ of degree $e_2$ in $x_n$. Thus taking $f = \prod_{e_1, e_2 \geq 0, e_1 + e_2 = d} f_{e_1, e_2}$ we win. □

**Lemma 25.5.** Let $f : X \to Y$ be a morphism of schemes. Assume

1. $Y$ is irreducible with generic point $\eta$.
2. $X_\eta$ is geometrically irreducible, and
3. $f$ is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

**First proof of Lemma 25.5** We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X_V & \xrightarrow{f} & X \\
\downarrow{f'} & & \downarrow{f} & & \\
Y' & \xrightarrow{g} & V & \xrightarrow{f} & Y
\end{array}$$

as in Lemma 22.7. Note that the generic fibre of $f'$ is the reduction of the geometric fibre of $f$ (see Lemma 22.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism $f'$. Then after shrinking $V$ all the fibres of $f'$ are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_Y X_V$ are geometrically irreducible. Hence by Lemma 25.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \text{Spec}(A)$ with $A$ a domain.
Let \( x \in X_\eta \) be the generic point. As \( X_\eta \) is geometrically irreducible and reduced we see that \( L = \kappa(x) \) is a finitely generated extension of \( K = \kappa(\eta) \) which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 43.4 and 43.6. In particular the field extension \( K \subset L \) is separable, see Algebra, Lemma 43.1. Hence we can find \( x_1, \ldots, x_r + 1 \in L \) which generate \( L \) over \( K \) and such that \( x_1, \ldots, x_r \) is a transcendence basis for \( L \) over \( K \), see Algebra, Lemma 41.3. Let \( P \in K(x_1, \ldots, x_r)[T] \) be the minimal polynomial for \( x_{r+1} \). Clearing denominators we may assume that \( P \) has coefficients in \( A[x_1, \ldots, x_r] \). Note that as \( L \) is geometrically reduced and geometrically irreducible over \( K \), the polynomial \( P \) is irreducible in \( \overline{K}[x_1, \ldots, x_r, T] \) where \( \overline{K} \) is the algebraic closure of \( K \). Denote 
\[
B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))
\]
and set \( X' = \text{Spec}(B') \). By construction the fraction field of \( B' \) is isomorphic to \( L = \kappa(x) \) as \( K \)-extensions. Hence there exists an open \( U \subset X \), and open \( U' \subset X' \) and a \( Y \)-isomorphism \( U \to U' \), see Morphisms, Lemma 48.7. Here is a diagram:
\[
\begin{array}{ccc}
X & \xleftarrow{\sim} & U \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\sim} & U' \\
& & \downarrow \\
& & X' = \text{Spec}(B')
\end{array}
\]
Note that \( U_\eta \subset X_\eta \) and \( U'_\eta \subset X'_\eta \) are dense opens. Thus after shrinking \( Y \) by applying Lemma 22.3 we obtain that \( U_\eta \) is dense in \( X_\eta \) and \( U'_\eta \) is dense in \( X'_\eta \) for all \( y \in Y \). Thus it suffices to prove the lemma for \( X' \to Y \) which is the content of Lemma 25.4.

**Second proof of Lemma 25.5** Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' \to X \) be the reduction of \( X \). Note that \( X' \to X \to Y \) factors through \( Y' \), see Schemes, Lemma 12.7. As \( Y' \to Y \) and \( X' \to X \) are universal homeomorphisms by Morphisms, Lemma 43.6 we see that it suffices to prove the lemma for \( X' \to Y' \). Thus we may assume that \( X \) and \( Y \) are reduced. In particular \( Y \) is integral, see Properties, Lemma 3.4. Thus by Morphisms, Proposition 26.1 there exists a nonempty affine open \( V \subset Y \) such that \( X_V \to V \) is flat and of finite presentation. After replacing \( Y \) by \( V \) we may assume, in addition to (1), (2), (3) that \( Y \) is integral affine, \( X \) is reduced, and \( f \) is flat and of finite presentation. In particular \( f \) is universally open, see Morphisms, Lemma 24.10.

Pick a nonempty affine open \( U \subset X \). Then \( U \to Y \) is flat and of finite presentation with geometrically irreducible generic fibre. The complement \( X_\eta \setminus U_\eta \) is nowhere dense. Thus after shrinking \( Y \) we may assume \( U_\eta \subset X_\eta \) is open dense for all \( y \in Y \), see Lemma 22.3. Thus we may replace \( X \) by \( U \) and we reduce to the case where \( Y \) is integral affine and \( X \) is reduced affine, flat and of finite presentation over \( Y \) with geometrically irreducible generic fibre \( X_\eta \).

Write \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \). Then \( A \) is a domain, \( B \) is reduced, \( A \to B \) is flat of finite presentation, and \( B_K \) is geometrically irreducible over the fraction field \( K \) of \( A \). In particular we see that \( B_K \) is a domain. Let \( L \) be the fraction field of \( B_K \). Note that \( L \) is a finitely generated field extension of \( K \) as \( B \) is an \( A \)-algebra of finite presentation. Let \( K \subset K' \) be a finite purely inseparable extension such that \((L \otimes_K K')_{\text{red}} \) is a separably generated field extension, see Algebra, Lemma 44.3. Choose \( x_1, \ldots, x_n \in K' \) which generate the field extension \( K' \) over \( K \), and
such that \( x_i^{q_i} \in A \) for some prime power \( q_i \) (proof existence \( x_i \) omitted). Let \( A' \) be the \( A \)-subalgebra of \( K' \) generated by \( x_1, \ldots, x_n \). Then \( A' \) is a finite \( A \)-subalgebra \( A' \subseteq K' \) whose fraction field is \( K' \). Note that \( \text{Spec}(A') \to \text{Spec}(A) \) is a universal homeomorphism, see Algebra, Lemma \[45.7\]. Hence it suffices to prove the result after base changing to \( \text{Spec}(A') \). We are going to replace \( A \) by \( A' \) and \( B \) by \( (B \otimes_A A')_{\text{red}} \) to arrive at the situation where \( L \) is a separably generated field extension of \( K \). Of course it may happen that \( (B \otimes_A A')_{\text{red}} \) is no longer flat, or of finite presentation over \( A' \), but this can be remedied by replacing \( A' \) by \( A'_f \) for a suitable \( f \in A' \), see Algebra, Lemma \[117.3\].

At this point we know that \( A \) is a domain, \( B \) is reduced, \( A \to B \) is flat and of finite presentation, \( B_K \) is a domain whose fraction field \( L \) is a separably generated field extension of the fraction field \( K \) of \( A \). By Algebra, Lemma \[41.3\] we may write \( L = K(x_1, \ldots, x_{r+1}) \) where \( x_1, \ldots, x_r \) are algebraically independent over \( K \), and \( x_{r+1} \) is separable over \( K(x_1, \ldots, x_r) \). After clearing denominators we may assume that the minimal polynomial \( P \in K(x_1, \ldots, x_r)[T] \) of \( x_{r+1} \) over \( K(x_1, \ldots, x_r) \) has coefficients in \( A[x_1, \ldots, x_r] \). Note that since \( L/K \) is separable and since \( L \) is geometrically irreducible over \( K \), the polynomial \( P \) is irreducible over the algebraic closure \( \overline{K} \) of \( K \). Denote

\[
B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).
\]

By construction the fraction fields of \( B \) and \( B' \) are isomorphic as \( K \)-extensions. Hence there exists an isomorphism of \( A \)-algebras \( B_h \cong B'_h \) for suitable \( h \in B \) and \( h' \in B' \), see Morphisms, Lemma \[48.7\]. In other words \( X \) and \( X' = \text{Spec}(B') \) have a common affine open \( U \). Here is a diagram:

\[
\begin{array}{ccc}
X = \text{Spec}(B) & \xrightarrow{f} & U = \overline{Y} = \text{Spec}(B') = X' \\
\downarrow & & \downarrow \\
Y = \text{Spec}(A) & & 
\end{array}
\]

After shrinking \( Y \) once more (by applying Lemma \[22.3\] to \( Z = X \setminus U \) in \( X \) and \( Z' = X' \setminus U \) in \( X' \)) we see that \( U_y \) is dense in \( X_y \) and \( U_y' \) is dense in \( X'_y \) for all \( y \in Y \). Thus it suffices to prove the lemma for \( X' \to Y \) which is the content of Lemma \[25.3\].

055A **Lemma 25.6.** Let \( f : X \to Y \) be a morphism of schemes. Let \( n_{X/Y} \) be the function on \( Y \) counting the numbers of geometrically irreducible components of fibres of \( f \) introduced in Lemma \[25.3\]. Assume \( f \) of finite type. Let \( y \in Y \) be a point. Then there exists a nonempty open \( V \subseteq \{y\} \) such that \( n_{X/Y} |_V \) is constant.

**Proof.** Let \( Z \) be the reduced induced scheme structure on \( \{y\} \). Let \( f_Z : X_Z \to Z \) be the base change of \( f \). Clearly it suffices to prove the lemma for \( f_Z \) and the generic point of \( Z \). Hence we may assume that \( Y \) is an integral scheme, see Properties, Lemma \[3.4\]. Our goal in this case is to produce a nonempty open \( V \subseteq Y \) such that \( n_{X/Y} |_V \) is constant.

We apply Lemma \[22.8\] to \( f : X \to Y \) and we get \( g : Y' \to V \subseteq Y \). As \( g : Y' \to V \) is surjective finite étale, in particular open (see Morphisms, Lemma \[34.13\]), it suffices to prove that there exists an open \( V' \subseteq Y' \) such that \( n_{X'/Y'} |_{V'} \) is constant, see...
Lemma 25.3. Thus we see that we may assume that all irreducible components of the generic fibre $X_\eta$ are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let $X_1$ be the closure of $X_{1,\eta}$ in $X$. After shrinking $Y$ we may assume that $X = \bigcup X_i$, see Lemma 22.5. After shrinking $Y$ some more we see that each fibre of $f$ has at least $n$ irreducible components, see Lemma 25.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking $Y$ some more we obtain that $X_{i,y}$ is geometrically irreducible for each $i$ and all $y \in Y$, see Lemma 25.5. Since $X_Y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

Lemma 25.7. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ counting the numbers of geometrically irreducible components of fibres of $f$ introduced in Lemma 25.3. Assume $f$ of finite presentation. Then the level sets

$$E_n = \{ y \in Y \mid n_{X/Y}(y) = n \}$$

of $n_{X/Y}$ are locally constructible in $Y$.

**Proof.** Fix $n$. Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E_n \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim} A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 25.3 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E_n$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in $Z$. Let $\xi \in Z$ be the generic point. Then Lemma 25.6 shows that $n_{X/Y}$ is constant in a neighbourhood of $\xi$ in $Z$. This clearly implies what we want. □

26. Connected components of fibres

Lemma 26.1. Let $f : X \to Y$ be a morphism of schemes. Assume $Y$ irreducible with generic point $\eta$ and $f$ of finite type. If $X_\eta$ has $n$ connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre $X_y$ has at least $n$ connected components.

**Proof.** As the question is purely topological we may replace $X$ and $Y$ by their reductions. In particular this implies that $Y$ is integral, see Properties, Lemma 3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of $X_\eta$ into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of $X$ whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking $Y$ we may assume that $Z_{i,j} = \emptyset$, see Lemma 22.1. After shrinking $Y$ some more we may assume that $X_Y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 22.5. Moreover, after shrinking $Y$ we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 26.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 24.10. Thus there exists an open neighbourhood $V$ of $\eta$ which is contained
in $f(X_i)$ for each $i$. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of $X_y$, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that $X_y$ has at least $n$ connected components. □

**Lemma 26.2.** Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of $f$. Then

$$
\{ y' \in Y' \mid \text{$X_{y'}$ is geometrically connected} \} = g^{-1}( \{ y \in Y \mid \text{$X_y$ is geometrically connected} \}).
$$

**Proof.** This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if $X_y$ is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 7.3. □

**Lemma 26.3.** Let $f : X \to Y$ be a morphism of schemes. Let

$$
n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}
$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where $K$ is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$
n_{X'/Y'} = n_{X/Y} \circ g
$$

where $X' \to Y'$ is the base change of $f$.

**Proof.** Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$
\begin{array}{ccc}
K & \longrightarrow & K' \\
\uparrow & & \uparrow \\
\kappa(y) & \longrightarrow & \kappa(y')
\end{array}
$$

of fields. The result follows as the morphisms of schemes

$$
(X_y)_K' \leftarrow (X_{y'})_{K'} = (X_y)_{K''} \longrightarrow (X_y)_K
$$

induce bijections between connected components, see Varieties, Lemma 7.6. □

**Lemma 26.4.** Let $f : X \to Y$ be a morphism of schemes. Assume

1. $Y$ is irreducible with generic point $\eta$,
2. $X_\eta$ is geometrically connected, and
3. $f$ is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

**Proof.** Choose a diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X_V \\
\downarrow & & \downarrow f \\
Y' & \longrightarrow & Y
\end{array}
$$

as in Lemma 22.8. Note that the generic fibre of $f'$ is geometrically connected (for example by Lemma 26.3). Suppose that the lemma holds for the morphism $f'$. This means that there exists a nonempty open $W \subset Y'$ such that every fibre
of $X' \to Y'$ over $W$ is geometrically connected. Then, as $g$ is an open morphism by Morphisms, Lemma 34.13 all the fibres of $f$ at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 26.3. In this way we see that we may assume that the irreducible components of the generic fibre $X_\eta$ are geometrically irreducible.

Let $Y'$ be the reduction of $Y$, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 26.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that $Y$ is irreducible and reduced (i.e., integral, see Properties, Lemma 3.4) and that the irreducible components of the generic fibre $X_\eta$ are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let $X_i$ be the closure of $X_{i,\eta}$ in $X$. After shrinking $Y$ we may assume that $X = \bigcup X_i$, see Lemma 22.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \amalg J$$

where $(i,j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i,j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking $Y$ we may assume that $Z_{i,j} = \emptyset$ for all $(i,j) \in I$, see Lemma 22.1. After shrinking $Y$ we obtain that $X_{i,y}$ is geometrically irreducible for each $i$ and all $y \in Y$, see Lemma 25.5. After shrinking $Y$ some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i,j) \in J$, see Morphisms, Proposition 26.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 24.10. We claim that

$$V = \bigcap_{(i,j) \in J} f(Z_{i,j})$$

works, i.e., that $X_y$ is geometrically connected for each $y \in V$. Namely, the fact that $X_\eta$ is connected implies that the equivalence relation generated by the pairs in $J$ has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only if $(i,j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. 

055H **Lemma 26.5.** Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ counting the numbers of geometrically connected components of fibres of $f$ introduced in Lemma 26.3. Assume $f$, of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \{y\}$ such that $n_{X/Y}|_V$ is constant.

**Proof.** Let $Z$ be the reduced induced scheme structure on $\{y\}$. Let $f_Z : X_Z \to Z$ be the base change of $f$. Clearly it suffices to prove the lemma for $f_Z$ and the generic point of $Z$. Hence we may assume that $Y$ is an integral scheme, see Properties, Lemma 3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 22.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 34.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X/Y'}|_{V'}$ is constant, see Lemma 25.3. Thus we see that we may assume that all irreducible components of the generic fibre $X_\eta$ are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma
8.15 This implies that also the connected components of \( X_\eta \) are geometrically connected.

At this point suppose that \( X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta} \) is the decomposition of the generic fibre into (geometrically) connected components. In particular \( n_{X/Y}(\eta) = n \). Let \( X_i \) be the closure of \( X_{i,\eta} \) in \( X \). After shrinking \( Y \) we may assume that \( X = \bigcup X_i \), see Lemma \( \ref{lem:decomposition-connected} \). After shrinking \( Y \) some more we see that each fibre of \( f \) has at least \( n \) connected components, see Lemma \( \ref{lem:connected-components} \). Hence \( n_{X/Y}(y) \geq n \) for all \( y \in Y \). After shrinking \( Y \) some more we obtain that \( X_{i,y} \) is geometrically connected for each \( i \) and all \( y \in Y \), see Lemma \( \ref{lem:geometric-connectedness} \). Since \( X_y = \bigcup X_{i,y} \) this shows that \( n_{X/Y}(y) \leq n \) and finishes the proof. \( \square \)

**Lemma 26.6.** Let \( f : X \rightarrow Y \) be a morphism of schemes. Let \( n_{X/Y} \) be the function on \( Y \) counting the numbers of geometric connected components of fibres of \( f \) introduced in Lemma \( \ref{lem:connected-components} \). Assume \( f \) of finite presentation. Then the level sets

\[
E_n = \{ y \in Y \mid n_{X/Y}(y) = n \}
\]

of \( n_{X/Y} \) are locally constructible in \( Y \).

**Proof.** Fix \( n \). Let \( y \in Y \). We have to show that there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( E_n \cap V \) is constructible in \( V \). Thus we may assume that \( Y \) is affine. Write \( Y = \text{Spec}(A) \) and \( A = \text{colim}_i A_i \) as a directed limit of finite type \( \mathbf{Z} \)-algebras. By Limits, Lemma \( \ref{lem:limit-finite-type} \) we can find an \( i \) and a morphism \( f_i : X_i \rightarrow \text{Spec}(A_i) \) of finite presentation whose base change to \( Y \) recovers \( f \). By Lemma \( \ref{lem:connected-components} \) it suffices to prove the lemma for \( f_i \). Thus we reduce to the case where \( Y \) is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma \( \ref{lem:local-constructible-criterion} \) to prove that \( E_n \) is constructible in case \( Y \) is a Noetherian scheme. To see this let \( Z \subset Y \) be an irreducible closed subscheme. We have to show that \( E_n \cap Z \) either contains a nonempty open subset or is not dense in \( Z \). Let \( \xi \in Z \) be the generic point. Then Lemma \( \ref{lem:geometric-connectedness} \) shows that \( n_{X/Y} \) is constant in a neighbourhood of \( \xi \) in \( Z \). This clearly implies what we want. \( \square \)

**Lemma 26.7.** Let \( f : X \rightarrow S \) be a morphism of schemes. Assume that

1. \( S \) is the spectrum of a discrete valuation ring,
2. \( f \) is flat,
3. \( X \) is connected,
4. the closed fibre \( X_s \) is reduced.

Then the generic fibre \( X_\eta \) is connected.

**Proof.** Write \( S = \text{Spec}(R) \) and let \( \pi \in R \) be a uniformizer. To get a contradiction assume that \( X_\eta \) is disconnected. This means there exists a nontrivial idempotent \( e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta}) \). Let \( U = \text{Spec}(A) \) be any affine open in \( X \). Note that \( \pi \) is a nonzerodivisor on \( A \) as \( A \) is flat over \( R \), see More on Algebra, Lemma \( \ref{lem:flatness} \) for example. Then \( e|_U \) corresponds to an element \( e \in A[1/\pi] \). Let \( z \in A \) be an element such that \( e = z/\pi^n \) with \( n \geq 0 \) minimal. Note that \( z^2 = \pi^n z \). This means that \( z \mod \pi A \) is nilpotent if \( n > 0 \). By assumption \( A/\pi A \) is reduced, and hence minimality of \( n \) implies \( n = 0 \). Thus we conclude that \( e \in A ! \). In other words \( e \in \Gamma(X, \mathcal{O}_X) \). As \( X \) is connected it follows that \( e \) is a trivial idempotent which is a contradiction. \( \square \)
27. Connected components meeting a section

055M The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

055L **Situation 27.1.** Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of $f$. For every $y \in Y$ we denote $X_y^0$ the connected component of $X_y$ containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X_y^0$.

055M **Lemma 27.2.** Let $f : X \to Y$, $s : Y \to X$ be as in **Situation 27.1**. If $g : Y' \to Y$ is any morphism, consider the base change diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g'^{-1}(X^0))$.

**Proof.** Let $y' \in Y'$ with image $y \in Y$. We may think of $X_y^0$ as a closed subscheme of $X_y$, see for example Morphisms, Definition 25.3. As $s(y) \in X_y^0$ we conclude from Varieties, Lemma 7.14 that $X_y^0$ is a geometrically connected scheme over $\kappa(y)$. Hence $X_y \times_Y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$. Thus $X_y^0 \times_Y y' \subset (X'_{y'})^0$. The other inclusion $X_y^0 \times_Y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in $X_y$ is a connected subset of $X_y$ which contains $s(y)$.

055N **Lemma 27.3.** Let $f : X \to Y$, $s : Y \to X$ be as in **Situation 27.1**. Assume $f$ of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \{y\}$ such that the inverse image of $X^0$ in the base change $X_V$ is open and closed in $X_V$.

**Proof.** Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\{y\}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of $f$ and $s$. By Lemma 27.2 we have $(X_Z)^0 = (X_Z)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of $Z$. In other words we have reduced the problem to the case where $Y$ is an integral scheme (see Properties, Lemma 3.4) with generic point $\eta$. Our goal is to show that after shrinking $Y$ the subset $X^0$ becomes an open and closed subset of $X$.

Note that the scheme $X_\eta$ is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X_\eta^0 \cup T_\eta$ for some open and closed subset $T_\eta$ of $X_\eta$. Next, let $T \subset X$ be the closure of $T_\eta$ and let $X^{00} \subset X$ be the closure of $X_\eta^0$. Note that $T_\eta$, resp. $X_\eta^0$ is the generic fibre of $T$, resp. $X^{00}$, see discussion preceding Lemma 22.5. Moreover, that lemma implies that after shrinking $Y$ we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X_\eta^0 = \emptyset$. Hence after shrinking $Y$ we may assume that $T \cap X^{00} = \emptyset$, see Lemma 22.1. In particular $X^{00}$ is open in $X$. Note that $X_\eta^0$ is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 7.14. Thus after shrinking $Y$ we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 26.4. At this point it follows that the fibres $X^{00}_y$ are open, closed, and connected subsets of $X_y$ containing $\sigma(y)$. It follows that $X^{00} = X^{00}_y$ and we win. □
Lemma 27.4. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 27.1. If $f$ is of finite presentation then $X^0$ is locally constructible in $X$.

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood $U$ of $x$ such that $X^0 \cap U$ is constructible in $U$. This reduces us to the case where $Y$ is affine. Write $Y = \Spec(A)$ and $A = \colim A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \Spec(A_i)$ of finite presentation, endowed with a section $s_i : \Spec(A_i) \to X_i$ whose base change to $Y$ recovers $f$ and the section $s$. By Lemma 27.2 it suffices to prove the lemma for $f_i, s_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

Assume $Y$ is a Noetherian affine scheme. Since $f$ is of finite presentation, i.e., of finite type, we see that $X$ is a Noetherian scheme too, see Morphisms, Lemma 14.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of $Z$ or is not dense in $Z$, see Topology, Lemma 16.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 27.3 there exists a nonempty open subset $V \subset \{y\}$ such that $X^0 \cap X_V$ is open and closed in $X_V$. Since $f(Z) \subset \{y\}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of $Z$. It follows that $X^0 \cap W$ is open and closed in $W$. Since $W$ is irreducible we see that $X^0 \cap W$ is either empty or equal to $W$. This proves the lemma.

Lemma 27.5. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 27.1. Let $y \in Y$ be a point. Assume

1. $f$ is of finite presentation and flat, and
2. the fibre $X_y$ is geometrically reduced.

Then $X^0$ is a neighbourhood of $X^0_y$ in $X$.

Proof. We may replace $Y$ with an affine open neighbourhood of $y$. Write $Y = \Spec(A)$ and $A = \colim A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \Spec(A_i)$ of finite presentation, endowed with a section $s_i : \Spec(A_i) \to X_i$ whose base change to $Y$ recovers $f$ and the section $s$. After possibly increasing $i$ we may also assume that $f_i$ is flat, see Limits, Lemma 8.7. Let $y_i$ be the image of $y$ in $Y_i$. Note that $X_{y_i} = (X_{i,y_i}) \times_{Y_i} y_i$. Hence $X_{i,y_i}$ is geometrically reduced, see Varieties, Lemma 6.6. By Lemma 27.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

Assume $Y$ is the spectrum of a Noetherian ring. Since $f$ is of finite presentation, i.e., of finite type, we see that $X$ is a Noetherian scheme too, see Morphisms, Lemma 14.6. Let $x \in X^0$ be a point lying over $y$. By Topology, Lemma 16.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through $x$ the intersection $X^0 \cap Z$ is dense in $Z$. In particular it suffices to prove that the generic point $x' \in Z$ is in $X^0$. By Properties, Lemma 5.10 we can find a discrete valuation ring $R$ and a morphism $\Spec(R) \to X$ which maps the special point to $x$ and the generic point to $x'$. We are going to think of $\Spec(R)$ as a scheme over $Y$ via the composition $\Spec(R) \to X \to Y$. By Lemma 27.2 we have that $(X^0_R)$ is the inverse image of $X^0$. By construction we have a second section $t : \Spec(R) \to X_R$ (besides the base change $s_R$ of $s$) of the structure morphism $X_R \to \Spec(R)$ such that $t(\eta_R)$ is a point of $X_R$ which maps to $x'$ and $t(0_R)$ is a point of $X_R$ which maps to $x$. Note
that \( t(0_R) \) is in \((X_R)^0\) and that \( t(\eta_R) \twoheadrightarrow t(0_R) \). Thus it suffices to prove that this implies that \( t(\eta_R) \in (X_R)^0 \). Hence it suffices to prove the lemma in the case where \( Y \) is the spectrum of a discrete valuation ring and \( y \) its closed point.

Assume \( Y \) is the spectrum of a discrete valuation ring and \( y \) is its closed point. Our goal is to prove that \( X^0 \) is a neighbourhood of \( X_y^0 \). Note that \( X_y^0 \) is open and closed in \( X_y \) as \( X_y \) has finitely many irreducible components. Hence the complement \( C = X_y \setminus X_y^0 \) is closed in \( X \). Thus \( U = X \setminus C \) is an open neighbourhood of \( X_y^0 \) and \( U^0 = X^0 \). Hence it suffices to prove the result for the morphism \( U \to Y \). In other words, we may assume that \( X_y \) is connected. Suppose that \( X \) is disconnected, say \( X = X_1 \amalg \ldots \amalg X_n \) is a decomposition into connected components. Then \( s(Y) \) is completely contained in one of the \( X_i \). Say \( s(Y) \subset X_1 \). Then \( X^0 \subset X_1 \). Hence we may replace \( Y \) by \( X_1 \) and assume that \( X \) is connected. At this point Lemma \ref{lemma-connectedness} implies that \( X_\eta \) is connected, i.e., \( X^0 = X \) and we win.

\[\begin{array}{l}
\text{Lemma 27.6.} \quad \text{Let } f : X \to Y, \ s : Y \to X \text{ be as in Situation } 27.1. \text{ Assume }\\
\quad (1) \ f \text{ is of finite presentation and flat, and} \\
\quad (2) \ \text{all fibres of } f \ \text{are geometrically reduced.} \\
\text{Then } X^0 \text{ is open in } X.
\end{array}\]

\textbf{Proof.} This is an immediate consequence of Lemma \ref{lemma-flatness}. \hfill \square

**28. Dimension of fibres**

\[\begin{array}{l}
\text{Lemma 28.1.} \quad \text{Let } f : X \to Y \text{ be a morphism of schemes. Assume } Y \text{ irreducible with generic point } \eta \text{ and } f \text{ of finite type. If } X_\eta \text{ has dimension } n, \text{ then there exists a nonempty open } V \subset Y \text{ such that for all } y \in V \text{ the fibre } X_y \text{ has dimension } n.
\end{array}\]

\textbf{Proof.} Let \( Z = \{ x \in X \mid \dim_x(X_{f(x)}) > n \} \). By Morphisms, Lemma \ref{lemma-finite-type-open} this is a closed subset of \( X \). By assumption \( Z_\eta = \emptyset \). Hence by Lemma \ref{lemma-open-irreducible} we may shrink \( Y \) and assume that \( Z = \emptyset \). Let \( Z' = \{ x \in X \mid \dim_x(X_{f(x)}) = n - 1 \} = \{ x \in X \mid \dim_x(X_{f(x)}) = n \} \). As before this is a closed subset of \( X \). By assumption we have \( Z' _\eta \neq \emptyset \). Hence after shrinking \( Y \) we may assume that \( Z' \to Y \) is surjective, see Lemma \ref{lemma-surface-surface}. Hence we win. \hfill \square

\[\begin{array}{l}
\text{Lemma 28.2.} \quad \text{Let } f : X \to Y \text{ be a morphism of finite type. Let }
\quad n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}
\text{be the function which associates to } y \in Y \text{ the dimension of } X_y. \text{ If } g : Y' \to Y \text{ is a morphism then }
\quad n_{X'/Y'} = n_{X/Y} \circ g
\text{where } X' \to Y' \text{ is the base change of } f.
\end{array}\]

\textbf{Proof.} This follows from Morphisms, Lemma \ref{lemma-base-change-dimension}. \hfill \square

\[\begin{array}{l}
\text{Lemma 28.3.} \quad \text{Let } f : X \to Y \text{ be a morphism of schemes. Let } n_{X/Y} \text{ be the function on } Y \text{ giving the dimension of fibres of } f \text{ introduced in Lemma } 28.2. \text{ Assume } f \text{ of finite presentation. Then the level sets }
\quad E_n = \{ y \in Y \mid n_{X/Y}(y) = n \}
\text{of } n_{X/Y} \text{ are locally constructible in } Y.
\end{array}\]
Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E_n \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim} A_i$ as a directed limit of finite type $\mathbb{Z}$-algebras. By Limits, Lemma 10.1, we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 28.2, it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E_n$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \setminus Z$ either contains a nonempty open subset or is not dense in $Z$. Let $\xi \in Z$ be the generic point. Then Lemma 28.1 shows that $n_{X/Y}$ is constant in a neighbourhood of $\xi$ in $Z$. This implies what we want. \qed

**Lemma 28.4.** Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Then $n_{X/Y}$ is lower semi-continuous.

**Proof.** Let $W \subset X$, $W = \coprod_{d \geq 0} U_d$ be the open constructed in Lemmas 20.7 and 20.9. Let $y \in Y$ be a point. If $n_{X/Y}(y) = \dim(X_y) = n$, then $y$ is in the image of $U_n \to Y$. By Morphisms, Lemma 24.10 we see that $f(U_n)$ is open in $Y$. Hence there is an open neighbourhood of $y$ where $n_{X/Y}$ is $\geq n$. \qed

**Lemma 28.5.** Let $f : X \to Y$ be a proper morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Then $n_{X/Y}$ is upper semi-continuous.

**Proof.** Let $Z_d = \{ x \in X \mid \dim_x(X_{f(x)}) > d \}$. Then $Z_d$ is a closed subset of $X$ by Morhisms, Lemma 27.4. Since $f$ is proper, $f(Z_d)$ is closed. Since $y \in f(Z_d) \iff n_{X/Y}(y) > d$ we see that the lemma is true. \qed

**Lemma 28.6.** Let $f : X \to Y$ be a proper, flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Then $n_{X/Y}$ is locally constant.

**Proof.** Immediate consequence of Lemmas 28.4 and 28.5. \qed

### 29. Theorem of the cube

The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

**Lemma 29.1.** Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : T \to S$ consider the base change diagram

\[
\begin{array}{ccc}
X_T & \longrightarrow & X \\
\downarrow p & & \downarrow f \\
T & \longrightarrow & S
\end{array}
\]

Assume $\mathcal{O}_T \to p_* \mathcal{O}_{X_T}$ is an isomorphism for all $g : T \to S$. Then there exists an immersion $j : Z \to S$ of finite presentation such that a morphism $g : T \to S$ factors through $Z$ if and only if there exists a finite locally free $\mathcal{O}_T$-module $\mathcal{N}$ with $p^* \mathcal{N} \cong q^* \mathcal{E}$. 
Proof. Observe that the fibres $X_s$ of $f$ are connected by our assumption that $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Thus the rank of $E$ is constant on the fibres. Since $f$ is open (Morphisms, Lemma 24.10) and closed we conclude that there is a decomposition $S = \bigsqcup S_r$ of $S$ into open and closed subschemes such that $E$ has constant rank $r$ on the inverse image of $S_r$. Thus we may assume $E$ has constant rank $r$. We will denote $\mathcal{E} = \mathcal{Hom}(E, \mathcal{O}_X)$ the dual rank $r$ module.

By cohomology and base change (more precisely by Derived Categories of Schemes, Lemma 27.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of $S$ and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}'$. Since there is never any cohomology in degrees $< 0$, we see that $E$ and $E'$ have (locally) tor-amplitude in $[0, b]$ for some $b$. Observe that for any $g : T \to S$ we have $p_*(q^*E) = H^0(Lg^*E)$ and $p_*(q^*E') = H^0(Lg^*E')$. Let $j : Z \to S$ and $j' : Z' \to S$ be immersions of finite presentation constructed in Derived Categories of Schemes, Lemma 28.4 for $E$ and $E'$ with $a = 0$ and $r = r'$; these are roughly speaking characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are finite locally free modules compatible with pullback.

Let $g : T \to S$ be a morphism. If there exists an $N$ as in the lemma, then, using the projection formula Cohomology, Lemma 49.2 we see that the modules

$$p_*(q^*L) \cong p_*(p^*N) \cong N \otimes_{\mathcal{O}_T} p_*\mathcal{O}_{XT} \cong N'$$

are finite locally free modules of rank $r$ and remain so after any further base change $T' \to T$. Hence in this case $T \to S$ factors through $j$ and through $j'$. Thus we may replace $S$ by $Z \times_S Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}'$ are finite locally free $\mathcal{O}_S$-modules of rank $r$ whose formation commutes with arbitrary change of base (small detail omitted).

In this situation if $g : T \to S$ be a morphism and there exists an $N$ as in the lemma, then the map (cup product in degree 0)

$$p_*(q^*E) \otimes_{\mathcal{O}_T} p_*(q^*E') \longrightarrow \mathcal{O}_T$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on $T$ we may choose a basis of sections $\sigma_1, \ldots, \sigma_r$ in $p_*(q^*E)$ and $\tau_1, \ldots, \tau_r$ in $p_*(q^*E')$ whose products satisfy $\sigma_i \tau_j = \delta_{ij}$. Thinking of $\sigma_i$ as a section of $q^*E$ on $X_T$ and $\tau_j$ as a section of $q^*E'$ on $X_T$, we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X_T}^r \rightarrow q^*E$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : q^*E \longrightarrow \mathcal{O}_{X_T}^r$$

In other words, we see that $p^*p_*q^*E \cong q^*E$. But the condition that the cup product is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. \qed

The lemma above in particular tells us, that if a vector bundle is trivial on fibres for a proper flat family of proper spaces, then it is the pull back of a vector bundle. Let’s spell this out a bit.

**Lemma 29.2.** Let $f : X \to S$ be a flat, proper morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and this remains true after arbitrary base change. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume
(1) \( \mathcal{E}|_{X_s} \) is isomorphic to \( O_{X_s}^{\oplus r_s} \) for all \( s \in S \), and
(2) \( S \) is reduced.

Then \( \mathcal{E} = f^*\mathcal{N} \) for some finite locally free \( O_S \)-module \( \mathcal{N} \).

**Proof.** Namely, in this case the locally closed immersion \( j : Z \rightarrow S \) of Lemma 29.1 is bijective and hence a closed immersion. But since \( S \) is reduced, \( j \) is an isomorphism. \( \square \)

**Lemma 29.3.** Let \( f : X \rightarrow S \) be a proper flat morphism of finite presentation. Let \( \mathcal{L} \) be an invertible \( O_X \)-module. Assume

(1) \( S \) is the spectrum of a valuation ring,
(2) \( \mathcal{L} \) is trivial on the generic fibre \( X_\eta \) of \( f \),
(3) the closed fibre \( X_0 \) of \( f \) is integral,
(4) \( H^0(X_\eta, O_{X_\eta}) \) is equal to the function field of \( S \).

Then \( \mathcal{L} \) is trivial.

**Proof.** Write \( S = \text{Spec}(A) \). We will first prove the lemma when \( A \) is a discrete valuation ring (as this is the case most often used in practice). Let \( \pi \in A \) be a uniformizer. Take a trivializing section \( s \in \Gamma(X_\eta, \mathcal{L}_\eta) \). After replacing \( s \) by \( \pi^n s \) if necessary we can assume that \( s \in \Gamma(X, \mathcal{L}) \). If \( s|_{X_0} = 0 \), then we see that \( s \) is divisible by \( \pi \) (because \( X_0 \) is the scheme theoretic fibre and \( X \) is flat over \( A \)). Thus we may assume that \( s|_{X_0} \) is nonzero. Then the zero locus \( Z(s) \) of \( s \) is contained in \( X_0 \) but does not contain the generic point of \( X_\eta \) (because \( X_0 \) is integral). This means that the \( Z(s) \) has codimension \( \geq 2 \) in \( X \) which contradicts Divisors, Lemma 15.3 unless \( Z(s) = \emptyset \) as desired.

Proof in the general case. Since the valuation ring \( A \) is coherent (Algebra, Example 89.2) we see that \( H^0(X, \mathcal{L}) \) is a coherent \( A \)-module, see Derived Categories of Schemes, Lemma 30.1. Equivalently, \( H^0(X, \mathcal{L}) \) is a finitely presented \( A \)-module (Algebra, Lemma 89.4). Since \( H^0(X, \mathcal{L}) \) is torsion free (by flatness of \( X \) over \( A \)), we see from More on Algebra, Lemma 111.3 that \( H^0(X, \mathcal{L}) = A^{\oplus n} \) for some \( n \). By flat base change (Cohomology of Schemes, Lemma 5.2) we have

\[
K = H^0(X_\eta, O_{X_\eta}) \cong H^0(X_\eta, \mathcal{L}_\eta) = H^0(X, \mathcal{L}) \otimes_A K
\]

where \( K \) is the fraction field of \( A \). Thus \( n = 1 \). Pick a generator \( s \in H^0(X, \mathcal{L}) \). Let \( m \subset A \) be the maximal ideal. Then \( \kappa = A/m = \text{colim} A/\pi \) where this is a filtered colimit over nonzero \( \pi \in m \) (here we use that \( A \) is a valuation ring). Thus \( X_0 = \lim X \times_S \text{Spec}(A/\pi) \). If \( s|_{X_0} \) is zero, then for some \( \pi \) we see that \( s \) restricts to zero on \( X \times_S \text{Spec}(A/\pi) \), see Limits, Lemma 47. But if this happens, then \( \pi^{-1}s \) is a global section of \( \mathcal{L} \) which contradicts the fact that \( s \) is a generator of \( H^0(X, \mathcal{L}) \). Thus \( s|_{X_0} \) is not zero. Let \( Z(s) \subset X \) be the zero scheme of \( s \). Since \( s|_{X_0} \) is not zero and since \( X_0 \) is integral, we see that \( Z(s)_0 \subset X_0 \) is an effective Cartier divisor. Since \( f \) is proper and \( S \) is local, every point of \( Z(s) \) specializes to a point of \( Z(s)_0 \). Thus by Divisors, Lemma 18.9 part (3) we see that \( Z(s) \) is a relative effective Cartier divisor, in particular \( Z(s) \to S \) is flat. Hence if \( Z(s) \) were nonempty, then \( Z(s)_\eta \) would be nonempty which contradicts the fact that \( s|_{X_\eta} \) is a trivialization of \( \mathcal{L}_\eta \). Thus \( Z(s) = \emptyset \) as desired. \( \square \)

**Lemma 29.4.** Let \( f : X \rightarrow S \) and \( \mathcal{E} \) be as in Lemma 29.1 and in addition assume \( \mathcal{E} \) is an invertible \( O_X \)-module. If moreover the geometric fibres of \( f \) are integral, then \( Z \) is closed in \( S \).
**Lemma 29.5.** Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{f} & & \downarrow{f} \\
S & & S
\end{array}
\]

with \(f' : X' \to S\) and \(f : X \to S\) satisfying the hypotheses of Lemma 29.1. Let \(\mathcal{L}\) be an invertible \(\mathcal{O}_X\)-module and let \(\mathcal{L}'\) be the pullback to \(X'\). Let \(Z \subset S\), resp. \(Z' \subset S\) be the locally closed subscheme constructed in Lemma 29.1 for \((f, \mathcal{L})\), resp. \((f', \mathcal{L}')\) so that \(Z \subset Z'\). If \(s \in Z\) and

\[
H^1(X_s, \mathcal{O}) \longrightarrow H^1(X_s', \mathcal{O})
\]

is injective, then \(Z \cap U = Z' \cap U\) for some open neighbourhood \(U\) of \(s\).

**Proof.** We may replace \(S\) by \(Z'\). After shrinking \(S\) to an affine open neighbourhood of \(s\) we may assume that \(\mathcal{L}' = \mathcal{O}_{X'}\). Let \(E = Rf_*\mathcal{L}\) and \(E' = Rf'_*\mathcal{L}' = Rf'_*\mathcal{O}_{X'}\).

These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 27.4). In particular we see that

\[
E \otimes_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})
\]

The second equality because \(s \in Z\). Set \(h_i = \dim_{\kappa(s)} H^i(X_s, \mathcal{O}_{X_s})\). After shrinking \(S\) we can represent \(E\) by a complex

\[
\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots
\]

see More on Algebra, Lemma 71.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 9.7). Similarly, we may assume \(E'\) is represented by a complex

\[
\mathcal{O}_S \to \mathcal{O}_S^{\oplus h'_1} \to \mathcal{O}_S^{\oplus h'_2} \to \ldots
\]

where \(h'_i = \dim_{\kappa(s)} H^i(X'_s, \mathcal{O}_{X'_s})\). By functoriality of cohomology we have a map

\[
E \longrightarrow E'
\]

in \(D(\mathcal{O}_S)\) whose formation commutes with change of base. Since the complex representing \(E\) is a finite complex of finite free modules and since \(S\) is affine, we can choose a map of complexes

\[
\begin{array}{cccccc}
\mathcal{O}_S & \xrightarrow{d} & \mathcal{O}_S^{\oplus h_1} & \xrightarrow{b} & \mathcal{O}_S^{\oplus h_2} & \longrightarrow & \ldots \\
& \downarrow{a} & \downarrow{b} & \downarrow{c} & & & \\
\mathcal{O}_S & \xrightarrow{d'} & \mathcal{O}_S^{\oplus h'_1} & \xrightarrow{c} & \mathcal{O}_S^{\oplus h'_2} & \longrightarrow & \ldots
\end{array}
\]

representing the given map \(E \to E'\). Since \(s \in Z\) we see that the trivializing section of \(\mathcal{L}_s\) pulls back to a trivializing section of \(\mathcal{L}'_s = \mathcal{O}_{X'_s}\). Thus \(a \otimes \kappa(s)\) is an isomorphism, hence after shrinking \(S\) we see that \(a\) is an isomorphism. Finally, we use the hypothesis that \(H^1(X_s, \mathcal{O}) \to H^1(X'_s, \mathcal{O})\) is injective, to see that there
exists a $h_1 \times h_1$ minor of the matrix defining $b$ which maps to a nonzero element in $\kappa(s)$. Hence after shrinking $S$ we may assume that $b$ is injective. However, since $\mathcal{L}' = \mathcal{O}_X$, we see that $d' = 0$. It follows that $d = 0$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over $X$. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking $S$ a bit further. □

Lemma 29.6. Consider $n$ commutative diagrams of schemes

$$
\begin{array}{ccc}
X_i & \rightarrow & X \\
\downarrow f_i & & \downarrow f \\
S & \leftarrow & \\
\end{array}
$$

with $f_i : X_i \rightarrow S$ and $f : X \rightarrow S$ satisfying the hypotheses of Lemma 29.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}_i$ be the pullback to $X_i$. Let $Z \subset S$, resp. $Z_i \subset S$ be the locally closed subscheme constructed in Lemma 29.1 for $(f, \mathcal{L})$, resp. $(f_i, \mathcal{L}_i)$ so that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \rightarrow \bigoplus_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$$

is injective, then $Z \cap U = (\bigcap_{i=1,\ldots,n} Z_i) \cap U$ (scheme theoretic intersection) for some open neighbourhood $U$ of $s$.

Proof. This lemma is a variant of Lemma 29.5 and we strongly urge the reader to read that proof first; this proof is basically a copy of that proof with minor modifications. It follows from the description of (scheme valued) points of $Z$ and the $Z_i$ that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$ where we take the scheme theoretic intersection. Thus we may replace $S$ by the scheme theoretic intersection $\bigcap_{i=1,\ldots,n} Z_i$. After shrinking $S$ to an affine open neighbourhood of $s$ we may assume that $\mathcal{L}_i = \mathcal{O}_{X_i}$ for $i = 1, \ldots, n$. Let $E = Rf_* \mathcal{L}$ and $E_i = Rf_{i,*} \mathcal{L}_i = Rf_{i,*} \mathcal{O}_{X_i}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 27.4). In particular we see that

$$E \otimes_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_{i,j} = \dim_{\kappa(s)} H^j(X_i, \mathcal{O}_{X_i})$. After shrinking $S$ we can represent $E$ by a complex

$$\mathcal{O}_S \rightarrow \mathcal{O}_S^{\oplus h_{1,1}} \rightarrow \mathcal{O}_S^{\oplus h_{2,2}} \rightarrow \ldots$$

see More on Algebra, Lemma 71.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 3.7). Similarly, we may assume $E_i$ is represented by a complex

$$\mathcal{O}_S \rightarrow \mathcal{O}_S^{\oplus h_{1,1}} \rightarrow \mathcal{O}_S^{\oplus h_{2,2}} \rightarrow \ldots$$

where $h_{i,j} = \dim_{\kappa(s)} H^j(X_{i,s}, \mathcal{O}_{X_{i,s}})$. By functoriality of cohomology we have a map

$$E \rightarrow E_i$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing $E$ is a finite complex of finite free modules and since $S$ is affine, we
can choose a map of complexes

\[
\begin{array}{c}
\mathcal{O}_S \xrightarrow{d} \mathcal{O}^{\oplus h_1}_S \xrightarrow{b_1} \mathcal{O}^{\oplus h_2}_S \xrightarrow{c_1} \cdots \\
\mathcal{O}_S \xrightarrow{d_i} \mathcal{O}^{\oplus h_{i,1}}_S \xrightarrow{c_i} \mathcal{O}^{\oplus h_{i,2}}_S \xrightarrow{c_i} \cdots
\end{array}
\]

representing the given map \( E \to E_i \). Since \( s \in Z \) we see that the trivializing section of \( \mathcal{L}_s \) pulls back to a trivializing section of \( \mathcal{L}_{i,s} = \mathcal{O}_{X_{i,s}} \). Thus \( a_i \otimes \kappa(s) \) is an isomorphism, hence after shrinking \( S \) we see that \( a_i \) is an isomorphism. Finally, we use the hypothesis that \( H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1, \ldots, n} H^1(X_{i,s}, \mathcal{O}) \) is injective, to see that there exists a nontrivial element \( \kappa_i \) which maps to a nontrivial element in \( \kappa(s) \). Hence after shrinking \( S \) we may assume that \((b_1, \ldots, b_n) : \mathcal{O}^{\oplus h_1}_S \to \bigoplus_{i=1, \ldots, n} \mathcal{O}^{\oplus h_{i,1}}_S \) is injective. However, since \( \mathcal{L}_i = \mathcal{O}_{X_i} \) we see that \( d_i = 0 \) for \( i = 1, \ldots, n \). It follows that \( d = 0 \) because \( (b_1, \ldots, b_n) \circ d = (\oplus d_i) \circ (a_1, \ldots, a_n) \). In this way we see that the trivializing section of \( \mathcal{L}_s \) lifts to a section of \( \mathcal{L} \) over \( X \). A straightforward topological argument (omitted) shows that this means that \( \mathcal{L} \) is trivial after possibly shrinking \( S \) a bit further.

**Lemma 29.7.** Let \( f : X \to S \) and \( g : Y \to S \) be morphisms of schemes satisfying the hypotheses of Lemma 29.4. Let \( \sigma : S \to X \) and \( \tau : S \to Y \) be sections of \( f \) and \( g \). Let \( s \in S \). Let \( \mathcal{L} \) be an invertible sheaf on \( X \times_S Y \). If \((1 \times \tau)^* \mathcal{L} \) on \( X \times_S Y \), \( (\sigma \times 1)^* \mathcal{L} \) on \( Y \), and \( \mathcal{L}|_{(X \times_S Y)_s} \) are trivial, then there is an open neighbourhood \( U \) of \( s \) such that \( \mathcal{L} \) is trivial over \((X \times_S Y)_U\).

**Proof.** By Künneth (Varieties, Lemma 29.1) the map

\[
H^1(X_s \times_{\text{Spec}(\kappa(s))} Y_s, \mathcal{O}) \to H^1(X_s, \mathcal{O}) \oplus H^1(Y_s, \mathcal{O})
\]

is injective. Thus we may apply Lemma 29.6 to the two morphisms

\[
1 \times \tau : X \to X \times_S Y \quad \text{and} \quad \sigma \times 1 : Y \to X \times_S Y
\]

to conclude.

**Theorem 29.8** (Theorem of the cube). Let \( k \) be a field. Let \( X, Y, Z \) be varieties with \( k \)-rational points \( x, y, z \). Let \( \mathcal{L} \) be an invertible module on \( X \times Y \times Z \). If

1. \( \mathcal{L} \) is trivial over \( x \times Y \times Z \), \( x \times y \times Z \), and \( X \times y \times z \), and
2. \( X \) and \( Y \) are geometrically integral and proper over \( k \),

then \( \mathcal{L} \) is trivial.

**Proof.** Since \( X \) and \( Y \) are geometrically integral and proper over \( k \) the product \( X \times_k Y \) is geometrically integral and proper over \( k \). This implies that \( H^0(X \times Y, \mathcal{O}_{X \times Y}) = k \) and that the same remains true after any base change. Thus we may apply Lemma 29.1 to the morphism

\[
p : X \times Y \times Z \to Z
\]

and the invertible module \( \mathcal{L} \) to get a locally closed subscheme \( Z' \subset Z \) such that \( \mathcal{L}|_{X \times Y \times Z'} \) is the pullback of an invertible module \( \mathcal{N} \) on \( Z' \). By Lemma 29.4 we see that \( Z' \subset Z \) is a closed subscheme. Hence if \( Z' \) contains an open neighbourhood of \( z \), then \( Z' = Z \) and we see that \( \mathcal{L} = p^* \mathcal{N} \). Restricting to \( x \times y \times Z \) we find that \( \mathcal{N} \cong \mathcal{O}_Z \) and \( \mathcal{L} \) is trivial. To get the desired open neighbourhood of \( z \) apply
Lemma \[29.7\] to the morphism \(p\), the point \(z\), and the sections \(\sigma : Z \to X \times Z\) and \(\tau : Z \to Y \times Z\) given by \(x\) and \(y\).

### 30. Limit arguments

05FA Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

**Lemma 30.1.** Let \(f : X \to S\) be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram

\[
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow f_0 & & \downarrow f \\
S_0 & \leftarrow & S
\end{array}
\]

such that

1. \(X_0, S_0\) are affine schemes,
2. \(S_0\) of finite type over \(\mathbb{Z}\),
3. \(f_0\) is finite of finite type.

**Proof.** Write \(S = \text{Spec}(A)\) and \(X = \text{Spec}(B)\). As \(f\) is of finite presentation we see that \(B\) is of finite presentation as an \(A\)-algebra, see Morphisms, Lemma \[20.2\]. Thus the lemma follows from Algebra, Lemma \[126.18\].

05FC **Lemma 30.2.** Let \(f : X \to S\) be a morphism of affine schemes, which is of finite presentation. Let \(\mathcal{F}\) be a quasi-coherent \(\mathcal{O}_X\)-module of finite presentation. Then there exists a diagram as in Lemma \[30.1\] such that there exists a coherent \(\mathcal{O}_{X_0}\)-module \(\mathcal{F}_0\) with \(g^*\mathcal{F}_0 = \mathcal{F}\).

**Proof.** Write \(S = \text{Spec}(A)\), \(X = \text{Spec}(B)\), and \(\mathcal{F} = \mathcal{M}\). As \(f\) is of finite presentation we see that \(B\) is of finite presentation as an \(A\)-algebra, see Morphisms, Lemma \[20.2\]. As \(\mathcal{F}\) is of finite presentation over \(\mathcal{O}_X\) we see that \(M\) is of finite presentation as a \(B\)-module, see Properties, Lemma \[16.2\]. Thus the lemma follows from Algebra, Lemma \[126.18\].

05FD **Lemma 30.3.** Let \(f : X \to S\) be a morphism of affine schemes, which is of finite presentation. Let \(\mathcal{F}\) be a quasi-coherent \(\mathcal{O}_X\)-module of finite presentation and flat over \(S\). Then we may choose a diagram as in Lemma \[30.2\] and sheaf \(\mathcal{F}_0\) such that in addition \(\mathcal{F}_0\) is flat over \(S_0\).

**Proof.** Write \(S = \text{Spec}(A)\), \(X = \text{Spec}(B)\), and \(\mathcal{F} = \mathcal{M}\). As \(f\) is of finite presentation we see that \(B\) is of finite presentation as an \(A\)-algebra, see Morphisms, Lemma \[20.2\]. As \(\mathcal{F}\) is of finite presentation over \(\mathcal{O}_X\) we see that \(M\) is of finite presentation as a \(B\)-module, see Properties, Lemma \[16.2\]. As \(\mathcal{F}\) is flat over \(S\) we see that \(M\) is flat over \(A\), see Morphisms, Lemma \[24.2\]. Thus the lemma follows from Algebra, Lemma \[163.1\].

05FE **Lemma 30.4.** Let \(f : X \to S\) be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma \[30.1\] such that in addition \(f_0\) is flat.

**Proof.** This is a special case of Lemma \[30.3\].
Lemma 30.5. Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ is smooth.

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and as $f$ is smooth we see that $B$ is smooth as an $A$-algebra, see Morphisms, Lemma 32.2. Hence the lemma follows from Algebra, Lemma 137.14. 

Lemma 30.6. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically reduced fibres.

Proof. Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \leftarrow & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 24.5 the set $E \subset S_0$ of points where the fibre of $f_0$ is geometrically reduced is a constructible subset. By Lemma 24.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in $E$ for some $i$. After replacing $S_0$ by $\text{Spec}(A_i)$ and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ are geometrically reduced. 

Lemma 30.7. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically irreducible fibres.

Proof. Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \leftarrow & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 25.7 the set $E \subset S_0$ of points where the fibre of $f_0$ is geometrically irreducible is a constructible subset. By Lemma 25.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in $E$ for some $i$. After replacing $S_0$ by $\text{Spec}(A_i)$ and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ are geometrically irreducible. 

Lemma 30.8. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically connected fibres.

Proof. Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \leftarrow & S
\end{array}
$$
of affine schemes with \( X_0 \rightarrow S_0 \) a finite type morphism of schemes of finite type over \( \mathbb{Z} \). By Lemma 26.6 the set \( E \subset S_0 \) of points where the fibre of \( f_0 \) is geometrically connected is a constructible subset. By Lemma 26.2 we have \( h(S) \subset E \). Write \( S_0 = \text{Spec}(A_0) \) and \( S = \text{Spec}(A) \). Write \( A = \text{colim}_i A_i \) as a direct colimit of finite type \( A_0 \)-algebras. By Limits, Lemma 4.10 we see that \( \text{Spec}(A_i) \rightarrow S_0 \) has image contained in \( E \) for some \( i \). After replacing \( S_0 \) by \( \text{Spec}(A_i) \) and \( X_0 \) by \( X_0 \times_{S_0} \text{Spec}(A_i) \) we see that all fibres of \( f_0 \) are geometrically connected. □

**Lemma 30.9.** Let \( d \geq 0 \) be an integer. Let \( f : X \rightarrow S \) be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension \( d \). Then there exists a diagram as in Lemma 30.7 such that in addition all fibres of \( f_0 \) have dimension \( d \).

**Proof.** Apply Lemma 30.1 to get a cartesian diagram

\[
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow f_0 & & \downarrow f \\
S_0 & \leftarrow & S
\end{array}
\]

of affine schemes with \( X_0 \rightarrow S_0 \) a finite type morphism of schemes of finite type over \( \mathbb{Z} \). By Lemma 28.3 the set \( E \subset S_0 \) of points where the fibre of \( f_0 \) has dimension \( d \) is a constructible subset. By Lemma 28.2 we have \( h(S) \subset E \). Write \( S_0 = \text{Spec}(A_0) \) and \( S = \text{Spec}(A) \). Write \( A = \text{colim}_i A_i \) as a direct colimit of finite type \( A_0 \)-algebras. By Limits, Lemma 4.10 we see that \( \text{Spec}(A_i) \rightarrow S_0 \) has image contained in \( E \) for some \( i \). After replacing \( S_0 \) by \( \text{Spec}(A_i) \) and \( X_0 \) by \( X_0 \times_{S_0} \text{Spec}(A_i) \) we see that all fibres of \( f_0 \) have dimension \( d \). □

**Lemma 30.10.** Let \( f : X \rightarrow S \) be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 29.1). Then there exists a diagram as in Lemma 30.7 such that in addition \( f_0 \) is standard syntomic.

**Proof.** This lemma is a copy of Algebra, Lemma 135.12. □

**Lemma 30.11.** (Noetherian approximation and combining properties.) Let \( P, Q \) be properties of schemes which are stable under base change. Let \( f : X \rightarrow S \) be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams

\[
\begin{array}{ccc}
X_1 & \leftarrow & X \\
\downarrow f_1 & & \downarrow f \\
S_1 & \leftarrow & S
\end{array}
\quad
\begin{array}{ccc}
X_2 & \leftarrow & X \\
\downarrow f_2 & & \downarrow f \\
S_2 & \leftarrow & S
\end{array}
\]

of affine schemes, with \( S_1, S_2 \) of finite type over \( \mathbb{Z} \) and \( f_1, f_2 \) of finite type such that \( f_1 \) has property \( P \) and \( f_2 \) has property \( Q \). Then we can find a cartesian diagram

\[
\begin{array}{ccc}
X_0 & \leftarrow & X \\
\downarrow f_0 & & \downarrow f \\
S_0 & \leftarrow & S
\end{array}
\]

of affine schemes with \( S_0 \) of finite type over \( \mathbb{Z} \) and \( f_0 \) of finite type such that \( f_0 \) has both property \( P \) and property \( Q \).
**Definition 31.1.** Let $S$ be a scheme. Let $s \in S$ be a point.

1. An **étale neighbourhood of** $(S, s)$ is a pair $(U, u)$ together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.

2. A **morphism of étale neighbourhoods** $f : (V, v) \to (U, u)$ of $(S, s)$ is simply a morphism of $S$-schemes $f : V \to U$ such that $f(v) = u$.

3. An **elementary étale neighbourhood** is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

The notion of an elementary étale neighbourhood has many different names in the literature, for example these are sometimes called “étale neighbourhoods” ([Mil80, Page 36] or “strongly étale” ([GR71, Page 108]). Here we follow the convention of the paper [GR71] by calling them elementary étale neighbourhoods.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then $f$ is automatically étale, see Morphisms, Lemma [34.18]. Hence it turns $(V, v)$ into an étale neighbourhood of $(U, u)$. Of course, since the composition of étale morphisms is étale (Morphisms, Lemma [34.3]) we see that conversely any étale neighbourhood $(V, v)$ of $(U, u)$ is an étale neighbourhood of $(S, s)$ as well. We also remark that if $U \subset S$ is an open neighbourhood of $s$, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma [34.9]). We will use these remarks without further mention throughout this section.

Note that $\kappa(s) \subset \kappa(u)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma [34.15].

**Lemma 31.2.** Let $S$ be a scheme. Let $s \in S$. Let $\kappa(s) \subset k$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(s) \subset \kappa(u)$ is isomorphic to $\kappa(s) \subset k$.  

**Proof.** The given pair of diagrams correspond to cocartesian diagrams of rings

\[
\begin{array}{ccc}
B_1 & \to & B \\
\uparrow & & \uparrow \\
A_1 & \to & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
B_2 & \to & B \\
\uparrow & & \uparrow \\
A_2 & \to & A
\end{array}
\]

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of $A$ containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both $A_1$ and $A_2$ are finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type $A_0$-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as $A$-algebras. As $A$ is the directed colimit of its finite type $A_0$-subalgebras, by Limits, Lemma [10.1] we may assume after enlarging $A_0$ that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as $A_0$-algebras. Since properties $P$ and $Q$ are assumed stable under base change we conclude that setting $S_0 = \text{Spec}(A_0)$ and

$X_0 = X_1 \times_{S_1} S_0 = \text{Spec}(B_{0,1}) \cong \text{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0$

works. \qed
Proof. We may assume $S$ is affine. In this case the lemma follows from Algebra, Lemma \[142.15\].

057A Lemma 31.3. Let $S$ be a scheme, and let $s$ be a point of $S$. The category of étale neighborhoods has the following properties:

1. Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of $s$ in $S$. Then there exists a third étale neighborhood $(U, u)$ and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.

2. Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of $s$. Assume $h_1, h_2$ induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood $(U'', u'')$ and a morphism $h : (U'', u'') \to (U, u)$ which equalizes $h_1$ and $h_2$, i.e., such that $h_1 \circ h = h_2 \circ h$.

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both $U_1$ and $U_2$ because étale morphisms are preserved under base change, see Morphisms, Lemma \[34.4\]. There is a point of $U$ mapping to both $u_1$ and $u_2$ for example by the description of points of a fibre product in Schemes, Lemma \[17.5\]. For part (2), define $U''$ as the fibre product

$$
\begin{array}{ccc}
U'' & \longrightarrow & U \\
\downarrow & & \downarrow_{(h_1, h_2)} \\
U' & \Delta & U' \times_S U'.
\end{array}
$$

Since $h_1$ and $h_2$ induce the same map of residue fields $\kappa(u') \to \kappa(u)$, there exists a point $u'' \in U''$ lying over $u'$ with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since $U'$ is étale over $S$, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas \[34.4\] and \[34.3\]). Hence the vertical arrow $(h_1, h_2)$ is étale by Morphisms, Lemma \[34.18\]. Therefore $U''$ is étale over $U'$ by base change, and hence also étale over $S$ (because compositions of étale morphisms are étale). Thus $(U'', u'')$ is a solution to the problem. \[Q.E.D.\]

057B Lemma 31.4. Let $S$ be a scheme, and let $s$ be a point of $S$. The category of elementary étale neighborhoods of $(S, s)$ is cofiltered (see Categories, Definition \[20.1\]).

Proof. This is immediate from the definitions and Lemma \[31.3\]. \[Q.E.D.\]

05KS Lemma 31.5. Let $S$ be a scheme. Let $s \in S$. Then we have

$$
\mathcal{O}^h_{S, s} = \text{colim}_{(U, u)} \mathcal{O}(U)
$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods $(U, u)$ of $(S, s)$.

Proof. Let $\text{Spec}(A) \subset S$ be an affine neighbourhood of $s$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to $s$. With these choices we have canonical isomorphisms $\mathcal{O}_{S, s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods $(U, u)$ such that $U$ is affine and $U \to S$ factors through $\text{Spec}(A)$. In other words, we see that the right hand side is equal to $\text{colim}_{(B, \mathfrak{q})} B$ where the colimit is over étale $A$-algebras $B$ endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Thus the lemma follows from Algebra, Lemma \[151.7\]. \[Q.E.D.\]

We can lift étale neighbourhoods of points on fibres to the total space.
Let \( X \to S \) be a morphism of schemes. Let \( x \in X \) with image \( s \in S \). Let \((V,v) \to (X_s,x)\) be an étale neighbourhood. Then there exists an étale neighbourhood \((U,u) \to (X,x)\) such that there exists a morphism \((U_s,u) \to (V,v)\) of étale neighbourhoods of \((X_s,x)\) which is an open immersion.

**Proof.** We may assume \( X, V, \) and \( S \) affine. Say the morphism \( X \to S \) is given by \( A \to B \) the point \( x \) by a prime \( q \subset B \), the point \( s \) by \( p = A \cap q \), and the morphism \( V \to X_s \) by \( B \otimes_A \kappa(p) \to C \). Since \( \kappa(p) \) is a localization of \( A/p \) there exists an \( f \in A, f \not\in p \) and an étale ring map \( B \otimes_A (A/p)_f \to D \) such that

\[
C = (B \otimes_A \kappa(p)) \otimes_{B \otimes_A (A/p)} D
\]

See Algebra, Lemma \[142.3\] part (9). After replacing \( A \) by \( A_f \) and \( B \) by \( B_f \) we may assume \( D \) is étale over \( B \otimes_A A/p = B/pB \). Then we can apply Algebra, Lemma \[142.10\] This proves the lemma. \( \square \)

### 32. Étale neighbourhoods and Artin approximation

In this section we prove results of the form: if two pointed schemes have isomorphic complete local rings, then they have isomorphic étale neighbourhoods. We will rely on Popescu’s theorem, see Smoothing Ring Maps, Theorem \[12.1\].

**Lemma 32.1.** Let \( S \) be a locally Noetherian scheme. Let \( X, Y \) be schemes locally of finite type over \( S \). Let \( x \in X \) and \( y \in Y \) be points lying over the same point \( s \in S \). Assume \( O_{S,s} \) is a G-ring. Assume further we are given a local \( O_{S,s} \)-algebra map

\[
\varphi : O_{Y,y} \to O_{X,x}^\wedge
\]

between the complete local rings. For every \( N \geq 1 \) there exists an elementary étale neighbourhood \((U,u) \to (X,x)\) and an \( S \)-morphism \( f : U \to Y \) mapping \( u \) to \( y \) such that the diagram

\[
\begin{array}{ccc}
O_{X,x}^\wedge & \longrightarrow & O_{U,u}^\wedge \\
\varphi \downarrow & & \downarrow \\
O_{Y,y} & \longrightarrow & O_{U,u}
\end{array}
\]

commutes modulo \( m_u^N \).

**Proof.** The question is local on \( X \) hence we may assume \( X, Y, S \) are affine. Say \( S = \text{Spec}(R) \), \( X = \text{Spec}(A) \), \( Y = \text{Spec}(B) \). Write \( B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \).

Let \( p \subset A \) be the prime ideal corresponding to \( x \). The local ring \( O_{X,x} = A_p \) is a G-ring by More on Algebra, Proposition \[49.10\]. The map \( \varphi \) is a map

\[
B_q^\wedge \to A_p^\wedge
\]

where \( q \subset B \) is the prime corresponding to \( y \). Let \( a_1, \ldots, a_n \in A_p^\wedge \) be the images of \( x_1, \ldots, x_n \) via \( R[x_1, \ldots, x_n] \to B \to B_q^\wedge \to A_p^\wedge \). Then we can apply Smoothing Ring Maps, Lemma \[13.4\] to get an étale ring map \( A \to A' \) and a prime ideal \( p' \subset A' \) and \( b_1, \ldots, b_n \in A' \) such that \( \kappa(p) = \kappa(p') \), \( a_i - b_i \in (p')^N(A_p^\wedge) \), and \( f_i(b_1, \ldots, b_n) = 0 \) for \( j = 1, \ldots, n \). This determines an \( R \)-algebra map \( B \to A' \) by sending the class of \( x_i \) to \( b_i \in A' \). This finishes the proof by taking \( U = \text{Spec}(A') \to \text{Spec}(B) \) as the morphism \( f \) and \( u = p' \). \( \square \)
Lemma 32.2. Let $S$ be a locally Noetherian scheme. Let $X$, $Y$ be schemes locally of finite type over $S$. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume we have an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

between the complete local rings. Then for every $N \geq 1$ there exists morphisms $(X, x) \leftarrow (U, u) \to (Y, y)$

of pointed schemes over $S$ such that both arrows define elementary étale neighbourhoods and such that the diagram

$$\mathcal{O}_{U,u}^\wedge \quad \xymatrix{ & \mathcal{O}_{U,u}^\wedge \ar[ld] \ar[rd]^\varphi & \\
\mathcal{O}_{Y,y}^\wedge & & \mathcal{O}_{X,x}^\wedge \ar[lu] \ar[ru]}
$$

commutes modulo $m_n^N$.

Proof. We may assume $N \geq 2$. Apply Lemma 32.1 to get $(U, u) \to (X, x)$ and $f : (U, u) \to (Y, y)$. We claim that $f$ is étale at $u$ which will finish the proof. In fact, we will show that the induced map $\mathcal{O}_{Y,y}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism. Having proved this, Lemma 32.1 will show that $f$ is smooth at $u$ and of course $f$ is unramified at $u$ as well, so Morphisms, Lemma 34.5 tells us $f$ is étale at $u$. For a local ring $(R, m)$ we set $\text{Gr}_m(R) = \bigoplus_{n \geq 0} m^n/m^{n+1}$. To prove the claim we look at the induced diagram of graded rings

$$\begin{array}{cccc}
\text{Gr}_m(\mathcal{O}_{U,u}) & & & \\
\text{Gr}_m(\mathcal{O}_{Y,y}) & \xymatrix{ & \text{Gr}_m(\mathcal{O}_{U,u}) \ar[ld] \ar[rd]^\varphi & \\
\text{Gr}_m(\mathcal{O}_{X,x})}
\end{array}
$$

Since $N \geq 2$ this diagram is actually commutative as the displayed graded algebras are generated in degree 1! By assumption the lower arrow is an isomorphism. By More on Algebra, Lemma 42.9 (for example) the map $\mathcal{O}_{Y,y}^\wedge \to \mathcal{O}_{U,u}^\wedge$ is an isomorphism and hence the north-west arrow in the diagram is an isomorphism. We conclude that $f$ induces an isomorphism $\text{Gr}_m(\mathcal{O}_{X,x}) \to \text{Gr}_m(\mathcal{O}_{U,u})$. Using induction and the short exact sequences

$$0 \to \text{Gr}_m^n(R) \to R/m^{n+1} \to R/m^n \to 0$$

for both local rings we conclude (from the snake lemma) that $f$ induces isomorphisms $\mathcal{O}_{Y,y}/m_n^y \to \mathcal{O}_{U,u}/m_n^u$ for all $n$ which is what we wanted to show. ∎

Lemma 32.3. Consider a diagram

$$\begin{array}{cccc}
X & Y \\
\downarrow & \downarrow \\
S & T
\end{array}
$$

with points

$$\begin{array}{cccc}
x & y \\
\downarrow & \downarrow \\
S & T
\end{array}
$$

where $S$ be a locally Noetherian scheme and the morphisms are locally of finite type. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map

$$\sigma : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$$
and a local $\mathcal{O}_{S,s}$-algebra map

$$\varphi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y,\sigma}$$

where $Y_\sigma = Y \times_{T,\sigma} \text{Spec}(\mathcal{O}_{S,s})$ and $y_\sigma$ is the unique point of $Y_\sigma$ lying over $y$. For every $N \geq 1$ there exists a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \times_S V \\
\downarrow & & \downarrow f \\
S & \longrightarrow & V
\end{array}
\begin{array}{ccc}
W & \longrightarrow & Y \times_{T,\tau} V \\
\downarrow & & \downarrow \\
Y & \longrightarrow & T
\end{array}

of schemes over $S$ and points $w \in W$, $v \in V$ such that

1. $v \mapsto s$, $\tau(v) = t$, $f(w) = (x, v)$, and $w \mapsto (y, v)$,
2. $(V, v) \to (S, s)$ is an elementary étale neighbourhood,
3. the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{S,s} & \longrightarrow & \mathcal{O}_{V,v} \\
\downarrow & & \downarrow \\
\mathcal{O}_{T,t} & \longrightarrow & \mathcal{O}_{V,v}
\end{array}
$$

commutes module $m_N^v$,
4. $(W, w) \to (Y \times_{T,\tau} V, (y, v))$ is an elementary étale neighbourhood,
5. the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{Y_\sigma,y_\sigma} \\
\downarrow & & \downarrow \\
\mathcal{O}_{X,z} & \longrightarrow & \mathcal{O}_{X \times_S V, (x, v)}
\end{array}
\begin{array}{ccc}
\mathcal{O}_{Y_\sigma,y_\sigma} / m_N^{y_\sigma} & \longrightarrow & \mathcal{O}_{Y \times_{T,\tau} V, (y, v)} / m_N^{y,v} \\
\downarrow & & \downarrow \cong \\
\mathcal{O}_{W,w} / m_N^w & \longrightarrow & \mathcal{O}_{W,\sigma} / m_N^w
\end{array}
$$

commutes. The equality comes from the fact that $Y_\sigma$ and $Y \times_{T,\tau} V$ are canonically isomorphic over $\mathcal{O}_{V,v} / m_N^v = \mathcal{O}_{S,s} / m_N^s$ by parts (2) and (3).

**Proof.** After replacing $X$, $S$, $T$, $Y$ by affine open subschemes we may assume the diagram in the statement of the lemma comes from applying $\text{Spec}$ to a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \uparrow \text{with primes} \\
R & \longrightarrow & C
\end{array}
\begin{array}{ccc}
P_A & \longrightarrow & P_B \\
\downarrow P_R & & \downarrow P_C
\end{array}
$$

of Noetherian rings and finite type ring maps. In this proof every ring $E$ will be a Noetherian $R$-algebra endowed with a prime ideal $p_E$ lying over $p_R$ and all ring maps will be $R$-algebra maps compatible with the given primes. Moreover, if we write $E^\wedge$ we mean the completion of the localization of $E$ at $p_E$. We will also use without further mention that an étale ring map $E_1 \to E_2$ such that $\kappa(p_{E_1}) = \kappa(p_{E_2})$ induces an isomorphism $E_1^\wedge = E_2^\wedge$ by More on Algebra, Lemma 42.9.

With this notation $\sigma$ and $\varphi$ correspond to ring maps

$$\sigma : C \to R^\wedge \quad \text{and} \quad \varphi : A \longrightarrow (B \otimes_{C,\sigma} R^\wedge)^\wedge$$
Here is a picture

$$A \xrightarrow{\varphi} B \xrightarrow{} B \otimes_{C,\sigma} R^\wedge \xrightarrow{} (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

Observe that $R^\wedge$ is a G-ring by More on Algebra, Proposition 49.6. Thus $B \otimes_{C,\sigma} R^\wedge$ is a G-ring by More on Algebra, Proposition 49.10. By Lemma 32.1 (translated into algebra) there exists an étale ring map $B \otimes_{C,\sigma} R^\wedge \to B'$ inducing an isomorphism $\kappa(p_{B \otimes_{C,\sigma} R^\wedge}) \to \kappa(p_{B'})$ and an $R$-algebra map $A \to B'$ such that the composition $A \to B' \to (B')^\wedge = (B \otimes_{C,\sigma} R^\wedge)^\wedge$ is the same as $\varphi$ modulo $(p_{B \otimes_{C,\sigma} R^\wedge})^N$. Thus we may replace $\varphi$ by this composition because the only way $\varphi$ enters the conclusion is via the commutativity requirement in part (5) of the statement of the lemma. Picture:

$$A \xrightarrow{} B' \xrightarrow{} (B')^\wedge$$

Next, we use that $R^\wedge$ is a filtered colimit of smooth $R$-algebras (Smoothing Ring Maps, Theorem 12.1) because $R_{p_R}$ is a G-ring by assumption. Since $C$ is of finite presentation over $R$ we get a factorization

$$C \to R' \to R^\wedge$$

for some $R \to R'$ smooth, see Algebra, Lemma 126.3. After increasing $R'$ we may assume there exists an étale $B \otimes_{C} R'$-algebra $B''$ whose base change to $B \otimes_{C,\sigma} R^\wedge$ is $B'$, see Algebra, Lemma 142.3. Then $B'$ is the filtered colimit of these $B''$ and we conclude that after increasing $R'$ we may assume there is an $R$-algebra map $A \to B''$ such that $A \to B'' \to B'$ is the previously constructed map (same reference as above). Picture

$$A \xrightarrow{} B'' \xrightarrow{} B' \xrightarrow{} (B')^\wedge$$

and

$$B' = B'' \otimes_{(B \otimes_{C} R')} (B \otimes_{C,\sigma} R^\wedge)$$
This means that we may replace $C$ by $R'$, $\sigma : C \to R^\wedge$ by $R' \to R^\wedge$, and $B$ by $B''$ so that we simplify to the diagram

$$
\begin{array}{cccccc}
A & \longrightarrow & B & \longrightarrow & B \otimes_{C,\sigma} R^\wedge & \longrightarrow \\
| & & | & & | & \\
R & \rightrightarrows & C & \longrightarrow & \sigma \longrightarrow & R^\wedge
\end{array}
$$

with $\varphi$ equal to the composition of the horizontal arrows followed by the canonical map from $B \otimes_{C,\sigma} R^\wedge$ to its completion. The final step in the proof is to apply Lemma 32.4 (or its proof) one more time to $\text{Spec}(C)$ and $\text{Spec}(R)$ over $\text{Spec}(R)$ and the map $C \to R^\wedge$. The lemma produces a ring map $C \to D$ such that $R \to D$ is étale, such that $\kappa(p_R) = \kappa(p_D)$, and such that

$$C \to D \to D^\wedge = R^\wedge$$

is equal to $\sigma : C \to R^\wedge$ modulo $(p_{R^\wedge})^N$. Then we can take

$$V = \text{Spec}(D) \text{ and } W = \text{Spec}(B \otimes_C D)$$

as our solution to the problem posed by the lemma. Namely the diagram

$$\begin{array}{cccccc}
A & \longrightarrow & B \otimes_{C,\sigma} R^\wedge & \longrightarrow & B \otimes_{C,\sigma} R^\wedge / (p_{R^\wedge})^N & \longrightarrow \\
| & & | & & | & \\
A & \longrightarrow & A \otimes_R D & \longrightarrow & B \otimes_R D & \longrightarrow & B \otimes_C D / (p_D)^N
\end{array}$$

commutes because $C \to D \to D^\wedge = R^\wedge$ is equal to $\sigma$ modulo $(p_{R^\wedge})^N$. This proves part (5) and the other properties are immediate from the construction. \qed

0CAX Let $T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ be a local $\mathcal{O}_{S,s}$-algebra map. For every $N \geq 1$ there exists a finite type morphism $(T', t') \to (T, t)$ such that $\sigma$ factors through $\mathcal{O}_{T',t'} \to \mathcal{O}_{S,s}$, and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the maps $\sigma$ and $\sigma'$ agree modulo $m_s^N$.

Proof. We may assume $S$ and $T$ are affine. Say $S = \text{Spec}(R)$ and $T = \text{Spec}(C)$. Let $c_1, \ldots, c_n \in C$ be generators of $C$ as an $R$-algebra. Let $p \subset R$ be the prime ideal corresponding to $s$. Say $p = (f_1, \ldots, f_m)$. After replacing $R$ by a principal localization (to clear denominators in $R_p$) we may assume there exist $r_1, \ldots, r_n \in R$ and $a_{i,1} \in \mathcal{O}_{S,s}$ where $I = (i_1, \ldots, i_m)$ with $\sum i_j = N$ such that

$$\sigma(c_i) = r_i + \sum_{I} a_{i,I} f_{I1} \cdots f_{Im}$$

in $\mathcal{O}_{S,s}^\wedge$. Then we consider

$$C' = C[t_{i,1}]/(c_i - r_i - \sum_{I} t_{i,I} f_{I1} \cdots f_{Im})$$

with $p' = pC' + (t_{i,1})$ and factorization of $\sigma : C \to \mathcal{O}_{S,s}$ through $C'$ given by sending $t_{i,1}$ to $a_{i,1}$. Taking $T' = \text{Spec}(C')$ works because any $\sigma'$ as in the statement of the lemma will send $c_i$ to $r_i$ modulo the maximal ideal to the power $N$. \qed

0CAY Let $Y \to T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ be a local $\mathcal{O}_{S,s}$-algebra map. There exists a finite type morphism $(T', t') \to (T, t)$ such that $\sigma$ factors through...
$\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^{\wedge}$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the closed immersions

$$Y \times_{T,Y} \text{Spec}(\mathcal{O}_{S,s}^{\wedge}) = Y_s \to Y = Y \times_{T,Y} \text{Spec}(\mathcal{O}_{S,s}^{\wedge})$$

have isomorphic conormal algebras.

Proof. A useful observation is that $\kappa(s) = \kappa(t)$ by the existence of $\sigma$. Observe that the statement makes sense as the fibres of $Y_s$ and $Y_{s'}$ over $s \in \text{Spec}(\mathcal{O}_{S,s}^{\wedge})$ are both canonically isomorphic to $Y_t$. We will think of the property “$\sigma'$ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$” as a constraint on $\sigma'$. If we have several such constraints, say given by $(T'_i, t'_i) \to (T, t)$, $i = 1, \ldots, n$ then we can combined them by considering $(T'_1 \times_T \ldots \times_T T'_n, (t'_1, \ldots, t'_n)) \to (T, t)$. We will use this without further mention in the following.

By Lemma [32.4] we can assume that any $\sigma'$ as in the statement of the lemma is the same as $\sigma$ modulo $m^2_s$. Note that the conormal algebra of $Y_t$ in $Y_s$ is just the quasi-coherent graded $\mathcal{O}_{Y_t}$-algebra

$$\bigoplus_{n \geq 0} m^n_s \mathcal{O}_{Y_s} / m^{n+1}_s \mathcal{O}_{Y_s}$$

and similarly for $Y_{s'}$. Since $\sigma$ and $\sigma'$ agree modulo $m^2_s$ we see that these two algebras are the same in degrees 0 and 1. On the other hand, these conormal algebras are generated in degree 1 over degree 0. Hence if there is an isomorphism extending the isomorphism just constructed in degrees 0 and 1, then it is unique.

We may assume $S$ and $T$ are affine. Let $Y = Y_1 \cup \ldots \cup Y_n$ be an affine open covering. If we can construct $(T'_i, t'_i) \to (T, t)$ as in the lemma such that the desired isomorphism (see previous paragraph) exists for $Y_i \to T \to S$ and $\sigma$, then these glue by uniqueness to prove the result for $Y \to T$. Thus we may assume $Y$ is affine.

Write $S = \text{Spec}(R)$, $T = \text{Spec}(C)$, and $Y = \text{Spec}(B)$. Choose a presentation $B = C[x_1, \ldots, x_n] / (f_1, \ldots, f_m)$. Denote $R^{\wedge} = \mathcal{O}_{S,s}^{\wedge}$. Let $a_{kj} \in R^{\wedge}[x_1, \ldots, x_n]$ be polynomials such that

$$\sum_{j=1, \ldots, m} a_{kj} \sigma(f_j) = 0, \quad \text{for } k = 1, \ldots, K$$

is a set of generators for the module of relations among the $\sigma(f_j) \in R^{\wedge}[x_1, \ldots, x_n]$. Thus we have an exact sequence

$$(32.5.1) \quad R^{\wedge}[x_1, \ldots, x_n] \otimes_K R^{\wedge}[x_1, \ldots, x_n] \to R^{\wedge}[x_1, \ldots, x_n] \to B \otimes_{C, \sigma} R^{\wedge} \to 0$$

Let $c$ be an integer which works in the Artin-Rees lemma for both the first and the second map in this sequence and the ideal $m_{R^{\wedge}} R^{\wedge}[x_1, \ldots, x_n]$ as defined in More on Algebra, Section [4] Write

$$a_{kj} = \sum_{I \in \Omega} a_{kj,I} x^I$$

and

$$f_j = \sum_{I \in \Omega} f_{j,I} x^I$$

in multiindex notation where $a_{kj,I} \in R^{\wedge}$, $f_{j,I} \in C$, and $\Omega$ a finite set of multiindices. Then we see that

$$\sum_{j=1, \ldots, m, I, I' \in \Omega, I + I' = I''} a_{kj,I} \sigma(f_{j,I'}) = 0, \quad \text{I'' a multiindex}$$

in $R^{\wedge}$. Thus we take

$$C' = C[t_{jk,k}] / \left( \sum_{j=1, \ldots, m, I, I' \in \Omega, I + I' = I''} t_{jk,k} f_{j,I'} x^I' \right)$$

as defined in Section [4].
Then $\sigma$ factors through a map $\bar{\sigma} : C' \to R^\wedge$ sending $t_{kj,i}$ to $a_{jk,i}$. Thus $T' = \text{Spec}(C')$ comes with a point $t' \in T'$ such that $\sigma$ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$. Let $t_{kj} = \sum t_{kj,i} x^i$ in $C'[x_1, \ldots, x_n]$. Then we see that we have a complex

$$0CB0 \quad (32.5.2) \quad C'[x_1, \ldots, x_n] \otimes K \to C'[x_1, \ldots, x_n] \otimes m \to C'[x_1, \ldots, x_n] \to B \otimes_C C' \to 0$$

which is exact at $C'[x_1, \ldots, x_n]$ and whose base change by $\bar{\sigma}$ gives (32.5.1).

By Lemma 32.4 we can find a further morphism $(T'', t'') \to (T', t')$ such that $\bar{\sigma}$ factors through $\mathcal{O}_{T'', t''} \to \mathcal{O}_{T', t'}$ and such that if $\sigma' : C \to R^\wedge$ factors through $\mathcal{O}_{T'', t''}$, then the induced map $\bar{\sigma}' : C' \to R^\wedge$ agrees modulo $m_s^{s+1}$ with $\bar{\sigma}$. Thus if $\sigma'$ is such a map, then we obtain a complex

$$R^\wedge[x_1, \ldots, x_n] \otimes K \to R^\wedge[x_1, \ldots, x_n] \otimes m \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C, \sigma'} R^\wedge \to 0$$

over $R^\wedge[x_1, \ldots, x_n]$ by applying $\bar{\sigma}'$ to the polynomials $t_{kj}$ and $f_j$. In other words, this is the base change of the complex (32.5.2) by $\bar{\sigma}'$. The matrices defining this complex are congruent modulo $m_s^{s+1}$ to the matrices defining the complex (32.5.1) because $\bar{\sigma}$ and $\bar{\sigma}'$ are congruent modulo $m_s^{s+1}$. Since (32.5.1) is exact, we can apply More on Algebra, Lemma 42.9 to conclude that

$$\text{Gr}_{m_s}(B \otimes_{C, \sigma'} R^\wedge) \cong \text{Gr}_{m_s}(B \otimes_{C, \sigma} R^\wedge)$$

as desired. \hfill \Box

0CB1 Lemma 32.6. With notation as in Lemma 32.3 assume that $\varphi$ induces an isomorphism on completions. Then we can choose our diagram such that $f$ is étale.

Proof. We may assume $N \geq 2$ and we may replace $(T, t)$ with $(T', t')$ as in Lemma 32.5. Since $(V, v) \to (S, s)$ is an elementary étale neighbourhood, so is $(X \times_S V, (x, v)) \to (X, x)$. Thus $\mathcal{O}_{X, x} \to \mathcal{O}_{X \times_S V, (x, v)}$ induces an isomorphism on completions by More on Algebra, Lemma 12.3. We claim $\mathcal{O}_{X, x} \to \mathcal{O}_{W, w}$ induces an isomorphism on completions. Having proved this, Lemma 12.1 will show that $f$ is smooth at $w$ and of course $f$ is unramified at $u$ as well, so Morphisms, Lemma 34.5 tells us $f$ is étale at $w$.

First we use the commutativity in part (5) of Lemma 32.3 to see that for $i \leq N$ there is a commutative diagram

$$\begin{array}{ccc}
\text{Gr}^i_{m_x}(\mathcal{O}_{X, x}) & \xrightarrow{\varphi} & \text{Gr}^i_{m_y}(\mathcal{O}_{Y, y}) \\
\text{Gr}^i_{m_x}(\mathcal{O}_{X, x}) & \xrightarrow{\cong} & \text{Gr}^i_{m_{(x, v)}}(\mathcal{O}_{X \times_S V, (x, v)}) \\
\text{Gr}^i_{m_y}(\mathcal{O}_{Y, y}) & \xrightarrow{f^*_w} & \text{Gr}^i_{m_w}(\mathcal{O}_{W, w})
\end{array}$$

This implies that $f^*_w$ defines an isomorphism $\kappa(x) \to \kappa(w)$ on residue fields and an isomorphism $m_x/m_x^2 \to m_w/m_w^2$ on cotangent spaces. Hence $f^*_w$ defines a surjection $\mathcal{O}_{X, x} \to \mathcal{O}_{W, w}$ on complete local rings.

By Lemma 32.5 there is an isomorphism of $\text{Gr}_{m_x}(\mathcal{O}_{Y \times_S V, (y, v)})$ with $\text{Gr}_{m_x}(\mathcal{O}_{Y, y})$. This follows by taking stalks of the isomorphism of conormal sheaves at the point $y$. Since our local rings are Noetherian taking associated graded with respect to $m_x$ commutes with completion because completion with respect to an ideal is exact.
functor on finite modules over Noetherian rings. This produces the right vertical isomorphism in the diagram of graded rings

\[
\begin{align*}
\text{Gr}\, (\mathcal{O}_{W,w}^\wedge) & \quad \text{Gr}\, (\mathcal{O}_{Y \times T', V, (y,v)}^\wedge) \\
\uparrow & \quad \uparrow \\
\text{Gr}\, (\mathcal{O}_{X,x}^\wedge) & \quad \text{Gr}\, (\mathcal{O}_{Y, y'}^\wedge)
\end{align*}
\]

We do not claim the diagram commutes. By the result of the previous paragraph the left arrow is surjective. The other three arrows are isomorphisms. It follows that the left arrow is a surjective map between isomorphic Noetherian rings. Hence it is an isomorphism by Algebra, Lemma 30.10 (you can argue this directly using Hilbert functions as well). In particular \( \mathcal{O}_{X,x}^\wedge \to \mathcal{O}_{W,w}^\wedge \) must be injective as well as surjective which finishes the proof. \( \square \)

### 33. Étale neighbourhoods and branches

The number of (geometric) branches of a scheme at a point was defined in Properties, Section 15. In Varieties, Section 39 we related this to fibres of the normalization morphism. In this section we discuss a characterization of this number in terms of étale neighbourhoods.

**Lemma 33.1.** Let \( R = \text{colim} R_i \) be colimit of a directed system of rings whose transition maps are faithfully flat. Then the number of minimal primes of \( R \) taken as an element of \( \{0, 1, 2, \ldots, \infty\} \) is the supremum of the numbers of minimal primes of the \( R_i \).

**Proof.** If \( A \to B \) is a flat ring map, then \( \text{Spec}(B) \to \text{Spec}(A) \) maps minimal primes to minimal primes by going down (Algebra, Lemma 38.19). If \( A \to B \) is faithfully flat, then every minimal prime is the image of a minimal prime (by Algebra, Lemma 38.16 and 29.7). Hence the number of minimal primes of \( R_i \) is \( \geq \) the number of minimal primes of \( R_i' \) if \( i \leq i' \). By Algebra, Lemma 38.20 each of the maps \( R_i \to R \) is faithfully flat and we also see that the number of minimal primes of \( R \) is \( \geq \) the number of minimal primes of \( R_i \). Finally, suppose that \( q_1, \ldots, q_n \) are pairwise distinct minimal primes of \( R \). Then we can find an \( i \) such that \( R_i \cap q_1, \ldots, R_i \cap q_n \) are pairwise distinct (as sets and hence as prime ideals). This implies the lemma. \( \square \)

**Lemma 33.2.** Let \( X \) be a scheme and \( x \in X \) a point. Then

1. the number of branches of \( X \) at \( x \) is equal to the supremum of the number of irreducible components of \( U \) passing through \( u \) taken over elementary étale neighbourhoods \( (U, u) \to (X, x) \),
2. the number of geometric branches of \( X \) at \( x \) is equal to the supremum of the number of irreducible components of \( U \) passing through \( u \) taken over étale neighbourhoods \( (U, u) \to (X, x) \),
3. \( X \) is unibranch at \( x \) if and only if for every elementary étale neighbourhood \( (U, u) \to (X, x) \) there is exactly one irreducible component of \( U \) passing through \( u \), and
4. \( X \) is geometrically unibranch at \( x \) if and only if for every étale neighbourhood \( (U, u) \to (X, x) \) there is exactly one irreducible component of \( U \) passing through \( u \).
Proof. Parts (3) and (4) follow from parts (1) and (2) via Properties, Lemma 15.6.

Proof of (1). Let $\text{Spec}(A)$ be an affine open neighbourhood of $x$ and let $p \subset A$ be the prime ideal corresponding to $x$. We may replace $X$ by $\text{Spec}(A)$ and it suffices to consider affine elementary étale neighbourhoods $(U, u)$ in the supremum as they form a cofinal subsystem. Recall that the henselization $A^h_p$ is the colimit of the rings $B_q$ over the category of pairs $(B, q)$ where $B$ is an étale $A$-algebra and $q$ is a prime lying over $p$ with $\kappa(q) = \kappa(p)$, see Algebra, Lemma 151.7. These pairs $(B, q)$ correspond exactly to the affine elementary étale neighbourhoods $(U, u)$ by the correspondence between rings and affine schemes. Observe that irreducible components of $\text{Spec}(B)$ passing through $q$ are exactly the minimal prime ideals of $B_q$. The number of minimal primes of $A^h_p$ is the number of branches of $X$ at $x$ by Properties, Definition 15.4. Observe that the transition maps $B_q \to B'_q$ in the system are all flat. Since a flat local ring map is faithfully flat (Algebra, Lemma 38.17) we see that the lemma follows from Lemma 33.1.

Proof of (2). The proof is the same as the proof of (1), except that we use Algebra, Lemma 151.13. There is a tiny difference: given a separable algebraic closure $\kappa_s$ of $\kappa(x)$ for every étale neighbourhood $(U, u) \to (X, x)$ we can choose a $\kappa(x)$-embedding $\phi : \kappa(u) \to \kappa_s$ because $\kappa(u)/\kappa(x)$ is finite separable (Morphisms, Lemma 34.15). Hence we can look at the supremum over all triples $(U, u, \phi)$ where $(U, u) \to (X, x)$ is an affine étale neighbourhood and $\phi : \kappa(u) \to \kappa_s$ is a $\kappa(x)$-embedding. These triples correspond exactly to the triples in Algebra, Lemma 151.13 and the rest of the proof is exactly the same. □

We will need a relative variant of the lemma above.

Lemma 33.3. Let $X \to S$ be a morphism of schemes and $x \in X$ a point with image $s$. Then

1. The number of branches of the fibre $X_s$ at $x$ is equal to the supremum of the number of irreducible components of the fibre $U_s$ passing through $u$ taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,
2. The number of geometric branches of the fibre $X_s$ at $x$ is equal to the supremum of the number of irreducible components of the fibre $U_s$ passing through $u$ taken over étale neighbourhoods $(U, u) \to (X, x)$,
3. The fibre $X_s$ is unibranch at $x$ if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of the fibre $U_s$ passing through $u$, and
4. $X$ is geometrically unibranch at $x$ if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of $U_s$ passing through $u$.

Proof. Combine Lemmas 33.2 and 31.6. □

Lemma 33.4. Let $X \to S$ be a smooth morphism of schemes. Let $x \in X$ with image $s \in S$. Then

1. The number of geometric branches of $X$ at $x$ is equal to the number of geometric branches of $S$ at $s$.
In this section we explain a result that roughly states that smooth coverings of a scheme $S$ can be refined by étale coverings. The technique to prove this relies on a slicing argument.

**Lemma 34.1.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume

1. $f$ is smooth at $x$, and
2. the image $dh$ of $dh$ in 
\[ \Omega_{X,s,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \]
is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ such that $h$ comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in $U$ with $x \in D$ and $D \to S$ smooth.

**Proof.** As $f$ is smooth at $x$ we may assume, after replacing $X$ by an open neighbourhood of $x$ that $f$ is smooth. In particular we see that $f$ is flat and locally of finite presentation. By Lemma [21.1] we already know there exists an open neighbourhood $U \subset X$ of $x$ such that $h$ comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in $U$ with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma [31.15] we have a short exact sequence
\[ C_{D/U} \to i^* \Omega_{U/S} \to \Omega_{D/S} \to 0 \]
where $i : D \to U$ is the closed immersion and $C_{D/U}$ is the conormal sheaf of $D$ in $U$. As $D$ is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $C_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^* \Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator $h$ to the section $dh|_D$ of $i^* \Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes \kappa(x)$ we conclude that
\[ \dim_{\kappa(x)}(\Omega_{D/S,x} \otimes \kappa(x)) = \dim_{\kappa(x)}(\Omega_{U/S,x} \otimes \kappa(x)) - 1. \]
By Morphisms, Lemma [32.14] we see that $\Omega_{U/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim_x(D_s) = \dim_x(U_{s,x}) - 1$ by Algebra, Lemma [59.12] (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma [18.1]) and then using Morphisms, Lemma [27.1]. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at $x$ by Morphisms, Lemma [32.14] again. After shrinking $U$ we get that $D \to S$ is smooth and we win. $\square$
Lemma 34.2. Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Assume

1. \( f \) is smooth at \( x \), and
2. the map
   \[
   \Omega_{X,s}/\kappa(x) \to \Omega_{\kappa(s)/\kappa(s)}
   \]
   has a nonzero kernel.

Then there exists an affine open neighbourhood \( U \subset X \) of \( x \) and an effective Cartier divisor \( D \subset U \) containing \( x \) such that \( D \to S \) is smooth.

Proof. Write \( k = \kappa(s) \) and \( R = \mathcal{O}_{X,s} \). Denote \( m \) the maximal ideal of \( R \) and \( \kappa = R/m \) so that \( \kappa = \kappa(x) \). As formation of modules of differentials commutes with localization (see Algebra, Lemma 130.8) we have \( \Omega_{X,s} = \Omega_{R/k} \). By Algebra, Lemma 130.9 there is an exact sequence

\[
\frac{m}{m^2} \xrightarrow{d} \Omega_{R/k} \otimes R \kappa \to \Omega_{\kappa/k} \to 0.
\]

Hence if (2) holds, there exists an element \( h \in m \) such that \( dh \) is nonzero. Choose a lift \( h \in \mathcal{O}_{X,x} \) of \( h \) and apply Lemma 34.1. \( \square \)

Remark 34.3. The second condition in Lemma 34.2 is necessary even if \( x \) is a closed point of a positive dimensional fibre. An example is the following: Let \( k \) be a field of characteristic \( p > 0 \) which is imperfect. Let \( a \in k \) be an element which is not a \( p \)-th power. Let \( m = (x^p - a) \subset k[x, y] \). This corresponds to a closed point \( w \) of \( X = \mathbb{A}^2_k \). Set \( S = \mathbb{A}^1_k \) and let \( f : X \to S \) be the morphism corresponding to \( k[x] \to k[x, y] \). Then there does not exist any commutative diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
S & \to & X
\end{array}
\]

with \( g \) étale and \( w \) in the image of \( h \). This is clear as the residue field extension \( \kappa(f(w)) \subset \kappa(w) \) is purely inseparable, but for any \( s' \in S' \) with \( g(s') = f(w) \) the extension \( \kappa(f(w)) \subset \kappa(s') \) would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark 34.3 does not happen. Here is the precise result.

Lemma 34.4. Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Assume

1. \( f \) is smooth at \( x \), and
2. the residue field extension \( \kappa(s) \subset \kappa(x) \) is separable, and
3. \( x \) is not a generic point of \( X_s \).

Then there exists an affine open neighbourhood \( U \subset X \) of \( x \) and an effective Cartier divisor \( D \subset U \) containing \( x \) such that \( D \to S \) is smooth.

Proof. Write \( k = \kappa(s) \) and \( R = \mathcal{O}_{X,s} \). Denote \( m \) the maximal ideal of \( R \) and \( \kappa = R/m \) so that \( \kappa = \kappa(x) \). As formation of modules of differentials commutes with localization (see Algebra, Lemma 130.8) we have \( \Omega_{X,s} = \Omega_{R/k} \). By assumption (2) and Algebra, Lemma 139.4 the map

\[
d : \frac{m}{m^2} \to \Omega_{R/k} \otimes R \kappa(m)
\]
is injective. Assumption (3) implies that $m/m^2 \neq 0$. Thus there exists an element $\overline{h} \in m$ such that $d\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 34.1.

The subscheme $Z$ constructed in the following lemma is really a complete intersection in an affine open neighbourhood of $x$. If we ever need this we will explicitly formulate a separate lemma stating this fact.

**Lemma 34.5.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is smooth at $x$, and
2. $x$ is a closed point of $X_s$ and $\kappa(s) \subset \kappa(x)$ is separable.

Then there exists an immersion $Z \to X$ containing $x$ such that

1. $Z \to S$ is étale, and
2. $Z_s = \{x\}$ set theoretically.

**Proof.** We may and do replace $S$ by an affine open neighbourhood of $s$. We may and do replace $X$ by an affine open neighbourhood of $x$ such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_X(X_s)$.

The case $d = 0$. In this case we show that we may take $Z$ to be an open neighbourhood of $x$. Namely, if $d = 0$, then $X \to S$ is quasi-finite at $x$, see Morphisms, Lemma 28.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 53.2. Thus after replacing $X$ by $U$ we see that $X$ is quasi-finite and smooth over $S$, hence smooth of relative dimension 0 over $S$, hence étale over $S$. Moreover, the fibre $X_s$ is a finite discrete set. Hence after replacing $X$ by a further affine open neighbourhood of $x$ we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on $X_s$ is induced from the topology on $X$, see Schemes, Lemma 18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because $x$ is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 19.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions. Thus we may apply Lemma 34.2 to find a diagram

$$
\begin{array}{ccc}
D & \to & U \\
\downarrow & & \downarrow \\
 & \to & X \\
& \downarrow & \downarrow \\
 & & S
\end{array}
$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 50.12 (also $D_s \subset X_s$ is effective Cartier, see Divisors, Lemma 18.1) and then using Morphisms, Lemma 27.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof.

**Lemma 34.6.** Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of $f$. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a $S$-morphism $S' \to X$.

**First proof of Lemma 34.6** By assumption $X_s \neq \emptyset$. By Varieties, Lemma 25.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension.
of \(\kappa(s)\). Hence by Lemma \[34.5\] there exists an immersion \(Z \to X\) such that \(Z \to S\) is étale and such that \(x \in Z\). Take \((S', s') = (Z, x)\).

**Second proof of Lemma \[34.6\]** Pick a point \(x \in X\) with \(f(x) = s\). Choose a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{U} & A^n_s \\
\downarrow & & \downarrow \\
S & \xleftarrow{V} & \end{array}
\]

with \(\pi\) étale, \(x \in U\) and \(V = \text{Spec}(R)\) affine, see Morphisms, Lemma \[34.20\]. In particular \(s \in V\). The morphism \(\pi : U \to A^n_s\) is open, see Morphisms, Lemma \[34.13\]. Thus \(W = \pi(V) \cap A^n_s\) is a nonempty open subset of \(A^n_s\). Let \(w \in W\) be a point with \(\kappa(s) \subset \kappa(w)\) finite separable, see Varieties, Lemma \[25.5\]. By Algebra, Lemma \[113.1\] there exist \(d\) elements \(f_1, \ldots, f_d \in \kappa(s)[x_1, \ldots, x_d]\) which generate the maximal ideal corresponding to \(w\) in \(\kappa(s)[x_1, \ldots, x_d]\). After replacing \(R\) by a principal localization we may assume there are \(f_1, \ldots, f_d \in R[x_1, \ldots, x_d]\) which map to \(f_1, \ldots, f_d \in \kappa(s)[x_1, \ldots, x_d]\). Consider the \(R\)-algebra

\[R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)\]

and set \(S' = \text{Spec}(R')\). By construction we have a closed immersion \(j : S' \to A^n_s\) over \(V\). By construction the fibre of \(S' \to V\) over \(s\) is a single point \(s'\) whose residue field is finite separable over \(\kappa(s)\). Let \(q' \subset R'\) be the corresponding prime. By Algebra, Lemma \[135.11\] we see that \((R')_q\) is a relative global complete intersection over \(R\) for some \(g \in R', g \not\in q\). Thus \(S' \to V\) is flat and of finite presentation in a neighbourhood of \(s'\), see Algebra, Lemma \[135.14\]. By construction the scheme theoretic fibre of \(S' \to V\) over \(s\) is \(\text{Spec}(\kappa(s'))\). Hence it follows from Morphisms, Lemma \[34.15\] that \(S' \to S\) is étale at \(s'\). Set

\[S'' = U \times_{\pi, A^n_s, j} S'.\]

By construction there exists a point \(s'' \in S''\) which maps to \(s'\) via the projection \(p : S'' \to S'\). Note that \(p\) is étale as the base change of the étale morphism \(\pi\), see Morphisms, Lemma \[34.4\]. Choose a small affine neighbourhood \(S''' \subset S''\) of \(s''\) which maps into the nonempty open neighbourhood of \(s' \in S'\) where the morphism \(S' \to S\) is étale. Then the étale neighbourhood \((S''', s'') \to (S, s)\) is a solution to the problem posed by the lemma.

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

**Lemma 34.7.** Let \(S\) be a scheme. Let \(\mathcal{U} = \{S_i \to S\}_{i \in I}\) be a smooth covering of \(S\), see Topologies, Definition \[5.1\]. Then there exists an étale covering \(\mathcal{V} = \{T_j \to S\}_{j \in J}\) (see Topologies, Definition \[4.7\]) which refines (see Sites, Definition \[8.7\]) \(\mathcal{U}\).

**Proof.** For every \(s \in S\) there exists an \(i \in I\) such that \(s\) is in the image of \(S_i \to S\). By Lemma \[34.6\] we can find an étale morphism \(g_s : T_s \to S\) such that \(s \in g_s(T_s)\) and such that \(g_s\) factors through \(S_i \to S\). Hence \(\{T_s \to S\}\) is an étale covering of \(S\) that refines \(\mathcal{U}\).

**Lemma 34.8.** Let \(f : X \to S\) be a smooth morphism of schemes. Then there exists an étale covering \(\{U_i \to X\}_{i \in I}\) such that \(U_i \to S\) factors as \(U_i \to V_i \to S\).
where $V_i \to S$ is étale and $U_i \to V_i$ is a smooth morphism of affine schemes, which has a section, and has geometrically connected fibres.

**Proof.** Let $s \in S$. By Varieties, Lemma 25.6 the set of closed points $x \in X_s$ such that $\kappa(x)/\kappa(s)$ is separable is dense in $X_s$. Thus it suffices to construct an étale morphism $U \to X$ with $x$ in the image such that $U \to S$ factors in the manner described in the lemma. To do this, choose an immersion $Z \to X$ passing through $x$ such that $Z \to S$ is étale (Lemma 34.5). After replacing $S$ by $Z$ and $X$ by $Z \times_S X$ we see that we may assume $X \to S$ has a section $\sigma : S \to X$ with $\sigma(s) = x$. Then we can first replace $S$ by an affine open neighbourhood of $s$ and next replace $X$ by an affine open neighbourhood of $x$. Then finally, we consider the subset $X^0 \subset X$ of Section 27. By Lemmas 27.6 and 27.4 this is a retrocompact open subscheme containing $\sigma$ such that the fibres $X^0 \to S$ are geometrically connected. If $X^0$ is not affine, then we choose an affine open $U \subset X^0$ containing $x$. Since $X^0 \to S$ is smooth, the image of $U$ is open. Choose an affine open neighbourhood $V \subset S$ of $s$ contained in $\sigma^{-1}(U)$ and in the image of $U \to S$. Finally, the reader sees that $U \cap f^{-1}(V) \to V$ has all the desired properties. For example $U \cap f^{-1}(V)$ is equal to $U \times_S V$ is affine as a fibre product of affine schemes. Also, the geometric fibres of $U \cap f^{-1}(V) \to V$ are nonempty open subschemes of the irreducible fibres of $X^0 \to S$ and hence connected. Some details omitted. \(\Box\)

The following lemma should probably be somewhere else.

**Lemma 34.9.** Let $S$ be a scheme. Let $Y \to X$ be a closed immersion of schemes smooth over $S$. For every $y \in Y$ there exist integers $0 \leq m, n$ and a commutative diagram

$$
\begin{array}{ccc}
Y & \leftarrow & V \\
\downarrow & & \downarrow \\
X & \leftarrow & U \xrightarrow{\pi} A^{m+n}_S \\
\end{array}
$$

where $U \subset X$ is open, $V = Y \cap U$, $\pi$ is étale, and $V = \pi^{-1}(A^m_S)$.

**Proof.** The question is local on $X$ hence we may replace $X$ by an open neighbourhood of $y$. Since $Y \to X$ is a regular immersion by Divisors, Lemma 22.11 we may assume $X = \text{Spec}(A)$ is affine and there exists a regular sequence $f_1, \ldots, f_n \in A$ such that $Y = V(f_1, \ldots, f_n)$. After shrinking $X$ (and hence $Y$) further we may assume there exists an étale morphism $Y \to A^m_S$, see Morphisms, Lemma 34.20. Let $g_1, \ldots, g_m$ in $\mathcal{O}_Y(Y)$ be the coordinate functions of this étale morphism. Choose lifts $g_1, \ldots, g_m \in A$ of these functions and consider the morphism

$$(g_1, \ldots, g_m, f_1, \ldots, f_n) : X \to A^{m+n}_S$$

over $S$. This is a morphism of schemes locally of finite presentation over $S$ and hence is locally of finite presentation (Morphisms, Lemma 20.11). The restriction of this morphism to $A^m_S \subset A^{m+n}_S$ is étale by construction. Thus, in order to show that $X \to A^{m+n}_S$ is étale at $y$ it suffices to show that $X \to A^{m+n}_S$ is flat at $y$, see Morphisms, Lemma 34.15. Let $s \in S$ be the image of $y$. It suffices to check that $X_s \to A^{m+n}_S$ is flat at $y$, see Theorem 16.2. Let $z \in A^{m+n}_S$ be the image of $y$. The local ring map

$$\mathcal{O}_{A^{m+n}_S, z} \to \mathcal{O}_{X_s, y}$$
is flat by Algebra, Lemma \[127.1\]. Namely, schemes smooth over fields are regular and regular rings are Cohen-Macaulay, see Varieties, Lemma \[25.3\] and Algebra, Lemma \[105.3\]. Thus both source and target are regular local rings (and hence CM). The source and target have the same dimension: namely, we have \(\dim(\mathcal{O}_{Y_s, y}) = \dim(\mathcal{O}_{X_s, y})\) by More on Algebra, Lemma \[43.2\], we have \(\dim(\mathcal{O}_{A^n_s, z}) = n + \dim(\mathcal{O}_{A^n_s, z})\), and we have \(\dim(\mathcal{O}_{X_s, y}) = n + \dim(\mathcal{O}_{Y_s, y})\) because \(\mathcal{O}_{Y_s, y}\) is the quotient of \(\mathcal{O}_{X_s, y}\) by the regular sequence \(f_1, \ldots, f_n\) of length \(n\) (see Divisors, Remark \[22.5\]). Finally, the fibre ring of the displayed arrow is finite over \(\kappa(z)\) since \(Y_s \to A^n_s\) is étale at \(y\). This finishes the proof.

\[\square\]

35. Finite free locally dominates étale

In this section we explain a result that roughly states that étale coverings of a scheme \(S\) can be refined by Zariski coverings of finite locally free covers of \(S\).

**Lemma 35.1.** Let \(S\) be a scheme. Let \(s \in S\). Let \(f : (U, u) \to (S, s)\) be an étale neighbourhood. There exists an affine open neighbourhood \(s \in V \subset S\) and a surjective, finite locally free morphism \(\pi : T \to V\) such that for every \(t \in \pi^{-1}(s)\) there exists an open neighbourhood \(t \in W_t \subset T\) and a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{h_t} & U \\
\downarrow \pi & & \downarrow f \\
V & \xrightarrow{\pi} & S
\end{array}
\]

with \(h_t(t) = u\).

**Proof.** The problem is local on \(S\) hence we may replace \(S\) by any open neighbourhood of \(s\). We may also replace \(U\) by an open neighbourhood of \(u\). Hence, by Morphisms, Lemma \[34.14\] we may assume that \(U \to S\) is a standard étale morphism of affine schemes. In this case the lemma (with \(V = S\)) follows from Algebra, Lemma \[142.17\].

**Lemma 35.2.** Let \(f : U \to S\) be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism \(\pi : T \to S\) and a finite open covering \(T = T_1 \cup \ldots \cup T_n\) such that each \(T_i \to S\) factors through \(U \to S\). Diagram:

\[
\begin{array}{ccc}
\bigcup T_i & \xrightarrow{\pi} & U \\
\downarrow f & & \downarrow f \\
S & \xrightarrow{f} & S
\end{array}
\]

where the south-west arrow is a Zariski-covering.

**Proof.** This is a restatement of Algebra, Lemma \[142.18\].

**Remark 35.3.** In terms of topologies Lemmas \[35.1\] and \[35.2\] mean the following. Let \(S\) be any scheme. Let \(\{f_i : U_i \to S\}\) be an étale covering of \(S\). There exists a Zariski open covering \(S = \bigcup V_j\), for each \(j\) a finite locally free, surjective morphism \(W_j \to V_j\), and for each \(j\) a Zariski open covering \(\{W_{j,k} \to W_j\}\) such that the family \(\{W_{j,k} \to S\}\) refines the given étale covering \(\{f_i : U_i \to S\}\). What does this mean in
practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form \( \{ \pi : T \to S \} \) with \( \pi \) finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that \( \text{Br}(X) = \text{Br}'(X) \) for an affine scheme \( X \), see [Hoo82].

36. Étale localization of quasi-finite morphisms

Now we come to a series of lemmas around the theme “quasi-finite morphisms become finite after étale localization”. The general idea is the following. Suppose given a morphism of schemes \( f : X \to S \) and a point \( s \in S \). Let \( \varphi : (U, u) \to (S, s) \) be an étale neighbourhood of \( s \) in \( S \). Consider the fibre product \( X_U = U \times_S X \) and the basic diagram

\[
\begin{array}{ccc}
V & \to & X_U \\
\downarrow & & \downarrow f \\
U & \varphi & \to & S \\
\end{array}
\]

where \( V \subset X_U \) is open. Is there some standard model for the morphism \( f_U : X_U \to U \), or for the morphism \( V \to U \) for suitable opens \( V \)? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

**Lemma 36.1.** Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \). Set \( s = f(x) \). Assume that

1. \( f \) is locally of finite type, and
2. \( x \in X_s \) is isolated.

Then there exist

(a) an elementary étale neighbourhood \( (U, u) \to (S, s) \),
(b) an open subscheme \( V \subset X_U \) (see [36.0.1]) such that

(i) \( V \to U \) is a finite morphism,
(ii) there is a unique point \( v \) of \( V \) mapping to \( u \) in \( U \), and
(iii) the point \( v \) maps to \( x \) under the morphism \( X_U \to X \), inducing \( \kappa(x) = \kappa(v) \).

Moreover, for any elementary étale neighbourhood \( (U', u') \to (U, u) \) setting \( V' = U' \times_U V \subset X_{U'} \), the triple \( (U', u', V') \) satisfies the properties (i), (ii), and (iii) as well.

**Proof.** Let \( Y \subset X \), \( W \subset S \) be affine opens such that \( f(Y) \subset W \) and such that \( x \in Y \). Note that \( x \) is also an isolated point of the fibre of the morphism \( f|_Y : Y \to W \). If we can prove the theorem for \( f|_Y : Y \to W \), then the theorem follows for \( f \). Hence we reduce to the case where \( f \) is a morphism of affine schemes. This case is Algebra, Lemma [1.42.21].

In the preceding and following lemma we do not assume that the morphism \( f \) is separated. This means that the opens \( V \), \( V_i \) created in them are not necessarily closed in \( X_U \). Moreover, if we choose the neighbourhood \( U \) to be affine, then each \( V_i \) is affine, but the intersections \( V_i \cap V_j \) need not be affine (in the nonseparated case).

\[ ^{\text{In the presence of (1) this means that } f \text{ is quasi-finite at } x, \text{ see Morphisms, Lemma [19.6]}} \]
Lemma 36.2. Let \( f : X \to S \) be a morphism of schemes. Let \( x_1, \ldots, x_n \in X \) be points having the same image \( s \) in \( S \). Assume that

1. \( f \) is locally of finite type, and
2. \( x_i \in X_s \) is isolated for \( i = 1, \ldots, n \).

Then there exist

(a) an elementary étale neighbourhood \((U, u) \to (S, s)\),
(b) for each \( i \) an open subscheme \( V_i \subset X_U \),

such that for each \( i \) we have

(i) \( V_i \to U \) is a finite morphism,
(ii) there is a unique point \( v_i \) of \( V_i \) mapping to \( u \) in \( U \), and
(iii) the point \( v_i \) maps to \( x_i \) in \( X \) and \( \kappa(x_i) = \kappa(v_i) \).

Proof. We will use induction on \( n \). Namely, suppose \((U, u) \to (S, s)\) and \( V_i \subset X_U \), \( i = 1, \ldots, n - 1 \) work for \( x_1, \ldots, x_{n-1} \). Since \( \kappa(s) = \kappa(u) \) the fibre \((X_U)_u = X_v \). Hence there exists a unique point \( x'_n \in X_u \subset X_U \) corresponding to \( x_n \in X_s \). Also \( x'_n \) is isolated in \( X_u \). Hence by Lemma 36.1 there exists an elementary étale neighbourhood \((U', u') \to (U, u)\) and an open \( V_n \subset X_{U'} \) which works for \( x'_n \) and hence for \( x_n \). By the final assertion of Lemma 36.1 the open subschemes \( V'_i = U' \times_U V_i \) for \( i = 1, \ldots, n - 1 \) still work with respect to \( x_1, \ldots, x_{n-1} \). Hence we win. \( \square \)

If we allow a nontrivial field extension \( \kappa(s) \subset \kappa(u) \), i.e., general étale neighbourhoods, then we can split the points as follows.

Lemma 36.3. Let \( f : X \to S \) be a morphism of schemes. Let \( x_1, \ldots, x_n \in X \) be points having the same image \( s \) in \( S \). Assume that

1. \( f \) is locally of finite type, and
2. \( x_i \in X_s \) is isolated for \( i = 1, \ldots, n \).

Then there exist

(a) an étale neighbourhood \((U, u) \to (S, s)\),
(b) for each \( i \) an integer \( m_i \) and open subschemes \( V_{i,j} \subset X_U \), \( j = 1, \ldots, m_i \)

such that we have

(i) each \( V_{i,j} \to U \) is a finite morphism,
(ii) there is a unique point \( v_{i,j} \) of \( V_{i,j} \) mapping to \( u \) in \( U \) with \( \kappa(u) \subset \kappa(v_{i,j}) \) finite purely inseparable,
(iii) if \( v_{i,j} = v_{i',j'} \), then \( i = i' \) and \( j = j' \), and
(iv) the points \( v_{i,j} \) map to \( x_i \) in \( X \) and no other points of \((X_U)_u\) map to \( x_i \).

Proof. This proof is a variant of the proof of Algebra, Lemma [142.23] in the language of schemes. By Morphisms, Lemma [19.6] the morphism \( f \) is quasi-finite at each of the points \( x_i \). Hence \( \kappa(s) \subset \kappa(x_i) \) is finite for each \( i \) (Morphisms, Lemma [19.5]). For each \( i \), let \( \kappa(s) \subset L_i \subset \kappa(x_i) \) be the subfield such that \( L_i/\kappa(s) \) is separable, and \( \kappa(x_i)/L_i \) is purely inseparable. Choose a finite Galois extension \( \kappa(s) \subset L \) such that there exist \( \kappa(s) \)-embeddings \( L_i \to L \) for \( i = 1, \ldots, n \). Choose an étale neighbourhood \((U, u) \to (S, s)\) such that \( L \cong \kappa(u) \) as \( \kappa(s) \)-extensions (Lemma [31.2]).

Let \( y_{i,j}, \ j = 1, \ldots, m_i \) be the points of \( X_U \) lying over \( x_i \in X \) and \( u \in U \). By Schemes, Lemma [17.5] these points \( y_{i,j} \) correspond exactly to the primes in the
rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of $L$ (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair $i,j$. Also, by Morphisms, Lemma 02LN, for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all $i,j$.

Apply Lemma 36.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. □

**Lemma 36.4.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

1. $f$ is locally of finite type,
2. $f$ is separated, and
3. $x_1, \ldots, x_n$ are pairwise distinct isolated points of $X_s$.

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V_1 \amalg \ldots \amalg V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over $u$ are singletons $\{v_i\}$, each $v_i$ maps to $x_i$ with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over $u$ contains no points mapping to any of the $x_i$.

**Proof.** Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 36.2 Since $X_U \to U$ is separated (Schemes, Lemma 21.12) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 42.11) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 39.7. Hence $V_i \cap V_j$ is a closed subset of $V_i$ which does not contain $v_i$. Hence the image of $V_i \cap V_j$ in $U$ is a closed set (because $V_i \to U$ proper) not containing $u$. After shrinking $U$ we may therefore assume that $V_i \cap V_j = \emptyset$ for all $i, j$. This gives the decomposition as in the lemma. □

Here is the variant where we reduce to purely inseparable field extensions.

**Lemma 36.5.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

1. $f$ is locally of finite type,
2. $f$ is separated, and
3. $x_1, \ldots, x_n$ are pairwise distinct isolated points of $X_s$.

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg \coprod_{i=1}^n \bigcup_{j=1}^{m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over $u$ are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to $x_i$, $\kappa(u) \subset \kappa(v_{i,j})$ is purely inseparable, and the fibre of $W \to U$ over $u$ contains no points mapping to any of the $x_i$.

**Proof.** This is proved in exactly the same way as the proof of Lemma 36.4 except that it uses Lemma 36.3 instead of Lemma 36.2. □

The following version may be a little easier to parse.
Lemma 36.6. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

1. $f$ is locally of finite type,
2. $f$ is separated, and
3. $X_s$ has at most finitely many isolated points.

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre $W_u$ of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 36.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of $X_s$ and setting $V = \bigcup V_i$. \qed

37. Étale localization of integral morphisms

Some variants of the results of Section 36 for the case of integral morphisms.

Lemma 37.1. Let $R \to S$ be an integral ring map. Let $p \subset R$ be a prime ideal. Assume

1. there are finitely many primes $q_1, \ldots, q_n$ lying over $p$, and
2. for each $i$ the maximal separable subextension $\kappa(q_i)/\kappa(q_i)_\text{sep}/\kappa(p)$ (Fields, Lemma 14.6) is finite over $\kappa(p)$.

Then there exists an étale ring map $R \to R'$ and a prime $p'$ lying over $p$ such that

$$S \otimes_R R' = A_1 \times \cdots \times A_m$$

with $R' \to A_j$ integral having a unique prime $\mathfrak{r}_j$ over $p'$ such that $\kappa(\mathfrak{r}_j)/\kappa(p')$ is purely inseparable.

First proof. This proof uses Algebra, Lemma 142.23. Namely, choose a generator $\theta_i \in \kappa(q_i)_\text{sep}$ of this field over $\kappa(p)$ (Fields, Lemma 19.1). The spectrum of the fibre ring $S \otimes_R \kappa(p)$ is finite discrete with points corresponding to $q_1, \ldots, q_n$. By the Chinese remainder theorem (Algebra, Lemma 14.4) we see that $S \otimes_R \kappa(p) \to \prod \kappa(q_i)$ is surjective. Hence after replacing $R$ by $R_q$ for some $q \in R$, $q \not\in p$ we may assume that $(0, \ldots, 0, \theta_i, 0, \ldots, 0) \in \prod \kappa(q_i)$ is the image of some $x_i \in S$. Let $S' \subset S$ be the $R$-subalgebra generated by our $x_i$. Since Spec($S$) $\to$ Spec($S'$) is surjective (Algebra, Lemma 35.17) we conclude that $q'_i = S' \cap q_i$ are the primes of $S'$ over $p$. By our choice of $x_i$ we conclude these primes are distinct that and $\kappa(q'_i)_\text{sep} = \kappa(q_i)_\text{sep}$. In particular the field extensions $\kappa(q_i)/\kappa(q'_i)$ are purely inseparable. Since $R \to S'$ is finite we may apply Algebra, Lemma 142.23 and we get $R \to R'$ and $p'$ and a decomposition

$$S' \otimes_R R' = A'_1 \times \cdots \times A'_m \times B'$$

with $R' \to A'_j$ integral having a unique prime $\mathfrak{r}'_j$ over $p'$ such that $\kappa(\mathfrak{r}'_j)/\kappa(p')$ is purely inseparable and such that $B'$ does not have a prime lying over $p'$. Since $R' \to B'$ is finite (as $R \to S'$ is finite) we can after localizing $R'$ at some $g' \in R'$, $g' \not\in p'$ assume that $B' = 0$. Via the map $S' \otimes_R R' \to S \otimes_R R'$ we get the corresponding decomposition for $S$. \qed
Second proof. This proof uses strict henselization. First, assume $R$ is strictly henselization with maximal ideal $p$. Then $S/pS$ has finitely many primes corresponding to $q_1, \ldots, q_n$, each maximal, each with purely inseparable residue field over $\kappa(p)$. Hence $S/pS$ is equal to $\prod(S/pS)_{q_i}$. By More on Algebra, Lemma 11.6 we can lift this product decomposition to a product composition of $S$ as in the statement.

In the general case, let $R^{sh}$ be the strict henselization of $R_p$. Then we can apply the result of the first paragraph to $R^{sh} \to S \otimes_R R^{sh}$. Consider the $m$ mutually orthogonal idempotents in $S \otimes_R R^{sh}$ corresponding to the product decomposition. Since $R^{sh}$ is a filtered colimit of étale ring maps $(R, p) \to (R', p')$ by Algebra, Lemma 151.13 we see that these idempotents descend to some $R'$ as desired. □

38. Zariski’s Main Theorem

In this section we prove Zariski’s main theorem as reformulated by Grothendieck. Often when we say “Zariski’s main theorem” in this content we mean either of Lemma 38.1, Lemma 38.2, or Lemma 38.3. In most texts people refer to the last of these as Zariski’s main theorem.

We have already proved the algebraic version in Algebra, Theorem 122.12 and we have already restated this algebraic version in the language of schemes, see Morphisms, Theorem 53.1. The version in this section is more subtle; to get the full result we use the étale localization techniques of Section 36 to reduce to the algebraic case.

**Lemma 38.1.** Let $f : X \to S$ be a morphism of schemes. Assume $f$ is of finite type and separated. Let $S'$ be the normalization of $S$ in $X$, see Morphisms, Definition 51.3. Picture:

\[
\begin{array}{c}
X \\
\downarrow f \\
S
\end{array}
\quad \begin{array}{c}
\downarrow f' \\
S'
\end{array}
\]

Then there exists an open subscheme $U' \subset S'$ such that

1. $(f')^{-1}(U') \to U'$ is an isomorphism, and
2. $(f')^{-1}(U') \subset X$ is the set of points at which $f$ is quasi-finite.

**Proof.** By Morphisms, Lemma 53.2 the subset $U \subset X$ of points where $f$ is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over $S$ (as an open subscheme of $X$) and for $v \in V$ the residue field extension $\kappa(v) \supset \kappa(\nu(v))$ is algebraic (as $V \subset S'$ and $S'$ is integral over $S$). Hence the fibres of $V \to S$ are discrete (Morphisms, Lemma 19.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 19.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since $x$ was arbitrary we see that (a), (b), and (c) are true.
Let \( s = f(x) \). Let \( (T, t) \to (S, s) \) be an elementary étale neighbourhood. Denote by a subscript \( T \) the base change to \( T \). Let \( y = (x, t) \in X_T \) be the unique point in the fibre \( X_T \) lying over \( x \). Note that \( U_T \subset X_T \) is the set of points where \( f_T \) is quasi-finite, see Morphisms, Lemma \[19.13\]. Note that
\[
X_T \xrightarrow{f_T} S'_T \xrightarrow{\nu_T} T
\]
is the normalization of \( T \) in \( X_T \), see Lemma \[17.2\]. Suppose that the claim holds for \( y \in U_T \subset X_T \to S'_T \to T \), i.e., suppose that we can find an open neighbourhood \( f'_T(y) \in V' \subset S'_T \) such that \((f'_T)^{-1}V' \to V'\) is an isomorphism. The morphism \( S'_T \to S' \) is étale hence the image \( V \subset S' \) of \( V' \) is open. Observe that \( f'(x) \in V \) as \( f'_T(y) \in V' \). Observe that
\[
(f'_T)^{-1}V' \longrightarrow (f')^{-1}(V)
\]
is a fibre square (as \( S'_T \times_{S'} X = X_T \)). Since the left vertical arrow is an isomorphism and \( \{V' \to V\} \) is an étale covering, we conclude that the right vertical arrow is an isomorphism by Descent, Lemma \[20.17\]. In other words, the claim holds for \( x \in U \subset X \to S' \to S \).

By the result of the previous paragraph we may replace \( S \) by an elementary étale neighbourhood of \( s = f(x) \) in order to prove the claim. Thus we may assume there is a decomposition
\[
X = V \amalg W
\]
into open and closed subschemes where \( V \to S \) is finite and \( x \in V \), see Lemma \[36.4\]. Since \( X \) is a disjoint union of \( V \) and \( W \) over \( S \) and since \( V \to S \) is finite we see that the normalization of \( S \) in \( X \) is the morphism
\[
X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S
\]
where \( W' \) is the normalization of \( S \) in \( W \), see Morphisms, Lemmas \[51.10\], \[42.4\], and \[51.12\]. The claim follows and we win. □

02LR Lemma \[38.2\]. Let \( f : X \to S \) be a morphism of schemes. Assume \( f \) is quasi-finite and separated. Let \( S' \) be the normalization of \( S \) in \( X \), see Morphisms, Definition \[37.3\]. Picture:

\[
\begin{array}{ccc}
X & \longrightarrow & S' \\
\downarrow f' & & \downarrow \nu \\
\downarrow f & & \\
S & \longrightarrow & S
\end{array}
\]

Then \( f' \) is a quasi-compact open immersion and \( \nu \) is integral. In particular \( f \) is quasi-affine.

Proof. This follows from Lemma \[38.1\]. Namely, by that lemma there exists an open subscheme \( U' \subset S' \) such that \((f')^{-1}(U') = X \) (!) and \( X \to U' \) is an isomorphism! In other words, \( f' \) is an open immersion. Note that \( f' \) is quasi-compact as \( f \) is quasi-compact and \( \nu : S' \to S \) is separated (Schemes, Lemma \[21.14\]). It follows that \( f \) is quasi-affine by Morphisms, Lemma \[12.3\]. □
Lemma 38.3 (Zariski’s Main Theorem). Let $f : X \to S$ be a morphism of schemes. Assume $f$ is quasi-finite and separated and assume that $S$ is quasi-compact and quasi-separated. Then there exists a factorization

$$
\begin{array}{ccc}
X & \to & T \\
\downarrow f & & \downarrow \pi \\
S & \to &
\end{array}
$$

where $j$ is a quasi-compact open immersion and $\pi$ is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 38.2. By Properties, Lemma 22.13 we can write $\nu_* O_{S'} = \colim_{i \in I} A_i$ as a directed colimit of finite quasi-coherent $O_X$-algebras $A_i \subset \nu_* O_{S'}$. Then $\pi_i : T_i = \text{Spec}_S(A_i) \to S$ is a finite morphism for each $i$. Note that the transition morphisms $T_i' \to T_i$ are affine and that $S' = \lim T_i$. By Limits, Lemma 4.11 there exists an $i$ and an open subscheme $U_i \subset T_i$ whose inverse image in $T$ equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of $U_i$ in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits, Lemma 2.2. By Limits, Lemma 4.16 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large $i'$ but we don’t need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms, Lemma 3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. $\square$

Lemma 38.4. With notation and hypotheses as in Lemma 38.3. Assume moreover that $f$ is locally of finite presentation. Then we can choose the factorization such that $T$ is finite and of finite presentation over $S$.

Proof. By Limits, Lemma 9.8 we can write $T = \lim T_i$ where all $T_i$ are finite and of finite presentation over $Y$ and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits, Lemma 4.11 there exists an $i$ and an open subscheme $U_i \subset T_i$ whose inverse image in $T$ is $X$. By Limits, Lemma 4.16 we see that $X \cong U_i$ for large enough $i$. Replacing $T$ by $T_i$ finishes the proof. $\square$

39. Applications of Zariski’s Main Theorem, I

A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

Lemma 39.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

1. $f$ is finite,
2. $f$ is proper with finite fibres,
3. $f$ is proper and locally quasi-finite,
4. $f$ is universally closed, separated, locally of finite type and has finite fibres.

Proof. We have (1) implies (2) by Morphisms, Lemmas 42.11, 19.10, and 42.10. We have (2) implies (3) by Morphisms, Lemma 19.7. We have (3) implies (4) by the definition of proper morphisms and Morphisms, Lemmas 19.9 and 19.10. Assume (3). Pick $s \in S$. By Morphisms, Lemma 19.7 we see that all the finitely many points of $X_s$ are isolated in $X_s$. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \amalg W$ as in Lemma 36.6. Note that
$W_u = \emptyset$ because all points of $X_s$ are isolated. Since $f$ is universally closed we see that the image of $W$ in $U$ is a closed set not containing $u$. After shrinking $U$ we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over $U$. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover $S$ such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 4.1. Hence it is an fpqc covering, see Topologies, Lemma 9.6. Hence we conclude $f$ is finite by Descent, Lemma 20.23.

As a consequence we have the following useful results.

**Lemma 39.2.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that $\overline{f^{-1}\{s\}}$ is finite. Then there exists an open neighbourhood $V \subset S$ of $s$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

**Proof.** The morphism $f$ is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 19.7. By Morphisms, Lemma 53.2 the set of points at which $f$ is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since $f$ is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of $s$ with $f(Z) \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 19.9), hence has finite fibres (Morphisms, Lemma 19.10), hence is finite by Lemma 39.1.

**Lemma 39.3.** Consider a commutative diagram of schemes

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow f & & \downarrow g \\
S & \xrightarrow{g} & Y
\end{array}
$$

Let $s \in S$. Assume

1. $X \to S$ is a proper morphism,
2. $Y \to S$ is separated and locally of finite type, and
3. the image of $X_s \to Y_s$ is finite.

Then there is an open subspace $U \subset S$ containing $s$ such that $X_U \to Y_U$ factors through a closed subscheme $Z \subset Y_U$ finite over $U$.

**Proof.** Let $Z \subset Y$ be the scheme theoretic image of $h$, see Morphisms, Section 6. By Morphisms, Lemma 39.10 the morphism $X \to Z$ is surjective and $Z \to S$ is proper. Thus $X_s \to Z_s$ is surjective. We see that either (3) implies $Z_s$ is finite. Hence $Z \to S$ is finite in an open neighbourhood of $s$ by Lemma 39.2.

**40. Applications of Zariski’s Main Theorem, II**

In this section we give a few more consequences of Zariski’s main theorem to the structure of quasi-finite morphisms.

**Lemma 40.1.** Let $f : X \to Y$ be a separated, locally quasi-finite morphism with $Y$ affine. Then every finite set of points of $X$ is contained in an open affine of $X$.

**Proof.** Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 38.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 29.5.
Lemma 40.2. Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.

Proof. If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 12.3. Thus the problem is local on $X$ and we may assume that $X$ is affine.

Assume $X$ is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with $V_j$ affine. This is possible since $f$ is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of $X$. We see from Morphisms, Lemmas 19.10 and 49.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose $U_\eta$ such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of $X$ is dense in $X$. Thus we see there exists a dense open $W = \bigcup U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace $X$ by an affine open subscheme of $W$ and assume that each $V_j \to X$ is finite.

Assume $X$ is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with $V_j$ affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$
\Delta_{ij} = \left( V_i \cap V_j \right) \setminus V_i \cap V_j \cap V_j.
$$

This is a nowhere dense closed subset of $V_j$ because it is the boundary of the open subset $V_i \cap V_j$ in $V_j$. By Morphisms, Lemma 46.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of $X$. By Topology, Lemma 21.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of $X$. Thus $U = X \setminus T$ is a dense open subset of $X$. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V'_j = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$
f' = f|_{Y'} : Y' \to U'
$$

of $f$. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V'_j$ is an affine open covering, each $V'_j \to U'$ is finite, and $V'_i \cap V'_j$ is (open and) closed both in $V'_i$ and $V'_j$. Hence $V'_i \cap V'_j$ is affine, and the map

$$
\mathcal{O}(V'_i) \otimes_{\mathcal{O}} \mathcal{O}(V'_j) \to \mathcal{O}(V'_i \cap V'_j)
$$

is surjective. This implies that $Y'$ is separated, see Schemes, Lemma 21.7. Finally, consider the commutative diagram

$$
\begin{array}{ccc}
\prod_{j=1,\ldots,m} V'_j & \longrightarrow & Y' \\
 & \searrow & \\
 & U' &
\end{array}
$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 39.8 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 39.1 and we win. \qed
Lemma 40.3. Let \( f : X \to S \) be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets
\[
\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S
\]
such that with \( S_r = Z_r \setminus Z_{r-1} \) the stratification \( S = \coprod_{r=0}^n S_r \) is characterized by the following universal property: Given \( g : T \to S \) the projection \( X \times_S T \to T \) is finite locally free of degree \( r \) if and only if \( g(T) \subset S_r \) (set theoretically).

Proof. Let \( n \) be an integer bounding the degree of the fibres of \( X \to S \). By Morphisms, Lemma 54.6 we see that any base change has degrees of fibres bounded by \( n \) also. In particular, all the integers \( r \) that occur in the statement of the lemma will be \( \leq n \). We will prove the lemma by induction on \( n \). The base case is \( n = 0 \) which is obvious.

We claim the set of points \( s \in S \) with \( \deg_{X_S}(X_s) = n \) is an open subset \( S_n \subset S \) and that \( X \times_S S_n \to S_n \) is finite locally free of degree \( n \). Namely, suppose that \( s \in S \) is such a point. Choose an elementary étale morphism \((U, u) \to (S, s)\) and a decomposition \( U \times_S X = W \amalg V \) as in Lemma 36.6. Since \( V \to U \) is finite, flat, and locally of finite presentation, we see that \( V \to U \) is finite locally free, see Morphisms, Lemma 46.2. After shrinking \( U \) to a smaller neighbourhood of \( u \) we may assume \( V \to U \) is finite locally free of some degree \( d \), see Morphisms, Lemma 46.5. As \( u \mapsto s \) and \( W_u = \emptyset \) we see that \( d = n \). Since \( n \) is the maximum degree of a fibre we see that \( W = \emptyset \)! Thus \( U \times_S X \to U \) is finite locally free of degree \( n \).

By Descent, Lemma 20.30 we conclude that \( X \to S \) is finite locally free of degree \( n \) over \( \text{Im}(U \to S) \) which is an open neighbourhood of \( s \) (Morphisms, Lemma 34.13). This proves the claim.

Let \( S' = S \setminus S_n \) endowed with the reduced induced scheme structure and set \( X' = X \times_S S' \). Note that the degrees of fibres of \( X' \to S' \) are universally bounded by \( n - 1 \). By induction we find a stratification \( S' = S_0 \amalg \ldots \amalg S_{n-1} \) adapted to the morphism \( X' \to S' \). We claim that \( S = \coprod_{r=0}^n S_r \) works for the morphism \( X \to S \). Let \( g : T \to S \) be a morphism of schemes and assume that \( X \times_S T \to T \) is finite locally free of degree \( r \). As remarked above this implies that \( r \leq n \). If \( r = n \), then it is clear that \( T \to S \) factors through \( S_n \). If \( r < n \), then \( g(T) \subset S' = S \setminus S_d \) (set theoretically) hence \( T_{\text{red}} \to S \) factors through \( S' \), see Schemes, Lemma 12.7. Note that \( X \times_S T_{\text{red}} \to T_{\text{red}} \) is also finite locally free of degree \( r \) as a base change. By the universal property of the stratification \( S' = \coprod_{r=0}^{n-1} S_r \) we see that \( g(T) = g(T_{\text{red}}) \) is contained in \( S_r \). Conversely, suppose that we have \( g : T \to S \) such that \( g(T) \subset S_r \) (set theoretically). If \( r = n \), then \( g \) factors through \( S_n \) and it is clear that \( X \times_S T \to T \) is finite locally free of degree \( n \) as a base change. If \( r < n \), then \( X \times_S T \to T \) is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to \( T_{\text{red}} \) is finite locally free of degree \( r \). Since \( T_{\text{red}} \to T \) is a universal homeomorphism, we conclude that \( X \times_S T_{\text{red}} \to X \times_S T \) is a universal homeomorphism too and hence \( X \times_S T \to T \) is universally closed (as this is true for the finite morphism \( X \times_S T_{\text{red}} \to T_{\text{red}} \)). It follows that \( X \times_S T \to T \) is finite, for example by Lemma 39.1. Then we can use Morphisms, Lemma 46.2 to see that \( X \times_S T \to T \) is finite locally free. Finally, the degree is \( r \) as all the fibres have degree \( r \). \( \square \)
Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets
$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S$$
such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \bigsqcup S_r$ is characterized by the following universal property: Given a morphism $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree $r$ if and only if $g(T) \subset S_r$ (set theoretically). Moreover, the inclusion maps $S_r \to S$ are quasi-compact.

**Proof.** The question is local on $S$, hence we may assume that $S$ is affine. By Morphisms, Lemma 05.10 the fibres of $f$ are universally bounded in this case. Hence the existence of the stratification follows from Lemma 40.3.

We will show that $U_r = S \setminus Z_r \to S$ is quasi-compact for each $r \geq 0$. This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that $U_r \to U_{r-1}$ is quasi-compact. Choose an affine open $W \subset U_{r-1}$. Write $W = \text{Spec}(A)$. Then $Z_r \cap W = V(I)$ for some ideal $I \subset A$ and $X \times_S \text{Spec}(A/I) \to \text{Spec}(A/I)$ is finite locally free of degree $r$. Note that $A/I = \text{colim} A/I_i$ where $I_i \subset I$ runs through the finitely generated ideals. By Limits, Lemma 55.5 we see that $X \times_S \text{Spec}(A/I_i) \to \text{Spec}(A/I_i)$ is finite locally free of degree $r$ for some $i$. (This uses that $X \to S$ is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence $\text{Spec}(A/I_i) \to \text{Spec}(A) = W$ factors (set theoretically) through $Z_r \cap W$. It follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of $A$. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. \hfill \square

Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes
$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$
such that a morphism $\text{Spec}(k) \to S$ factors through $U_d$ if and only if $X \times_S \text{Spec}(k)$ has degree $\geq d$ over $k$.

**Proof.** The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on $S$ and assume $S$ is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 40.4. Since $U_d = \bigcup_{W} U_d(W)$ the result follows. \hfill \square

Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram
$$\begin{array}{ccc}
V & \longrightarrow & X \\
\pi \downarrow & & \downarrow f \\
U & \longrightarrow & S,
\end{array}$$
a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_* \mathcal{O}_V$ whose image contains $g|_V$. 

\newcommand{\Delta}{\Delta}
Proof. We may assume $X$ and $S$ affine. We obtain a filtration $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 40.3 and 40.4. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\text{Im}(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that $T$ is the support of $g$ as a section of $\mathcal{O}_X$ (Modules, Definition 5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let $r$ be the smallest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of $T$ such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace $S$ by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \text{Spec}(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 28.1). Since the support of $g$ is contained in $f^{-1}V(I)$ by our choice of $r$ we see that there exists an integer $N$ such that $a_j^Ng = 0$ for $j = 1, \ldots, m$. Replacing $a_j$ by $a_j^r$ we may assume that $Ig = 0$. For any $A$-module $M$ write $M[I]$ for the $I$-torsion of $M$, i.e., $M[I] = \{m \in M \mid Im = 0\}$. Write $X = \text{Spec}(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of $Z_r$, the $A/I$-module $B/IB$ is finite locally free of rank $r$. Hence after replacing $S$ by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/I)^{\oplus r}$ as $A/I$-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo $I$ to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \to B[I]$$

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the $A$-module $A[I]$ and the map $\mathcal{F}^{\oplus r} \to \pi_* \mathcal{O}_V$ the one corresponding to $A[I]^{\oplus r} \subset A^{\oplus r} \to B$. \square

Lemma 40.7. Let $U \to X$ be a surjective étale morphism of schemes. Assume $X$ is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through $U$. In fact, we may assume $Y \to X$ is finite and of finite presentation.

Proof. Since $X$ is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \bigsqcup U_i \to X$ is surjective. After replacing $U$ by $U'$, we see that we may assume $U$ is affine. In particular $U \to X$ is separated (Morphisms, Lemma 17.2). Then there exists an integer $d$ bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 54.10). We will prove the lemma by induction on $d$ for all quasi-compact and separated schemes $U$ mapping surjective and étale onto $X$. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 38.2 and we obtain a factorization

$$
\begin{array}{ccc}
U & \to & Y \\
\downarrow j & & \downarrow \pi \\
X & \to & \\
\end{array}
$$

with $\pi$ integral and $j$ a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in $Y$. Note that

$$U \times_X Y = U \amalg W$$
where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 21.10 and open because it is étale as a morphism between schemes étale over $Y$) and the second summand is the (open and closed) complement. The image $V \subset Y$ of $W$ is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of $Y$ for example by Lemma 40.3 (the degree of the fibres of a quasi-compact separated étale morphism does not go up under specialization).

Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with $Z$ a scheme, which Zariski locally factors through $W$. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 38.2). After replacing $Z'$ by the scheme theoretic closure of $Z$ in $Z'$ we may assume that $Z$ is scheme theoretically dense in $Z'$. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \to X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through $U$. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then $z$ maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of $z$ on which the morphism factors through $U$. If $z \in Z$, then we have a neighbourhood $\Omega \subset Z$ which factors through $W \subset U \times_X Y$ and hence through $U$. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through $U$. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $U$. We can write $Y = \lim_i Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 7.2. For large enough $i$ we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in $Y$ recovers $V_j$, see Limits, Lemma 4.11. For even larger $i$ the morphisms $V_{i,j} \to U$ over $X$ come from morphisms $V_{i,j} \to U$ over $X$, see Limits, Proposition 6.1. This finishes the proof.

\[\Box\]

41. Application to morphisms with connected fibres

In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

**Lemma 41.1.** Consider a diagram of morphisms of schemes

$$\begin{array}{ccc}
Z & \xrightarrow{\sigma} & X \\
& \searrow & \downarrow \\
& & Y
\end{array}$$

an a point $y \in Y$. Assume

1. $X \to Y$ is of finite presentation and flat,
2. $Z \to Y$ is finite locally free,
3. $Z_y \neq \emptyset$,
4. all fibres of $X \to Y$ are geometrically reduced, and
5. $X_y$ is geometrically connected over $\kappa(y)$.
Then there exists a quasi-compact open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

**Proof.** In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 24.8, 20.4, 46.4, 24.6, 20.3, 46.3, 24.10 and 42.11

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section $s$ coming from $\sigma$. Let $X^0_Z$ denote the subset of $X_Z$ defined in Situation 27.1. By Lemma 27.6 it is an open subset of $X_Z$.

The pullback $X_{Z \times_Y Z}$ of $X$ to $Z \times_Y Z$ comes equipped with two sections $s_0, s_1$, namely the base changes of $s$ by $pr_0, pr_1 : Z \times_Y Z \to Z$. The construction of Situation 27.1 gives two subsets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$. By Lemma 27.2 these are the inverse images of $X^0_Z$ under the morphisms $1_X \times pr_0, 1_X \times pr_1 : X_{Z \times_Y Z} \to X$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As $X_y$ is geometrically connected, we see that the fibres of $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ over each $z_i$ agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z \times_Y Z})^0_{s_0}.$$  

Since the sets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ are open in $X_{Z \times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z \times_Y Z})^0_{s_0}.$$  

Then it follows directly from the construction in Situation 27.1 that

$$p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_1}$$

where $p : X_{Z \times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an affine open neighbourhood $V \subset Y$ of $y$ such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace $Y$ by $V$. After we do this we see that $X^0_Z \subset Y_Z$ is an open such that

$$(1_X \times pr_0)^{-1}(X^0_Z) = (1_X \times pr_1)^{-1}(X^0_Z).$$

This means that the image $X^0 \subset X$ of $X^0_Z$ is an open such that $(X_Z \to X)^{-1}(X^0) = X^0_Z$, see Descent, Lemma 10.6. Finally, $X^0$ is quasi-compact because $X^0_Z$ is quasi-compact by Lemma 27.4 (use that at this point $Y$ is affine, hence $X$ is quasi-compact and quasi-separated, hence locally constructible is the same as constructible and in particular quasi-compact; details omitted). In this way we see that $X^0$ has all the desired properties.  \[\square\]

**Lemma 41.2.** Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume

1. $h$ is flat and of finite presentation,
2. all fibres of $h$ are geometrically reduced, and
3. $T$ is geometrically connected over $\kappa(s)$.

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) all fibres of $V \to S'$ are geometrically connected,
(b) $V_s' = T \times_s s'$.

**Proof.** The problem is clearly local on $S$, hence we may replace $S$ by an affine open neighbourhood of $s$. The topology on $Y_s$ is induced from the topology on $Y$, see Schemes, Lemma 18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of $h$ to $V$ is quasi-compact (as $S$ affine and $V$ quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing $Y$ by $V$ we may assume, in addition to (1) and (2) that $Y_s = T$ and $S$ affine.

Pick a closed point $y \in Y_s$ such that $h$ is Cohen-Macaulay at $y$, see Lemma 20.7. By Lemma 21.4 there exists a diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
& \searrow & \\
& & S
\end{array}
$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{y\}$. Apply Lemma 36.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over $s$. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 46.3. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of $h$. Also $Y_{s'} = Y_s$ is geometrically connected. Apply Lemma 41.1 to $Z' \to Y_{S'}$ over $S'$ to get $V \subset Y_{S'}$ quasi-compact open satisfying (2) whose fibres over $S'$ are either empty or geometrically connected. As $V \to S'$ is open (Morphisms, Lemma 24.10), after replacing $S'$ by an affine open neighbourhood of $s'$ we may assume $V \to S'$ is surjective, whence (1) holds. \qed

**Lemma 41.3.** Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation and flat with geometrically reduced fibres. Then there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $X_i \to S$ factors as $X_i \to S_i \to S$ where $S_i \to S$ is étale and $X_i \to S_i$ is flat of finite presentation with geometrically connected and geometrically reduced fibres.

**Proof.** Pick a point $x \in X$ with image $s \in S$. We will produce a diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & S' \times_S X \\
& \downarrow & \downarrow \\
& & X
\end{array}
$$

and points $s' \in S'$, $x' \in X'$, $y \in S' \times_S X$ such that $x'$ maps to $x$, $(S', s') \to (S, s)$ is an étale neighbourhood, $(X', x') \to (S' \times_S X, y)$ is an étale neighbourhood and $X' \to S'$ has geometrically connected fibres. If we can do this for every $x \in X$, then the lemma follows (with members of the covering given by the collection of étale morphisms $X' \to X$ so produced). The first step is the replace $X$ and $S$ by affine open neighbourhoods of $x$ and $s$ which reduces us to the case that $X$ and $S$ are affine (and hence $f$ of finite presentation).

\footnote{The proof actually gives an open $X' \subset S' \times_S X$.}
Choose a separable algebraic extension \( \overline{k} \) of \( \kappa(s) \). Denote \( X_T \) the base change of \( X_s \). Choose a point \( \mathfrak{p} \) in \( X_T \) mapping to \( x \in X_s \). Choose a connected quasi-compact open neighbourhood \( \mathcal{V} \subset X_{k'} \) of \( \mathfrak{p} \). (This is possible because any scheme locally of finite type over a field is locally connected as a locally Noetherian topological space.) By Varieties, Lemma 7.9 we can find a finite separable extension \( k'/\kappa(s) \) and a quasi-compact open \( V' \subset X_{k'} \) whose base change is \( \mathcal{V} \). In particular \( V' \) is geometrically connected over \( k' \), see Varieties, Lemma 7.7. By Lemma 31.2 we can find an étale neighbourhood \((S', s') \to (S, s) \) such that \( \kappa(s') \) is isomorphic to \( k' \) as an extension of \( \kappa(s) \). Denote \( x' \in (S' \times_S X)_{s'} = X_{k'} \) the image of \( \mathfrak{p} \). Thus after replacing \((S, s) \) by \((S', s') \) and \((X, x) \) by \((S' \times_S X, x') \) we reduce to the case handled in the next paragraph.

Assume there is a quasi-compact open \( V \subset X_s \) which contains \( x \) and is geometrically irreducible. Then we can apply Lemma 41.2 to find an affine étale neighbourhood \((S', s') \to (S, s) \) and a quasi-compact open \( X' \subset S' \times_S X \) such that \( X' \to S' \) has geometrically connected fibres and such that \( X' \) contains a point mapping to \( x \). This finishes the proof. \( \square \)

**Lemma 41.4.** Let \( h : X \to S \) be a morphism of schemes. Let \( s \in S \) be a point. Let \( T \subset Y_s \) be an open subscheme. Assume

\begin{enumerate}
\item \( h \) is of finite presentation,
\item \( h \) is normal, and
\item \( T \) is geometrically irreducible over \( \kappa(s) \).
\end{enumerate}

Then we can find an affine elementary étale neighbourhood \((S', s') \to (S, s) \) and a quasi-compact open \( V \subset Y_{s'} \) such that

\begin{enumerate}
\item all fibres of \( V \to S' \) are geometrically integral,
\item \( V_{s'} = T \times_s s' \).
\end{enumerate}

**Proof.** Apply Lemma 41.2 to find an affine elementary étale neighbourhood \((S', s') \to (S, s) \) and a quasi-compact open \( V \subset Y_{s'} \) such that all fibres of \( V \to S' \) are geometrically connected and \( V_{s'} = T \times_s s' \). As \( V \) is an open of the base change of \( h \) all fibres of \( V \to S' \) are geometrically normal, see Lemma 18.2 In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if \( t \in S' \) then \( V_t \) is of finite type over \( \kappa(t) \) and hence \( V_t \times_{\kappa(t)} \kappa(\overline{t}) \) is of finite type over \( \overline{\kappa(t)} \) hence Noetherian. By choice of \( S' \to S \) the scheme \( V_t \times_{\kappa(t)} \kappa(\overline{t}) \) is connected. Hence \( V_t \times_{\kappa(t)} \kappa(\overline{t}) \) is irreducible by Properties, Lemma 7.6 and we win. \( \square \)

42. Application to the structure of finite type morphisms

**Lemma 42.1.** Let \( f : X \to S \) be a morphism. Let \( x \in X \) and set \( s = f(x) \). Assume that \( f \) is locally of finite type and that \( n = \dim_x(X_s) \). Then there exists a
and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$ we have

1. $h : Y' \to S'$ is smooth of relative dimension $n$,
2. $g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,
3. $\pi$ is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and
4. $\kappa(y')$ is a purely transcendental extension of $\kappa(s)$.

Moreover, if $f$ is locally of finite presentation then $\pi$ is of finite presentation.

**Proof.** The problem is local on $X$ and $S$, hence we may assume that $X$ and $S$ are affine. By Algebra, Lemma [124.3] after replacing $X$ by a standard open neighbourhood of $x$ in $X$ we may assume there is a factorization

$$X \longrightarrow A^n_S \longrightarrow S$$

such that $\pi$ is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma [36.1] there exists an elementary étale neighbourhood

$$(Y, y) \to (A^n_S, \pi(x))$$

and an open $X' \subset X \times_{A^n_S} Y$ which contains a unique point $x'$ lying over $y$ such that $X' \to Y$ is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma [20.11]. □

**Lemma 42.2.** Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that $f$ is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram

$$X \begin{array}{c} g \downarrow \downarrow \pi \downarrow \downarrow \downarrow \downarrow h \end{array} X' \begin{array}{c} x \downarrow \downarrow \downarrow \end{array} Y \begin{array}{c} \downarrow \downarrow \downarrow \end{array} s' \begin{array}{c} e \downarrow \downarrow \downarrow \end{array} X' \begin{array}{c} x \downarrow \downarrow \downarrow \end{array} y' \begin{array}{c} \downarrow \downarrow \downarrow \end{array} S' \begin{array}{c} \downarrow \downarrow \downarrow \end{array} S$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

1. $h : Y' \to S'$ is smooth of relative dimension $n$,
2. all fibres of $Y' \to S'$ are geometrically integral,
3. $g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,
4. $\pi$ is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,
5. $\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and
6. $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.

Moreover, if $f$ is locally of finite presentation, then $\pi$ is of finite presentation.
**Proof.** The question is local on $S$, hence we may replace $S$ by an affine open neighbourhood of $s$. Next, we apply Lemma 42.1 to get a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\pi} & Y' \\
\downarrow & & \downarrow \\
S & \xleftarrow{h} & S' \\
\end{array}
\]

where $h$ is smooth of relative dimension $n$ and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on $X$ also, we may replace $Y$ by an affine neighbourhood of $y$ (and $X'$ by the inverse image of this under $\pi$). As $S$ is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let $T$ be the connected component of $Y_s$ containing $y$. As $Y_s$ is Noetherian we see that $T$ is open. We also see that $T$ is geometrically connected over $\kappa(s)$ by Varieties, Lemma 7.14. Since $T$ is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 25.4. We conclude that $T$ is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 7.6). Finally, note that the smooth morphism $h$ is normal by Lemma [18.3]. At this point we have verified all assumption of Lemma 41.4 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 41.4 we obtain $e : S' \to S$, $s' \in S'$, $Y'$ as in the commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\pi} & Y' \\
\downarrow & & \downarrow \\
S & \xleftarrow{h} & S' \\
\end{array}
\]

where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over $S'$ are geometrically irreducible, such that $Y_{S'} = T$ via the identification $Y_s = Y_{S', s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over $y$ with residue field equal to $\kappa(y)$ which maps to $s'$ in $S'$. Similarly, let $(x', s') \in X' \times_S Y' \subset X' \times_S S'$ be the unique point over $x'$ with residue field equal to $\kappa(x')$ lying over $s'$. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. \qed

**Lemma 42.3.** Assumption and notation as in Lemma 42.2. In addition to properties (1) - (6) we may also arrange it so that

(7) $S'$, $Y'$, $X'$ are affine.

**Proof.** Note that if $Y'$ is affine, then $X'$ is affine as $\pi$ is finite. Choose an affine open neighbourhood $U \subset S'$ of $s'$. Choose an affine open neighbourhood $V \subset h^{-1}(U')$ of $y'$. Let $W' = h(V')$. This is an open neighbourhood of $s'$ in $S'$, see Morphisms, Lemma [32.10] contained in $U'$. Choose an affine open neighbourhood $U'' \subset W'$ of $s'$. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By
We can use the above étale localization techniques to prove the following result:

\[ \text{Let } \pi : X \to Y \text{ be a finite morphism. Let } x \in X \text{ with } y = \pi(x) \text{ such that } \pi^{-1}(\{y\}) = \{x\}. \]

Then

1. For every neighbourhood \( U \subset X \) of \( x \) in \( X \), there exists a neighbourhood \( V \subset Y \) of \( y \) such that \( \pi^{-1}(V) \subset U \).
2. The ring map \( O_{Y,y} \to O_{X,x} \) is finite.
3. If \( \pi \) is of finite presentation, then \( O_{Y,y} \to O_{X,x} \) is of finite presentation.
4. For any quasi-coherent \( O_X \)-module \( F \) we have \( F_x = \pi_*F_y \) as \( O_{Y,y} \)-modules.

**Proof.** The first assertion is purely topological; use that \( \pi \) is a continuous and closed map such that \( \pi^{-1}(\{y\}) = \{x\} \). To prove the second and third parts we may assume \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \). Then \( A \to B \) is a finite ring map and \( y \) corresponds to a prime \( \mathfrak{p} \) of \( A \) such that there exists a unique prime \( \mathfrak{q} \) of \( B \) lying over \( \mathfrak{p} \). Then \( B_{\mathfrak{q}} = B_{\mathfrak{p}} \), see Algebra, Lemma \[40.11\]. In other words, the map \( A_{\mathfrak{p}} \to B_{\mathfrak{p}} \) is equal to the map \( A_{\mathfrak{p}} \to B_{\mathfrak{p}} \) you get from localizing \( A \to B \) at \( \mathfrak{p} \). Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that \( F = \tilde{M} \) for some \( B \)-module \( M \). Then \( F = M_{\mathfrak{q}} \) and \( \pi_*F_y = M_{\mathfrak{p}} \). By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that \( F_x = \pi_*F_y \) directly. \( \Box \)

### 43. Application to the fppf topology

We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology “generated by” Zariski coverings and by coverings of the form \( \{f : T \to S\} \) where \( f \) is surjective finite locally free.

**Lemma 43.1.** Let \( S \) be a scheme. Let \( \{S_i \to S\}_{i \in I} \) be an fppf covering. Then there exist

1. an étale covering \( \{S'_i \to S\} \),
2. surjective finite locally free morphisms \( V_a \to S'_a \),

such that the fppf covering \( \{V_a \to S\} \) refines the given covering \( \{S_i \to S\} \).

**Proof.** We may assume that each \( S_i \to S \) is locally quasi-finite, see Lemma \[21.6\].

Fix a point \( s \in S \). Pick an \( i \in I \) and a point \( s_i \in S_i \) mapping to \( s \). Choose an elementary étale neighbourhood \( (S', s) \to (S, s) \) such that there exists an open

\[ S_i \times_S S' \supset V \]

which contains a unique point \( v \in V \) mapping to \( s \in S' \) and such that \( V \to S' \) is finite, see Lemma \[36.1\]. Then \( V \to S' \) is finite locally free, because it is finite and because \( S_i \times_S S' \to S' \) is flat and locally of finite presentation as a base change of the morphism \( S_i \to S \), see Morphisms, Lemmas \[20.4\], \[24.8\] and \[16.2\]. Hence \( V \to S' \) is open, and after shrinking \( S' \) we may assume that \( V \to S' \) is surjective finite locally free. Since we can do this for every point of \( S \) we conclude that \( \{S_i \to S\} \) can be refined by a covering of the form \( \{V_a \to S\}_{a \in A} \) where each \( V_a \to S \) factors as \( V_a \to S'_a \to S \) with \( S'_a \to S \) étale and \( V_a \to S'_a \) surjective finite locally free. \( \Box \)
Lemma 43.2. Let $S$ be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fpqc covering. Then there exist

1. a Zariski open covering $S = \bigcup U_j$,
2. surjective finite locally free morphisms $W_j \to U_j$,
3. Zariski open coverings $W_j = \bigcup_{i} W_{j,i}$,
4. surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fpqc covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. Let $\{V_a \to S\}_{a \in A}$ be the fpqc covering found in Lemma 43.1. In other words, this covering refines $\{S_i \to S\}$ and each $V_a \to S$ factors as $V_a \to S'_{a} \to S$ with $S'_{a} \to S$ étale and $V_a \to S'_{a}$ surjective finite locally free.

By Remark 35.3 there exists a Zariski open covering $S = \bigcup U_j$, for each $j$ a finite locally free, surjective morphism $W_j \to U_j$, and for each $j$ a Zariski open covering $\{W_{j,i} \to W_j\}$ such that the family $\{W_{j,i} \to S\}$ refines the étale covering $\{S'_{a} \to S\}$, i.e., for each pair $j, k$ there exists an $a(j, k)$ and a factorization $W_{j,i} \to S'_{a} \to S$ of the morphism $W_{j,K} \to S$. Set $T_{j,k} = W_{j,k} \times S'_{a} V_a$ and everything is clear. \hfill \qed

Lemma 43.3. Let $S$ be a scheme. If $U \subset S$ is open and $V \to U$ is a surjective integral morphism, then there exists a surjective integral morphism $V' \to S$ with $V' \times_S U$ isomorphic to $V$ as schemes over $U$.

Proof. Let $V' \to S$ be the normalization of $S$ in $U$, see Morphisms, Section 51. By construction $V' \to S$ is integral. By Morphisms, Lemmas 51.6 and 51.12 we see that the inverse image of $U$ in $V'$ is $V$. Let $Z$ be the reduced induced scheme structure on $S \setminus U$. Then $V' = V' \amalg Z$ works. \hfill \qed

Lemma 43.4. Let $S$ be a quasi-compact and quasi-separated scheme. If $U \subset S$ is a quasi-compact open and $V \to U$ is a surjective finite morphism, then there exists a surjective finite morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to $V$ as schemes over $U$.

Proof. By Zariski’s Main Theorem (Lemma 38.3) we can assume $V$ is a quasi-compact open in a scheme $V'$ finite over $S$. After replacing $V'$ by the scheme theoretic image of $V$ we may assume that $V$ is dense in $V'$. It follows that $V' \times_S U = V$ because $V \to V' \times_S U$ is closed as $V$ is finite over $U$. Let $Z$ be the reduced induced scheme structure on $S \setminus U$. Then $V' = V' \amalg Z$ works. \hfill \qed

Lemma 43.5. Let $S$ be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fpqc covering. Then there exists a surjective integral morphism $S' \to S$ and an open covering $S' = \bigcup U'_a$ such that for each $a$ the morphism $U'_a \to S$ factors through $S_i$ for some $i$.

Proof. Choose $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, and $T_{j,k} \to W_{j,k}$ as in Lemma 43.2. By Lemma 43.3 we can extend $W_j \to U_j$ to a surjective integral morphism $W_j \to S$. After this we can extend $T_{j,k} \to W_{j,k}$ to a surjective integral morphism $T_{j,k} \to W_{j,k}$. We set $T_j$ equal to the product of all the schemes $T_{j,k}$ over $W_j$ (Limits, Lemma 3.1). Then we set $S'$ equal to the product of all the schemes $T_j$ over $S$. If $x \in S'$, then there is a $j$ such that the image of $x$ in $S'$ lies in $U_j$. Hence there is a $k$ such that the image of $x$ under the projection $S' \to W_j$ lies in $W_{j,k}$. Hence under the projection $S' \to T_j \to T_{j,k}$ the point $x$ ends up in $T_{j,k}$. And $T_{j,k} \to S$ factors through $S_i$ for some $i$. Finally, the morphism $S' \to S$ is integral and surjective by Limits, Lemmas 3.3 and 3.2. \hfill \qed
Lemma 43.6. Let $S$ be a quasi-compact and quasi-separated scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exists a surjective finite morphism $S' \to S$ of finite presentation and an open covering $S' = \bigcup U'_\alpha$ such that for each $\alpha$ the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some $i$.

Proof. Let $Y \to X$ be the integral surjective morphism found in Lemma 43.5. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $S_{i(j)}$. We can write $Y = \lim Y_\lambda$ with $Y_\lambda \to X$ finite and of finite presentation, see Limits, Lemma 7.2. For large enough $\lambda$ we can find affine opens $V_{\lambda,j} \subset Y_\lambda$ whose inverse image in $Y$ recovers $V_j$, see Limits, Lemma 4.11. For even larger $\lambda$ the morphisms $V_j \to S_{i(j)}$ over $X$ come from morphisms $V_{\lambda,j} \to S_{i(j)}$ over $X$, see Limits, Proposition 6.1. Setting $S' = Y_\lambda$ for this $\lambda$ finishes the proof. □

Lemma 43.7. An fppf covering of schemes is a ph covering.

Proof. Let $\{T_i \to T\}$ be an fppf covering of schemes, see Topologies, Definition 7.1. Observe that $T_i \to T$ is locally of finite type. Let $U \subset T$ be an affine open. It suffices to show that $\{T_i \times_T U \to U\}$ can be refined by a standard ph covering, see Topologies, Definition 8.4. This follows immediately from Lemma 43.6 and the fact that a finite morphism is proper (Morphisms, Lemma 42.11). □

Remark 43.8. As a consequence of Lemma 43.7 we obtain a comparison morphism $\epsilon : (\text{Sch}/S)_{ph} \to (\text{Sch}/S)_{fppf}$

This is the morphism of sites given by the identity functor on underlying categories (with suitable choices of sites as in Topologies, Remark 11.1). The functor $\epsilon_*$ is the identity on underlying presheaves and the functor $\epsilon^{-1}$ associated to an fppf sheaf its ph sheafification. By composition we can in addition compare the ph topology with the syntomic, smooth, étale, and Zariski topologies.

44. Quasi-projective schemes

The term “quasi-projective scheme” has not yet been defined. A possible definition could be a scheme which has an ample invertible sheaf. However, if $X$ is a scheme over a base scheme $S$, then we say that $X$ is quasi-projective over $S$ if the morphism $X \to S$ is quasi-projective (Morphisms, Definition 38.1). Since the identity morphism of any scheme is quasi-projective, we see that a scheme quasi-projective over $S$ doesn’t necessarily have an ample invertible sheaf. For this reason it seems better to leave the term “quasi-projective scheme” undefined.

Lemma 44.1. Let $S$ be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. $X \to S$ is quasi-projective,
2. $X \to S$ is H-quasi-projective,
3. there exists a quasi-compact open immersion $X \to X'$ of schemes over $S$ with $X' \to S$ projective,
4. $X \to S$ is of finite type and $X$ has an ample invertible sheaf, and
5. $X \to S$ is of finite type and there exists an $f$-very ample invertible sheaf.

Proof. The implication $(2) \Rightarrow (1)$ is Morphisms, Lemma 38.5. The implication $(1) \Rightarrow (2)$ is Morphisms, Lemma 41.10. The implication $(2) \Rightarrow (3)$ is Morphisms, Lemma 41.11.
Assume \( X \subseteq X' \) is as in (3). In particular \( X \to S \) is of finite type. By Morphisms, Lemma \[41.11\] the morphism \( X \to S \) is H-projective. Thus there exists a quasi-compact immersion \( i : X \to P^n_S \). Hence \( \mathcal{L} = i^*\mathcal{O}_{P^n_S}(1) \) is \( f \)-very ample. As \( X \to S \) is quasi-compact we conclude from Morphisms, Lemma \[36.2\] that \( \mathcal{L} \) is \( f \)-ample. Thus \( X \to S \) is quasi-projective by definition.

The implication (4) \( \Rightarrow \) (2) is Morphisms, Lemma \[37.3\].

Assume the equivalent conditions (1), (2), (3) hold. Choose an immersion \( i : X \to P^n_S \) over \( S \). Let \( L \) be an ample invertible sheaf on \( S \). To finish the proof we will show that \( N = f^*\mathcal{L} \otimes \mathcal{O}_X i^*\mathcal{O}_{P^n_S}(1) \) is ample on \( X \). By Properties, Lemma \[26.14\] we reduce to the case \( X = P^n_S \). Let \( s \in \Gamma(S, \mathcal{L} \otimes^d) \) be a section such that the corresponding open \( S_s = \text{Spec}(A) \) is affine. Say \( S_s = \text{Spec}(A) \). Recall that \( P^n_S \) is the projective bundle associated to \( \mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n \), see Constructions, Lemma \[21.5\] and its proof. Let \( s_i \in \Gamma(P^n_S, \mathcal{O}(1)) \) be the global section corresponding to the section \( T_i \) of \( \mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n \). Then we see that \( X f^*s_i \otimes s_i^{\otimes n} \) is affine because it is equal to \( \text{Spec}(A[T_0/T_i, \ldots, T_n/T_i]) \). This proves that \( N \) is ample by definition.

The equivalence of (1) and (5) follows from Morphisms, Lemmas \[36.2\] and \[37.5\]. \( \square \)

**Lemma 44.2.** Let \( S \) be a scheme which has an ample invertible sheaf. Let \( \text{QP}_S \) be the full subcategory of the category of schemes over \( S \) satisfying the equivalent conditions of Lemma \[44.1\].

1. if \( S' \to S \) is a morphism of schemes and \( S' \) has an ample invertible sheaf, then base change determines a functor \( \text{QP}_S \to \text{QP}_{S'} \),
2. if \( X \in \text{QP}_S \) and \( Y \in \text{QP}_X \), then \( Y \in \text{QP}_S \),
3. the category \( \text{QP}_S \) is closed under fibre products,
4. the category \( \text{QP}_S \) is closed under finite disjoint unions,
5. if \( X \to S \) is projective, then \( X \in \text{QP}_S \),
6. if \( X \to S \) is quasi-affine of finite type, then \( X \in \text{QP}_S \),
7. if \( X \to S \) is quasi-finite and separated, then \( X \in \text{QP}_S \),
8. if \( X \to S \) is a quasi-compact immersion, then \( X \in \text{QP}_S \),
9. add more here.

**Proof.** Part (1) follows from Morphisms, Lemma \[38.2\].

Part (2) follows from the fourth characterization of Lemma \[44.1\].

If \( X \to S \) and \( Y \to S \) are quasi-projective, then \( X \times_S Y \to Y \) is quasi-projective by Morphisms, Lemma \[38.2\]. Hence (3) follows from (2).

If \( X = Y \amalg Z \) is a disjoint union of schemes and \( \mathcal{L} \) is an invertible \( \mathcal{O}_X \)-module such that \( \mathcal{L}|_Y \) and \( \mathcal{L}|_Z \) are ample, then \( \mathcal{L} \) is ample (details omitted). Thus part (4) follows from the fourth characterization of Lemma \[44.1\].

Part (5) follows from Morphisms, Lemma \[41.10\].

Part (6) follows from Morphisms, Lemma \[38.7\].

Part (7) follows from part (6) and Lemma \[38.2\].

Part (8) follows from part (7) and Morphisms, Lemma \[19.16\]. \( \square \)

The following lemma doesn’t really belong in this section, but there does not seem to be a good spot for it anywhere else.
Lemma 4.3. Let $X$ be a quasi-affine scheme. Let $f : U \to X$ be an integral morphism. Then $U$ is quasi-affine and the diagram

$$
\begin{array}{ccc}
U & \longrightarrow & \text{Spec}(\Gamma(U, \mathcal{O}_U)) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Spec}(\Gamma(X, \mathcal{O}_X))
\end{array}
$$

is cartesian.

Proof. The scheme $U$ is quasi-affine because integral morphisms are affine, affine morphisms are quasi-affine, a scheme is quasi-affine if and only if the structure morphism to $\text{Spec}(\mathbb{Z})$ is quasi-affine, and compositions of quasi-affine morphisms are quasi-affine. The first two statements follow immediately from the definition and the third is Morphisms, Lemma 12.4. Set $U' = X \times_{\text{Spec}(\Gamma(X, \mathcal{O}_X))} \text{Spec}(\Gamma(U, \mathcal{O}_U))$ and consider the extended diagram

$$
\begin{array}{ccc}
U & \longrightarrow & U' \\
\downarrow & j & \downarrow \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Spec}(\Gamma(X, \mathcal{O}_X))
\end{array}
$$

The morphism $j$ is closed by Morphisms, Lemma 39.7 combined with the fact that an integral morphism is universally closed (Morphisms, Lemma 42.7) and the fact that the vertical arrows are in the diagram are separated. On the other hand, $j$ is open because the horizontal arrows in the diagram of the lemma are open by Properties, Lemma 18.4. Thus $j$ identifies $U$ with an open and closed subscheme of $U'$. If $U \neq U'$ then $U$ isn’t dense in $U'$ and a fortiori not dense in the spectrum of $\Gamma(U, \mathcal{O}_U)$. However, the scheme theoretic image of $U$ in $\text{Spec}(\Gamma(U, \mathcal{O}_U))$ is $\text{Spec}(\Gamma(U, \mathcal{O}_U))$ because any ideal in $\Gamma(U, \mathcal{O}_U)$ cutting out a closed subscheme through which $U$ factors would have to be zero. Hence $U$ is dense in $\text{Spec}(\Gamma(U, \mathcal{O}_U))$ for example by Morphisms, Lemma 6.3. Thus $U = U'$ and we win. \qed

45. Projective schemes

This section is the analogue of Section 44 for projective morphisms.

Lemma 45.1. Let $S$ be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. $X \to S$ is projective,
2. $X \to S$ is $H$-projective,
3. $X \to S$ is quasi-projective and proper,
4. $X \to S$ is $H$-quasi-projective and proper,
5. $X \to S$ is proper and $X$ has an ample invertible sheaf,
6. $X \to S$ is proper and there exists an $f$-ample invertible sheaf,
7. $X \to S$ is proper and there exists an $f$-very ample invertible sheaf,
8. there is a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ with $\mathcal{A}_1$ a finite type $\mathcal{O}_S$-module such that $X = \text{Proj}_S(\mathcal{A})$.

Proof. Observe first that in each case the morphism $f$ is proper, see Morphisms, Lemmas 41.3 and 41.5. Hence it suffices to prove the equivalence of the notions
in case \( f \) is a proper morphism. We will use this without further mention in the following.

The equivalences \((1) \iff (3)\) and \((2) \iff (4)\) are Morphisms, Lemma 41.13.

The implication \((2) \implies (1)\) is Morphisms, Lemma 41.3.

The implications \((1) \implies (2)\) and \((3) \implies (4)\) are Morphisms, Lemma 41.16.

The implication \((1) \implies (7)\) is immediate from Morphisms, Definitions 41.1 and 36.1.

The conditions \((3)\) and \((6)\) are equivalent by Morphisms, Definition 38.1.

Thus \((1) – (4), (6)\) are equivalent and imply \((7)\). By Lemma 44.1 conditions \((3), (5), \) and \((7)\) are equivalent. Thus we see that \((1) – (7)\) are equivalent.

By Divisors, Lemma 30.5 we see that \((8)\) implies \((1)\). Conversely, if \((2)\) holds, then we can choose a closed immersion

\[ i : X \longrightarrow \mathbf{P}^n_S = \text{Proj}_S(O_S[T_0, \ldots, T_n]). \]

See Constructions, Lemma 21.5 for the equality. By Divisors, Lemma 31.1 we see that \( X \) is the relative Proj of a quasi-coherent graded quotient algebra \( \mathcal{A} \) of \( O_S[T_0, \ldots, T_n] \). Then \( \mathcal{A} \) satisfies the conditions of \((8)\).

\[ \square \]

**Lemma 45.2.** Let \( S \) be a scheme which has an ample invertible sheaf. Let \( P_S \) be the full subcategory of the category of schemes over \( S \) satisfying the equivalent conditions of Lemma 45.1.

1. if \( S' \to S \) is a morphism of schemes and \( S' \) has an ample invertible sheaf, then base change determines a functor \( P_S \to P_{S'} \),
2. if \( X \in P_S \) and \( Y \in P_X \), then \( Y \in P_S \),
3. the category \( P_S \) is closed under fibre products,
4. the category \( P_S \) is closed under finite disjoint unions,
5. if \( X \to S \) is finite, then \( X \) is in \( P_S \),
6. add more here.

**Proof.** Part (1) follows from Morphisms, Lemma 41.9.

Part (2) follows from the fifth characterization of Lemma 45.1 and the fact that compositions of proper morphisms are proper (Morphisms, Lemma 39.4).

If \( X \to S \) and \( Y \to S \) are projective, then \( X \times_S Y \to Y \) is projective by Morphisms, Lemma 41.9. Hence (3) follows from (2).

If \( X = Y \amalg Z \) is a disjoint union of schemes and \( \mathcal{L} \) is an invertible \( O_X \)-module such that \( \mathcal{L}|_Y \) and \( \mathcal{L}|_Z \) are ample, then \( \mathcal{L} \) is ample (details omitted). Thus part (4) follows from the fifth characterization of Lemma 45.1.

Part (5) follows from Morphisms, Lemma 42.16. \[ \square \]

Here is a slightly different type of result.

**Lemma 45.3.** Let \( f : X \to Y \) be a proper morphism of schemes. Let \( \mathcal{L} \) be an invertible \( O_X \)-module. Let \( y \in Y \) be a point such that \( \mathcal{L}|_y \) is ample on \( X_y \). Then there is an open neighbourhood \( V \subset Y \) of \( y \) such that \( \mathcal{L}|_{f^{-1}(V)} \) is ample on \( f^{-1}(V)/V \).
In this section we clarify the relationship between the Proj and the spectrum of a graded ring.

Let $R$ be a ring. Let $A$ be a graded $R$-algebra, see Algebra, Section \[55\]. For $m \geq 0$ we denote $A_{\geq m} = \bigoplus_{d \geq m} A_d$. Consider the graded ring

$$B = \bigoplus_{d \geq 0} A_{\geq d}$$

For $d' \geq d$ and $a \in A_{d'}$, let us denote $a^{(d)} \in B$ the element in $B_d$ corresponding to $a$. Let us denote $\sigma : A \to B$ and $\psi : A \to B$ the two obvious ring maps: if $a \in A_d$, then $\sigma(a) = a^{(0)}$ and $\psi(a) = a^{(d)}$. Then $\psi$ is a graded ring map and $\sigma$ turns $B$ into a graded algebra over $A$. There is also a surjective graded ring map $\tau : B \to A$ which for $d' \geq d$ and $a \in A_{d'}$ sends $a^{(d)}$ to 0 if $d' > d$ and to $a$ if $d' = d$.

Affine schemes and spectra. We set $X = \text{Spec}(A)$. The irrelevant ideal $A_+$ cuts out a closed subscheme $Z = V(A_+) = \text{Spec}(A/A_+) = \text{Spec}(A_0)$. Set $U = X \setminus Z$.

$$U \to X \to Z$$

Projective schemes and Proj. Set $P = \text{Proj}(A)$. We may and do view $P$ as a scheme over $\text{Spec}(A_0) = Z$. Set $L = \text{Proj}(B)$. We may and do view $L$ as a scheme over $\text{Spec}(B_0) = \text{Spec}(A) = X$; observe that the identification of $B_0$ with $A$ is given by $\sigma$. The surjection $\tau$ defines a closed immersion $0 : P \to L$. Since $A \xrightarrow{\sigma} B \to A$ is equal to the map $A \to A_0 \to A$ we conclude that

$$
\begin{array}{ccc}
\quad & P & \to & L \\
\downarrow & & \downarrow & \\
Z & \to & X
\end{array}
$$

is commutative.

We claim that $\psi$ defines a morphism $L \to P$. To see this, by Constructions, Lemma \[11.1\] it suffices to check $\psi(A_+) \not\subseteq \mathfrak{p}$ for every homogeneous prime ideal $\mathfrak{p} \subseteq B$ with $B_+ \not\subseteq \mathfrak{p}$. First, pick $g \in B_+$ homogeneous $g \not\subseteq \mathfrak{p}$. Then we can write $g$ as a finite sum $g = \sum a_i^{(d_i)}$ with $a_i \in A_{d_i}$ for some $d_i \geq d$. We conclude that there exist $d' \geq d$ and $a \in A_{d'}$ such that $a^{(d)} \not\subseteq \mathfrak{p}$. Then

$$(a^{(d)})^{d'} = (a^{d'})^{(d'd)} = a^{(d)}(a^{d'-1})^{(d(d'-1))} = \psi(a)(a^{d'-1})^{(d(d'-1))}$$
(the notation leaves something to be desired) is not in \( p \). Hence \( \psi(a) \not\in p \), proving the claim. Thus we can extend our diagram above to a commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{0} & L \\
\downarrow & & \downarrow \pi \\
Z & \xrightarrow{\tau} & X \\
\end{array}
\]

where \( X \to Z \) is given by \( A_0 \to A \). Since \( \tau \circ \psi = \text{id}_A \) we see \( \pi \circ 0 = \text{id}_P \).

Observe that \( \pi \) is an affine morphism. This is clear from the construction in Constructions, Lemma 11.1. In fact, if \( f \in A_d \) for some \( d > 0 \), then setting \( g = \psi(f) \) we have \( \pi^{-1}(D_+(f)) = D_+(g) \). In this case we have the following equality of homogeneous parts

\[
(B[1/g])_{m'} = \bigoplus_{m \geq m'} (A[1/f])_m
\]

This isomorphism is compatible with further localization. Taking \( m' = 0 \) we see that \( \pi_* O_L \) is the direct sum of \( O_P(m) \) for \( m \geq 0 \). We conclude \( L \) is identified with the relative spectrum:

\[
L = \text{Spec}_P \left( \bigoplus_{m \geq 0} O_P(m) \right)
\]

In particular \( L \to P \) is a con\[\text{d}^\big]\ see Constructions, Section \[\text{d}\] Moreover, it is clear that \( 0 : P \to L \) is the vertex of the cone.

Let \( f \in A_d \) for some \( d > 0 \) and \( g = \psi(f) \in B_d \) as in the previous paragraph. Looking at the structure of the ring maps

\[
\begin{array}{ccc}
A_0 & \xrightarrow{\psi} & (B[1/g])_0 \\
\downarrow & & \downarrow \sigma \\
(A[1/f])_0 & \xrightarrow{\tau} & (A[1/f])_0
\end{array}
\]

some computations\[\text{d}\] in graded rings will show that

1. \( \sigma(A_0)(B[1/g])_0 \subset \text{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0) \),
2. \( \sigma(f) \in (B[1/g])_0 \) is a nonzerodivisor,
3. \( \sigma(f)(B[1/g])_0 = \sigma(A_d)(B[1/g])_0 \) as ideals,
4. \( \sigma(f)(B[1/g])_0 \) and \( \text{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0) \) have the same radical,
5. if \( d = 1 \), then \( \sigma(f)(B[1/g])_0 = \text{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0) \).

We see in particular that

\[
0(D_+(f)) = V(\sigma(f)) \subset D_+(g) = \text{Spec}((B[1/g])_0)
\]

set theoretically. In other words, the ideal generated by \( \sigma(A_d) \) cuts out an effective Cartier divisor on \( D_+(g) \) which is set theoretically equal to the image of the closed immersion \( 0 : P \to L \).

\[\text{d}\] Often \( L \) is a line bundle over \( P \), see below.

\[\text{d}\] Parts (1) and (2) are clear. To see (3), note that if \( a \in A_d \), then \( \sigma(a) = \sigma(f) \psi(a/f) \). For (4) note that \( b/g^m \) is in the kernel of \( \tau \) if and only if \( b \in A_{\geq md} \) maps to zero in \( A_{md} \). Thus it suffices to show if \( m' > md \) and \( a \in A_{m'} \), then some power of \( a^{(md)}/g^m \) is in the ideal generated by \( \sigma(f) \). Take \( e \) such that \( em' - emd \geq d \). Then

\[
(a^{(md)}/g^m)^e = (a^e)\cdot g^{em} = (fa^e)\cdot g^{emd+d} = \sigma(f)\cdot (a^e)\cdot g^{emd+d} = g^{em+1}
\]

as desired (apologies for the terrible notation). To see (5) argue as before and note that \( a^{(md)}/g^m = \sigma(f)\cdot a^{(md+1)}/g^{m+1} \) if \( d = 1 \).
We claim that \( L \to X \) is an isomorphism over \( U \). Namely, if \( f \in A_d \) for some \( d > 0 \), then
\[
\text{Spec}(A_f) \times_X L = \text{Proj}(A_f \otimes_A B) = \text{Proj}(B_{\sigma(f)})
\]
For each \( e \) we have \((B_{\sigma(f)})_e = A_f \otimes B_e = A_f \otimes A_{A_e} = A_f\), the final equality induced by the injection \( A_{A_e} \subset A \). Hence \( B_{\sigma(f)} \cong A_f[T] \) with \( T \) in degree 1. This proves the claim as \( \text{Proj}(A_f[T]) \to \text{Spec}(A_f) \) is an isomorphism. From now on we identify \( U \) with the corresponding open of \( L \).

The identification made in the previous paragraph lets us consider the restriction \( \pi|_U : U \to P \). Pick \( f \in A_d \) for some \( d > 0 \) and \( g = \psi(f) \in B_d \) as we have done above several times. Then
\[
U \cap \pi^{-1}(D_{+}(f)) = U \cap D_{+}(g)
\]
is the complement of the zero locus of \( \sigma(f) \in (B[1/g])_0 \) via the identification of \( D_{+}(g) \) with the spectrum of \( (B[1/g])_0 \). This is assertion (4) above. Therefore \( U \cap D_{+}(g) \) is affine and
\[
\mathcal{O}_L(U \cap D_{+}(g)) = (B[1/g])_0[1/\sigma(f)] = \bigoplus_{m \in \mathbb{Z}} (A[1/f])_m
\]
where the last equal sign is the natural extension of the identification \( (B[1/g])_0 = \bigoplus_{m \geq 0} (A[1/f])_m \) made above. Exactly as we did before with \( \pi : L \to P \) we conclude that \( \pi|_U : U \to P \) is affine and
\[
U = \text{Spec}_P \left( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_P(m) \right)
\]
as schemes over \( P \).

Summarising the above, our constructions produce a commutative diagram
\[
\begin{array}{ccc}
\text{Spec}_P \left( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_P(m) \right) & \to & L = \text{Spec}_P \left( \bigoplus_{m \geq 0} \mathcal{O}_P(m) \right) \\
\sigma & & \pi \\
U & \to & X & \to & Z
\end{array}
\]
of schemes where \( \pi \) is a cone whose zero section \( 0 : P \to L \) maps set theoretically onto the inverse image of \( Z \) in \( L \).

Let \( W \subset P \) be the largest open such that \( \mathcal{O}_P(1)|_W \) is invertible and the natural maps induce isomorphisms \( \mathcal{O}_P(m)|_W \cong \mathcal{O}_P(1)^{\otimes m}|_W \) for all \( m \in \mathbb{Z} \), i.e., the open of Constructions, Lemma \( \ref{lem:10.4} \) for \( d = 1 \). Then we see that \( L|_W = \pi^{-1}(W) \to W \) is a vector bundle (Constructions, Section \( \ref{sec:6} \) of rank 1, namely,
\[
L|_W = \mathcal{V}(\mathcal{O}_P(1)|_W)
\]
in Grothendieckian notation. This is immediate from the above showing that \( L|_W \) is equal to the relative spectrum of the symmetric algebra over \( \mathcal{O}_W \) on \( \mathcal{O}_P(1)|_W \). Then clearly the morphism \( 0|_W : W \to L|_W \) is the zero section of this vector bundle. In particular \( 0(W) \) is an effective Cartier divisor on \( L|_W \). Moreover, the open \( U|_W = (\pi|_U)^{-1}(W) \) is the complement of the zero section.

If \( A \) is generated by \( f_1, \ldots, f_r \in A_1 \) over \( A_0 \), then \( (f_1, \ldots, f_r)^m = A_{\geq m} \) for all \( m \geq 0 \) and hence our \( B \) above is the Rees algebra for \( A_+ = (f_1, \ldots, f_r) \). Thus in this case \( L \to X \) is the blowup of \( Z \) and \( W = P \) where \( W \) is as in the preceding paragraph.
If \( P \) is quasi-compact, then for \( d \) sufficiently divisible, the closed subscheme \( D \subset L \) cut out by \( \sigma(A_d)O_L \) is an effective Cartier divisor, \( 0 : P \to L \) factors through \( D \), and \( 0(P) = D \) set theoretically. This follows from Constructions, Lemma 8.9 and (1), (2), (3), and (4) proved above. (Take any \( d \) divisible by the lcm of the degrees of the elements found in the lemma.)

We continue to assume \( P \) is quasi-compact. Let \( F \) be a quasi-coherent \( O_P \)-module. Let us set \( F_U = \pi_* F|_U \). Then we have

\[
R\Gamma(U, F_U) = \bigoplus_{m \in \mathbb{Z}} R\Gamma(P, F \otimes_{O_P} O_P(m))
\]

Moreover, this direct sum decomposition is functorial in \( F \) and the induced \( A \)-module structure on the right is the same as the \( A \)-module structure on the left coming from \( U \subset X \). To prove the formula, since \( \pi|_U \) is affine and \( (\pi|_U)_* O_U = \bigoplus_{m \in \mathbb{Z}} O_P(m) \) we get

\[
R(\pi|_U)_* F_U = (\pi|_U)_* F_U
= (\pi|_U)_*(\pi|_U)^* F
= F \otimes_{O_P} \bigoplus_{m \in \mathbb{Z}} O_P(m)
= \bigoplus_{m \in \mathbb{Z}} F \otimes_{O_P} O_P(m)
\]

By Leray we find that \( R\Gamma(U, F_U) = R\Gamma(P, R(\pi|_U)_* F_U) \), see Cohomology, Lemma 13.6. The proof is finished because taking cohomology commutes with direct sums in this case, see Derived Categories of Schemes, Lemma 4.2. This is where we use that \( P \) is quasi-compact; \( P \) is separated by Constructions, Lemma 8.8.

**Lemma 46.1.** Let \( R \) be a ring. Let \( P \) be a proper scheme over \( R \) and let \( L \) be an ample invertible \( O_P \)-module. Set \( A = \bigoplus_{m \geq 0} \Gamma(P, L^\otimes m) \). Then \( P = \text{Proj}(A) \) and diagram (46.0.1) becomes the diagram

\[
\begin{array}{ccc}
\text{Spec}_P \left( \bigoplus_{m \in \mathbb{Z}} L^\otimes m \right) & \longrightarrow & L = \text{Spec}_P \left( \bigoplus_{m \geq 0} L^\otimes m \right) \\
U & \longrightarrow & X \\
\sigma \downarrow & & \downarrow \\
& & Z
\end{array}
\]

having the properties explained above.

**Proof.** We have \( P = \text{Proj}(A) \) by Morphisms, Lemma 41.17. Moreover, by Properties, Lemma 28.2 via this identification we have \( O_P(m) = L^\otimes m \) for all \( m \in \mathbb{Z} \).

47. Closed points in fibres

Some of the material in this section is taken from the preprint [OP10].

**Lemma 47.1.** Let \( f : X \to S \) be a morphism of schemes. Let \( Z \subset X \) be a closed subscheme. Let \( s \in S \). Assume

1. \( S \) is irreducible with generic point \( \eta \),
2. \( X \) is irreducible,
3. \( f \) is dominant,
4. \( f \) is locally of finite type,
5. \( \dim(X_s) \leq \dim(X_\eta) \),
6. \( Z \) is locally principal in \( X \), and
The assertion that at this point is the following:

\[ \eta \]

to cut out by some element which is locally of finite type over the field \( X \) of a generic point of any irreducible component of \( X \) of \( \dim \) that \( X \), see Algebra, Lemmas 113.4 and 113.5. In view of Morphisms, Lemma 113.4 and condition (5) we conclude that \( \dim_{\kappa}(X_s) = d \) for every \( x \in X_s \).

In the Noetherian case the assertion can be proved as follows. If the lemma does not hold there exists \( x \in Z_s \) which is a generic point of an irreducible component of \( Z_s \), but not a generic point of any irreducible component of \( X_s \). Then we see that \( \dim_{\kappa}(Z_s) \leq d - 1 \), because \( \dim_{\kappa}(X_s) = d \) and in a neighbourhood of \( x \) in \( X_s \) the closed subscheme \( Z_s \) does not contain any of the irreducible components of \( X_s \). Hence after replacing \( X \) by an open neighbourhood of \( x \) we may assume that \( \dim_{\kappa}(Z_{f(z)}) \leq d - 1 \) for all \( z \in Z \), see Morphisms, Lemma 27.4. Let \( \xi' \in Z \) be a generic point of an irreducible component of \( Z \) and set \( s' = f(\xi) \). As \( Z \neq X \) is locally principal we see that \( \dim(\mathcal{O}_{X,\xi}) = 1 \), see Algebra, Lemma 59.10 (this is where we use \( X \) is Noetherian). Let \( \xi \in X \) be the generic point of \( X \) and let \( \xi_1 \) be a generic point of any irreducible component of \( X_{s'} \) which contains \( \xi' \). Then we see that we have the specializations

\[ \xi \leadsto \xi_1 \leadsto \xi' \]

As \( \dim(\mathcal{O}_{X,\xi}) = 1 \) one of the two specializations has to be an equality. By assumption \( \xi' \neq \eta \), hence the first specialization is not an equality. Hence \( \xi' = \xi_1 \) is a generic point of an irreducible component of \( X_{s'} \). Applying Morphisms, Lemma 27.4 one more time this implies \( \dim_{\kappa}(Z_{s'}) = \dim_{\kappa}(X_{s'}) \geq \dim(\kappa) = d \) which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point \( x \in X \), lying over \( s \) such that \( x \) is a generic point of an irreducible component of \( Z_s \), but not a generic point of any of the irreducible components of \( X_s \). Let \( U \subset S \) be an affine neighbourhood of \( s \) and let \( V \subset X \) be an affine neighbourhood of \( x \) with \( f(V) \subset U \). Write \( U = \text{Spec}(A) \) and \( V = \text{Spec}(B) \) so that \( f|_V \) is given by a ring map \( A \to B \). Let \( q \subset B \), resp. \( p \subset A \) be the prime corresponding to \( x \), resp. \( s \). After possibly shrinking \( V \) we may assume \( Z \cap V \) is cut out by some element \( g \in B \). Denote \( K \) the fraction field of \( A \). What we know at this point is the following:

1. \( A \subset B \) is a finitely generated extension of domains,
2. the element \( g \otimes 1 \) is invertible in \( B \otimes_A K \),
3. \( d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(p)) \),
4. \( g \otimes 1 \) is not a unit of \( B \otimes_A \kappa(p) \), and
5. \( g \otimes 1 \) is not in any of the minimal primes of \( B \otimes_A \kappa(p) \).
We prove each of the conditions in turn. Part (1) holds by construction. For part enough then (1) – (5) also hold for the system $K_Z$. Pick elements $a, a'_j$ such that defines $I$ and the injective map $A \to A[1, \ldots, X_n] \to B \to 0$ which defines $I$ where the second map sends $X_i$ to $x_i$. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(p)$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(p)[X_1, \ldots, X_n] \to B \otimes_A \kappa(p)$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I, j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(p)$. Write $h_j = \sum_{i=1}^{m} a_{j,i}X_i$. As soon as $A_0$ contains all $a_{j,i}$ we get to the situation where $B_0 \otimes_{A_0} K_0 \otimes_K 0 = B \otimes_A K$ and $B_0 \otimes_{A_0} \kappa(p_0) \otimes_{\kappa(p_0)} \kappa(p) = B \otimes_A \kappa(p)$.

By either Morphisms, Lemma [27.3] or Algebra, Lemma [115.5] we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(p_0) \subset B \otimes_A \kappa(p)$ each minimal prime of $B_0 \otimes_{A_0} \kappa(p_0)$ lies under a minimal prime of $B \otimes_A \kappa(p)$ by Algebra, Lemma [29.6]. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above.

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma [47.3].

Lemma 47.2. Let $A \to B$ be a local homomorphism of local rings, and $g \in m_B$. Assume

(1) $A$ and $B$ are domains and $A \subset B$,
(2) $B$ is essentially of finite type over $A$,
(3) $g$ is not contained in any minimal prime over $m_AB$, and
(4) $\dim(B/m_A B) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B)) = \text{trdeg}_A(B)$.

Then $A \subset B/gB$, i.e., the generic point of $\text{Spec}(A)$ is in the image of the morphism $\text{Spec}(B/gB) \to \text{Spec}(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma [29.6]. To start the proof let $C$ be an $A$-algebra of finite type and $q$ a prime of $C$ such that $B = C_q$. Of course we may assume that $C$ is a domain and that $q \subset C$. After replacing $C$ by a localization we see that $\dim(C/m_A C) = \dim(B/m_A B) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B))$, see Morphisms, Lemma [27.1]. Setting $K$ equal to the fraction field of $A$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $\text{Spec}(C) \to \text{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_q \otimes_A K$. As $C_q$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \notin q$. Thus
after replacing $C$ by $C_h$ we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of $C$ to make sure that the minimal primes of $C/m_A C$ correspond one-to-one with the minimal primes of $B/m_AB$. At this point we apply Lemma 47.1 to $X = \text{Spec}(C) \to \text{Spec}(A) = S$ and the locally closed subscheme $Z = \text{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(m_A)$ has to contain an irreducible component of $X \otimes \kappa(m_A) = \text{Spec}(C/m_A C)$. But this contradicts the assumption that $g$ is not contained in any prime minimal over $m_AB$. The lemma follows.

**Lemma 47.3.** Let $A \to B$ be a local homomorphism of local rings. Assume

1. $A$ and $B$ are domains and $A \subset B$,
2. $B$ is essentially of finite type over $A$, and
3. $B$ is flat over $A$.

Then we have

$$\dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B)) = \text{trdeg}_A(B).$$

**Proof.** Let $C$ be an $A$-algebra of finite type and $q$ a prime of $C$ such that $B = C_q$. We may assume $C$ is a domain. We have $\dim_q(C/m_A C) = \dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B))$, see Morphisms, Lemma 27.1. Setting $K$ equal to the fraction field of $A$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_q(C/m_A C) = \dim(C \otimes_A K)$. Choose a valuation ring $A'$ in $K$ dominating $A$, see Algebra, Lemma 49.2. Set $C' = C \otimes_A A'$. Choose a prime $q'$ of $C'$ lying over $q$; such a prime exists because $C'/m_{A'} C' = C/m_A C \otimes_{\kappa(m_A)} \kappa(m_{A'})$ which proves that $C/m_A C \to C'/m_{A'} C'$ is faithfully flat. This also proves that $\dim_q(C/m_A C) = \dim_q(C'/m_{A'} C')$, see Algebra, Lemma 115.6. Note that $B' = C'_q$ is a localization of $B \otimes_A A'$. Hence $B'$ is flat over $A'$. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence $B'$ is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_q(C'/m_{A'} C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_q(C/m_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where $A$ is a valuation ring.

Let $A \subset B$ be as in the lemma with $A$ a valuation ring. As before write $B = C_q$ for some domain $C$ of finite type over $A$. By Algebra, Lemma 124.9 we obtain $\dim(C/m_A C) = \dim(C \otimes_A K)$ and we win.

**Lemma 47.4.** Let $f : X \to S$ be a morphism of schemes. Let $x \leadsto x'$ be a specialization of points in $X$. Set $s = f(x)$ and $s' = f(x')$. Assume

1. $x'$ is a closed point of $X_{s'}$, and
2. $f$ is locally of finite type.

Then the set

$$\{ x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \leadsto x_1 \leadsto x' \}$$

is dense in the closure of $x$ in $X_s$.

**Proof.** We apply Schemes, Lemma 20.4 to the specialization $x \leadsto x'$. This produces a morphism $\varphi : \text{Spec}(B) \to X$ where $B$ is a valuation ring such that $\varphi$ maps the generic point to $x$ and the closed point to $x'$. We may also assume that $\kappa(x)$ is the fraction field of $B$. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see
Algebra, Lemma 49.6, which dominates the image of $\mathcal{O}_{S,x'} \rightarrow \kappa(s)$. Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(B) & \longrightarrow & X_A \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & S
\end{array}
$$

The generic (resp. closed) point of $B$ maps to a point $x_A$ (resp. $x'_A$) of $X_A$ lying over the generic (resp. closed) point of $\text{Spec}(A)$. Note that $x'_A$ is a closed point of the special fibre of $X_A$ by Morphisms, Lemma 19.4. Note that the generic fibre of $X_A \rightarrow \text{Spec}(A)$ is isomorphic to $X_s$. Thus we have reduced the lemma to the case where $S$ is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of $X_\eta$ specializing to $x$ and $x$ is closed in its fibre, see Morphisms, Lemma 19.6 and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\{x\}$ of $X$, see Schemes, Definition 12.5. To prove the lemma we may replace $X$ by $X'$ as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that $X$ is an integral scheme and that $x$ is its generic point. In addition, we may replace $X$ by an affine neighbourhood of $x'$. Thus we have $X = \text{Spec}(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 124.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to “avoid”). As $X_\eta$ is of finite type over a field we see that $W$ has finitely many irreducible components $W = W_1 \cup \ldots \cup W_r$. Let $q_j \subset B$, $j = 1, \ldots , r$ be the corresponding prime ideals. Let $q \subset B$ be the maximal ideal corresponding to the point $x'$. Let $p_1, \ldots , p_s \subset B$ be the minimal primes lying over $m_{A,B}$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension $> 0$ (see above) hence we see that $p_i \neq q$ for all $i$. Now, pick an element $g \in q$ such that $g \notin q_j$ for all $j$ and $g \notin p_i$ for all $i$, see Algebra, Lemma 14.2. Denote $Z \subset X$ the locally principal closed subscheme defined by $g$. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of $Z$ into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing $X$ by a smaller affine neighbourhood we may assume that $x' \in Z_i$ for each $i = 1, \ldots , n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 17.1 we conclude that $Z_\eta \neq \emptyset$. In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension $\geq 2$ in $X_\eta$ whereas $\text{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 59.10. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \rightarrow S$ and the specialization $x_1 \rightsquigarrow x'$. This produces a closed point $x_2$ of $Z_{1,\eta}$ not contained in $W$ which specializes to $x'$. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point $x_2$ is closed in $X_\eta$, and $x_2 \notin W$ as desired. □
Remark 47.5. The proof of Lemma 47.4 actually shows that there exists a sequence of specializations
\[ x \twoheadrightarrow x_1 \twoheadrightarrow x_2 \twoheadrightarrow \ldots \twoheadrightarrow x_d \twoheadrightarrow x' \]
where all \( x_i \) are in the fibre \( X_s \), each specialization is immediate, and \( x_d \) is a closed point of \( X_s \). The integer \( d = \text{trdeg}_{k(x)}(\kappa(x)) = \dim(\overline{\{x\}}) \) where the closure is taken in \( X_s \). Moreover, the points \( x_i \) can be chosen to avoid any closed subset of \( X_s \) which does not contain the point \( x \).

Examples, Section 33 shows that the following lemma is false if \( A \) is not assumed Noetherian.

Lemma 47.6. Let \( \varphi : A \to B \) be a local ring map of local rings. Let \( V \subset \text{Spec}(B) \) be an open subscheme which contains at least one prime not lying over \( m_A \). Assume \( A \) is Noetherian, \( \varphi \) essentially of finite type, and \( A/m_A \subset B/m_B \) is finite. Then there exists a \( q \in V \), \( m_A \neq q \cap A \) such that \( A \to B/q \) is the localization of a quasi-finite ring map.

Proof. Since \( A \) is Noetherian and \( A \to B \) is essentially of finite type, we know that \( B \) is Noetherian too. By Properties, Lemma 6.4 the topological space \( \text{Spec}(B) \setminus \{m_B\} \) is Jacobson. Hence we can choose a closed point \( q \) which is contained in the nonempty open
\[ V \setminus \{q \subset B \mid m_A = q \cap A\}. \]
(Nonempty by assumption, open because \( \{m_A\} \) is a closed subset of \( \text{Spec}(A) \).) Then \( \text{Spec}(B/q) \) has two points, namely \( m_B \) and \( q \) and \( q \) does not lie over \( m_A \). Write \( B/q = C/m \) for some finite type \( A \)-algebra \( C \) and prime ideal \( m \). Then \( A \to C \) is quasi-finite at \( m \) by Algebra, Lemma 121.2. Hence by Algebra, Lemma 122.13 we see that after replacing \( A \) by a principal localization the ring map \( A \to C \) is quasi-finite.

Lemma 47.7. Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) with image \( s \in S \). Let \( U \subset X \) be an open subscheme. Assume \( f \) locally of finite type, \( S \) locally Noetherian, \( x \) a closed point of \( X_s \), and assume there exists a point \( x' \in U \) with \( x' \twoheadrightarrow x \) and \( f(x') \neq s \). Then there exists a closed subscheme \( Z \subset X \) such that (a) \( x \in Z \), (b) \( f|_Z : Z \to S \) is quasi-finite at \( x \), and (c) there exists a \( z \in Z, z \in U, z \twoheadrightarrow x \) and \( f(z) \neq s \).

Proof. This is a reformulation of Lemma 47.6. Namely, set \( A = \mathcal{O}_{S,s} \) and \( B = \mathcal{O}_{X,x} \). Denote \( V \subset \text{Spec}(B) \) the inverse image of \( U \). The ring map \( f^* : A \to B \) is essentially of finite type. By assumption there exists at least one point of \( V \) which does not map to the closed point of \( \text{Spec}(A) \). Hence all the assumptions of Lemma 47.6 hold and we obtain a prime \( q \subset B \) which does not lie over \( m_A \) and such that \( A \to B/q \) is the localization of a quasi-finite ring map. Let \( z \in X \) be the image of the point \( q \) under the canonical morphism \( \text{Spec}(B) \to X \). Set \( Z = \{z\} \) with the induced reduced scheme structure. As \( z \twoheadrightarrow x \) we see that \( x \in Z \) and \( \mathcal{O}_{Z,x} = B/q \). By construction \( Z \to S \) is quasi-finite at \( x \).

Remark 47.8. We can use Lemma 47.6 or its variant Lemma 47.7 to give an alternative proof of Lemma 47.4 in case \( S \) is locally Noetherian. Here is a rough sketch. Namely, first replace \( S \) by the spectrum of the local ring at \( s' \). Then we may use induction on \( \dim(S) \). The case \( \dim(S) = 0 \) is trivial because then \( s' = s \).
Replace $X$ by the reduced induced scheme structure on $\{x\}$. Apply Lemma 47.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing $x$. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at $x'$ and such that $f(z) \neq s'$. Then $z$ is a closed point of $X_{f(z)}$, and $z \mapsto x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since $x$ is the generic point of $X$ we see $x \mapsto z$, hence $s = f(x) \mapsto f(z)$. Apply the induction hypothesis to $s \mapsto f(z)$ and $z \mapsto f(z)$ to win.

Lemma 47.9. Suppose that $f : X \to S$ is locally of finite type, $S$ locally Noetherian, $x \in X$ a closed point of its fibre $X_s$, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in U$, then the conclusions of Lemma 47.7 hold.

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace $X$ and $S$ by affine neighbourhoods of $x$ and $s$. Then $X$ is Noetherian, in particular $U$ is quasi-compact (see Morphisms, Lemma 14.6 and Topology, Lemmas 9.2 and 12.13). Hence there exists a specialization $x' \mapsto x$ with $x' \in U$ (see Morphisms, Lemma 6.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. $\square$

48. Stein factorization

Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

Lemma 48.1. Let $S$ be a scheme. Let $f : X \to S$ be a universally closed and quasi-separated morphism. There exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & S' \\
\downarrow f & & \downarrow \pi \\
S & & 
\end{array}
$$

with the following properties:

1. the morphism $f'$ is universally closed, quasi-compact, quasi-separated, and surjective,
2. the morphism $\pi : S' \to S$ is integral,
3. we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
4. we have $S' = \text{Spec}_S(f_*\mathcal{O}_X)$, and
5. $S'$ is the normalization of $S$ in $X$, see Morphisms, Definition 51.3.

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms, Lemma 39.9 the morphism $f$ is quasi-compact. Hence the normalization $S'$ of $S$ in $X$ is defined (Morphisms, Definition 51.3) and we have the factorization $X \to S' \to S$. By Morphisms, Lemma 51.11 we have (2), (4), and (5). The morphism $f'$ is universally closed by Morphisms, Lemma 39.7. It is quasi-compact by Schemes, Lemma 21.14 and quasi-separated by Schemes, Lemma 21.13.

To show the remaining statements we may assume the base scheme $S$ is affine, say $S = \text{Spec}(R)$. Then $S' = \text{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral $R$-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 24.1 and hence equal to $A$). This proves (3).
Let us show that $f'$ is surjective. As $f'$ is universally closed (see above) the image of $f'$ is a closed subset $V(I) \subset S' = \text{Spec}(A)$. Pick $h \in I$. Then $h|_X = f'(h)$ is a global section of the structure sheaf of $X$ which vanishes at every point. As $X$ is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|_X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words $I$ is contained in the Jacobson radical ideal of $A$ and we conclude that $V(I) = S'$ as desired. 

\textbf{Lemma 48.2.} In Lemma 48.4 assume in addition that $f$ is locally of finite type. Then for $s \in S$ the fibre $\pi^{-1}(\{s\}) = \{s_1, \ldots, s_n\}$ is finite and the field extensions $\kappa(s_i)/\kappa(s)$ are finite.

\textbf{Proof.} Recall that there are no specializations among the points of $\pi^{-1}(\{s\})$, see Algebra, Lemma 35.20. As $f'$ is surjective, we find that $|X_s| \to \pi^{-1}(\{s\})$ is surjective. Observe that $X_s$ is a quasi-separated scheme of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus $X_s$ is Noetherian (Morphisms, Lemma 14.6). A topological argument (omitted) now shows that $\pi^{-1}(\{s\})$ is finite. For each $i$ we can pick a finite type point $x_i \in X_s$ mapping to $s_i$ (Morphisms, Lemma 15.7). We conclude that $\kappa(s_i)/\kappa(s)$ is finite: $x_i$ can be represented by a morphism $\text{Spec}(k_i) \to X_s$ of finite type (by our definition of finite type points) and hence $\text{Spec}(k_i) \to s = \text{Spec}(\kappa(s))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(s)$ is finite (Morphisms, Lemma 15.1).

\textbf{Lemma 48.3.} Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then $X_s$ is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre $X_u$.

\textbf{Proof.} If $X_s$ is geometrically connected, then any base change of it is connected. On the other hand, suppose that $X_s$ is not geometrically connected. Then by Varieties, Lemma 7.11 we see that $X_s \times_{\text{Spec}(\kappa(s))} \text{Spec}(k)$ is disconnected for some finite separable field extension $\kappa(s) \subset k$. By Lemma 31.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k$. In this case $X_u$ is disconnected.

\textbf{Theorem 48.4} (Stein factorization; Noetherian case). Let $S$ be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & S' \\
\downarrow f & & \downarrow \pi \\
S & & \\
\end{array}
$$

with the following properties:

1. the morphism $f'$ is proper with geometrically connected fibres,
2. the morphism $\pi : S' \to S$ is finite,
3. we have $f_*\mathcal{O}_X = \mathcal{O}_{S'}$,
4. we have $S' = \text{Spec}_S(f_*\mathcal{O}_X)$, and
5. $S'$ is the normalization of $S$ in $X$, see Morphisms, Definition 31.3.

\textbf{Proof.} Let $f = \pi \circ f'$ be the factorization of Lemma 48.1. Note that besides the conclusions of Lemma 48.1 we also have that $f'$ is separated (Schemes, Lemma 36.23.2).…

and finite type (Morphisms, Lemma 14.8). Hence \( f' \) is proper. By Cohomology of Schemes, Proposition 19.1 we see that \( f_*\mathcal{O}_X \) is a coherent \( \mathcal{O}_S \)-module. Hence we see that \( \pi \) is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of \( f' \) are geometrically connected. It is clear from the discussion above that we may replace \( S \) by \( S' \), and we may therefore assume that \( S \) is Noetherian, affine, \( f : X \to S \) is proper, and \( f_*\mathcal{O}_X = \mathcal{O}_S \). Let \( s \in S \) be a point of \( S \). We have to show that \( X_s \) is geometrically connected. By Lemma 48.3 we see that it suffices to show \( X_u \) is connected for every étale neighbourhood \((U, u) \to (S, s)\). We may assume \( U \) is affine. Thus \( U \) is Noetherian (Morphisms, Lemma 14.6), the base change \( f_U : X_U \to U \) is proper (Morphisms, Lemma 39.3, and also \((f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U \) (Cohomology of Schemes, Lemma 5.2). Hence after replacing \((f : X \to S, s)\) by the base change \((f_U : X_U \to U, u)\) it suffices to prove that the fibre \( X_u \) is connected.

At this point we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 20.7. It tells us that \( \mathcal{O}_{S, s}^{\text{et}} = \lim H^0(X_n, \mathcal{O}_{X_n}) \) where \( X_n \) is the \( n \)th infinitesimal neighbourhood of \( X_s \). Since the underlying topological space of \( X_n \) is equal to that of \( X_s \) we see that if \( X_s = T_1 \amalg T_2 \) is a disjoint union of nonempty open and closed subschemes, then similarly \( X_n = T_{1,n} \amalg T_{2,n} \) for all \( n \). And this in turn means \( H^0(X_n, \mathcal{O}_{X_n}) \) contains a nontrivial idempotent \( e_{1,n} \), namely the function which is identically 1 on \( T_{1,n} \) and identically 0 on \( T_{2,n} \). It is clear that \( e_{1,n+1} \) restricts to \( e_{1,n} \) on \( X_n \). Hence \( e_1 = \lim e_{1,n} \) is a nontrivial idempotent of the limit. This contradicts the fact that \( \mathcal{O}_{S, s}^{\text{et}} \) is a local ring. Thus the assumption was wrong, i.e., \( X_s \) is connected, and we win. \( \square \)

**Theorem 48.5** (Stein factorization; general case). *Let \( S \) be a scheme. Let \( f : X \to S \) be a proper morphism. There exists a factorization*

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & S' \\
\downarrow{f} & & \downarrow{\pi} \\
S' \\
\end{array}
\]

*with the following properties:*

1. the morphism \( f' \) is proper with geometrically connected fibres,
2. the morphism \( \pi : S' \to S \) is integral,
3. we have \( f'_*\mathcal{O}_X = \mathcal{O}_{S'} \),
4. \( S' = \text{Spec}_{S}(f_*\mathcal{O}_X) \), and
5. \( S' \) is the normalization of \( S \) in \( X \), see Morphisms, Definition 51.3.

**Proof.** We may apply Lemma 48.1 to get the morphism \( f' : X \to S' \). Note that besides the conclusions of Lemma 48.1 we also have that \( f' \) is separated (Schemes, Lemma 21.13) and finite type (Morphisms, Lemma 14.8). Hence \( f' \) is proper. At this point we have proved all of the statements except for the statement that \( f' \) has geometrically connected fibres.

We may assume that \( S = \text{Spec}(R) \) is affine. Set \( R' = \Gamma(X, \mathcal{O}_X) \). Then \( S' = \text{Spec}(R') \). Thus we may replace \( S \) by \( S' \) and assume that \( S = \text{Spec}(R) \) is affine \( R = \Gamma(X, \mathcal{O}_X) \). Next, let \( s \in S \) be a point. Let \( U \to S \) be an étale morphism of affine schemes and let \( u \in U \) be a point mapping to \( s \). Let \( U \to S \) be the base
change of $X$. By Lemma 48.3 it suffices to show that the fibre of $X_U \to U$ over $u$ is connected. By Cohomology of Schemes, Lemma 5.2 we see that $\Gamma(X_U, \mathcal{O}_{X_U}) = \Gamma(U, \mathcal{O}_U)$. Hence we have to show: Given $S = \text{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X, \mathcal{O}_X) = R$ and $s \in S$ is a point, the fibre $X_s$ is connected.

By Limits, Lemma 13.3 we can write $(X \to S) = \lim (X_i \to S_i)$ with $X_i \to S_i$ proper and of finite presentation and $S_i$ Noetherian. For $i$ large enough $S_i$ is affine (Limits, Lemma 4.13). Say $S_i = \text{Spec}(R_i)$. Let $R_i' = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that we have ring maps $R_i \to R_i' \to R$. Namely, we have the first because $X_i$ is a scheme over $R_i$ and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \colim R_i'$ by Limits, Lemma 4.7.

Then

\[
\begin{array}{ccc}
X & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
S & \longrightarrow & S_i'
\end{array}
\]

is commutative with $S_i' = \text{Spec}(R_i')$. Let $s_i' \in S_i'$ be the image of $s$. We have $X_s = \lim X_{i,s'}$ because $X = \lim X_i$, $S = \lim S_i'$, and $\kappa(s) = \colim \kappa(s_i')$. Now let $X_s = U \amalg V$ with $U$ and $V$ open and closed. Then $U, V$ are the inverse images of opens $U_i, V_i$ in $X_{i,s'}$ (Limits, Lemma 4.11). By Theorem 48.4 the fibres of $X_i \to S_i'$ are connected, hence either $U$ or $V$ is empty. This finishes the proof. \qed

Here is an application.

**Lemma 48.6.** Let $f : X \to S$ be a morphism of schemes. Assume

1. $f$ is proper,
2. $S$ is integral with generic point $\xi$,
3. $S$ is normal,
4. $X$ is reduced,
5. every generic point of an irreducible component of $X$ maps to $\xi$,
6. we have $H^0(X_{\xi}, \mathcal{O}) = \kappa(\xi)$.

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and $f$ has geometrically connected fibres.

**Proof.** Apply Theorem 48.5 to get a factorization $X \to S' \to S$. It is enough to show that $S' = S$. This will follow from Morphisms, Lemma 52.8. Namely, $S'$ is reduced because $X$ is reduced (Morphisms, Lemma 51.8). The morphism $S' \to S$ is integral by the theorem cited above. Every generic point of $S'$ lies over $\xi$ by Morphisms, Lemma 51.9 and assumption (5). On the other hand, since $S'$ is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre $S'_\xi$ is the spectrum of $H^0(X_{\xi}, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence $S'$ is an integral scheme with function field equal to the function field of $S$. This finishes the proof. \qed

Here is another application.

**Lemma 48.7.** Let $X \to S$ be a flat proper morphism of finite presentation. Let $n_{X/S}$ be the function on $S$ counting the numbers of geometric connected components of fibres of $f$ introduced in Lemma 26.7. Then $n_{X/S}$ is lower semi-continuous.

**Proof.** Let $s \in S$. Set $n = n_{X/S}(s)$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at $s$ is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood $V$ of $s$.
such that \( n_{X/S}|_V \geq n \). Let \( X \to S' \to S \) be the Stein factorization as in Theorem \[48.5\] By Lemma \[48.2\] there are finitely many points \( s'_1, \ldots, s'_m \in S' \) lying over \( s \) and the extensions \( \kappa(s'_i)/\kappa(s) \) are finite. Then Lemma \[37.1\] tells us that after replacing \( S \) by an étale neighbourhood of \( s \) we may assume \( S = V_1 \amalg \ldots \amalg V_m \) as a scheme with \( s'_i \in V_i \) and \( \kappa(s'_i)/\kappa(s) \) purely inseparable. Then the schemes \( X'_i \) are geometrically connected over \( \kappa(s) \), hence \( m = n \). The schemes \( X_i = (f')^{-1}(V_i) \), \( i = 1, \ldots, n \) are flat and of finite presentation over \( S \). Hence the image of \( X_i \to S \) is open (Morphisms, Lemma \[24.10\]). Thus in a neighbourhood of \( s \) we see that \( n_{X/S} \) is at least \( n \).

\[\blacksquare\]

**Proof.** We will denote \( Y \) a reference for the Lemma 48.9.

A final application.

Then the function \( n_{X/S} : S \to \mathbb{Z} \) counting the numbers of geometric connected components of fibres of \( f \) is locally constant.

**Proof.** By Lemma \[48.7\] the function \( n_{X/S} \) is lower semicontinuous. For \( s \in S \) consider the \( \kappa(s) \)-algebra

\[
A = H^0(X_s, \mathcal{O}_{X_s})
\]

By Varieties, Lemma \[9.3\] and the fact that \( X_s \) is geometrically reduced \( A \) is finite product of finite separable extensions of \( \kappa(s) \). Hence \( A \otimes_{\kappa(s)} \kappa(\overline{s}) \) is a product of \( \beta_0(s) = \dim_{\kappa(s)} H^0(E \otimes^L \kappa(s)) \) copies of \( \kappa(\overline{s}) \). Thus \( X_s \) has \( \beta_0(s) = \dim_{\kappa(s)} A \) connected components. In other words, we have \( n_{X/S} = \beta_0 \) as functions on \( S \). Thus \( n_{X/S} \) is upper semi-continuous by Derived Categories of Schemes, Lemma \[29.1\] This finishes the proof. \[\blacksquare\]

A final application.

**Lemma 48.9.** Let \((A, I)\) be a henselian pair. Let \( X \to \text{Spec}(A) \) be separated and of finite type. Set \( X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I) \). Let \( Y \subset X_0 \) be an open and closed subscheme such that \( Y \to \text{Spec}(A/I) \) is proper. Then there exists an open and closed subscheme \( W \subset X \) which is proper over \( A \) with \( W \times_{\text{Spec}(A)} \text{Spec}(A/I) = Y \).

**Proof.** We will denote \( T \to T_0 \) the base change by \( \text{Spec}(A/I) \to \text{Spec}(A) \). By Chow's lemma (in the form of Limits, Lemma \[12.1\]) there exists a surjective proper morphism \( \varphi : X' \to X \) such that \( X' \) admits an immersion into \( \mathbb{P}^n_A \). Set \( Y' = \varphi^{-1}(Y) \). This is an open and closed subscheme of \( X'_0 \). Suppose the lemma holds for \((X', Y')\). Let \( W' \subset X' \) be the open and closed subscheme proper over \( A \) such that \( Y' = W'_0 \). By Morphisms, Lemma \[39.7\] \( W = \varphi(W') \subset X \) and \( Q = \varphi(X' \setminus W') \subset X \) are closed subsets and by Morphisms, Lemma \[39.8\] \( W \) is proper over \( A \). The image of \( W \cap Q \) in \( \text{Spec}(A) \) is closed. Since \((A, I)\) is henselian, if \( W \cap Q \) is nonempty, then we find that \( W \cap Q \) has a point lying over \( \text{Spec}(A/I) \). This is impossible as \( W'_0 = Y' = \varphi^{-1}(Y) \). We conclude that \( W \) is an open and closed subscheme of \( X \) proper over \( A \) with \( W_0 = Y \). Thus we reduce to the case described in the next paragraph.

Assume there exists an immersion \( j : X \to \mathbb{P}^n_A \) over \( A \). Let \( \overline{X} \) be the scheme theoretic image of \( j \). Since \( j \) is a quasi-compact morphism (Schemes, Lemma \[21.14\]) we see that \( j : X \to \overline{X} \) is an open immersion (Morphisms, Lemma \[7.7\]). Hence the base change \( j_0 : X_0 \to \overline{X}_0 \) is an open immersion as well. Thus \( j_0(Y) \subset \overline{X}_0 \) is open.

It is also closed by Morphisms, Lemma \[39.7\] Suppose that the lemma holds for A reference for the case of an adic Noetherian base is [DG67, III, Proposition 5.5.1]
(X, j_0(Y)). Let \( W \subset X \) be the corresponding open and closed subscheme proper over \( A \) such that \( j_0(Y) = W_0 \). Then \( T = W \setminus j(X) \) is closed in \( W \), hence has closed image in \( \text{Spec}(A) \) by properness of \( W \) over \( A \). Since \( (A, I) \) is henselian, we find that if \( T \) is nonempty, then there is a point of \( T \) mapping into \( \text{Spec}(A/I) \). This is impossible because \( j_0(Y) = W_0 \) is contained in \( j(X) \). Hence \( W \) is contained in \( j(X) \) and we can set \( W \subset X \) equal to the unique open and closed subscheme mapping isomorphically to \( W \) via \( j \). Thus we reduce to the case described in the next paragraph.

Assume \( X \subset \mathbb{P}^n_A \) is a closed subscheme. Then \( X \to \text{Spec}(A) \) is a proper morphism. Let \( Z = X_0 \setminus Y \). This is an open and closed subscheme of \( X_0 \) and \( X_0 = Y \amalg Z \). Let \( X \to X' \to \text{Spec}(A) \) be the Stein factorization as in Theorem 48.5. Let \( Y' \subset X'_0 \) and \( Z' \subset X'_0 \) be the images of \( Y \) and \( Z \). Since the fibres of \( X \to Z \) are geometrically connected, we see that \( Y' \cap Z' = \emptyset \). Hence \( X'_0 = Y' \amalg Z' \) as \( X \to X' \) is surjective. Since \( X' \to \text{Spec}(A) \) is integral, we see that \( X' \) is the spectrum of an \( A \)-algebra integral over \( A \). Recall that open and closed subsets of spectra correspond 1-to-1 with idempotents in the corresponding ring, see Algebra, Lemma 20.3. Hence by More on Algebra, Lemma 11.6 we see that we may write \( X' = W' \amalg V' \) with \( W' \) and \( V' \) open and closed and with \( Y' = W'_0 \) and \( Z' = V'_0 \). Let \( W \) be the inverse image in \( X \) to finish the proof.

49. Descending separated locally quasi-finite morphisms

In this section we show that “separated locally quasi-finite morphisms satisfy descent for fppf-coverings”. See Descent, Definition 33.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [GR71, Lemma 5.7.1]. It can also be found in [Mu95], and [ABD+66, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

**Lemma 49.1.** Let \( S \) be a scheme. Let \( \{X_i \to S\}_{i \in I} \) be an fppf covering, see Topologies, Definition 7.1. Let \((V_i/X_i, \varphi_{ij})\) be a descent datum relative to \( \{X_i \to S\} \). If each morphism \( V_i \to X_i \) is separated and locally quasi-finite, then the descent datum is effective.

**Proof.** Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 21.12 and Morphisms, Lemma 19.13 Hence Descent, Lemma 33.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single \( \{X \to S\} \) flat surjective morphism of finite presentation of affines. Say \( X = \text{Spec}(A) \) and \( S = \text{Spec}(R) \) so that \( R \to A \) is a faithfully flat ring map. Let \((V, \varphi)\) be a descent datum relative to \( X \) over \( S \) and assume that \( \pi : V \to X \) is separated and locally quasi-finite.
Let $W^1 \subset V$ be any affine open. Consider $W = \text{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

\[
\begin{array}{ccc}
W^1 \times_S X & \xrightarrow{\varphi} & \varphi(W^1 \times_S X) \\
\downarrow & & \downarrow \\
V \times_S X & \xrightarrow{\varphi} & X \times_S V \\
\downarrow & & \downarrow \\
X^1 \times_S X & \xrightarrow{1} & X \times_S X \\
\downarrow \text{pr}_0 & & \downarrow \text{pr}_1 \\
W^1 & \xrightarrow{\varphi} & V \\
\end{array}
\]

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma \ref{mm}). Hence we conclude that $W$ is open. Moreover, it is also clearly the case that $W$ is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for $\varphi$. Hence we obtain a new descent datum $(W, \varphi')$ by restricting $\varphi$ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma \ref{mm}. Thus by Descent, Lemma \ref{mm} we see that the descent datum $(W, \varphi')$ is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens $W_i$ which are stable for the descent morphism $\varphi$. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data $(W, \varphi')$ is effective. This means the original descent datum is effective by glueing the schemes obtained from descending the opens $W_i$, see Descent, Lemma \ref{mm} \hfill \Box

50. Relative finite presentation

05GX Let $R \to A$ be a finite type ring map. Let $M$ be an $A$-module. In More on Algebra, Section \ref{moreon} we defined what it means for $M$ to be finitely presented relative to $R$. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

05H1 \textbf{Definition} 50.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is \textit{finitely presented relative to $S$} or \textit{of finite presentation relative to $S$} if there exists an affine open covering $S = \bigcup V_i$ and for every $i$ an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to $S$, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to $S$.

09T7 \textbf{Lemma} 50.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

1. $\mathcal{F}$ is \textit{of finite presentation relative to $S$,}.
2. for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is \textit{finitely presented relative to $\mathcal{O}_S(V)$.}
Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to $V$.

**Proof.** The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 50.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 74.8 it suffices to find a standard open covering $U = \bigcup U_k$ of $U$ such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick $i$ such that $f(u) \in V_i$ and then pick $j$ such that $u \in U_{ij}$. By Schemes, Lemma 11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in $V'$ and $V_i$. Then $f^{-1} V' \cap U$, resp. $f^{-1} V' \cap U_{ij}$ are standard affine opens of $U$, resp. $U_{ij}$. Applying the lemma again we can find $u \in U' \subset f^{-1} V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1} V' \cap U$ and $f^{-1} V' \cap U_{ij}$. Thus $U'$ is also a standard affine open of $U$ and $U_{ij}$. By More on Algebra, Lemma 74.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 74.5 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 74.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. \[ \square \]

**Lemma 50.3.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

1. If $f$ is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to $S$ if and only if $\mathcal{F}$ is of finite presentation.

2. The morphism $f$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to $S$.

**Proof.** Follows immediately from the definitions, see discussion following More on Algebra, Definition 74.2. \[ \square \]

**Lemma 50.4.** Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme $S$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite presentation relative to $S$ if and only if $\pi_* \mathcal{F}$ is of finite presentation relative to $S$.

**Proof.** Translation of the result of More on Algebra, Lemma 74.3 into the language of schemes. \[ \square \]

**Lemma 50.5.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes, set $X' = X \times_S S'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X'$. If $\mathcal{F}$ is of finite presentation relative to $S$, then $\mathcal{F}'$ is of finite presentation relative to $S'$.

**Proof.** Translation of the result of More on Algebra, Lemma 74.3 into the language of schemes. \[ \square \]

**Lemma 50.6.** Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $f : X \to Y$ is locally of finite presentation and $\mathcal{G}$ of finite presentation relative to $S$, then $f^* \mathcal{G}$ is of finite presentation relative to $S$. \[ \square \]
Proof. Translation of the result of More on Algebra, Lemma 74.6 into the language of schemes.

**Lemma 50.7.** Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Y \to S$ is locally of finite presentation and $\mathcal{F}$ is of finite presentation relative to $Y$, then $\mathcal{F}$ is of finite presentation relative to $S$.

Proof. Translation of the result of More on Algebra, Lemma 74.7 into the language of schemes.

**Lemma 50.8.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules.

1. If $\mathcal{F}', \mathcal{F}''$ are finitely presented relative to $S$, then so is $\mathcal{F}$.
2. If $\mathcal{F}'$ is a finite type $\mathcal{O}_X$-module and $\mathcal{F}$ is finitely presented relative to $S$, then $\mathcal{F}''$ is finitely presented relative to $S$.

Proof. Translation of the result of More on Algebra, Lemma 74.9 into the language of schemes.

**Lemma 50.9.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F} \oplus \mathcal{F}'$ is finitely presented relative to $S$, then so are $\mathcal{F}$ and $\mathcal{F}'$.

Proof. Translation of the result of More on Algebra, Lemma 74.10 into the language of schemes.

51. Relative pseudo-coherence

This section is the analogue of More on Algebra, Section 75 for schemes. We strongly urge the reader to take a look at that section first. Although we have developed the material in this section and the material on pseudo-coherent complexes in Cohomology, Sections 42, 43, 44, and 45 for arbitrary complexes of $\mathcal{O}_X$-modules, if $X$ is a scheme then working exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$ greatly simplifies many of the lemmas and arguments, often reducing the problem at hand immediately to the algebraic counterpart. Moreover, one of the first thing we do is to show that being relatively pseudo-coherent implies the cohomology sheaves are quasi-coherent, see Lemma 51.3. Hence, on a first reading we suggest the reader work exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$.

**Lemma 51.1.** Let $X \to S$ be a finite type morphism of affine schemes. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. The following are equivalent

1. for some closed immersion $i : X \to \mathbb{A}^n_\mathbb{R}$ the object $Ri_* E$ of $D(\mathcal{O}_{\mathbb{A}^n_\mathbb{R}})$ is $m$-pseudo-coherent, and
2. for all closed immersions $i : X \to \mathbb{A}^n_\mathbb{R}$ the object $Ri_* E$ of $D(\mathcal{O}_{\mathbb{A}^n_\mathbb{R}})$ is $m$-pseudo-coherent.

Proof. Say $S = \text{Spec}(R)$ and $X = \text{Spec}(A)$. Let $i$ correspond to the surjection $\alpha : R[x_1, \ldots, x_n] \to A$ and let $X \to \mathbb{A}^m_\mathbb{R}$ correspond to $\beta : R[y_1, \ldots, y_m] \to A$. Choose
\[ f_j \in R[x_1, \ldots, x_n] \text{ with } \alpha(f_j) = \beta(y_j) \text{ and } g_i \in R[y_1, \ldots, y_m] \text{ with } \beta(g_i) = \alpha(x_i). \]

Then we get a commutative diagram
\[
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{y_j \mapsto f_j} & R[x_1, \ldots, x_n] \\
\downarrow_{x_i \mapsto g_i} & & \downarrow \\
R[y_1, \ldots, y_m] & \rightarrow & A
\end{array}
\]
corresponding to the commutative diagram of closed immersions
\[
\begin{array}{ccc}
A_S^{n+m} & \xleftarrow{\alpha} & A_S^n \\
\uparrow & & \uparrow \\
A_S^m & \xleftarrow{\beta} & X
\end{array}
\]

Thus it suffices to show that under a closed immersion
\[ f : A_S^n \rightarrow A_S^{n+m} \]
an object \( E \) of \( D(O_{A_S^n}) \) is \( m \)-pseudo-coherent if and only if \( Rf_* E \) is \( m \)-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 11.5 and the fact that \( f_* O_{A^n_S} \) is a pseudo-coherent \( O_{A^{n+m}_S} \)-module. The pseudo-coherence of \( f_* O_{A^n_S} \) is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 9.2 and More on Algebra, Lemma 75.3.

Recall that if \( f : X \rightarrow S \) is a morphism of scheme which is locally of finite type, then for every pair of affine opens \( U \subset X \) and \( V \subset S \) such that \( f(U) \subset V \), the ring map \( O_S(V) \rightarrow O_X(U) \) is of finite type (Morphisms, Lemma 14.2). Hence there always exist closed immersions \( U \rightarrow A^n_S \) and the following definition makes sense.

\[ \text{Definition 51.2.} \quad \text{Let } f : X \rightarrow S \text{ be a morphism of schemes which is locally of finite type. Let } E \text{ be an object of } D(O_X). \text{ Let } F \text{ be an } O_X \text{-module. Fix } m \in \mathbb{Z}. \]

(1) We say \( E \) is \( m \)-pseudo-coherent relative to \( S \) if there exists an affine open covering \( S = \bigcup V_i \) and for each \( i \) an affine open covering \( f^{-1}(V_i) = \bigcup U_{ij} \) such that the equivalent conditions of Lemma 51.1 are satisfied for each of the pairs \( (U_{ij} \rightarrow V_i, E|_{U_{ij}}) \).

(2) We say \( E \) is pseudo-coherent relative to \( S \) if \( E \) is \( m \)-pseudo-coherent relative to \( S \) for all \( m \in \mathbb{Z} \).

(3) We say \( F \) is \( m \)-pseudo-coherent relative to \( S \) if \( F \) viewed as an object of \( D(O_X) \) is \( m \)-pseudo-coherent relative to \( S \).

(4) We say \( F \) is pseudo-coherent relative to \( S \) if \( F \) viewed as an object of \( D(O_X) \) is pseudo-coherent relative to \( S \).

If \( X \) is quasi-compact and \( E \) is \( m \)-pseudo-coherent relative to \( S \) for some \( m \), then \( E \) is bounded above. If \( E \) is pseudo-coherent relative to \( S \), then \( E \) has quasi-coherent cohomology sheaves.

\[ \text{Lemma 51.3.} \quad \text{Let } f : X \rightarrow S \text{ be a morphism of schemes which is locally of finite type. If } E \text{ in } D(O_X) \text{ is } m \text{-pseudo-coherent relative to } S, \text{ then } H^i(E) \text{ is a quasi-coherent } O_X \text{-module for } i > m. \text{ If } E \text{ is pseudo-coherent relative to } S, \text{ then } E \text{ is an object of } D_{QCoh}(O_X). \]
Let the lower left arrow be an open immersion and the lower right arrow be a closed immersion. Choose an affine open covering of $X$, and we may assume that there exists an closed immersion $i : X \to \mathbb{A}^d_S$ such that $R_i E$ is $m$-pseudo-coherent on $\mathbb{A}^d_S$. By Derived Categories of Schemes, Lemma 9.1 this means that $H^q(R_i E)$ is quasi-coherent for $q > m$. Since $i_*$ is an exact functor, we have $i_* H^q(E) = H^q(R_i E)$ is quasi-coherent on $\mathbb{A}^d_S$. By Morphisms, Lemma 4.1 this implies that $H^q(E)$ is quasi-coherent as desired (strictly speaking it implies there exists some quasi-coherent $\mathcal{O}_X$-module $F$ such that $i_* F = i_* H^q(E)$ and then Modules, Lemma 13.1 tells us that $F \cong H^q(E)$ hence the result).

Next, we prove the condition of relative pseudo-coherence localizes well.

**Lemma 51.4.** Let $S$ be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let $E$ be an object of $D(\mathcal{O}_X)$. If the equivalent conditions of Lemma 51.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 51.1 are satisfied for the pair $(U \to S, E_U)$.

**Proof.** Write $S = \text{Spec}(R)$, $V = D(f)$, $X = \text{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 51.1 are satisfied for the pair $(X \to V, E)$. Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to A^n_v \to \mathbb{A}^n_{S}$$

and we can apply Derived Categories of Schemes, Lemma 11.5 to conclude that the pushforward $E'$ of $E$ to $\mathbb{A}^n_{S}$ is $m$-pseudo-coherent.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $R[x_0, \ldots, x_{n+1}] \to R[x_0, \ldots, x_n, 1/g']$. We obtain

$$\begin{array}{ccc}
X & \xrightarrow{g'} & \mathbb{A}^n_{S} \\
\downarrow & & \downarrow \\
U & \to & \mathbb{A}^{n+2}_{S}
\end{array}$$

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'_|D(g')$ to $\mathbb{A}^{n+2}_{S}$ is $m$-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbb{A}^{n+2}_{S}$ we conclude the lemma is true.

**Lemma 51.5.** Let $X \to S$ be a finite type morphism of affine schemes. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent

1. the equivalent conditions of Lemma 51.1 hold for the pairs $(U_i \to S, E|_{U_i})$,
2. the equivalent conditions of Lemma 51.1 hold for the pair $(X \to S, E)$.

**Proof.** The implication $(2) \Rightarrow (1)$ is Lemma 51.4. Assume $(1)$. Say $S = \text{Spec}(R)$ and $X = \text{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in $A$. Consider the surjections

$$R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.$$
which sends $y_i$ to $f_i$ and $z_i$ to $g_i$. Note that $R[x_i, y_i, z_i]/(\sum y_iz_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of $E$ to $T = \text{Spec}(R[x_i, y_i, z_i]/(\sum y_iz_i - 1))$ is $m$-pseudo-coherent, see Derived Categories of Schemes, Lemma 11.5. For each $i_0$ it suffices to prove the restriction of this pushforward to $W_{i_0} = \text{Spec}(R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_iz_i - 1))$ is $m$-pseudo-coherent. Note that there is a commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & U_{i_0} \\
\downarrow & & \downarrow \\
T & \leftarrow & W_{i_0}
\end{array}
$$

which implies that the pushforward of $E$ to $T$ restricted to $W_{i_0}$ is the pushforward of $E|_{U_{i_0}}$ to $W_{i_0}$. Since $R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_iz_i - 1)$ is isomorphic to a polynomial ring over $R$ this proves what we want. \qed

09UJ Lemma 51.6. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E$ be an object of $D(O_X)$. Fix $m \in \mathbb{Z}$. The following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $S$, 
2. for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 51.1 are satisfied for the pair $(U \to V, E|_U)$. 

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is $m$-pseudo-coherent relative to $V$.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 51.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 51.5 it suffices to find a standard open covering $U = \bigcup U_{i_0}$ of $U$ such that the equivalent conditions of Lemma 51.1 are satisfied for the pairs $(U_{i_0} \to V, E|_{U_{i_0}})$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that the equivalent conditions of Lemma 51.1 are satisfied for the pair $(U' \to V, E|_{U'})$. Pick $i$ such that $f(u) \in V_i$ and then pick $j$ such that $u \in U_{ij}$. By Schemes, Lemma 11.3 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in $V'$ and $V_i$. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of $U$, resp. $U_{ij}$. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus $U'$ is also a standard affine open of $U$ and $U_{ij}$. By Lemma 51.4 the assumption that the equivalent conditions of Lemma 51.1 are satisfied for the pair $(U_{ij} \to V, E|_{U_{ij}})$ implies that the equivalent conditions of Lemma 51.1 are satisfied for the pair $(U' \to V, E|_{U'})$. \qed

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative $m$-pseudo-coherence to the notion defined in More on Algebra, Definition 75.4. We will use the fact that for an affine scheme $U = \text{Spec}(A)$ the functor $R\Gamma(U, -)$ induces an equivalence between $D_{QCoh}(O_U)$ and $D(A)$, see Derived Categories of Schemes, Lemma 3.5. This functor is compatible with pullbacks: if $E$ is an object of $D_{QCoh}(O_U)$ and $A \to B$ is a ring map corresponding to a morphism of affine schemes $g : V = \text{Spec}(B) \to \text{Spec}(A) = U$, then $R\Gamma(V, Lg^*E) = R\Gamma(U, E) \otimes_B^L B$. See Derived Categories of Schemes, Lemma 3.8.
Lemma 51.7. Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. Let \( E \) be an object of \( \text{DQCoh}(\mathcal{O}_X) \). Fix \( m \in \mathbb{Z} \). The following are equivalent

1. \( E \) is \( m \)-pseudo-coherent relative to \( S \),
2. there exists an affine open covering \( S = \bigcup V_i \) and for each \( i \) an affine open covering \( f^{-1}(V_i) = \bigcup U_{ij} \) such that the complex of \( \mathcal{O}_X(U_{ij}) \)-modules \( R\Gamma(U_{ij}, E) \) is \( m \)-pseudo-coherent relative to \( \mathcal{O}_S(V_i) \), and
3. for every affine opens \( U \subset X \) and \( V \subset S \) with \( f(U) \subset V \) the complex of \( \mathcal{O}_X(U) \)-modules \( R\Gamma(U, E) \) is \( m \)-pseudo-coherent relative to \( \mathcal{O}_S(V) \).

Proof. Let \( U \) and \( V \) be as in (2) and choose a closed immersion \( i : U \to \mathbb{A}^n_V \). A formal argument, using Lemma 51.6, shows it suffices to prove that \( R\Gamma_i(E|_U) \) is \( m \)-pseudo-coherent if and only if \( R\Gamma(U, E) \) is \( m \)-pseudo-coherent relative to \( \mathcal{O}_S(V) \).

Say \( U = \text{Spec}(A) \), \( V = \text{Spec}(R) \), and \( \mathbb{A}^n_V = \text{Spec}(R[x_1, \ldots, x_n]) \). By the remarks preceding the lemma, \( E|_U \) is quasi-isomorphic to the complex of quasi-coherent sheaves on \( U \) associated to the object \( R\Gamma(U, E) \) of \( D(A) \). Note that \( R\Gamma(U, E) = R\Gamma_i(\mathbb{A}^n_V, E|_U) \) as \( i \) is a closed immersion (and hence \( i_* \) is exact). Thus \( R\Gamma_i(E|_U) \) is associated to \( R\Gamma(U, E) \) viewed as an object of \( D(R[x_1, \ldots, x_n]) \). We conclude as \( m \)-pseudo-coherence of \( R\Gamma_i(E|_U) \) is equivalent to \( m \)-pseudo-coherence of \( R\Gamma(U, E) \) in \( D(R[x_1, \ldots, x_n]) \) by Derived Categories of Schemes, Lemma 9.2 which is equivalent to \( R\Gamma(U, E) \) is \( m \)-pseudo-coherent relative to \( R = \mathcal{O}_S(V) \) by definition.

Lemma 51.8. Let \( i : X \to Y \) morphism of schemes locally of finite type over a base scheme \( S \). Assume that \( i \) induces a homeomorphism of \( X \) with a closed subset of \( Y \). Let \( E \) be an object of \( \text{DQCoh}(\mathcal{O}_X) \). Then \( E \) is \( m \)-pseudo-coherent relative to \( S \) if and only if \( Ri_* E \) is \( m \)-pseudo-coherent relative to \( S \).

Proof. By Morphisms, Lemma 43.4 the morphism \( i \) is affine. Thus we may assume \( S, Y \), and \( X \) are affine. Say \( S = \text{Spec}(R) \), \( Y = \text{Spec}(A) \), and \( X = \text{Spec}(B) \). The condition means that \( A/\text{rad}(A) \to B/\text{rad}(B) \) is surjective; here \( \text{rad}(A) \) and \( \text{rad}(B) \) denote the Jacobson radical of \( A \) and \( B \). As \( B \) is of finite type over \( A \), we can find \( b_1, \ldots, b_m \in \text{rad}(B) \) which generate \( B \) as an \( A \)-algebra. Say \( b_j^N = 0 \) for all \( j \). Consider the diagram of rings

\[
\begin{array}{ccc}
R[x_1, y_j]/(y_j^N) & \xrightarrow{} & R[x_1, y_j] \\
| & & | \\
R[x_1] & \xrightarrow{} & A
\end{array}
\]

which translates into a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{} & T \\
| & & | \\
| & & | \\
Y & \xrightarrow{} & \mathbb{A}^{n+m}_S
\end{array}
\]

of affine schemes. By Lemma 51.6 we see that \( E \) is \( m \)-pseudo-coherent relative to \( S \) if and only if its pushforward to \( \mathbb{A}^{n+m}_S \) is \( m \)-pseudo-coherent. By Derived Categories of Schemes, Lemma 11.3 we see that this is true if and only if its pushforward to \( T \) is \( m \)-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to \( \mathbb{A}^n_S \) is \( m \)-pseudo-coherent. Again by Lemma 51.6 this holds if and only if \( Ri_* E \) is \( m \)-pseudo-coherent relative to \( S \).
Lemma 51.9. Let \( \pi : X \to Y \) be a finite morphism of schemes locally of finite type over a base scheme \( S \). Let \( E \) be an object of \( D_{QCoh}(\mathcal{O}_X) \). Then \( E \) is \( m \)-pseudo-coherent relative to \( S \) if and only if \( R\pi_*E \) is \( m \)-pseudo-coherent relative to \( S \).

Proof. Translation of the result of More on Algebra, Lemma 75.5 into the language of schemes. Observe that \( R\pi_* \) indeed maps \( D_{QCoh}(\mathcal{O}_X) \) into \( D_{QCoh}(\mathcal{O}_Y) \) by Derived Categories of Schemes, Lemma 4.1. To do the translation use Lemma 51.6.

Lemma 51.10. Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. Let \( (E, E', E'') \) be a distinguished triangle of \( D(\mathcal{O}_X) \). Let \( m \in \mathbb{Z} \).

1. If \( E \) is \((m+1)\)-pseudo-coherent relative to \( S \) and \( E' \) is \( m \)-pseudo-coherent relative to \( S \) then \( E'' \) is \( m \)-pseudo-coherent relative to \( S \).
2. If \( E, E'' \) are \( m \)-pseudo-coherent relative to \( S \), then \( E' \) is \( m \)-pseudo-coherent relative to \( S \).
3. If \( E' \) is \((m+1)\)-pseudo-coherent relative to \( S \) and \( E'' \) is \( m \)-pseudo-coherent relative to \( S \), then \( E \) is \((m+1)\)-pseudo-coherent relative to \( S \).

Moreover, if two out of three of \( E, E', E'' \) are pseudo-coherent relative to \( S \), the so is the third.

Proof. Immediate from Lemma 51.6 and Cohomology, Lemma 43.4.

Lemma 51.11. Let \( X \to S \) be a morphism of schemes which is locally of finite type. Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. Then

1. \( \mathcal{F} \) is \( m \)-pseudo-coherent relative to \( S \) for all \( m > 0 \),
2. \( \mathcal{F} \) is 0-pseudo-coherent relative to \( S \) if and only if \( \mathcal{F} \) is a finite type \( \mathcal{O}_X \)-module,
3. \( \mathcal{F} \) is \((-1)\)-pseudo-coherent relative to \( S \) if and only if \( \mathcal{F} \) is quasi-coherent and finitely presented relative to \( S \).

Proof. Part (1) is immediate from the definition. To see part (3) we may work locally on \( X \) (both properties are local). Thus we may assume \( X \) and \( S \) are affine. Choose a closed immersion \( i : X \to \mathbb{A}^n_{\mathbb{A}} \). Then we see that \( \mathcal{F} \) is \((-1)\)-pseudo-coherent relative to \( S \) if and only if \( i_*\mathcal{F} \) is \((-1)\)-pseudo-coherent, which is true if and only if \( i_*\mathcal{F} \) is an \( \mathcal{O}_{\mathbb{A}^n_{\mathbb{A}}} \)-module of finite presentation, see Cohomology, Lemma 43.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 11.2. By Morphisms, Lemma 4.1 we see that \( \mathcal{F} \) is quasi-coherent if and only if \( i_*\mathcal{F} \) is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved.

Lemma 51.12. Let \( X \to S \) be a morphism of schemes which is locally of finite type. Let \( m \in \mathbb{Z} \). Let \( E, K \) be objects of \( D(\mathcal{O}_X) \). If \( E \oplus K \) is \( m \)-pseudo-coherent relative to \( S \) so are \( E \) and \( K \).

Proof. Follows from Cohomology, Lemma 43.6 and the definitions.

Lemma 51.13. Let \( X \to S \) be a morphism of schemes which is locally of finite type. Let \( m \in \mathbb{Z} \). Let \( \mathcal{F}^i \) be a (locally) bounded above complex of \( \mathcal{O}_X \)-modules such that \( \mathcal{F}^i \) is \((m-i)\)-pseudo-coherent relative to \( S \) for all \( i \). Then \( \mathcal{F}^\bullet \) is \( m \)-pseudo-coherent relative to \( S \).

Proof. Follows from Cohomology, Lemma 43.7 and the definitions.
Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(O_X)$. If $E$ is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to $S$ for all $i$, then $E$ is $m$-pseudo-coherent relative to $S$.

**Proof.** Follows from Cohomology, Lemma \[43.8\] and the definitions. \hfill $\Box$

Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(O_X)$ which is $m$-pseudo-coherent relative to $S$. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote $E'$ the derived pullback of $E$ to $X'$. If $S'$ and $X$ are Tor independent over $S$, then $E'$ is $m$-pseudo-coherent relative to $S'$.

**Proof.** The problem is local on $X$ and $X'$ hence we may assume $X$, $S$, $S'$, and $X'$ are affine. Choose a closed immersion $i : X \to A^n_S$ and denote $i' : X' \to A^n_{S'}$, the base change to $S'$. Denote $g : X' \to X$ and $g' : A^n_{S'} \to A^n_S$ the projections, so $E' = Lg^*E$. Since $X$ and $S'$ are tor-independent over $S$, the base change map (Cohomology, Remark \[28.3\]) induces an isomorphism

$$Ri'_*(Lg^*E) = L(g')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at $x'$ is the map

$$E_x \otimes_{O_{A^n_{S'}},x} O_{A^n_{S'},x'} \longrightarrow E_x \otimes_{O_{X,x}} O_{X',x'}$$

coming from the closed immersions $i$ and $i'$. Note that the source is quasi-isomorphic to a localization of $E_x \otimes_{O_{S,s}} O_{S',s'}$ which is isomorphic to the target as $O_{X',x'}$ is isomorphic to (the same) localization of $O_{X,x} \otimes_{O_{S,s}} O_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma \[43.3\]. \hfill $\Box$

Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base $S$. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(O_Y)$. Assume

1. $O_X$ is pseudo-coherent relative to $Y$\[^{10}\] and
2. $E$ is $m$-pseudo-coherent relative to $S$.

Then $Lf^*E$ is $m$-pseudo-coherent relative to $S$.

**Proof.** The problem is local on $X$. Thus we may assume $X$, $Y$, and $S$ are affine. Arguing as in the proof of More on Algebra, Lemma \[75.13\] we can find a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & A^n_Y \\
\downarrow f & & \downarrow j \\
Y & \xrightarrow{p} & A^n_S \\
\end{array}
$$

Observe that

$$Ri_*Lf^*E = Ri_*Li^*Lp^*E = Lp^*E \otimes_{O_{A^n_Y}} Ri_*O_X$$

by Cohomology, Lemma \[49.4\]. By assumption and the fact that $Y$ is affine, we can represent $Ri_*O_X = i_*O_X$ by a complexes of finite free $O_{A^n_Y}$-modules $F^*$, with $F^q = 0$ for $q > 0$ (details omitted; use Derived Categories of Schemes, Lemma \[9.2\] and More on Algebra, Lemma \[75.7\]). By assumption $E$ is bounded above, say

\[^{10}\]This means $f$ is pseudo-coherent, see Definition \[52.2\].
$H^q(E) = 0$ for $q > a$. Represent $E$ by a complex $E^\bullet$ of $\mathcal{O}_Y$-modules with $E_i = 0$ for $q > a$. Then the derived tensor product above is represented by $\text{Tot}(p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^\bullet)$.

Since $j$ is a closed immersion, the functor $j_*$ is exact and $Rj_*$ is computed by applying $j_*$ to any representing complex of sheaves. Thus we have to show that $j_* \text{Tot}(p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^\bullet)$ is $m$-pseudo-coherent as a complex of $\mathcal{O}_{\mathbb{A}^n_S}$-modules. Note that $\text{Tot}(p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^\bullet)$ has a filtration by subcomplexes with successive quotients $p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^q[-q]$. Note that for $q \ll 0$ the complexes $p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^q[-q]$ have zero cohomology in degrees $\leq m$ and hence are $m$-pseudo-coherent. Hence, applying Lemma 51.10 and induction, it suffices to show that $p^*E^\bullet \otimes_{\mathcal{O}_{\mathbb{A}^n_Y}} \mathcal{F}^q$ is pseudo-coherent relative to $S$ which follows from Lemma 51.15 for instance.

**Lemma 51.17.** Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base $S$. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(\mathcal{O}_X)$. Assume $\mathcal{O}_Y$ is pseudo-coherent relative to $S$. Then the following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $Y$, and
2. $E$ is $m$-pseudo-coherent relative to $S$.

**Proof.** The question is local on $X$, hence we may assume $X$, $Y$, and $S$ are affine. Arguing as in the proof of More on Algebra, Lemma 75.13 we can find a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{A}^m_Y \xrightarrow{j} \mathbb{A}^{n+m}_S \\
\downarrow{f} & & \downarrow{p} \\
Y & \xrightarrow{p} & \mathbb{A}^n_S \\
\end{array}
$$

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to $S$ implies that $\mathcal{O}_{\mathbb{A}^n_Y}$ is pseudo-coherent relative to $\mathbb{A}^n_S$ (by flat base change; this can be seen by using for example Lemma 51.15). This in turn implies that $j_* \mathcal{O}_{\mathbb{A}^n_Y}$ is pseudo-coherent as an $\mathcal{O}_{\mathbb{A}^{n+m}_S}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma [11.5].

**Lemma 51.18.** Let

$$
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow{f} & & \downarrow{} \\
S & & \\
\end{array}
$$

be a commutative diagram of schemes. Assume $i$ is a closed immersion and $P \to S$ flat and locally of finite presentation. Let $E$ be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $S$,
2. $Ri_*E$ is $m$-pseudo-coherent relative to $S$, and
3. $Ri_*E$ is $m$-pseudo-coherent on $P$.

---

This means $Y \to S$ is pseudo-coherent, see Definition [52.2]
The equivalence of (1) and (2) is Lemma 51.9. The equivalence of (2) and (3) follows from Lemma 51.17 applied to \( \text{id} : P \to P \) provided we can show that \( \mathcal{O}_P \) is pseudo-coherent relative to \( S \). This follows from More on Algebra, Lemma 76.3 and the definitions. \[ \square \]

52. Pseudo-coherent morphisms

Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 62, 75, and 76. For now the only thing you need to know is that a ring map \( A \to B \) is pseudo-coherent if and only if \( B = A[x_1, \ldots , x_n]/I \) and \( B \) as an \( A[x_1, \ldots , x_n] \)-module has a resolution by finite free \( A[x_1, \ldots , x_n] \)-modules.

Lemma 52.1. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent

(1) there exist an affine open covering \( S = \bigcup V_j \) and for each \( j \) an affine open covering \( f^{-1}(V_j) = \bigcup U_{ji} \) such that \( \mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ji}) \) is a pseudo-coherent ring map,

(2) for every pair of affine opens \( U \subset X, V \subset S \) such that \( f(U) \subset V \) the ring map \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is pseudo-coherent, and

(3) \( f \) is locally of finite type and \( \mathcal{O}_X \) is pseudo-coherent relative to \( S \).

Proof. To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 13.1 for the property of being a pseudo-coherent ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 75.12, 75.11, and 75.16.

If (1) holds, then \( f \) is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma 51.7 and the definitions that \( \mathcal{O}_X \) is pseudo-coherent relative to \( S \). Conversely, if (3) holds, then we see that for every \( U \) and \( V \) as in (2) the ring \( \mathcal{O}_X(U) \) is of finite type over \( \mathcal{O}_S(V) \) and \( \mathcal{O}_X(U) \) is a module pseudo-coherent relative to \( \mathcal{O}_S(V) \), see Lemmas 51.6 and 51.7. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold. \[ \square \]

Definition 52.2. A morphism of schemes \( f : X \to S \) is called pseudo-coherent if the equivalent conditions of Lemma 52.1 are satisfied. In this case we also say that \( X \) is pseudo-coherent over \( S \).

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 52.3. A flat base change of a pseudo-coherent morphism is pseudo-coherent.

Proof. This translates into the following algebra result: Let \( A \to B \) be a pseudo-coherent ring map. Let \( A \to A' \) be flat. Then \( A' \to B \otimes_A A' \) is pseudo-coherent. This follows from the more general More on Algebra, Lemma 75.12. \[ \square \]

Lemma 52.4. A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.

Proof. This translates into the following algebra result: If \( A \to B \to C \) are composable pseudo-coherent ring maps then \( A \to C \) is pseudo-coherent. This follows from either More on Algebra, Lemma 75.13 or More on Algebra, Lemma 75.16. \[ \square \]
Lemma 52.5. A pseudo-coherent morphism is locally of finite presentation.

Proof. Immediate from the definitions.

Lemma 52.6. A flat morphism which is locally of finite presentation is pseudo-coherent.

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 76.4.

Lemma 52.7. Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme $S$. Then $f$ is pseudo-coherent.

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 75.15.

Lemma 52.8. Let $f : X \to S$ be a finite morphism of schemes. Then $f$ is pseudo-coherent if and only if $f_* \mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_S$-module.

Proof. Translated into algebra this lemma says the following: If $R \to A$ is a finite ring map, then $R \to A$ is pseudo-coherent as a ring map (which means by definition that $A$ as an $A$-module is pseudo-coherent relative to $R$) if and only if $A$ is pseudo-coherent as an $R$-module. This follows from the more general More on Algebra, Lemma 75.5.

Lemma 52.9. Let $f : X \to S$ be a morphism of schemes. If $S$ is locally Noetherian, then $f$ is pseudo-coherent if and only if $f$ is locally of finite type.

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with $R$ Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 62.18 that $A$ is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map.

Lemma 52.10. The property $P(f) =$ “$f$ is pseudo-coherent” is fpqc local on the base.

Proof. We will use the criterion of Descent, Lemma 19.4 to prove this. By Definition 52.2 being pseudo-coherent is Zariski local on the base. By Lemma 52.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 19.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an $A$-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then $C$ is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 62.16.

Lemma 52.11. Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if $I$ is pseudo-coherent as a $B$-module.

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that $B$ is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 52.6. Note that $A \to B/I$ is pseudo-coherent if and only if $B/I$ is
pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 62.12 we see this is equivalent to the condition that $B/I$ is pseudo-coherent as an $B$-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that $I$ is pseudo-coherent if and only if $B/I$ is (see More on Algebra, Lemma 62.7).

The following lemma will be obsoleted by the stronger Lemma 52.13.

**Lemma 52.12.** The property $P(f) =$ “$f$ is pseudo-coherent” is syntomic local on the source.

**Proof.** We will use the criterion of Descent, Lemma 23.4 to prove this. It follows from Lemmas 52.6 and 52.4 that being pseudo-coherent is preserved under pre-composing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 29.7 and 29.6). It is clear from Definition 52.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 23.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 11.1. Choose a closed immersion $X \to A^n_Y$. By Algebra, Lemma 135.18 we can find an affine open covering $X' = \bigcup_{i = 1}^n X'_i$ and syntomic morphisms $W_i \to A^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$
\begin{array}{ccc}
X'_i & \longrightarrow & W_i \\
\downarrow & & \downarrow \\
X & \longrightarrow & A^n_Y
\end{array}
$$

After replacing $X'$ by $\bigsqcup X'_i$ and setting $W = \bigsqcup W_i$ we obtain a fibre product diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & & \downarrow h \\
X & \longrightarrow & A^n_Y
\end{array}
$$

with $W \to A^n_Y$ flat of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to A^n_Y$ is open (see Morphisms, Lemma 24.10) and $X' \to X$ is surjective we can find $f \in \Gamma(A^n_Y, O)$ such that $X \subset D(f) \subset \operatorname{Im}(h)$. Write $Y = \operatorname{Spec}(R)$, $X = \operatorname{Spec}(A)$, $X' = \operatorname{Spec}(A')$ and $W = \operatorname{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$
A' = B/IB \leftarrow B \\
\downarrow \quad \downarrow \\
A = R[x_1, \ldots, x_n]/I \leftarrow R[x_1, \ldots, x_n]
$$

By Lemma 52.11 we see that $IB$ is pseudo-coherent as a $B$-module. The ring map $R[x_1, \ldots, x_n]/f \to Bf$ is faithfully flat by our choice of $f$ above. This implies that $I_f \subset R[x_1, \ldots, x_n]/f$ is pseudo-coherent, see More on Algebra, Lemma 62.10. Applying Lemma 52.11 one more time we see that $R \to A$ is pseudo-coherent. □
Lemma 52.13. The property $\mathcal{P}(f) =$“$f$ is pseudo-coherent” is fpqc local on the source.

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fpqc covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 52.2 there exist

1. a Zariski open covering $X = \bigcup U_j$,
2. surjective finite locally free morphisms $W_j \to U_j$,
3. Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
4. surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fpqc covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{g_i(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both $X_i$ and $T_{j,k}$ are pseudo-coherent over $X$ by Lemma 52.6. Hence $\psi_{j,k} : T_{j,k} \to X$ is pseudo-coherent, see Lemma 52.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_i(j,k)$, see Lemma 52.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 11.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that $X'$, $X$ and $S$ are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an $A$-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption that $R \to A'$ is pseudo-coherent means that $A'$ as an $A'$-module is pseudo-coherent relative to $R$. By More on Algebra, Lemma 75.5 this implies that $A'$ as an $A$-module is pseudo-coherent relative to $R$. Since $A' \cong A^{\oplus r}$ as an $A$-module we see that $A$ as an $A$-module is pseudo-coherent relative to $R$, see More on Algebra, Lemma 75.8. This by definition means that $R \to A$ is pseudo-coherent and we win. 

53. Perfect morphisms

In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and $B$ has finite tor dimension as an $A$-module.

Lemma 53.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent

1. there exist an affine open covering $S = \bigcup V_j$ and for each $j$ an affine open covering $f^{-1}(V_j) = \bigcup U_{ij}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and
2. for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.

Proof. Assume (1) and let $U, V$ be as in (2). It follows from Lemma 53.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property

of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 13.1. These properties follow from More on Algebra, Lemmas 63.11, 63.14, and 63.16. Some details omitted.

Definition 53.2. A morphism of schemes \( f : X \to S \) is called perfect if the equivalent conditions of Lemma 53.1 are satisfied. In this case we also say that \( X \) is perfect over \( S \).

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 53.3. A flat base change of a perfect morphism is perfect.

Proof. This translates into the following algebra result: Let \( A \to B \) be a perfect ring map. Let \( A \to A' \) be flat. Then \( A' \to B \otimes_A A' \) is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 52.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 63.14.

Lemma 53.4. A composition of perfect morphisms of schemes is perfect.

Proof. This translates into the following algebra result: If \( A \to B \to C \) are composable perfect ring maps then \( A \to C \) is perfect. We have seen this is the case for pseudo-coherent in Lemma 52.4 and its proof. By assumption there exist integers \( n, m \) such that \( B \) has tor dimension \( \leq n \) over \( A \) and \( C \) has tor dimension \( \leq m \) over \( B \). Then for any \( A \)-module \( M \) we have

\[
M \otimes_A^L C = (M \otimes_A^L B) \otimes_B^L C
\]

and the spectral sequence of More on Algebra, Example 60.4 shows that \( \text{Tor}_p^A(M, C) = 0 \) for \( p > n + m \) as desired.

Lemma 53.5. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. \( f \) is flat and perfect, and
2. \( f \) is flat and locally of finite presentation.

Proof. The implication (2) \( \Rightarrow \) (1) is More on Algebra, Lemma 76.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 52.5.

Lemma 53.6. Let \( f : X \to S \) be a morphism of schemes. Assume \( S \) is regular and \( f \) is locally of finite type. Then \( f \) is perfect.

Proof. See More on Algebra, Lemma 76.5.

Lemma 53.7. A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2 it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that \( I \subset A \) is an ideal generated by a Koszul-regular sequence \( f_1, \ldots, f_r \) of \( A \). Then \( A \to A/I \) is a perfect ring map. Since \( A \to A/I \) is surjective this is a presentation of \( A/I \) by a polynomial algebra over \( A \). Hence it suffices to see that \( A/I \) is pseudo-coherent as an \( A \)-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex
$K(A, f_1, \ldots, f_r)$ is a finite free resolution of $A/I$. Hence $A/I$ is a perfect complex of $A$-modules and we win. □

**Lemma 53.8.** Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{i} & Y
\end{array}
$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

**Proof.** We can factor $f$ as the composition

$$
X \to X \times_S Y \to Y
$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 32.9 we see that $X \times_S Y \to Y$ is perfect by Lemma 53.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 32.5. Hence $i$ is a section of a smooth morphism, therefore $i$ is a regular immersion, see Divisors, Lemma 22.8. This implies that $i$ is perfect, see Lemma 53.7. We conclude that $f$ is perfect because the composition of perfect morphisms is perfect, see Lemma 53.4. □

**Remark 53.9.** It is not true that a morphism between schemes $X, Y$ perfect over a base $S$ is perfect. An example is $S = \text{Spec}(k)$, $X = \text{Spec}(k)$, $Y = \text{Spec}(k[\{x\}]/(x^2))$ and $X \to Y$ the unique $S$-morphism.

**Lemma 53.10.** The property $\mathcal{P}(f) = "f \text{ is perfect}"$ is fpqc local on the base.

**Proof.** We will use the criterion of Descent, Lemma 19.4 to prove this. By Definition 53.2 being perfect is Zariski local on the base. By Lemma 53.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 19.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an $A$-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then $C$ is perfect as an $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 63.12. □

**Lemma 53.11.** Let $f : X \to S$ be a pseudo-coherent morphism of schemes. The following are equivalent

1. $f$ is perfect,
2. $\mathcal{O}_X$ locally has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules, and
3. for all $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.

**Proof.** The problem is local on $X$ and $S$. Hence we may assume that $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and $f$ corresponds to a pseudo-coherent ring map $A \to B$.

If (1) holds, then $B$ has finite tor dimension $d$ as $A$-module. Then $B_q$ has tor dimension $d$ as an $A_p$-module for all primes $q \subset B$ with $p = A \cap q$, see More on Algebra, Lemma 63.15. Then $\mathcal{O}_X$ has tor dimension $d$ as a sheaf of $f^{-1}\mathcal{O}_S$-modules by Cohomology, Lemma 44.5. Thus (1) implies (2).

By Cohomology, Lemma 44.5 (2) implies (3).

Assume (3). We cannot use More on Algebra, Lemma 63.15 to conclude as we are not given that the tor dimension of $B_q$ over $A_p$ is bounded independent of $q$. Choose
a presentation $A[x_1, \ldots, x_n] \to B$. Then $B$ is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. Let $q \subset A[x_1, \ldots, x_n]$ be a prime ideal lying over $p \subset A$. Then either $B_q$ is zero or by assumption it has finite tor dimension as an $A_p$-module. Since the fibres of $A \to A[x_1, \ldots, x_n]$ have finite global dimension, we can apply More on Algebra, Lemma 72.7 to $A_p \to A[x_1, \ldots, x_n]_q$ to see that $B_q$ is a perfect $A[x_1, \ldots, x_n]_q$-module. Hence $B$ is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 72.7. Thus $A \to B$ is a perfect ring map by definition. □

**Lemma 53.12.** Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a perfect proper morphism of schemes. Let $E \in D(O_X)$ be perfect. Then $Rf_*E$ is a perfect object of $D(O_S)$.

**Proof.** We claim that Derived Categories of Schemes, Lemma 24.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on $X$. Thus we may assume $X$ and $S$ affine and $E$ represented by a strictly perfect complex of $O_X$-modules. Thus it suffices to show that $O_X$ has finite tor dimension as a sheaf of $f^{-1}O_S$-modules. This is equivalent to being perfect by Lemma 53.11. □

**Lemma 53.13.** The property $\mathcal{P}(f) = "f \text{ is perfect}"$ is fpqc local on the source.

**Proof.** Let $\{g_i : X_i \to X\}_{i \in I}$ be an fpqc covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 52.13 we conclude that $f$ is pseudo-coherent. Hence by Lemma 53.11 it suffices to check that $O_{X,x}$ is an $O_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to $x$. Then we see that $O_{X,x_i}$ has finite tor dimension over $O_{S,f(x)}$ and that $O_{X,x} \to O_{X,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 63.17. □

**Lemma 53.14.** Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume

1. $X$ is locally Noetherian,
2. $j \circ i$ is a regular immersion, and
3. $i$ is perfect.

Then $i$ and $j$ are regular immersions.

**Proof.** Since $X$ (and hence $Y$) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 20.8. Thus the result follows from Divided Power Algebra, Lemma 7.3. □

### 54. Local complete intersection morphisms

In Divisors, Section 21 we have defined 4 different types of regular immersions: regular, Koszul-regular, $H_1$-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on $X$ factors as

$$
\begin{array}{c}
X \\
i \\
\downarrow \\
A^n_S \\
S
\end{array}
$$

where $i$ is a $*$-regular immersion for $* \in \{\emptyset, \text{Koszul}, H_1, \text{quasi}\}$. However, we don’t know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require $i$ to be a regular immersion. On the other hand, we want a
local complete intersection morphism to be perfect, which is only going to be true if $\star = \text{Koszul}$ or $\star = \emptyset$. Hence we will define a local complete intersection morphism or Koszul morphism to be a morphism of schemes $f : X \to S$ that locally on $X$ has a factorization as above with $i$ a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

**Lemma 54.1.** Let $S$ be a scheme. Let $U$, $P$, $P'$ be schemes over $S$. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over $S$. Assume $P$ and $P'$ smooth over $S$. Then the following are equivalent

1. $i$ is a Koszul-regular immersion in a neighbourhood of $x$, and
2. $i'$ is a Koszul-regular immersion in a neighbourhood of $x$.

**Proof.** Assume $i$ is a Koszul-regular immersion in a neighbourhood of $x$. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 22.9 that $j$ is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 22.12 that $i'$ is a Koszul-regular immersion. □

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to A^n_S$ followed by the projection $A^n_S \to S$ which is smooth. Picture

\[
\begin{array}{ccc}
X & \xleftarrow{i} & A^n_S = P \\
\downarrow{i} & & \downarrow{\pi} \\
S
\end{array}
\]

In fact you can do this with any affine open neighbourhood $U$ of $x$ in $X$, see Morphisms, Lemma 37.2.

**Definition 54.2.** Let $f : X \to S$ be a morphism of schemes.

1. Let $x \in X$. We say that $f$ is Koszul at $x$ if $f$ is of finite type at $x$ and there exists an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.
2. We say $f$ is a Koszul morphism, or that $f$ is a local complete intersection morphism if $f$ is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion $i$ followed by a smooth morphism $\pi$, whether or not $i$ is Koszul regular in a neighbourhood of $x$ is an intrinsic property of $f$ at $x$. Let us record this here explicitly as a lemma so that we can refer to it

**Lemma 54.3.** Let $f : X \to S$ be a local complete intersection morphism. Let $P$ be a scheme smooth over $S$. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over $S$. Then $i$ is a Koszul-regular immersion.

**Proof.** This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

**Lemma 54.4.** Let $f : X \to S$ be a local complete intersection morphism. Then
(1) \(f\) is locally of finite presentation,
(2) \(f\) is pseudo-coherent, and
(3) \(f\) is perfect.

**Proof.** Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that \(f\) factors as \(\pi \circ i\) where \(\pi\) is smooth and \(i\) is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 53.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 53.5. Finally a composition of perfect morphisms is perfect, see Lemma 53.4. \(\square\)

**Lemma 54.5.** Let \(f : X = \text{Spec}(B) \to S = \text{Spec}(A)\) be a morphism of affine schemes. Then \(f\) is a local complete intersection morphism if and only if \(A \to B\) is a local complete intersection homomorphism, see More on Algebra, Definition 32.2.

**Proof.** Follows immediately from the definitions. \(\square\)

Beware that a base change of a Koszul morphism is not Koszul in general.

**Lemma 54.6.** A flat base change of a local complete intersection morphism is a local complete intersection morphism.

**Proof.** Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.3. \(\square\)

**Lemma 54.7.** A composition of local complete intersection morphisms is a local complete intersection morphism.

**Proof.** Let \(g : Y \to S\) and \(f : X \to Y\) be local complete intersection morphisms. Let \(x \in X\) and set \(y = f(x)\). Choose an open neighbourhood \(V \subset Y\) of \(y\) and a factorization \(g|_V = \pi \circ i\) for some Koszul-regular immersion \(i : V \to P\) and smooth morphism \(\pi : P \to S\). Next choose an open neighbourhood \(U\) of \(x \in X\) and a factorization \(f|_U = \pi' \circ i'\) for some Koszul-regular immersion \(i' : U \to P'\) and smooth morphism \(\pi' : P' \to Y\). In fact, we may assume that \(P' = A^n_V\), see discussion preceding and following Definition 54.2. Picture:

\[
\begin{array}{ccc}
X & \leftarrow & U \to^{i'} P' = A^n_V \\
\downarrow & & \downarrow \\
Y & \leftarrow & V \to^{i} P \\
\downarrow & & \downarrow \\
S & \leftarrow & S
\end{array}
\]

Set \(P'' = A^n_P\). Then \(U \to P' \to P''\) is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely \(i'\) and the flat base change of \(i\) via \(P'' \to P\), see Divisors, Lemma 21.3 and Divisors, Lemma 21.7. Also \(P'' \to P \to S\) is smooth as a composition of smooth morphisms, see Morphisms, Lemma 32.4. Hence we conclude that \(X \to S\) is Koszul at \(x\) as desired. \(\square\)
Lemma 54.8. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. $f$ is flat and a local complete intersection morphism, and
2. $f$ is syntomic.

Proof. Working affine locally this is More on Algebra, Lemma 32.5. We also give a more geometric proof.

Assume (2). By Morphisms, Lemma 29.10 for every point $x$ of $X$ there exist affine open neighbourhoods $U$ of $x$ and $V$ of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = \text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \text{Spec}(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over $R$. By Algebra, Lemma 135.13 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_q$ for every prime $q \supseteq (f_1, \ldots, f_c)$. Consider the Koszul complex $K_* = K_*(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_*)$. By More on Algebra, Lemma 29.2 we see that $(H_i)_q = 0$, $i > 0$ for every $q$ as above. On the other hand, by More on Algebra, Lemma 28.6 we see that $H_i$ is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0$, $i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbb{A}^d_V$ followed by a smooth morphism as desired.

Assume (1). Then $f$ is a flat and locally of finite presentation (Lemma 54.4). Hence, according to Morphisms, Lemma 29.10 it suffices to show that the local rings $\mathcal{O}_{X, x}$ are local complete intersection rings. Choose, locally on $X$, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over $S$, see Divisors, Definition 21.2. Hence according to Divisors, Lemma 22.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of $X$ are local complete intersections.

Lemma 54.9. A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2 it suffices to prove the second statement. The second statement follows immediately from the definition.

Lemma 54.10. Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & & \end{array}
\]

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions.

Lemma 54.11. Let $f : X \to Y$ be a morphism of schemes. If $f$ is locally of finite type and $X$ and $Y$ are regular, then $f$ is a local complete intersection morphism.
Proof. We may assume there is a factorization $X \to \mathbb{A}^n_Y \to Y$ where the first arrow is an immersion. As $Y$ is regular also $\mathbb{A}^n_Y$ is regular by Algebra, Lemma 158.10. Hence $X \to \mathbb{A}^n_Y$ is a regular immersion by Divisors, Lemma 21.12. 

The following lemma is of a different nature.

09RL Lemma 54.12. Let

$$\xymatrix{ X \ar[r]^f \ar[d] & Y \ar[d] \\ S & S \ar[ur] }$$

be a commutative diagram of morphisms of schemes. Assume

1. $S$ is locally Noetherian,
2. $Y \to S$ is locally of finite type,
3. $f : X \to Y$ is perfect,
4. $X \to S$ is a local complete intersection morphism.

Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 29.7. Moreover, whether an ideal (resp. ideal sheaf) is regular may be checked on local rings (resp. stalks), see Algebra, Lemma 67.6 (resp. Divisors, Lemma 20.8).

The question is local. Hence we may assume $S, X, Y$ are affine. In this situation we may choose a commutative diagram

$$\xymatrix{ \mathbb{A}^{n+m}_S \ar[r] \ar[d] & X \ar[d] \\ \mathbb{A}^n_S \ar[r] & Y \ar[d] \\ S & S \ar[ur] }$$

whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings

$$\xymatrix{ J \ar[r] \ar[d] & \mathcal{O}_{\mathbb{A}^{n+m}_S, x} \ar[r] \ar[d] & \mathcal{O}_{X, x} \\ I \ar[r] & \mathcal{O}_{\mathbb{A}^n_S, f(x)} \ar[r] & \mathcal{O}_{Y, f(x)} }$$

where $J$ and $I$ are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal $J$ is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X, x}$ has finite tor dimension over $\mathcal{O}_{Y, f(x)}$. Hence we may apply Divided Power Algebra, Lemma 7.6 to conclude that $I$ and $J/I$ are generated by regular sequences. By our initial remarks, this finishes the proof. \[\square\]
Lemma 54.13. Let

\[
\begin{array}{c}
X \\
\downarrow f \\
\downarrow S \\
Y
\end{array}
\]

be a commutative diagram of morphisms of schemes. Assume \( S \) is locally Noetherian, \( Y \to S \) is locally of finite type, \( Y \) is regular, and \( X \to S \) is a local complete intersection morphism. Then \( f : X \to Y \) is a local complete intersection morphism and \( Y \to S \) is Koszul at \( f(x) \) for all \( x \in X \).

**Proof.** This is a special case of Lemma 54.12 in view of Lemma 53.6 (and Morphisms, Lemma 14.8). □

Lemma 54.14. Let \( i : X \to Y \) be an immersion. If

1. \( i \) is perfect,
2. \( Y \) is locally Noetherian, and
3. the conormal sheaf \( \mathcal{C}_{Z/X} \) is finite locally free,

then \( i \) is a regular immersion.

**Proof.** Translated into algebra, this is Divided Power Algebra, Proposition 11.3. □

Lemma 54.15. Let \( f : X \to Y \) be a perfect morphism of locally Noetherian schemes. The following are equivalent

1. \( f \) is a local complete intersection morphism,
2. \( NL_{X/Y} \) has tor-amplitude in \([-1, 0]\), and
3. \( NL_{X/Y} \) is perfect with tor-amplitude in \([-1, 0]\).

**Proof.** Translated into algebra this is Divided Power Algebra, Lemma 11.4. To do the translation use Lemmas 54.5 and 13.2 as well as Derived Categories of Schemes, Lemmas 3.5, 9.4 and 9.7. □

Lemma 54.16. Let \( f : X \to Y \) be a flat morphism of finite presentation. The following are equivalent

1. \( f \) is a local complete intersection morphism,
2. \( f \) is syntomic,
3. \( NL_{X/Y} \) has tor-amplitude in \([-1, 0]\), and
4. \( NL_{X/Y} \) is perfect with tor-amplitude in \([-1, 0]\).

**Proof.** Translated into algebra this is Divided Power Algebra, Lemma 11.6. To do the translation use Lemmas 54.5 and 13.2 as well as Derived Categories of Schemes, Lemmas 3.5, 9.4 and 9.7. □

The following lemma gives a characterization of smooth morphisms as flat morphisms whose diagonal is perfect.

Lemma 54.17. Let \( f : X \to Y \) be a finite type morphism of locally Noetherian schemes. Denote \( \Delta : X \to X \times_Y X \) the diagonal morphism. The following are equivalent

1. \( f \) is smooth,
2. \( f \) is flat and \( \Delta : X \to X \times_Y X \) is a regular immersion,
3. \( f \) is flat and \( \Delta : X \to X \times_Y X \) is a local complete intersection morphism,
(4) \( f \) is flat and \( \Delta : X \to X \times_Y X \) is perfect.

**Proof.** Assume (1). Then \( f \) is flat by Morphisms, Lemma 32.9. The projections \( X \times_Y X \to X \) are smooth by Morphisms, Lemma 32.5. Hence the diagonal is a section to a smooth morphism and hence a regular immersion, see Divisors, Lemma 22.8. Hence (1) \( \Rightarrow \) (2). The implication (2) \( \Rightarrow \) (3) is Lemma 54.9. The implication (3) \( \Rightarrow \) (4) is Lemma 54.4. The interesting implication (4) \( \Rightarrow \) (1) follows immediately from Divided Power Algebra, Lemma 10.2. \( \square \)

**Lemma 54.18.** The property \( P(f) = \"f \) is a local complete intersection morphism\" is fpqc local on the base.

**Proof.** Let \( f : X \to S \) be a morphism of schemes. Let \( \{S_i \to S\} \) be an fpqc covering of \( S \). Assume that each base change \( f_i : X_i \to S_i \) of \( f \) is a local complete intersection morphism. Note that this implies in particular that \( f \) is locally of finite type, see Lemma 54.4 and Descent, Lemma 20.10. Let \( x \in X \). Choose an open neighbourhood \( U \) of \( x \) and an immersion \( j : U \to A^n_S \) over \( S \) (see discussion preceding Definition 54.2). We have to show that \( j \) is a Koszul-regular immersion. Since \( f_i \) is a local complete intersection morphism, we see that the base change \( j_i : U \times_S S_i \to A^n_{S_i} \) is a Koszul-regular immersion, see Lemma 54.3. Because \( \{A^n_{S_i} \to A^n_S\} \) is a fpqc covering we see from Descent, Lemma 20.32 that \( j \) is a Koszul-regular immersion as desired. \( \square \)

**Lemma 54.19.** The property \( P(f) = \"f \) is a local complete intersection morphism\" is syntomic local on the source.

**Proof.** We will use the criterion of Descent, Lemma 23.4 to prove this. It follows from Lemmas 54.8 and 54.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 54.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 23.4 it suffices to prove the following: Suppose \( X' \to X \to Y \) are morphisms of affine schemes with \( X' \to X \) syntomic and \( X' \to Y \) a local complete intersection morphism. Then \( X \to Y \) is a local complete intersection morphism. To see this, note that in any case \( X \to Y \) is of finite presentation by Descent, Lemma 21.1. Choose a closed immersion \( X \to A^n_Y \). By Algebra, Lemma 135.18 we can find an affine open covering \( X' = \bigsqcup_{i=1}^n X'_i \) and syntomic morphisms \( W_i \to A^n_Y \) lifting the morphisms \( X'_i \to X \), i.e., such that there are fibre product diagrams

\[
\begin{array}{ccc}
X'_i & \longrightarrow & W_i \\
\downarrow & & \downarrow \\
X & \longrightarrow & A^n_Y \\
\end{array}
\]

After replacing \( X' \) by \( \bigsqcup X'_i \) and setting \( W = \bigsqcup W_i \) we obtain a fibre product diagram of affine schemes

\[
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & & \downarrow^h \\
X & \longrightarrow & A^n_Y \\
\end{array}
\]
with \( h : W \to A^n_Y \) syntomic and \( X' \to Y \) still a local complete intersection morphism. Since \( W \to A^n_Y \) is open (see Morphisms, Lemma 24.10) and \( X' \to X \) is surjective we see that \( X \) is contained in the image of \( W \to A^n_Y \). Choose a closed immersion \( W \to A^{n+m}_Y \) over \( A^n_Y \). Now the diagram looks like

\[
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & & \downarrow \scriptstyle{h} \\
X & \longrightarrow & A^n_Y
\end{array}
\]

Because \( h \) is syntomic and hence a local complete intersection morphism (see above) the morphism \( W \to A^{n+m}_Y \) is a Koszul-regular immersion. Because \( X' \to Y \) is a local complete intersection morphism the morphism \( X' \to A^{n+m}_Y \) is a Koszul-regular immersion. We conclude from Divisors, Lemma 21.8 that \( X' \to W \) is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 20.32) we conclude that \( X \to A^n_Y \) is a Koszul-regular immersion which is what we had to show.

\[\text{Lemma 54.20.} \quad \text{Let } S \text{ be a scheme. Let } f : X \to Y \text{ be a morphism of schemes over } S. \text{ Assume both } X \text{ and } Y \text{ are flat and locally of finite presentation over } S. \text{ Then the set}
\]

\[\{ x \in X \mid f \text{ Koszul at } x \}\]

\[\text{is open in } X \text{ and its formation commutes with arbitrary base change } S' \to S.\]

\[\text{Proof.} \quad \text{The set is open by definition (see Definition 54.2). Let } S' \to S \text{ be a morphism of schemes. Set } X' = S' \times_S X, Y' = S' \times_S Y, \text{ and denote } f' : X' \to Y' \text{ the base change of } f. \text{ Let } x' \in X' \text{ be a point such that } f' \text{ is Koszul at } x'. \text{ Denote } s' \in S', x \in X, y' \in Y', y \in Y, s \in S \text{ the image of } x'. \text{ Note that } f \text{ is locally of finite presentation, see Morphisms, Lemma 20.11. Hence we may choose an affine neighbourhood } U \subset X \text{ of } x \text{ and an immersion } i : U \to A^n_Y. \text{ Denote } U' = S' \times_S U \text{ and } i' : U' \to A^n_{Y'}, \text{ the base change of } i. \text{ The assumption that } f' \text{ is Koszul at } x' \text{ implies that } i' \text{ is a Koszul-regular immersion in a neighbourhood of } x', \text{ see Lemma 54.3. The scheme } X' \text{ is flat and locally of finite presentation over } S' \text{ as a base change of } X \text{ (see Morphisms, Lemmas 24.8 and 20.4). Hence } i' \text{ is a relative } H_1 \text{-regular immersion over } S' \text{ in a neighbourhood of } x \text{ (see Divisors, Definition 22.2). Thus the base change } i''_{s'} : U'_{s'} \to A^n_{Y'} \text{ is a } H_1 \text{-regular immersion in an open neighbourhood of } x', \text{ see Divisors, Lemma 22.1 and the discussion following Divisors, Definition 22.2. Since } s' = \text{Spec}(\kappa(s')) \to \text{Spec}(\kappa(s)) = s \text{ is a surjective flat universally open morphism (see Morphisms, Lemma 22.4) we conclude that the base change } i_s : U_s \to A^n_Y \text{ is an } H_1 \text{-regular immersion in a neighbourhood of } x, \text{ see Descent, Lemma 20.32. Finally, note that } A^n_Y \text{ is flat and locally of finite presentation over } S, \text{ hence Divisors, Lemma 22.7 implies that } i \text{ is a (Koszul-)regular immersion in a neighbourhood of } x \text{ as desired.}\]

\[\text{Lemma 54.21.} \quad \text{Let } f : X \to Y \text{ be a local complete intersection morphism of schemes. Then } f \text{ is unramified if and only if } f \text{ is formally unramified and in this case the conormal sheaf } C_{X/Y} \text{ is finite locally free on } X.\]

\[\text{Proof.} \quad \text{The first assertion follows immediately from Lemma 6.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the} \]
conormal sheaf of $f$ we choose, locally on $X$, a factorization of $f$ as $f = p \circ i$ where $i: X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma 11.13 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as $i$ is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 21.5.

**Lemma 54.22.** Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 7.12 is short exact.

**Proof.** The question is local on $Z$ hence we may assume there exists a factorization $Z \to \mathbb{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{i'} & \mathbb{A}^n_Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i} & Y \\
\end{array}
\begin{array}{ccc}
& \mathbb{A}^n_X & \\
\downarrow & \downarrow & \\
& X & \\
\end{array}
$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbb{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 21.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbb{A}^n_Y/\mathbb{A}^n_X} \to \mathcal{C}_{Z/\mathbb{A}^n_X} \to \mathcal{C}_{Z/\mathbb{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbb{A}^n_Y/\mathbb{A}^n_X}$ by Lemma 1.7 and note that the diagram

$$
\begin{array}{ccc}
(i')^*\mathcal{C}_{\mathbb{A}^n_Y/\mathbb{A}^n_X} & \cong & \mathcal{C}_{Z/\mathbb{A}^n_X} \\
\downarrow & & \downarrow \\
i^*\mathcal{C}_{Y/X} & \mathcal{C}_{Z/X} & \\
\end{array}
$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma.

\[\square\]

**55. Exact sequences of differentials and conormal sheaves**

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

In the sequences below each of the maps are as constructed in either Morphisms, Lemma 31.8 or Lemma 7.5. Let $g: Z \to Y$ and $f: Y \to X$ be morphisms of schemes.

1. There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Morphisms, Lemma 31.9. If $g: Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 11.1.

2. If $g$ is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 7.10. If $f \circ g: Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 11.12.
If \( g \) and \( f \circ g \) are formally unramified, then there is a canonical exact sequence
\[
C_{Z/X} \rightarrow C_{Z/Y} \rightarrow g^* \Omega_{Y/X} \rightarrow 0,
\]
see Lemma 7.11. If \( f : Y \rightarrow X \) is formally smooth, then this sequence is a short exact sequence, see Lemma 11.13.

If \( g \) and \( f \) are formally unramified, then there is a canonical exact sequence
\[
g^* C_{Y/X} \rightarrow C_{Z/X} \rightarrow C_{Z/Y} \rightarrow 0.
\]
see Lemma 7.12. If \( g : Z \rightarrow Y \) is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 54.22.

56. Weakly étale morphisms

A ring homomorphism \( A \rightarrow B \) is weakly étale if both \( A \rightarrow B \) and \( B \otimes_A B \rightarrow B \) are flat, see More on Algebra, Definition 94.1. The analogous notion for morphisms of schemes is the following.

Definition 56.1. A morphism of schemes \( X \rightarrow Y \) is weakly étale or absolutely flat if both \( X \rightarrow Y \) and the diagonal morphism \( X \rightarrow X \times_Y X \) are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if \( X \rightarrow Y \) is weakly étale, then \( L_{X/Y} = 0 \), as follows from Cotangent, Lemma 8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 9). In this section we stick with the basics.

Lemma 56.2. Let \( f : X \rightarrow Y \) be a morphism of schemes. The following are equivalent

1. \( X \rightarrow Y \) is weakly étale, and
2. for every \( x \in X \) the ring map \( \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x} \) is weakly étale.

Proof. Observe that under both assumptions (1) and (2) the morphism \( f \) is flat. Thus we may assume \( f \) is flat. Let \( x \in X \) with image \( y = f(x) \) in \( Y \). There are canonical maps of rings
\[
\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \rightarrow \mathcal{O}_{X,x}
\]
where the first map is a localization (hence flat) and the second map is a surjection (hence an epimorphism of rings). Condition (1) means that for all \( x \) the second arrow is flat. Condition (2) is that for all \( x \) the composition is flat. These conditions are equivalent by Algebra, Lemma 38.4 and More on Algebra, Lemma 94.2. \( \square \)

Lemma 56.3. Let \( X \rightarrow Y \) be a morphism of schemes such that \( X \rightarrow X \times_Y X \) is flat. Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. If \( \mathcal{F} \) is flat over \( Y \), then \( \mathcal{F} \) is flat over \( X \).

Proof. Let \( x \in X \) with image \( y = f(x) \) in \( Y \). Since \( X \rightarrow X \times_Y X \) is flat, we see that \( \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \) is flat. Hence the result follows from More on Algebra, Lemma 94.2 and the definitions. \( \square \)

Lemma 56.4. Let \( f : X \rightarrow S \) be a morphism of schemes. The following are equivalent

1. The morphism \( f \) is weakly étale.
2. For every affine opens \( U \subset X, V \subset S \) with \( f(U) \subset V \) the ring map \( \mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U) \) is weakly étale.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J$, $i \in I_j$ is weakly étale.

(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of weakly étale, for all $j \in J$, $i \in I_j$.

Moreover, if $f$ is weakly étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is weakly-étale.

**Proof.** Suppose given open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$. Then $U \times_Y U \subset X \times_Y X$ is open (Schemes, Lemma 17.3) and the diagonal $\Delta_{U/V}$ of $f|_U : U \to V$ is the restriction $\Delta_{X/Y}|_U : U \to U \times_Y U$. Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 24.3), the final statement of the lemma is follows as well as the equivalence of (1) and (3). If $X$ and $Y$ are affine, then $X \to Y$ is weakly étale if and only if $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is weakly étale (use again Morphisms, Lemma 24.3). Thus (1) and (3) are also equivalent to (2) and (4). \qed

**Lemma 56.5.** Let $X \to Y \to Z$ be morphisms of schemes.

1. If $X \to X \times_Y X$ and $Y \to Y \times_Z Y$ are flat, then $X \to X \times_Z X$ is flat.
2. If $X \to Y$ and $Y \to Z$ are weakly étale, then $X \to Z$ is weakly étale.

**Proof.** Part (1) follows from the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal of $X$ over $Z$, the fact that

$$X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y,$$

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 24.8 and 24.6). Part (2) follows from part (1) and the fact (just used) that the composition of flat morphisms is flat. \qed

**Lemma 56.6.** Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of $X$.

1. If $X \to X \times_Y X$ is flat, then $X' \to X' \times_Y X'$ is flat.
2. If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.

**Proof.** Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_Y X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 24.8. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. \qed

**Lemma 56.7.** Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \times_Z X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times_Z Y
\end{array}
$$


The top horizontal arrow is flat and the vertical arrows are flat. Hence $X$ is flat over $Y \times_Z Y$. By Morphisms, Lemma 24.13 we see that $Y$ is flat over $Y \times_Z Y$. □

**Lemma 56.8.** Let $f : X \to Y$ be a weakly étale morphism of schemes. Then $f$ is formally unramified, i.e., $\Omega_{X/Y} = 0$.

**Proof.** Recall that $f$ is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 6.7. Via Lemma 56.4 and Morphisms, Lemma 31.6 this follows from the case of rings which is More on Algebra, Lemma 94.12. □

**Lemma 56.9.** Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases

1. $X \to Y$ is a flat monomorphism,
2. $X \to Y$ is an open immersion,
3. $X \to Y$ is flat and unramified,
4. $X \to Y$ is étale.

**Proof.** If (1) holds, then $\Delta_{X/Y}$ is an isomorphism (Schemes, Lemma 23.2), hence certainly $f$ is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 33.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 34.5), hence (4) follows from (3). □

**Lemma 56.10.** Let $f : X \to Y$ be a morphism of schemes. If $Y$ is reduced and $f$ weakly étale, then $X$ is reduced.

**Proof.** Via Lemma 56.4 this follows from the case of rings which is More on Algebra, Lemma 94.8. □

The following lemma uses a nontrivial result about weakly étale ring maps.

**Lemma 56.11.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

1. $f$ is weakly étale, and
2. for $x \in X$ the local ring map $O_{Y,f(x)} \to O_{X,x}$ induces an isomorphism on strict henselizations.

**Proof.** Let $x \in X$ be a point with image $y = f(x)$ in $Y$. Choose a separable algebraic closure $\kappa^{sep}$ of $\kappa(x)$. Let $O_{X,x}^{sh}$ be the strict henselization corresponding to $\kappa^{sep}$ and $O_{Y,y}^{sh}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in $\kappa^{sep}$. Consider the commutative diagram

$$
\begin{array}{ccc}
O_{X,x} & \longrightarrow & O_{X,x}^{sh} \\
\uparrow & & \uparrow \\
O_{Y,y} & \longrightarrow & O_{Y,y}^{sh}
\end{array}
$$

local homomorphisms of local rings, see Algebra, Lemma 151.12. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 94.14 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 44.14. Assume $f$ weakly étale. By Lemma 56.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 94.9 and 94.11 the right vertical arrow is weakly étale.
étale. By More on Algebra, Theorem 94.25 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}_{Y,y}^{sh} \to \mathcal{O}_{X,x}^{sh}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{sh}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 94.10. Thus (2) implies (1) by Lemma 56.2. □

Lemma 56.12. Let $f : X \to Y$ be a morphism of schemes. If $Y$ is a normal scheme and $f$ weakly étale, then $X$ is a normal scheme.

Proof. By More on Algebra, Lemma 44.6 a scheme $S$ is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 56.11. □

Lemma 56.13. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. If $X$, $Y$ are weakly étale over $S$, then $f$ is weakly étale.

Proof. We will use Morphisms, Lemmas 24.8 and 24.6 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an extension. Thus Lemma 56.3 implies, that since $X$ is flat over $X \times_S X$ it follows that $X$ is flat over $X \times_Y X$. □

The following is a scheme theoretic generalization of the observation that a field extension that is simultaneously separable and purely inseparable must be an isomorphism.

Lemma 56.14. Let $f : X \to Y$ be a morphism of schemes. If $f$ is weakly étale and a universal homeomorphism, it is an isomorphism.

Proof. Since $f$ is a universal homeomorphism, the diagonal $\Delta : X \to X \times_Y X$ is a surjective closed immersion by Morphisms, Lemmas 43.4 and 10.2. Since $\Delta$ is also flat, we see that $\Delta$ must be an isomorphism by Morphisms, Lemma 25.1. In other words, $f$ is a monomorphism (Schemes, Lemma 23.2). Since $f$ is a universal homeomorphism it is certainly quasi-compact. Hence by Descent, Lemma 22.1 we find that $f$ is an isomorphism. □

The following is a weakly étale generalization of Étale Morphisms, Lemma 14.3.

Lemma 56.15. Let $U \to X$ be a weakly étale morphism of schemes where $X$ is a scheme in characteristic $p$. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 35.6. The morphism $F_{U/X}$ is weakly étale as a morphism between schemes weakly étale over $X$ by Lemma 56.13. Hence $F_{U/X}$ is an isomorphism by Lemma 56.14. □

57. Reduced fibre theorem

In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in a straightforward manner from Epp’s result on eliminating ramification, see More on Algebra, Theorem 104.19.
Let $A$ be a Dedekind domain with fraction field $K$. Let $X$ be a scheme flat and of finite type over $A$. Let $L$ be a finite extension of $K$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a Dedekind domain (Algebra, Lemma \[119.18\]). Let $X_B = X \times_{\Spec(A)} \Spec(B)$ be the base change. Then $X_B \to \Spec(B)$ is of finite type (Morphisms, Lemma \[14.4\]). Hence $X_B$ is Noetherian (Morphisms, Lemma \[14.6\]). Thus the normalization $\nu : Y \to X_B$ exists (see Morphisms, Definition \[52.1\] and the discussion following). Picture

\[\begin{array}{ccc}
Y & \xrightarrow{\nu} & X_B \\
\downarrow & & \downarrow \\
\Spec(B) & \xrightarrow{} & \Spec(A)
\end{array}\]

We sometimes call $Y$ the normalized base change of $X$. In general the morphism $\nu$ may not be finite. But if $A$ is a Nagata ring (a condition that is virtually always satisfied in practice) then $\nu$ is finite and $Y$ is of finite type over $B$, see Morphisms, Lemmas \[52.10\] and \[17.1\].

Taking the normalized base change commutes with composition. More precisely, if $K \subset L \subset M$ are finite extensions of fields with integral closures $A \subset B \subset C$ then the normalized base change $Z$ of $Y \to \Spec(B)$ relative to $L \subset M$ is equal to the normalized base change of $X \to \Spec(A)$ relative to $K \subset M$.

\[\textbf{Theorem 57.1.}\] Let $A$ be a Dedekind ring with fraction field $K$. Let $X$ be a scheme flat and of finite type over $A$. Assume $A$ is a Nagata ring. There exists a finite extension $K \subset L$ such that the normalized base change $Y$ is smooth over $\Spec(B)$ at all generic points of all fibres.

\[\textbf{Proof.}\] During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \Spec(A)$ is smooth commutes with base change, see Morphisms, Lemma \[32.15\].

We first choose a finite extension $K \subset L$ such that $(X_L)_{\text{red}}$ is geometrically reduced over $L$, see Varieties, Lemma \[6.11\]. Since $Y \to (X_B)_{\text{red}}$ is birational we see applying Varieties, Lemma \[6.8\] that $Y_L$ is geometrically reduced over $L$ as well. Hence $Y_L \to \Spec(L)$ is smooth on a dense open $V \subset Y_L$ by Varieties, Lemma \[25.7\].

Thus the smooth locus $U \subset Y$ of the morphism $Y \to \Spec(B)$ is open (by Morphisms, Definition \[32.1\]) and is dense in the generic fibre. Replacing $A$ by $B$ and $X$ by $Y$ we reduce to the case treated in the next paragraph.

Assume $X$ is normal and the smooth locus $U \subset X$ of $X \to \Spec(A)$ is dense in the generic fibre. This implies that $U$ is dense in all but finitely many fibres, see Lemma \[22.3\].

Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \Spec(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Let $A_i$ be the localization of $A$ at the maximal ideal corresponding to the image of $x_i$ in $\Spec(A)$. By More on Algebra, Proposition \[104.21\] there exist finite extensions $K \subset K_i$ which are solutions for the extension of discrete valuation rings $A_i \to \mathcal{O}_i$. Let $K \subset L$ be a finite extension dominating all of the extensions $K \subset K_i$. Then $K \subset L$ is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma \[103.3\].

Consider the diagram \[57.0.1\] with the extension $L/K$ we just produced. Note that $U_B \subset X_B$ is smooth over $B$, hence normal (for example use Algebra, Lemma
Thus $Y \to X_B$ is an isomorphism over $U_B$. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \text{Spec}(B)$ lying over the maximal ideal $m \subset B$. Assume that $y \notin U_B$. Then $y$ maps to one of the points $x_i$. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because $K \subset L$ is a solution for $A_i \to \mathcal{O}_i$ we see that $B_m \to \mathcal{O}_{Y,y}$ is formally smooth in the $m_y$-adic topology (this is the definition of being a 'solution'). In other words, $m_{\mathcal{O}_{Y,y}} = m_y$ and the residue field extension is separable, see More on Algebra, Lemma 100.5. Hence the local ring of the fibre at $y$ is $\kappa(y)$ (details omitted). This finishes the proof. □

**Lemma 57.2** (Variant over curves). Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume $S$ is Nagata, integral with function field $K$, and regular of dimension 1. Then there exists a finite extension $L/K$ such that in the diagram

\[
\begin{array}{ccc}
Y & \to & X \\
\downarrow g & & \downarrow f \\
T & \to & S
\end{array}
\]

the morphism $g$ is smooth at all generic points of fibres. Here $T$ is the normalization of $S$ in $\text{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

**Proof.** Choose a finite affine open covering $S = \bigcup \text{Spec}(A_i)$. Then $K$ is equal to the fraction field of $A_i$ for all $i$. Let $X_i = X \times_S \text{Spec}(A_i)$. Choose $L_i/K$ as in Theorem 57.1 for the morphism $X_i \to \text{Spec}(A_i)$. Let $B_i \subset L_i$ be the integral closure of $A_i$ and let $Y_i$ be the normalized base change of $X$ to $B_i$. Let $L/K$ be a finite extension dominating each $L_i$. Let $T_i \subset T$ be the inverse image of $\text{Spec}(A_i)$. For each $i$ we get a commutative diagram

\[
\begin{array}{ccc}
g^{-1}(T_i) & \to & Y_i \\
\downarrow & & \downarrow \\
T_i & \to & \text{Spec}(B_i) \to \text{Spec}(A_i)
\end{array}
\]

and in fact the left hand square is a normalized base change as discussed at the beginning of the section. In the proof of Theorem 57.1 we have seen that the smooth locus of $Y \to T$ contains the inverse image in $g^{-1}(T_i)$ of the set of points where $Y_i$ is smooth over $B_i$. This proves the lemma. □

**Lemma 57.3** (Variant with separable extension). Let $A$ be a Dedekind ring with fraction field $K$. Let $X$ be a scheme flat and of finite type over $A$. Assume $A$ is a Nagata ring and that for every generic point $\eta$ of an irreducible component of $X$ the field extension $K \subset \kappa(\eta)$ is separable. Then there exists a finite separable extension $K \subset L$ such that the normalized base change $Y$ is smooth over $\text{Spec}(B)$ at all generic points of all fibres.

**Proof.** This is proved in exactly the same manner as Theorem 57.1 with a few minor modifications. The most important change is to use More on Algebra, Lemma 104.22 instead of More on Algebra, Proposition 104.21. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented)
morphism like \( X \to \text{Spec}(A) \) is smooth commutes with base change, see Morphisms, Lemma \[\text{52.15}\].

Since \( X \) is flat over \( A \) every generic point \( \eta \) of \( X \) maps to the generic point of \( \text{Spec}(A) \). After replacing \( X \) by its reduction we may assume \( X \) is reduced. In this case \( X_K \) is geometrically reduced over \( K \) by Varieties, Lemma \[\text{6.8}\]. Hence \( X_K \to \text{Spec}(K) \) is smooth on a dense open by Varieties, Lemma \[\text{25.7}\]. Thus the smooth locus \( U \subset X \) of the morphism \( X \to \text{Spec}(A) \) is open (by Morphisms, Definition \[\text{32.1}\], and is dense in the generic fibre. This reduces us to the situation of the following paragraph.

Assume \( X \) is normal and the smooth locus \( U \subset X \) of \( X \to \text{Spec}(A) \) is dense in the generic fibre. This implies that \( U \) is dense in all but finitely many fibres, see Lemma \[\text{22.3}\]. Let \( x_1, \ldots, x_r \in X \setminus U \) be the finitely many generic points of irreducible components of \( X \setminus U \) which are moreover generic points of irreducible components of fibres of \( X \to \text{Spec}(A) \). Set \( \mathcal{O}_i = \mathcal{O}_{X,x_i} \). Observe that the fraction field of \( \mathcal{O}_i \) is the residue field of a generic point of \( X \). Let \( A_i \) be the localization of \( A \) at the maximal ideal corresponding to the image of \( x_i \) in \( \text{Spec}(A) \). We may apply More on Algebra, Lemma \[\text{104.22}\] and we find finite separable extensions \( K \subset K_i \) which are solutions for \( A_i \to \mathcal{O}_i \). Let \( K \subset L \) be a finite separable extension dominating all of the extensions \( K \subset K_i \). Then \( K \subset L \) is still a solution for \( A_i \to \mathcal{O}_i \) by More on Algebra, Lemma \[\text{103.3}\].

Consider the diagram (\[\text{57.0.1}\]) with the extension \( L/K \) we just produced. Note that \( U_B \subset X_B \) is smooth over \( B \), hence normal (for example use Algebra, Lemma \[\text{158.9}\]). Thus \( Y \to X_B \) is an isomorphism over \( U_B \). Let \( y \in Y \) be a generic point of an irreducible component of a fibre of \( Y \to \text{Spec}(B) \) lying over the maximal ideal \( m \subset B \). Assume that \( y \not\in U_B \). Then \( y \) maps to one of the points \( x_i \). It follows that \( \mathcal{O}_{Y,y} \) is a local ring of the integral closure of \( \mathcal{O}_i \) in \( R(X) \otimes_K L \) (details omitted). Hence because \( K \subset L \) is a solution for \( A_i \to \mathcal{O}_i \) we see that \( B_m \to \mathcal{O}_{Y,y} \) is formally smooth (this is the definition of being a 'solution'). In other words, \( m \mathcal{O}_{Y,y} = m_y \) and the residue field extension is separable. Hence the local ring of the fibre at \( y \) is \( \kappa(y) \). This implies the fibre is smooth over \( \kappa(m) \) at \( y \) for example by Algebra, Lemma \[\text{139.5}\]. This finishes the proof. \( \square \)

**Lemma 57.4** (Variant with separable extensions over curves). Let \( f : X \to S \) be a flat, finite type morphism of schemes. Assume \( S \) is Nagata, integral with function field \( K \), and regular of dimension 1. Assume the field extensions \( K \subset \kappa(\eta) \) are separable for every generic point \( \eta \) of an irreducible component of \( X \). Then there exists a finite separable extension \( L/K \) such that in the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\nu} & X \times_S T \\
\downarrow g & & \downarrow f \\
T & \xrightarrow{f} & S
\end{array}
\]

the morphism \( g \) is smooth at all generic points of fibres. Here \( T \) is the normalization of \( S \) in \( \text{Spec}(L) \) and \( \nu : Y \to X \times_S T \) is the normalization.

**Proof.** This follows from Lemma \[\text{57.3}\] in exactly the same manner that Lemma \[\text{57.2}\] follows from Theorem \[\text{57.1}\]. \( \square \)
58. Ind-quasi-affine morphisms

0AP5 A bit of theory to be used later.

0AP6 **Definition 58.1.** A scheme $X$ is *ind-quasi-affine* if every quasi-compact open of $X$ is quasi-affine. Similarly, a morphism of schemes $X \to Y$ is *ind-quasi-affine* if $f^{-1}(V)$ is ind-quasi-affine for each affine open $V$ in $Y$.

An example of an ind-quasi-affine scheme is an open of an affine scheme. If $X = \bigcup_{i \in I} U_i$ is a union of quasi-affine opens such that any two $U_i$ are contained in a third, then $X$ is ind-quasi-affine. An ind-quasi-affine scheme $X$ is separated because any two affine opens $U, V$ are contained in a separated open subscheme of $X$, namely $U \cup V$. Similarly an ind-quasi-affine morphism is separated.

0F1U **Lemma 58.2.** For a morphism of schemes $f : X \to Y$, the following are equivalent:

1. $f$ is ind-quasi-affine,
2. for every affine open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.
3. for some cover $\{ V_j \}_{j \in J}$ of $Y$ by quasi-compact and quasi-separated open subschemes $V_j \subset Y$, every $j \in J$, and every quasi-compact open subscheme $U \subset f^{-1}(V_j)$, the induced morphism $U \to V_j$ is quasi-affine.
4. for every quasi-compact and quasi-separated open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.

In particular, the property of being an ind-quasi-affine morphism is Zariski local on the base.

**Proof.** The equivalence (1) $\iff$ (2) follows from the definitions and Morphisms, Lemma 12.3. For (2) $\implies$ (4), let $U$ and $V$ be as in (4). By Schemes, Lemma 21.14, the induced morphism $U \to V$ is quasi-compact. Thus, for every affine open $V' \subset V$, the fiber product $V' \times_V U$ is quasi-compact, so, by (2), the induced map $V' \times_V U \to V'$ is quasi-affine. Thus, $U \to V$ is also quasi-affine by Morphisms, Lemma 12.3. This argument also gives (3) $\implies$ (4): indeed, keeping the same notation, those affine opens $V' \subset V$ that lie in one of the $V_j$ cover $V$, so one needs to argue that the quasi-compact map $V' \times_V U \to V'$ is quasi-affine. However, by (3), the composition $V' \times_V U \to V' \to V_j$ is quasi-affine and, by Schemes, Lemma 21.13, the map $V' \to V_j$ is quasi-separated. Thus, $V' \times_V U \to V'$ is quasi-affine by Morphisms, Lemma 12.8. The final implications (4) $\implies$ (2) and (4) $\implies$ (3) are evident. □

0F1V **Lemma 58.3.** The property of being an ind-quasi-affine morphism is stable under composition.

**Proof.** Let $f : X \to Y$ and $g : Y \to Z$ be ind-quasi-affine morphisms. Let $V \subset Z$ and $U \subset f^{-1}(g^{-1}(V))$ be quasi-compact opens such that $V$ is also quasi-separated. The image $f(U)$ is a quasi-compact subset of $g^{-1}(V)$, so it is contained in some quasi-compact open $W \subset g^{-1}(V)$ (a union of finitely many affines). We obtain a factorization $U \to W \to V$. The map $W \to V$ is quasi-affine by Lemma 58.2 so, in particular, $W$ is quasi-separated. Then, by Lemma 58.2 once more, $U \to W$ is quasi-affine as well. Consequently, by Morphisms, Lemma 12.3, the composition $U \to V$ is also quasi-affine, and it remains to apply Lemma 58.2 once more. □
Lemma 58.4. Any quasi-affine morphism is ind-quasi-affine. Any immersion is ind-quasi-affine.

Proof. The first assertion is immediate from the definitions. In particular, affine morphisms, such as closed immersions, are ind-quasi-affine. Thus, by Lemma 58.3 it remains to show that an open immersion is ind-quasi-affine. This, however, is immediate from the definitions. □

Lemma 58.5. If \( f : X \to Y \) and \( g : Y \to Z \) are morphisms of schemes such that \( g \circ f \) is ind-quasi-affine, then \( f \) is ind-quasi-affine.

Proof. By Lemma 58.2 we may work Zariski locally on \( Z \) and then on \( Y \), so we lose no generality by assuming that \( Z \), and then also \( Y \), is affine. Then any quasi-compact open of \( X \) is quasi-affine, so Lemma 58.2 gives the claim. □

Lemma 58.6. The property of being ind-quasi-affine is stable under base change.

Proof. Let \( f : X \to Y \) be an ind-quasi-affine morphism. For checking that every base change of \( f \) is ind-quasi-affine, by Lemma 58.2 we may work Zariski locally on \( Z \), so we assume that \( Y \) is affine. Furthermore, we may also assume that in the base change morphism \( Z \times_Y X \to X \) is an affine morphism, so, by Lemmas 58.3 and 58.4 the map \( X \times_Y Z \to Y \) is ind-quasi-affine. Then, by Lemma 58.5 the base change \( X \times_Y Z \to Z \) is ind-quasi-affine, as desired. □

Lemma 58.7. The property of being ind-quasi-affine is fpqc local on the base.

Proof. The stability of ind-quasi-affineness under base change supplied by Lemma 58.6 gives one direction. For the other, let \( f : X \to Y \) be a morphism of schemes and let \( \{ g_i : Y_i \to Y \} \) be an fpqc covering such that the base change \( f_i : X_i \to Y_i \) is ind-quasi-affine for all \( i \). We need to show \( f \) is ind-quasi-affine.

By Lemma 58.2 we may work Zariski locally on \( Y \), so we assume that \( Y \) is affine. Then we use stability under base change ensured by Lemma 58.6 to refine the cover and assume that it is given by a single affine, faithfully flat morphism \( g : Y' \to Y \). For any quasi-compact open \( U \subset X \), its \( Y' \)-base change \( U \times_Y Y' \subset X \times_Y Y' \) is also quasi-compact. It remains to observe that, by Descent, Lemma 20.20 the map \( U \to Y \) is quasi-affine if and only if so is \( U \times_Y Y' \to Y' \). □

Lemma 58.8. A separated locally quasi-finite morphism of schemes is ind-quasi-affine.

Proof. Let \( f : X \to Y \) be a separated locally quasi-finite morphism of schemes. Let \( V \subset Y \) be affine and \( U \subset f^{-1}(V) \) quasi-compact open. We have to show \( U \) is quasi-affine. Since \( U \to V \) is a separated quasi-finite morphism of schemes, this follows from Zariski’s Main Theorem. See Lemma 58.2. □

59. Pushouts in the category of schemes, II

This section is a continuation of Section 14. In this section we construct pushouts of \( Y \leftarrow Z \to X \) where \( Z \to X \) is affine and \( Z \to Y \) is integral and an additional condition is satisfied. Please see the detailed discussion in [Fer03].

Situation 59.1. Here \( S \) is a scheme and \( i : Z \to X \) and \( j : Z \to Y \) are morphisms of schemes over \( S \). We assume
(1) \(i\) is a closed immersion,
(2) \(j\) is an integral morphism of schemes,
(3) for \(y \in Y\) there exists an affine open \(U \subset X\) with \(j^{-1}(\{y\}) \subset i^{-1}(U)\).

**Lemma 59.2.** In Situation 59.1 then for \(y \in Y\) there exist affine opens \(U \subset X\) and \(V \subset Y\) with \(i^{-1}(U) = j^{-1}(V)\).

**Proof.** Let \(y \in Y\). Choose an affine open \(U \subset X\) such that \(j^{-1}(\{y\}) \subset i^{-1}(U)\) (possible by assumption). Choose an affine open \(V \subset Y\) neighbourhood of \(y\) such that \(j^{-1}(V) \subset i^{-1}(U)\). This is possible because \(j : Z \to Y\) is a closed morphism (Morphisms, Lemma 42.7) and \(i^{-1}(U)\) contains the fibre over \(y\). Since \(j\) is integral, the scheme theoretic fibre \(Z_y\) is the spectrum of an algebra integral over a field. By Limits, Lemma 11.6 we can find an \(\overline{f} \in \Gamma(i^{-1}(U), \mathcal{O}_{i^{-1}(U)})\) such that \(Z_y \subset D(\overline{f}) \subset j^{-1}(V)\). Since \(i^{-1}(U) : i^{-1}(U) \to U\) is a closed immersion of affines, we can choose an \(f \in \Gamma(U, \mathcal{O}_U)\) whose restriction to \(i^{-1}(U)\) is \(\overline{f}\). After replacing \(U\) by the principal open \(D(f) \subset U\) we find affine opens \(y \in V \subset Y\) and \(U \subset X\) with \(j^{-1}(\{y\}) \subset i^{-1}(U) \subset j^{-1}(V)\).

Now we (in some sense) repeat the argument. Namely, we choose \(g \in \Gamma(V, \mathcal{O}_V)\) such that \(y \in D(g)\) and \(j^{-1}(D(g)) \subset i^{-1}(U)\) (possible by the same argument as above). Then we can pick \(f \in \Gamma(U, \mathcal{O}_U)\) whose restriction to \(i^{-1}(U)\) is the pullback of \(g\) by \(i^{-1}(U) \to V\) (again possible by the same reason as above). Then we finally have affine opens \(y \in V' = D(g) \subset V \subset Y\) and \(U' = D(f) \subset U \subset X\) with \(j^{-1}(V') = i^{-1}(V')\). \(\square\)

**Proposition 59.3.** In Situation 59.1 the pushout \(Y \amalg_Z X\) exists in the category of schemes. Picture

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow j & & \downarrow a \\
Y & \xrightarrow{b} & Y \amalg_Z X
\end{array}
\]

The diagram is a fibre square, the morphism \(a\) is integral, the morphism \(b\) is a closed immersion, and

\[\mathcal{O}_{Y \amalg_Z X} = b_* \mathcal{O}_Y \times_{c_* \mathcal{O}_Z} a_* \mathcal{O}_X\]

as sheaves of rings where \(c = a \circ i = b \circ j\).

**Proof.** As a topological space we set \(Y \amalg_Z X\) equal to the pushout of the diagram in the category of topological spaces (Topology, Section 29). This is just the pushout of the underlying sets (Topology, Lemma 29.1) endowed with the quotient topology. On \(Y \amalg_Z X\) we have the maps of sheaves of rings

\[b_* \mathcal{O}_Y \to c_* \mathcal{O}_Z \leftarrow a_* \mathcal{O}_X\]

and we can define

\[\mathcal{O}_{Y \amalg_Z X} = b_* \mathcal{O}_Y \times_{c_* \mathcal{O}_Z} a_* \mathcal{O}_X\]

as the fibre product in the category of sheaves of rings. To prove that we obtain a scheme we have to show that every point has an affine open neighbourhood. This is clear for points not in the image of \(c\) as the image of \(c\) is a closed subset whose complement is isomorphic as a ringed space to \((Y \setminus j(Z)) \amalg (X \setminus i(Z))\).
A point in the image of $c$ corresponds to a unique $y \in Y$ in the image of $j$. By Lemma 59.2 we find affine opens $U \subset X$ and $V \subset Y$ with $y \in V$ and $i^{-1}(U) = j^{-1}(V)$. Since the construction of the first paragraph is clearly compatible with restriction to compatible open subschemes, to prove that it produces a scheme we may assume $X$, $Y$, and $Z$ are affine.

If $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, and $Z = \text{Spec}(C)$ are affine, then More on Algebra, Lemma 6.2 shows that $Y \amalg Z X = \text{Spec}(B \times_C A)$ as topological spaces. To finish the proof that $Y \times Z X$ is a scheme, it suffices to show that on $\text{Spec}(B \times_C A)$ the structure sheaf is the fibre product of the pushforwards. This follows by applying More on Algebra, Lemma 5.3 to principal affine opens of $\text{Spec}(B \times_C A)$.

The discussion above shows the scheme $Y \amalg Z X$ has an affine open covering $Y \amalg Z X = \bigcup W_i$ such that $U_i = a^{-1}(W_i)$, $V_i = b^{-1}(W_i)$, and $\Omega_i = c^{-1}(W_i)$ are affine open in $X$, $Y$, and $Z$. Thus $a$ and $b$ are affine. Moreover, if $A_i$, $B_i$, $C_i$ are the rings corresponding to $U_i$, $V_i$, $\Omega_i$, then $A_i \to C_i$ is surjective and $W_i$ corresponds to $A_i \times_C B_i$ which surjects onto $B_i$. Hence $b$ is a closed immersion. The ring map $A_i \times_C B_i \to A_i$ is integral by More on Algebra, Lemma 6.3 hence $a$ is integral. The diagram is cartesian because

$$C_i \cong B_i \otimes_{B_i \times_C A_i} A_i$$

This follows as $B_i \times_C A_i \to B_i$ and $A_i \to C_i$ are surjective maps whose kernels are the same.

Finally, we can apply Lemmas 14.1 and 14.2 to conclude our construction is a pushout in the category of schemes.

\begin{lemma}
In Situation 59.1. If $X$ and $Y$ are separated, then the pushout $Y \amalg Z X$ (Proposition 59.3) is separated. Same with “separated over $S$”, “quasi-separated”, and “quasi-separated over $S$”.
\end{lemma}

\begin{proof}
The morphism $Y \amalg Z X \to Y \amalg Z X$ is surjective and universally closed. Thus we may apply Morphisms, Lemma 39.11.
\end{proof}

\begin{lemma}
In Situation 59.1 assume $S$ is a locally Noetherian scheme and $X$, $Y$, and $Z$ are locally of finite type over $S$. Then the pushout $Y \amalg Z X$ (Proposition 59.3) is locally of finite type over $S$.
\end{lemma}

\begin{proof}
Looking on affine opens we recover the result of More on Algebra, Lemma 15.1.
\end{proof}

\begin{lemma}
In Situation 59.1 suppose given a commutative diagram

$$
\begin{array}{ccc}
Y' & \xleftarrow{j'} & Z' & \xrightarrow{\quad j\quad} & X' \\
\downarrow{\quad g\quad} & & \downarrow{\quad h\quad} & & \downarrow{\quad f\quad} \\
Y & \xleftarrow{\quad h\quad} & Z & \xrightarrow{\quad f\quad} & X
\end{array}
$$

with cartesian squares and $f, g, h$ separated and locally quasi-finite. Then

1. the pushouts $Y \amalg Z X$ and $Y' \amalg Z', X'$ exist,
2. $Y' \amalg Z', X' \to Y \amalg Z X$ is separated and locally quasi-finite, and

\end{lemma}
(3) the squares

\[
\begin{array}{ccc}
Y' & \rightarrow & Y' \amalg_X X' \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y \amalg_Z X \\
\end{array}
\]

are cartesian.

**Proof.** The pushout \( Y \amalg_Z X \) exists by Proposition 0ECL. To see that the pushout \( Y' \amalg_Z Z') \) exists, we check condition (3) of Situation 59.1 for \((X',Y',Z',i',j')\). Namely, let \( y' \in Y' \) and denote \( y \in Y \) the image. Choose \( U \subset X \) affine open with \( i'(j^{-1}(y)) \subset U \). Choose a quasi-compact open \( U' \subset X' \) contained in \( f^{-1}(U) \) containing the quasi-compact subset \( i'(j^{-1}(\{y'\})) \). By Lemma 58.8 we see that \( U' \) is quasi-affine. Since \( Z'_Y \) is the spectrum of an algebra integral over a field, we can apply Limits, Lemma 11.6 and we find there exists an affine open subscheme of \( U' \) containing \( i'(j^{-1}(\{y'\})) \) as desired.

Having verified existence we check the other assertions. Affine locally we are exactly in the situation of More on Algebra, Lemma 7.7 with \( B \rightarrow D \) and \( A' \rightarrow C' \) locally quasi-finite. In particular, the morphism \( Y' \amalg_Z X' \rightarrow Y \amalg_Z X \) is locally of finite type. The squares in of the diagram are cartesian by More on Algebra, Lemma 6.4. Since being locally quasi-finite can be checked on fibres (Morphisms, Lemma 19.6) we conclude that \( Y' \amalg_Z X' \rightarrow Y \amalg_Z X \) is locally quasi-finite.

We still have to check \( Y' \amalg_Z X' \rightarrow Y \amalg_Z X \) is separated. Observe that \( Y' \amalg X' \rightarrow Y' \amalg X' \) is universally closed and surjective by Proposition 59.3. Since also the morphism \( Y' \amalg X' \rightarrow Y \amalg X \) is separated (as it factors as \( Y' \amalg X' \rightarrow Y \amalg X \) we conclude by Morphisms, Lemma 39.11.

**Lemma 59.7.** In Situation 59.1, the category of schemes flat, separated, and locally quasi-finite over the pushout \( Y \amalg_Z X \) is equivalent to the category of \((X',Y',Z',i',j',f,g,h)\) as in Lemma 59.6 with \( f,g,h \) flat. Similarly with “flat” replaced with “étale”.

**Proof.** If we start with \((X',Y',Z',i',j',f,g,h)\) as in Lemma 59.6 with \( f,g,h \) flat or étale, then \( Y' \amalg_Z X' \rightarrow Y \amalg_Z X \) is flat or étale by More on Algebra, Lemma 7.7.

For the converse, let \( W \rightarrow Y \amalg_Z X \) be a separated and locally quasi-finite morphism. Set \( X' = W \times_{Y \amalg_Z X} X \), \( Y' = W \times_{Y \amalg Z X} Y \), and \( Z' = W \times_{Y \amalg Z X} Z \) with obvious morphisms \( i',j',f,g,h \). Form the pushout \( Y' \amalg_Z X' \). We obtain a morphism

\[
Y' \amalg_Z X' \rightarrow W
\]

of schemes over \( Y \amalg_Z X \) by the universal property of the pushout. If we do not assume that \( W \rightarrow Y \amalg_Z X \) is flat, then in general this morphism won’t be an isomorphism. (In fact, More on Algebra, Lemma 6.5 shows the displayed arrow is a closed immersion but not an isomorphism in general.) However, if \( W \rightarrow Y \times_Z X \) is flat, then it is an isomorphism by More on Algebra, Lemma 7.7.

Next, we discuss existence in the case where both morphisms are closed immersions.

---

\(^{12}\)To be precise \( X,Y,Z,Y \amalg_Z X,X',Y',Z',Y' \amalg_Z X' \) correspond to \( A',B,A,B',C',D,C,D' \).
Lemma 59.8. Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Then the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow & & \downarrow a \\
Y & \xrightarrow{b} & Y \amalg_Z X
\end{array}
\]

The diagram is a fibre square, the morphisms $a$ and $b$ are closed immersions, and there is a short exact sequence

\[0 \to \mathcal{O}_{Y \amalg_Z X} \to a_* \mathcal{O}_X \oplus b_* \mathcal{O}_Y \to c_* \mathcal{O}_Z \to 0\]

where $c = a \circ i = b \circ j$.

Proof. This is a special case of Proposition 59.3. Observe that hypothesis (3) in Situation 59.1 is immediate because the fibres of $j$ are singletons. Finally, reverse the roles of the arrows to conclude that both $a$ and $b$ are closed immersions. □

Lemma 59.9. Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes and let $\varphi : X' \times_{X,j} Z \to Y' \times_{Y,j} Z$ be an isomorphism of schemes over $Z$. Consider the morphism

\[h : X' \amalg_{X' \times_{X,i} Z} Y' \to X \amalg_Z Y\]

Then we have

1. $h$ is locally of finite type if and only if $f$ and $g$ are locally of finite type,
2. $h$ is flat if and only if $f$ and $g$ are flat,
3. $h$ is flat and locally of finite presentation if and only if $f$ and $g$ are flat and locally of finite presentation,
4. $h$ is smooth if and only if $f$ and $g$ are smooth,
5. $h$ is étale if and only if $f$ and $g$ are étale, and
6. add more here as needed.

Proof. We know that the pushouts exist by Lemma 59.8. In particular we get the morphism $h$. Hence we may replace all schemes in sight by affine schemes. In this case the assertions of the lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7. □

60. Relative morphisms

In this section we prove a representability result which we will use in Fundamental Groups, Section 5 to prove a result on the category of finite étale coverings of a scheme. The material in this section is discussed in the correct generality in Criteria for Representability, Section 10.

Let $S$ be a scheme. Let $Z$ and $X$ be schemes over $S$. Given a scheme $T$ over $S$ we can consider morphisms $b : T \times_S Z \to T \times_S X$ over $S$. Picture

\[
\begin{array}{ccc}
T \times_S Z & \xrightarrow{b} & T \times_S X \\
\downarrow & & \downarrow \\
& & \\
T & \xrightarrow{b} & T \times_S X \\
\downarrow & & \downarrow \\
& & \\
& & S
\end{array}
\]
Of course, we can also think of $b$ as a morphism $b : T \times_S Z \to X$ such that

$$
\begin{align*}
T \times_S Z & \longrightarrow Z \\
\downarrow & \\
T & \longrightarrow S
\end{align*}
$$

commutes. In this situation we can define a functor

$$
\text{Mor}_S(Z, X) : (\text{Sch}/S)^{\text{opp}} \to \text{Sets}, \quad T \mapsto \{ b \text{ as above} \}
$$

Here is a basic representability result.

**Lemma 60.1.** Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $\text{Mor}_S(Z, X)$ is representable by an affine scheme over $S$.

**Proof.** Write $S = \text{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over $R$. Choose a presentation

$$
\Gamma(X, \mathcal{O}_X) = R[[x_i]]/(\{f_k\}_{k \in K}).
$$

We will denote $\pi_i$ the image of $x_i$ in this quotient. Write

$$
P = R[[a_{ij}]]_{i \in I, 1 \leq j \leq m}.
$$

Consider the $R$-algebra map

$$
\Psi : R[[x_i]]_{i \in I} \to P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \mapsto \sum_j a_{ij} \otimes e_j.
$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements $c_{kj}$, $k \in K$, $1 \leq j \leq m$. We claim that $W = \text{Spec}(P/J)$ represents the functor $\text{Mor}_S(Z, X)$.

First, note that by construction $P/J$ is an $R$-algebra, hence a morphism $W \to S$. Second, by construction the map $\Psi$ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an $P/J$-algebra homomorphism

$$
P/J \otimes_R \Gamma(X, \mathcal{O}_X) \to P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)
$$

which determines a morphism $b_{\text{univ}} : W \times_S Z \to W \times_S X$. By the Yoneda lemma $b_{\text{univ}}$ determines a transformation of functors $W \to \text{Mor}_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to \text{Mor}_S(Z, X)(T)$ over any affine scheme $T$.

Suppose $T = \text{Spec}(R')$ is an affine scheme over $S$ and $b \in \text{Mor}_S(Z, X)(T)$. The structure morphism $T \to S$ defines an $R$-algebra structure on $R'$ and $b$ defines an $R'$-algebra map

$$
b^t : R' \otimes_R \Gamma(X, \mathcal{O}_X) \to R' \otimes_R \Gamma(Z, \mathcal{O}_Z).
$$

In particular we can write $b^t(1 \otimes \pi_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an $R$-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient $P/J$ by the construction of the ideal $J$ to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that $b$ is the pullback of $b_{\text{univ}}$. Some details omitted. □

**Lemma 60.2.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $\text{Mor}_S(Z, X)$ is representable by a scheme affine over $S$. 
Proof. Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z \times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset \text{Mor}_S(Z, X)$ be the subfunctor which assigns to $T/S$ the empty set if $T \to S$ does not factor through $U_i$ and $\text{Mor}_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma \ref{lemma-commutative-diamond} which proves the lemma. Condition (2)(a) follows from Lemma \ref{lemma-characterizing-pseudo-coherent-complexes} and the other two follow from straightforward arguments. □

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: $X$ is quasi-affine, $f$ is quasi-affine, $f$ is quasi-projective, $f$ is locally projective, there exists an ample invertible sheaf on $X$, there exists an $f$-ample invertible sheaf on $X$, or there exists an $f$-very ample invertible sheaf on $X$.

**Lemma 60.3.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume

1. $Z \to S$ is finite locally free, and
2. for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.

Then $\text{Mor}_S(Z, X)$ is representable by a scheme.

**Proof.** Consider the set $I$ of pairs $(U, V)$ where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through $V$. For $i \in I$ denote $(U_i, V_i)$ the corresponding pair. Set $F_i = \text{Mor}_{V_i}(Z_{V_i}, U_i)$. It is immediate that $F_i$ is a subfunctor of $\text{Mor}_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma \ref{lemma-commutative-diamond} which proves the lemma.

Condition (2)(a) follows from Lemma \ref{lemma-projective-finite}

To check condition (2)(b) consider $T/S$ and $b \in \text{Mor}_S(Z, X)$. Thinking of $b$ as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i,b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i,b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider $T/S$ and $b \in \text{Mor}_S(Z, X)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i,b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some $i$ where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since $p$ is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image $s$ of $t$ in $S$. Thus our condition on $X \to S$ exactly shows a suitable pair exists. □

**Lemma 60.4.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume $Z \to S$ is finite locally free and $X \to S$ is separated and locally quasi-finite. Then $\text{Mor}_S(Z, X)$ is representable by a scheme.

**Proof.** This follows from Lemmas \ref{lemma-characterizing-pseudo-coherent-complexes} and \ref{lemma-projective-finite} □

61. Characterizing pseudo-coherent complexes, III

In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. This is a continuation of Derived Categories of Schemes, Section...
A basic tool will be to reduce to the case of projective space using a derived version of Chow’s lemma, see Lemma 61.2.

**Lemma 61.1.** Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
Z' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X' & \longrightarrow & S'
\end{array}
\]

Let \( S \to S' \) be a morphism. Denote by \( X \) and \( Y \) the base changes of \( X' \) and \( Y' \) to \( S \). Assume \( Y' \to S' \) and \( Z' \to X' \) are flat. Then \( X \times_S Y \) and \( Z' \) are Tor independent over \( X' \times_{S'} Y' \).

**Proof.** The question is local, hence we may assume all schemes are affine (some details omitted). Observe that

\[
\begin{array}{ccc}
X \times_S Y & \longrightarrow & X' \times_{S'} Y' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X'
\end{array}
\]

is cartesian with flat vertical arrows. Write \( X = \text{Spec}(A) \), \( X' = \text{Spec}(A') \), \( X \times_S Y = \text{Spec}(A \otimes_A B) \). Write \( Z' = \text{Spec}(C') \). We have to show

\[ \text{Tor}_p^B(A \otimes_{A'} B', C') = 0, \quad \text{for } p > 0 \]

Since \( A' \to B' \) is flat we have \( A \otimes_{A'} B' = A \otimes_{A'} B' \). Hence

\[ (A \otimes_{A'} B') \otimes_{B'} C' = (A \otimes_{A'} B') \otimes_{B'} (A \otimes_{A'} C') = A \otimes_{A'} C' \]

The second equality by More on Algebra, Lemma 58.4. The last equality because \( A' \to C' \) is flat. This proves the lemma. \( \square \)

**Lemma 61.2** (Derived Chow’s lemma). Let \( A \) be a ring. Let \( X \) be a separated scheme of finite presentation over \( A \). Let \( x \in X \). Then there exist an open neighborhood \( U \subset X \) of \( x \), an \( n \geq 0 \), an open \( V \subset P^n_A \), a closed subscheme \( Z \subset X \times_A P^n_A \), a point \( z \in Z \), and an object \( E \) in \( D(\mathcal{O}_{X \times_A P^n_A}) \) such that

1. \( Z \to X \times_A P^n_A \) is of finite presentation,
2. \( b : Z \to X \) is an isomorphism over \( U \) and \( b(z) = x \),
3. \( c : Z \to P^n_A \) is a closed immersion over \( V \),
4. \( b^{-1}(U) = c^{-1}(V) \), in particular \( c(z) \in V \),
5. \( E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V} \),
6. \( E \) is pseudo-coherent and supported on \( Z \).

**Proof.** We can find a finite type \( \mathbf{Z} \)-subalgebra \( A' \subset A \) and a scheme \( X' \) separated and of finite presentation over \( A' \) whose base change to \( A \) is \( X \). See Limits, Lemmas 10.1 and 8.6. Let \( x' \in X' \) be the image of \( x \). If we can prove the lemma for \( x' \in X'/A' \), then the lemma follows for \( x \in X/A \). Namely, if \( U', n', V', Z', z', E' \) provide the solution for \( x' \in X'/A' \), then we can let \( U \subset X \) be the inverse image of \( U' \), let \( n = n' \), let \( V \subset P^n_A \) be the inverse image of \( V' \), let \( Z \subset X \times P^n \) be the scheme theoretic inverse image of \( Z' \), let \( z \in Z \) be the unique point mapping to \( x \), and let \( E \) be the derived pullback of \( E' \). Observe that \( E \) is pseudo-coherent by Cohomology,
Lemma 61.3. It only remains to check (5). To see this set \( W = b^{-1}(U) = c^{-1}(V) \)
and \( W' = (b')^{-1}(U) = (c')^{-1}(V') \) and consider the cartesian square:

\[
\begin{array}{ccc}
W & \longrightarrow & W' \\
(b,c) \downarrow & & \downarrow (b',c') \\
X \times_A V & \longrightarrow & X' \times_{A'} V'
\end{array}
\]

By Lemma 61.1 the schemes \( X \times_A V \) and \( W' \) are Tor independent over \( X' \times_{A'} V' \).
Hence the derived pullback of \( (b', c')_*\mathcal{O}_{W'} \) to \( X \times_A V \) is \( (b, c)_*\mathcal{O}_W \) by Derived Categories of Schemes, Lemma 21.3. This also uses that \( R(b', c')_*\mathcal{O}_{Z'} = (b', c')_*\mathcal{O}_{Z} \)
because \( (b', c') \) is a closed immersion and similarly for \( (b, c)_*\mathcal{O}_Z \). Since \( E'(U' \times_{A'} V') = (b', c')_*\mathcal{O}_{W'} \) we obtain \( E|_{U \times_A V} = (b, c)_*\mathcal{O}_W \) and (5) holds. This reduces us to the
situation described in the next paragraph.

Assume \( A \) is of finite type over \( \mathbf{Z} \). Choose an affine open neighbourhood \( U \subset X \)
of \( x \). Then \( U \) is of finite type over \( A \). Choose a closed immersion \( U \to \mathbf{A}^n_A \) and denote \( j : U \to \mathbf{P}^n_A \) the immersion we get by composing with the open immersion \( \mathbf{A}^n_A \to \mathbf{P}^n_A \).
Let \( Z \) be the scheme theoretic closure of \( (\text{id}_U, j) : U \to X \times_A \mathbf{P}^n_A \).

Since the projection \( X \times \mathbf{P}^n_A \to X \) is separated, we conclude from Morphisms, Lemma 6.8 that \( b : Z \to X \) is an isomorphism over \( U \). Let \( z \in Z \) be the unique point lying over \( x \).

Let \( Y \subset \mathbf{P}^n_A \) be the scheme theoretic closure of \( j \). Then it is clear that \( Z \subset X \times_A Y \)
is the scheme theoretic closure of \( (\text{id}_U, j) : U \to X \times_A Y \). As \( X \) is separated, the
morphism \( X \times_A Y \to Y \) is separated as well. Hence we see that \( Z \to Y \) is an
isomorphism over the open subscheme \( j(U) \subset Y \) by the same lemma we used above. Choose \( V \subset \mathbf{P}^n_A \) open with \( V \cap Y = j(U) \). Then we see that (3) and (4)
hold.

Because \( A \) is Noetherian we see that \( X \) and \( X \times_A \mathbf{P}^n_A \) are Noetherian schemes.
Hence we can take \( E = (b, c)_*\mathcal{O}_Z \) in this case, see Derived Categories of Schemes, Lemma 9.3. This finishes the proof. \( \square \)
by property (5). Since \( i \) is a closed immersion we have \( i_*\mathcal{O}_W = R i_*\mathcal{O}_W \). Using Derived Categories of Schemes, Lemma \([21.1]\) we can rewrite this as
\[
Rq_* R i_* L i^* L p^* K|_{X \times_A V} = R(q' \circ i)_* L b^* K|_W = R(U \to V)_* K|_U
\]
which is what we want. \( \square \)

0CSL \textbf{Lemma 61.4.} Let \( A \) be a ring. Let \( X \) be a scheme separated and of finite presentation over \( A \). Let \( K \in D_{Qcoh}(\mathcal{O}_X) \). If \( R \Gamma(X, E \otimes^L K) \) is pseudo-coherent in \( D(A) \) for every pseudo-coherent \( E \) in \( D(\mathcal{O}_X) \), then \( K \) is pseudo-coherent relative to \( A \).

\textbf{Proof.} Assume \( K \in D_{Qcoh}(\mathcal{O}_X) \) and \( R \Gamma(X, E \otimes^L K) \) is pseudo-coherent in \( D(A) \) for every pseudo-coherent \( E \) in \( D(\mathcal{O}_X) \). Let \( x \in X \). We will show that \( K \) is pseudo-coherent relative to \( A \) in a neighbourhood of \( x \) and this will prove the lemma.

Choose \( U, n, V, z, E \) as in Lemma \([61.2]\). Denote \( p : X \times \mathbb{P}^n \to X \) and \( q : X \times \mathbb{P}^n \to \mathbb{P}^n_A \) the projections. Then for any \( i \in \mathbb{Z} \) we have
\[
R \Gamma(\mathbb{P}^n_A, Rq_* (L^p K \otimes^L E) \otimes^L \mathcal{O}_{\mathbb{P}^n_A}(i))
\]
\[
= R \Gamma(X \times \mathbb{P}^n, L^p K \otimes^L E \otimes^L L q^* \mathcal{O}_{\mathbb{P}^n_A}(i))
\]
\[
= R \Gamma(X, K \otimes^L Rq_* (E \otimes^L L q^* \mathcal{O}_{\mathbb{P}^n_A}(i)))
\]
by Derived Categories of Schemes, Lemma \([21.1]\). By Derived Categories of Schemes, Lemma \([27.5]\) the complex \( Rq_* (E \otimes^L L q^* \mathcal{O}_{\mathbb{P}^n_A}(i)) \) is pseudo-coherent on \( X \). Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of \( D(A) \). By Derived Categories of Schemes, Lemma \([31.2]\) we conclude that \( Rq_* (L^p K \otimes^L E) \) is pseudo-coherent on \( \mathbb{P}^n_A \). By Lemma \([61.3]\) we have
\[
Rq_* (L^p K \otimes^L E)|_{X \times_A V} = R(U \to V)_* K|_U
\]
Since \( U \to V \) is a closed immersion into an open subscheme of \( \mathbb{P}^n_A \) this means \( K|_U \) is pseudo-coherent relative to \( A \) by Lemma \([51.18]\). \( \square \)

62. Descent finiteness properties of complexes

0CSM This section is the continuation of Derived Categories of Schemes, Section \([11]\).

0CSN \textbf{Lemma 62.1.} Let \( X \to S \) be locally of finite type. Let \( \{ f_i : X_i \to X \} \) be an fpqc covering of schemes. Let \( E \in D_{Qcoh}(\mathcal{O}_X) \). Let \( m \in \mathbb{Z} \). Then \( E \) is \( m \)-pseudo-coherent relative to \( S \) if and only if each \( L f_i^* E \) is \( m \)-pseudo-coherent relative to \( S \).

\textbf{Proof.} Assume \( E \) is \( m \)-pseudo-coherent relative to \( S \). The morphisms \( f_i \) are pseudo-coherent by Lemma \([52.6]\). Hence \( L f_i^* E \) is \( m \)-pseudo-coherent relative to \( S \) by Lemma \([51.16]\).

Conversely, assume that \( L f_i^* E \) is \( m \)-pseudo-coherent relative to \( S \) for each \( i \). Pick \( S = \bigcup U_j, W_j \to U_j, T_{i,j} \to W_{i,j}, \) and morphisms \( T_{i,j} \to X_{i,j} \) over \( S \) as in Lemma \([43.2]\). Since the morphism \( T_{i,j} \to X_{i} \) is flat and of finite presentation, we see that \( \alpha_{i,j} \) is pseudo-coherent by Lemma \([52.7]\). Hence
\[
\alpha_{i,j}^* L f_{i,j}^* E = L(T_{i,j} \to S)^* E
\]
is \( m \)-pseudo-coherent relative to \( S \) by Lemma \([51.16]\). Now we want to descend this property through the coverings \( \{ T_{i,j} \to W_{i,j}, W_j = \bigcup W_{i,j}, \{ W_j \to U_j \}, \) and \( S = \bigcup U_j \). Since for Zariski coverings the result is true (by the definition of
Let $\pi : X \rightarrow S$ be a morphism of schemes. Assume $T \rightarrow S$ is flat and locally of finite presentation and $X \rightarrow T$ locally of finite type. Let $E \in \mathcal{D}(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if $E$ is $m$-pseudo-coherent relative to $T$.

Proof. Locally on $X$ we can choose a closed immersion $i : X \rightarrow \mathbb{A}^n_\mathbb{Z}$. Then $\mathbb{A}^n_\mathbb{Z} \rightarrow S$ is flat and locally of finite presentation. Thus we may apply Lemma 51.17 to see the equivalence holds.

Lemma 62.3. Let $f : X \rightarrow S$ be locally of finite type. Let $\{S_i \rightarrow S\}$ be an fpqc covering of schemes. Denote $f_i : X_i \rightarrow S_i$ the base change of $f$ and $g_i : X_i \rightarrow X$ the projection. Let $E \in \mathcal{D}_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if each $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$.

Proof. This follows formally from Lemmas 62.1 and 62.2. Namely, if $E$ is $m$-pseudo-coherent relative to $S$, then $Lg_i^*E$ is $m$-pseudo-coherent relative to $S'$ (by the first lemma), hence $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$ (by the second). Conversely, if $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$, then $Lg_i^*E$ is $m$-pseudo-coherent relative to $S$ (by the second lemma), hence $E$ is $m$-pseudo-coherent relative to $S$ (by the first lemma).

63. Relatively perfect objects

This section is a continuation of the discussion in Derived Categories of Schemes, Section 52.

Lemma 63.1. Let $i : X \rightarrow X'$ be a finite order thickening of schemes. Let $K' \in \mathcal{D}(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then $K'$ is pseudo-coherent.

Proof. We first prove $K'$ has quasi-coherent cohomology sheaves. To do this, we may reduce to the case of a first order thickening, see Section 2. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out $X$. Tensoring the short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X'} \rightarrow i_*\mathcal{O}_X \rightarrow 0$$

with $K'$ we obtain a distinguished triangle

$$K' \otimes_{\mathcal{O}_{X'}} \mathcal{I} \rightarrow K' \rightarrow K' \otimes_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \rightarrow (K' \otimes_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = R_i*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes_{\mathcal{O}_X} \mathcal{I}) \rightarrow K' \rightarrow i_*K \rightarrow i_*(K \otimes_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology, Lemma 49.4 to identify the terms. Since $K$ is in $\mathcal{D}_{QCoh}(\mathcal{O}_X)$ we conclude that $K'$ is in $\mathcal{D}_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Schemes, Lemmas 9.1, 3.9 and 4.1.
Assume $K'$ is in $D_{Qcoh}(O_{X'})$. The question is local on $X'$ hence we may assume $X'$ is affine. Say $X' = \text{Spec}(A')$ and $X = \text{Spec}(A)$ with $A = A'/I$ and $I$ nilpotent. Then $K'$ comes from an object $M' \in D(A')$, see Derived Categories of Schemes, Lemma 3.5. Thus $M = M' \otimes_A^L A$ is a pseudo-coherent object of $D(A)$ by Derived Categories of Schemes, Lemma 9.2 and our assumption on $K$. Hence we can represent $M$ by a bounded above complex of finite free $A$-modules $E^*$, see More on Algebra, Lemma 62.5. By More on Algebra, Lemma 71.3 we conclude that $M'$ is pseudo-coherent as desired. \hfill \square

\begin{lemma}
Consider a cartesian diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow f & & \downarrow f' \\
Y & \longrightarrow & Y'
\end{array}
\]

of schemes. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(O_{X'})$. If $E = L\varphi'(E')$ is $Y$-perfect, then $E'$ is $Y'$-perfect.

\begin{proof}
Recall that being $Y$-perfect for $E$ means $E$ is pseudo-coherent and locally has finite tor dimension as a complex of $\mathcal{O}_Y$-modules (Derived Categories of Schemes, Definition 32.1). By Lemma 63.1 we find that $E'$ is pseudo-coherent. In particular, $E'$ is in $D_{Qcoh}(O_{X'})$, see Derived Categories of Schemes, Lemma 9.1. To prove that $E'$ locally has finite tor dimension we may work locally on $X'$. Hence we may assume $X', S', X, S$ are affine, say given by rings $A', R', A, R$. Then we reduce to the commutative algebra version by Derived Categories of Schemes, Lemma 32.3. The commutative algebra version in More on Algebra, Lemma 77.8. \hfill \square
\end{proof}

\begin{lemma}
Let $(R, I)$ be a pair consisting of a ring and an ideal $I$ contained in the Jacobson radical. Set $S = \text{Spec}(R)$ and $S_0 = \text{Spec}(R/I)$. Let $f : X \to S$ be proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(O_X)$ be pseudo-coherent. If the derived restriction $E_0$ of $E$ to $X_0$ is $S_0$-perfect, then $E$ is $S$-perfect.

\begin{proof}
Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. For each $i$ we can choose a closed immersion $U_i \to \mathbb{A}^d_S$. Set $U_{i,0} = S_0 \times_S U_i$. For each $i$ the complex $E_0|_{U_{i,0}}$ has tor amplitude in $[a_i, b_i]$ for some $a_i, b_i \in \mathbb{Z}$. Let $x \in X$ be a point. We will show that the tor amplitude of $E_x$ over $R$ is in $[a_i - d_i, b_i]$ for some $i$. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology, Lemma 44.5.

Since $f$ is proper $f(\{x\})$ is a closed subset of $S$. Since $I$ is contained in the Jacobson radical, we see that $f(\{x\})$ meeting the closed subset $S_0 \subset S$. Hence there is a specialization $x \rightsquigarrow x_0$ with $x_0 \in X_0$. Pick an $i$ with $x_0 \in U_i$, so $x_0 \in U_{i,0}$. We will fix $i$ for the rest of the proof. Write $U_i = \text{Spec}(A)$. Then $A$ is a flat, finitely presented $R$-algebra which is a quotient of a polynomial $R$-algebra in $d_i$-variables. The restriction $E|_{U_i}$ corresponds (by Derived Categories of Schemes, Lemma 3.5) to a pseudo-coherent object $K$ of $D(A)$. Observe that $E_0$ corresponds to $K \otimes_A^L A/I$. Let $q \subset q_0 \subset A$ be the prime ideals corresponding to $x \rightsquigarrow x_0$. Then $E_x = K_q$ and $K_q$ is a localization of $K_{q_0}$. Hence it suffices to show that $K_{q_0}$ has
tor amplitude in \( [a_i - d_i, b_i] \) as a complex of \( R \)-modules. Let \( I \subset p_0 \subset R \) be the prime ideal corresponding to \( f(x_0) \). Then we have
\[
K \otimes^L_R \kappa(p_0) = (K \otimes^L_R R/I) \otimes^L_{R/I} \kappa(p_0) = (K \otimes^L_A A/IA) \otimes^L_{A/IA} \kappa(p_0)
\]
the second equality because \( R \to A \) is flat. By our choice of \( a_i, b_i \) this complex has cohomology only in degrees in the interval \( [a_i, b_i] \). Thus we may finally apply More on Algebra, Lemma 27.9 to \( R \to A, q_0, p_0 \) and \( K \) to conclude. \( \square \)

64. Contracting rational curves

Lemma 64.1. Let \( f : X \to Y \) be a proper morphism of schemes. Let \( y \in Y \) be a point with \( \dim(X_y) \leq 1 \). If
(1) \( R^1f_*\mathcal{O}_X = 0 \), or more generally
(2) there is a morphism \( g : Y' \to Y \) such that \( y \) is in the image of \( g \) and such that \( R^0g_*\mathcal{O}_{X'} = 0 \) where \( f' : X' \to Y' \) is the base change of \( f \) by \( g \).

Then \( H^1(X_y, \mathcal{O}_{X_y}) = 0 \).

Proof. To prove the lemma we may replace \( Y \) by an open neighbourhood of \( y \).
Thus we may assume \( Y \) is affine and that all fibres of \( f \) have dimension \( \leq 1 \), see Morphisms, Lemma 27.4. In this case \( R^1f_*\mathcal{O}_X \) is a quasi-coherent \( \mathcal{O}_Y \)-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 17.3 and 17.2. The lemma follows immediately. \( \square \)

Lemma 64.2. Let \( f : X \to Y \) be a proper morphism of schemes. Let \( y \in Y \) be a point with \( \dim(X_y) \leq 1 \) and \( H^1(X_y, \mathcal{O}_{X_y}) = 0 \). Then there is an open neighbourhood \( V \subset Y \) of \( y \) such that \( R^1f_*\mathcal{O}_X|_V = 0 \) and the same is true after base change by any \( Y' \to V \).

Proof. To prove the lemma we may replace \( Y \) by an open neighbourhood of \( y \).
Thus we may assume \( Y \) is affine and that all fibres of \( f \) have dimension \( \leq 1 \), see Morphisms, Lemma 27.4. In this case \( R^1f_*\mathcal{O}_X \) is a quasi-coherent \( \mathcal{O}_Y \)-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 17.3 and 17.2. Say \( Y = \text{Spec}(A) \), \( y \) corresponds to the prime \( p \subset A \), and \( R^1f_*\mathcal{O}_X \) corresponds to the finite \( A \)-module \( M \). Then \( H^1(X_y, \mathcal{O}_{X_y}) = 0 \) means that \( pM_p = M_p \) by the statement on base change. By Nakayama’s lemma we conclude \( M_p = 0 \). Since \( M \) is finite, we find an \( f \in A, f \not\in p \) such that \( M_f = 0 \). Thus taking \( V \) the principal open \( D(f) \) we obtain the desired result. \( \square \)

Lemma 64.3. Let \( f : X \to Y \) be a proper morphism of schemes such that \( \dim(X_y) \leq 1 \) and \( H^1(X_y, \mathcal{O}_{X_y}) = 0 \) for all \( y \in Y \). Let \( \mathcal{F} \) be quasi-coherent on \( X \). Then
(1) \( R^pf_*\mathcal{F} = 0 \) for \( p > 1 \), and
(2) \( R^1f_*\mathcal{F} = 0 \) if there is a surjection \( f^*\mathcal{G} \to \mathcal{F} \) with \( \mathcal{G} \) quasi-coherent on \( Y \).

If \( Y \) is affine, then we also have
(3) \( H^p(X, \mathcal{F}) = 0 \) for \( p \not\in \{0,1\} \), and
(4) \( H^1(X, \mathcal{F}) = 0 \) if \( \mathcal{F} \) is globally generated.
Proof. The vanishing in (1) is Limits, Lemma \[17.2\]. To prove (2) we may work locally on \( Y \) and assume \( Y \) is affine. Then \( R^1 f_* \mathcal{F} \) is the quasi-coherent module on \( Y \) associated to the module \( H^1(X, \mathcal{F}) \). Here we use that \( Y \) is affine, quasi-coherence of higher direct images (Cohomology of Schemes, Lemma \[4.5\]), and Cohomology of Schemes, Lemma \[4.6\]. Since \( Y \) is affine, the quasi-coherent module \( G \) is globally generated, and hence so is \( f^* G \) and \( \mathcal{F} \). In this way we see that (4) implies (2).

Part (3) follows from (1) as well as the remarks on quasi-coherence of direct images just made. Thus all that remains is the prove (4). If \( \mathcal{F} \) is globally generated, then there is a surjection \( \bigoplus_{i \in I} O_X \to \mathcal{F} \). By part (1) and the long exact sequence of cohomology this induces a surjection on \( H^1(X, O_X) \). Since \( H^1(X, O_X) = 0 \) because \( R^1 f_* O_X = 0 \) by Lemma \[64.2\], and since \( H^1(X, -) \) commutes with direct sums (Cohomology, Lemma \[19.1\]) we conclude.

\[ \square \]

Lemma 64.4. Let \( f : X \to Y \) be a proper morphism of schemes. Assume

1. for all \( y \in Y \) we have \( \dim(X_y) \leq 1 \) and \( H^1(X_y, O_{X_y}) = 0 \), and
2. \( O_Y \to f_* O_X \) is surjective.

Then \( O_Y \to f'_* O_X \) is surjective for any base change \( f' : X' \to Y' \) of \( f \).

Proof. We may assume \( Y \) and \( Y' \) affine. Then we can choose a closed immersion \( Y' \to Y''' \) with \( Y'' \to Y \) a flat morphism of affines. By flat base change (Cohomology of Schemes, Lemma \[5.2\]) we see that the result holds for \( X'' \to Y'' \). Thus we may assume \( Y' \) is a closed subscheme of \( Y \). Let \( I \subset O_Y \) be the ideal cutting out \( Y' \).

Then there is a short exact sequence

\[
0 \to IO_X \to O_X \to O_{X'} \to 0
\]

where we view \( O_{X'} \) as a quasi-coherent module on \( X \). By Lemma \[64.3\] we have \( H^1(X, IO_X) = 0 \). It follows that

\[
H^0(Y, O_Y) \to H^0(Y, f_* O_X) = H^0(X, O_X) \to H^0(X, O_{X'})
\]

is surjective as desired. The first arrow is surjective as \( Y \) is affine and since we assumed \( O_Y \to f_* O_X \) is surjective and the second by the long exact sequence of cohomology associated to the short exact sequence above and the vanishing just proved.

\[ \square \]

Lemma 64.5. Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & 
\end{array}
\]

of morphisms of schemes. Let \( s \in S \) be a point. Assume

1. \( X \to S \) is locally of finite presentation and flat at points of \( X_s \),
2. \( f \) is proper,
3. the fibres of \( f_s : X_s \to Y_s \) have dimension \( \leq 1 \) and \( R^1 f_{s*} O_{X_s} = 0 \),
4. \( O_{Y_s} \to f_{s*} O_{X_s} \) is surjective.

Then there is an open \( Y_s \subset V \subset Y \) such that (a) \( f^{-1}(V) \) is flat over \( S \), (b) \( \dim(X_y) \leq 1 \) for \( y \in V \), (c) \( R^1 f_* O_X|_V = 0 \), (d) \( O_V \to f_* O_X|_V \) is surjective, and (b), (c), and (d) remain true after base change by any \( Y' \to V \).
Proof. Let \( y \in Y \) be a point over \( s \). It suffices to find an open neighbourhood of \( y \) with the desired properties. As a first step, we replace \( Y \) by the open \( V \) found in Lemma 64.2 so that \( R^1f_*\mathcal{O}_X \) is zero universally (the hypothesis of the lemma holds by Lemma 64.1). We also shrink \( Y \) so that all fibres of \( f \) have dimension \( \leq 1 \) (use Morphisms, Lemma 27.4 and properness of \( f \)). Thus we may assume we have (b) and (c) with \( V = Y \) and after any base change \( Y' \to Y \). Thus by Lemma 64.4 it now suffices to show (d) over \( Y \). We may still shrink \( Y \) further; for example, we may and do assume \( Y \) and \( S \) are affine.

By Theorem 15.1 there is an open subset \( U \subset X \) where \( X \to S \) is flat which contains \( X_s \) by hypothesis. Then \( f(X \setminus U) \) is a closed subset not containing \( y \). Thus after shrinking \( Y \) we may assume \( X \) is flat over \( S \).

Say \( S = \text{Spec}(R) \). Choose a closed immersion \( Y \to Y' \) where \( Y' \) is the spectrum of a polynomial ring \( R[x_1, \ldots, x_n] \) on a set \( E \). Denote \( f' : X \to Y' \) the composition of \( f \) with \( Y \to Y' \). Then the hypotheses (1) – (4) as well as (b) and (c) hold for \( f' \) and \( s \). If we show \( \mathcal{O}_Y \to f'_*\mathcal{O}_X \) is surjective in an open neighbourhood of \( y \), then the same is true for \( \mathcal{O}_Y \to f_*\mathcal{O}_X \). Thus we may assume \( Y \) is the spectrum of \( R[x; e \in E] \).

At this point \( X \) and \( Y \) are flat over \( S \). Then \( Y_s \) and \( X \) are tor independent over \( Y \). We urge the reader to find their own proof, but it also follows from Lemma 61.1 applied to the square with corners \( X, Y, S, S \) and its base change by \( s \to S \). Hence

\[
Rf_*\mathcal{O}_X = L(Y_s \to Y)^*Rf_*\mathcal{O}_X
\]

by Derived Categories of Schemes, Lemma 21.3. Because of the vanishing already established this implies \( f_*\mathcal{O}_X = (Y_s \to Y)^*f_*\mathcal{O}_X \). We conclude that \( \mathcal{O}_Y \to f_*\mathcal{O}_X \) is a map of quasi-coherent \( \mathcal{O}_Y \)-modules whose pullback to \( Y_s \) is surjective. We claim \( f_*\mathcal{O}_X \) is a finite type \( \mathcal{O}_Y \)-module. If true, then the cokernel \( \mathcal{F} \) of \( \mathcal{O}_Y \to f_*\mathcal{O}_X \) is a finite type quasi-coherent \( \mathcal{O}_Y \)-module such that \( \mathcal{F}_y \otimes \kappa(y) = 0 \). By Nakayama’s lemma (Algebra, Lemma 19.1) we have \( \mathcal{F}_y = 0 \). Thus \( \mathcal{F} \) is zero in an open neighbourhood of \( y \) (Modules, Lemma 9.9) and the proof is complete.

Proof of the claim. For a finite subset \( E' \subset E \) set \( Y' = \text{Spec}(R[x_1; e \in E']) \). For large enough \( E' \) the morphism \( f' : X \to Y \to Y' \) is proper, see Limits, Lemma 13.3. We fix \( E' \) and \( Y' \) in the following. Write \( R = \text{colim} \, R_i \) as the colimit of its finite type \( \mathbb{Z} \)-subalgebras. Set \( S_i = \text{Spec}(R_i) \) and \( Y'_i = \text{Spec}(R_i[x_1; e \in E']) \). For \( i \) large enough we can find a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X_i & \xrightarrow{f'_i} & Y'_i \\
\end{array}
\]

with cartesian squares such that \( X_i \) is flat over \( S_i \) and \( X_i \to Y'_i \) is proper. See Limits, Lemmas 10.1 8.7 and 13.1. The same argument as above shows \( Y' \) and \( X_i \) are tor independent over \( Y'_i \) and hence

\[
R\Gamma(X, \mathcal{O}_X) = R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes_{R_i, [x_1; e \in E']}^L R[x_1; e \in E']
\]

by the same reference as above. By Cohomology of Schemes, Lemma 19.2 the complex \( R\Gamma(X_i, \mathcal{O}_{X_i}) \) is pseudo-coherent in the derived category of the Noetherian ring \( R_i[x_1; e \in E'] \) (see More on Algebra, Lemma 62.18). Hence \( R\Gamma(X, \mathcal{O}_X) \) is
Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & & Y
\end{array}
\]

of morphisms of schemes. Assume \(X \to S\) is flat, \(f\) is proper, \(\dim(X_y) \leq 1\) for \(y \in Y\), and \(R^1 f_*\mathcal{O}_X = 0\). Then \(f_*\mathcal{O}_X\) is \(S\)-flat and formation of \(f_*\mathcal{O}_X\) commutes with arbitrary base change \(S' \to S\).

**Proof.** We may assume \(Y\) and \(S\) are affine, say \(S = \text{Spec}(A)\). To show the quasi-coherent \(\mathcal{O}_Y\)-module \(f_*\mathcal{O}_X\) is flat relative to \(S\) it suffices to show that \(H^0(X, \mathcal{O}_X)\) is flat over \(A\) (some details omitted). By Lemma [64.3](#) we have \(H^1(X, \mathcal{O}_X \otimes_A M) = 0\) for every \(A\)-module \(M\). Since also \(\mathcal{O}_X\) is flat over \(A\) we deduce the functor \(M \mapsto H^0(X, \mathcal{O}_X \otimes_A M)\) is exact. Moreover, this functor commutes with direct sums by Cohomology, Lemma [19.1](#) Then it is an exercise to see that \(H^0(X, \mathcal{O}_X \otimes_A M) = M \otimes_A H^0(X, \mathcal{O}_X)\) functorially in \(M\) and this gives the desired flatness. Finally, if \(S' \to S\) is a morphism of affines given by the ring map \(A \to A'\), then in the affine case just discussed we see that

\[
H^0(X \times_S S', \mathcal{O}_{X \times_S S'}) = H^0(X, \mathcal{O}_X \otimes_A A') = H^0(X, \mathcal{O}_X) \otimes_A A'
\]

This shows that formation of \(f_*\mathcal{O}_X\) commutes with any base change \(S' \to S\). Some details omitted.

**Lemma 64.7.** Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & & Y
\end{array}
\]

of morphisms of schemes. Let \(s \in S\) be a point. Assume

1. \(X \to S\) is locally of finite presentation and flat at points of \(X_s\),
2. \(Y \to S\) is locally of finite presentation,
3. \(f\) is proper,
4. the fibres of \(f_s : X_s \to Y_s\) have dimension \(\leq 1\) and \(R^1 f_{s,*}\mathcal{O}_{X_s} = 0\),
5. \(\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}\) is an isomorphism.

Then there is an open \(Y_s \subset V \subset Y\) such that (a) \(V\) is flat over \(S\), (b) \(f^{-1}(V)\) is flat over \(S\), (c) \(\dim(X_y) \leq 1\) for \(y \in V\), (d) \(R^1 f_*\mathcal{O}_X|_V = 0\), (e) \(\mathcal{O}_V \to f_*\mathcal{O}_X|_V\) is an isomorphism, and (a) – (e) remain true after base change of \(f^{-1}(V) \to V\) by any \(S' \to S\).

**Proof.** Let \(y \in Y_s\). We may always replace \(Y\) by an open neighbourhood of \(y\). Thus we may assume \(Y\) and \(S\) affine. We may also assume that \(X\) is flat over \(S\), \(\dim(X_y) \leq 1\) for \(y \in Y\), \(R^1 f_*\mathcal{O}_X = 0\) universally, and that \(\mathcal{O}_V \to f_*\mathcal{O}_X\) is surjective, see Lemma [64.5](#) (We won’t use all of this.)
Assume $S$ and $Y$ affine. Write $S = \lim S_i$ as a cofiltered of affine Noetherian schemes $S_i$. By Limits, Lemma 10.1 there exists an element $0 \in I$ and a diagram

$$
\begin{array}{ccc}
X_0 & \rightarrow & Y_0 \\
\downarrow f_0 & & \downarrow \\
S_0 & \rightarrow & S
\end{array}
$$

of finite type morphisms of schemes whose base change to $S$ is the diagram of the lemma. After increasing $0$ we may assume $Y_0$ is affine and $X_0 \rightarrow S_0$ proper, see Limits, Lemmas 13.1 and 13.13. Let $s_0 \in S_0$ be the image of $s$. As $Y_s$ is affine, we see that $R^1 f_{s,*} \mathcal{O}_{X_s} = 0$ is equivalent to $H^1(X_s, \mathcal{O}_{X_s}) = 0$. Since $X_s$ is the base change of $X_{0,s_0}$ by the faithfully flat map $\kappa(s_0) \rightarrow \kappa(s)$ we see that $H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0$ and hence $R^1 f_{0,*} \mathcal{O}_{X_{0,s_0}} = 0$. Similarly, as $\mathcal{O}_{Y_s} \rightarrow f_{s,*} \mathcal{O}_{X_s}$ is an isomorphism, so is $\mathcal{O}_{Y_{0,s_0}} \rightarrow f_{0,*} \mathcal{O}_{X_{0,s_0}}$. Since the dimensions of the fibres of $X_s \rightarrow Y_s$ are at most 1, the same is true for the morphism $X_{0,s_0} \rightarrow Y_{0,s_0}$. Finally, since $X \rightarrow S$ is flat, after increasing $0$ we may assume $X_0$ is flat over $S_0$, see Limits, Lemma 8.7. Thus it suffices to prove the lemma for $X_0 \rightarrow Y_0 \rightarrow S_0$ and the point $s_0$.

Combining the reduction arguments above we reduce to the case where $S$ and $Y$ affine, $S$ Noetherian, the fibres of $f$ have dimension $\leq 1$, and $R^1 f_* \mathcal{O}_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$
\mathcal{O}_{Y,y} \rightarrow (f_* \mathcal{O}_X)_y
$$

is an isomorphism. The claim implies the lemma. Namely, since $f_* \mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition 19.1) the claim means we can replace $Y$ by an open neighbourhood of $y$ and obtain an isomorphism $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$. Then we conclude that $Y$ is flat over $S$ by Lemma 64.6. Finally, the isomorphism $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ remains an isomorphism after any base change $S' \rightarrow S$ by the final statement of Lemma 64.6.

Proof of the claim. We already know that $\mathcal{O}_{Y,y} \rightarrow (f_* \mathcal{O}_X)_y$ is surjective (Lemma 64.5) and that $(f_* \mathcal{O}_X)_y$ is $\mathcal{O}_{S,y}$-flat (Lemma 64.6) and that the induced map

$$
\mathcal{O}_{Y,y} = \mathcal{O}_{Y,y}/m_y \mathcal{O}_{Y,y} \rightarrow (f_* \mathcal{O}_X)_y/m_y (f_* \mathcal{O}_X)_y \rightarrow (f_{s,*} \mathcal{O}_{X_s})_y
$$

is injective by the assumption in the lemma. Then it follows from Algebra, Lemma 98.1 that $\mathcal{O}_{Y,y} \rightarrow (f_* \mathcal{O}_X)_y$ is injective as desired. \qed

0E24 Lemma 64.8. Let $f : X \rightarrow Y$ be a proper morphism of Noetherian schemes such that $f_* \mathcal{O}_X = \mathcal{O}_Y$, such that the fibres of $f$ have dimension $\leq 1$, and such that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Then $f^* : \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is a bijection onto the subgroup of $\mathcal{L} \in \text{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$.

Proof. By the projection formula (Cohomology, Lemma 49.2) we see that $f_* f^* \mathcal{N} \cong \mathcal{N}$ for $\mathcal{N} \in \text{Pic}(Y)$. We claim that for $\mathcal{L} \in \text{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$ we have $\mathcal{N} = f_* \mathcal{L}$ is invertible and $\mathcal{L} \cong f^* \mathcal{N}$. This will finish the proof.

The $\mathcal{O}_Y$-module $\mathcal{N} = f_* \mathcal{L}$ is coherent by Cohomology of Schemes, Proposition 19.1. Thus to see that it is an invertible $\mathcal{O}_Y$-module, it suffices to check on stalks (Algebra, Lemma 77.2). Since the map from a Noetherian local ring to its completion is faithfully flat, it suffices to check the completion $(f_* \mathcal{L})_y^{\text{com}}$ is free (see Algebra, Section 96 and Lemma 77.5). For this we will use the theorem of formal functions as formulated in Cohomology of Schemes, Lemma 20.7. Since $f_* \mathcal{O}_X = \mathcal{O}_Y$ and hence
(f_*\mathcal{O}_X)_{y_n} \cong \mathcal{O}_{Y_y}, it suffices to show that \( L|_{X_n} \cong \mathcal{O}_{X_n} \) for each \( n \) (compatibly for varying \( n \)). By Lemma 1.1 we have an exact sequence

\[ H^1(X_y, m_y^n\mathcal{O}_X/m_y^{n+1}\mathcal{O}_X) \to \text{Pic}(X_{n+1}) \to \text{Pic}(X_n) \]

with notation as in the theorem on formal functions. Observe that we have a surjection

\[ \mathcal{O}_{X_y}^{\oplus r_n} \cong m_y^n/m_y^{n+1} \oplus_{\kappa(y)} \mathcal{O}_{X_y} \to m_y^n\mathcal{O}_X/m_y^{n+1}\mathcal{O}_X \]

for some integers \( r_n \geq 0 \). Since \( \dim(X_y) \leq 1 \) this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees \( \geq 2 \) coming from Cohomology, Proposition 20.7). Hence the \( H^1 \) in the sequence is zero and the transition maps \( \text{Pic}(X_{n+1}) \to \text{Pic}(X_n) \) are injective as desired.

We still have to show that \( f^*\mathcal{N} \cong L \). This is proved by the same method and we omit the details. \( \square \)

65. Affine stratifications

0F2R This material is taken from [RV04]. Please read a little bit about stratifications in Topology, Section 28 before reading this section.

If \( X \) is a scheme, then a stratification of \( X \) usually means a stratification of the underlying topological space of \( X \). The strata are locally closed subsets. We will view these strata as reduced locally closed subschemes of \( X \) using Schemes, Remark 12.6.

0F2S **Definition 65.1.** Let \( X \) be a scheme. An affine stratification is a locally finite stratification \( X = \bigsqcup_{i \in I} X_i \) whose strata \( X_i \) are affine and such that the inclusion morphisms \( X_i \to X \) are affine.

The condition that a stratification \( X = \bigsqcup X_i \) is locally finite is, in the presence of the condition that the inclusion morphisms \( X_i \to X \) are quasi-compact, equivalent to the condition that the strata are locally constructible subsets of \( X \), see Properties, Lemma 2.7.

The condition that \( X_i \to X \) is an affine morphism is independent on the scheme structure we put on the locally closed subset \( X_i \), see Lemma 3.1 Moreover, if \( X \) is separated (or more generally has affine diagonal) and \( X = \bigsqcup X_i \) is a locally finite stratification with affine strata, then the morphisms \( X_i \to X \) are affine. See Morphisms, Lemma 11.11 This allows us to disregard the condition of affineness of the inclusion morphisms \( X_i \to X \) in most cases of interest.

We are often interested in the case where the partially ordered index set \( I \) of the stratification is finite. Recall that the length of a partially ordered set \( I \) is the supremum of the lengths \( p \) of chains \( i_0 < i_1 < \ldots < i_p \) of elements of \( I \).

0F2T **Lemma 65.2.** Let \( X \) be a scheme. Let \( X = \bigsqcup_{i \in I} X_i \) be a finite affine stratification. There exists an affine stratification with index set \( \{0, \ldots, n\} \) where \( n \) is the length of \( I \).

**Proof.** Recall that we have a partial ordering on \( I \) such that the closure of \( X_i \) is contained in \( \bigcup_{j \leq i} X_j \) for all \( i \in I \). Let \( I' \subset I \) be the set of maximal indices of \( I \). If \( i \in I' \), then \( X_i \) is open in \( X \) because the union of the closures of the other strata is the complement of \( X_i \). Let \( U = \bigcup_{i \in I'} X_i \) viewed as an open subscheme of \( X \) so that \( U_{red} = \bigsqcup_{i \in I'} X_i \) as schemes. Then \( U \) is an affine scheme by Schemes, Lemma
The morphism $U \to X$ is affine as each $X_i \to X$, $i \in I'$ is affine by the same reasoning using Lemma 4.1. The complement $Z = X \setminus U$ endowed with the reduced induced scheme structure has the affine stratification $Z = \bigcup_{i \in I} X_i$. Here we use that a morphism of schemes $T \to Z$ is affine if and only if the composition $T \to X$ is affine; this follows from Morphisms, Lemmas 11.9 and 11.11. Observe that the partially ordered set $I \setminus I'$ has length exactly one less than the length of $I$. Hence by induction we find that $Z$ has an affine stratification $Z = Z_0 \cup \ldots \cup Z_{n-1}$ with index set $\{1, \ldots, n\}$. Setting $Z_n = U$ we obtain the desired stratification of $X$.

If a scheme $X$ has a finite affine stratification, then of course $X$ is quasi-compact. A bit less obvious is the fact that it forces $X$ to be quasi-separated as well.

0F2U Lemma 65.3. Let $X$ be a scheme. The following are equivalent

1. $X$ has a finite affine stratification, and
2. $X$ is quasi-compact and quasi-separated.

Proof. Let $X = \bigcup X_i$ be a finite affine stratification. Since each $X_i$ is affine hence quasi-compact, we conclude that $X$ is quasi-compact. Let $U, V \subset X$ be affine open. Then $U \cap X_i$ and $V \cap X_i$ are affine open in $X_i$ since $X_i \to X$ is an affine morphism. Hence $U \cap V \cap X_i$ is an affine open of the affine scheme $X_i$ (see Schemes, Lemma 21.7 for example). Therefore $U \cap V = \bigsqcup U \cap V \cap X_i$ is quasi-compact as a finite union of affine strata. We conclude that $X$ is quasi-separated by Schemes, Lemma 21.6.

Assume $X$ is quasi-compact and quasi-separated. We may use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove the assertion that $X$ has a finite affine stratification. If $X$ is empty, then it has an empty affine stratification. If $X$ is nonempty affine then it has an affine stratification with one stratum. Next, assume $X = U \cup V$ where $U$ is quasi-compact open, $V$ is affine open, and we have a finite affine stratifications $U = \bigcup_{i \in I} U_i$ and $U \cap V = \bigcup_{j \in J} W_j$. Denote $Z = X \setminus V$ and $Z' = X \setminus U$. Note that $Z$ is closed in $U$ and $Z'$ is closed in $V$. Observe that $U_i \cap Z$ and $U_i \cap W_j = U_i \times_U W_j$ are affine schemes affine over $U$. (Hints: use that $U_i \times_U W_j \to W_j$ is affine as a base change of $U_i \to U$, hence $U_i \cap W_j$ is affine, hence $U_i \cap W_j \to U_i$ is affine, hence $U_i \cap W_j \to U$ is affine.) It follows that

$$U = \bigsqcup_{i \in I} (U_i \cap Z) \amalg \bigsqcup_{(i,j) \in I \times J} (U_i \cap W_j)$$

is a finite affine stratification with partial ordering on $I \amalg J$ given by $i' \leq (i,j) \iff i' \leq i$ and $(i', j') \leq (i, j) \iff i' \leq i$ and $j' \leq j$. Observe that $(U_i \cap Z) \times_X V = \emptyset$ and $(U_i \cap W_j) \times_X V = U_i \cap W_j$ are affine. Hence the morphisms $U_i \cap Z \to X$ and $U_i \cap W_j \to X$ are affine because we can check affineness of a morphism locally on the target (Morphisms, Lemma 11.3) and we have affineness over both $U$ and $V$. To finish the proof we take the stratification above and we add one additional stratum, namely $Z'$, whose index we add as a minimal element to the partially ordered set.

0F2V Definition 65.4. Let $X$ be a nonempty quasi-compact and quasi-separated scheme. The affine stratification number is the smallest integer $n \geq 0$ such that the following equivalent conditions are satisfied

1. there exists a finite affine stratification $X = \bigsqcup_{i \in I} X_i$ where $I$ has length $n$,
Lemma 65.5. Let X be a separated scheme which has an open covering by n + 1 affines. Then the affine stratification number of X is at most n.

Proof. Say $X = U_0 \cup \ldots \cup U_n$ is an affine open covering. Set

$$X_i = (U_i \cup \ldots \cup U_n) \setminus (U_{i+1} \cup \ldots \cup U_n)$$

Then $X_i$ is affine as a closed subscheme of $U_i$. The morphism $X_i \to X$ is affine by Morphisms, Lemma 11.11. Finally, we have $X_i \subset X_i \cup X_{i-1} \cup \ldots X_0$. □

Lemma 65.6. Let X be a Noetherian scheme of dimension $\infty > d \geq 0$. Then the affine stratification number of X is at most d.

Proof. By induction on d. If $d = 0$, then X is affine, see Properties, Lemma 10.5. Assume $d > 0$. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of X (Properties, Lemma 5.7). We can cover X by affine opens containing $\eta_1, \ldots, \eta_n$, see Properties, Lemma 29.4. Since X is quasi-compact we can find a finite affine open covering $X = \bigcup_{j=1, \ldots, m} U_j$ with $\eta_1, \ldots, \eta_n \in U_j$ for all $j = 1, \ldots, m$. Choose an affine open $U \subset U_1 \cap \ldots \cap U_m$ containing $\eta_1, \ldots, \eta_n$ (possible by the lemma already quoted). Then the morphism $U \to X$ is affine because $U \to U_j$ is affine for all j, see Morphisms, Lemma 11.3. Let $Z = X \setminus U$. By construction $\dim(Z) < \dim(X)$. By induction hypothesis we can find an affine stratification $Z = \bigcup_{i \in \{0, \ldots, n\}} Z_i$ of Z with $n \leq \dim(Z)$. Setting $U = X_{n+1}$ and $X_i = Z_i$ for $i \leq n$ we conclude. □

Proposition 65.7. Let X be a nonempty quasi-compact and quasi-separated scheme with affine stratification number n. Then $H^p(X, F) = 0$, $p > n$ for every quasi-coherent $\mathcal{O}_X$-module F.

Proof. We will prove this by induction on the affine stratification number n. If $n = 0$, then X is affine and the result is Cohomology of Schemes, Lemma 2.2. Assume $n > 0$. By Definition 65.3 there is an affine scheme U and an affine open immersion $j : U \to X$ such that the complement Z has affine stratification number $n - 1$. As U and j are affine we have $H^p(X, j_*(\mathcal{F}|_U)) = 0$ for $p > 0$, see Cohomology of Schemes, Lemmas 2.4 and 2.3. Denote $\mathcal{K}$ and $\mathcal{Q}$ the kernel and cokernel of the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$. Thus we obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to \mathcal{Q} \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules (see Schemes, Section 24). A standard argument, breaking our exact sequence into short exact sequences and using the long exact cohomology sequence, shows it suffices to prove $H^p(X, \mathcal{K}) = 0$ and $H^p(X, \mathcal{Q}) = 0$ for $p \geq n$. Since $\mathcal{F} \to j_*(\mathcal{F}|_U)$ restricts to an isomorphism over U, we see that $\mathcal{K}$ and $\mathcal{Q}$ are supported on Z. By Properties, Lemma 22.3 we can write these modules as the filtered colimits of their finite type quasi-coherent submodules. Using the fact that cohomology of sheaves on X commutes with filtered colimits, see Cohomology, Lemma 19.1, we conclude it suffices to show that if $\mathcal{G}$ is a finite type quasi-coherent module whose support is contained in Z, then $H^p(X, \mathcal{G}) = 0$ for $p \geq n$. Let $Z' \subset X$ be the scheme theoretic support of $\mathcal{G} \oplus \mathcal{O}_Z$; we may and do think of $\mathcal{G}$ as a
Let $k$ be a field and let $X = \mathbb{P}^n_k$ be $n$-dimensional projective space over $k$. Lemma 65.5 applies to this by Constructions, Lemma 13.3. Hence the affine stratification number of $\mathbb{P}^n_k$ is at most $n$. On the other hand, we have nonzero cohomology in degree $n$ for some quasi-coherent modules on $\mathbb{P}^n_k$, see Cohomology of Schemes, Lemma 8.1. Using Proposition 65.7 we conclude that the affine stratification number of $\mathbb{P}^n_k$ is equal to $n$.

66. Universally open morphisms

Lemma 66.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

1. $f$ is universally open,
2. for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is open, and
3. for every $n$ the morphism $\mathbb{A}^n \times X \to \mathbb{A}^n \times S$ is open.

Proof. It is clear that (1) implies (2) and (2) implies (3). Let us prove that (3) implies (1). Suppose that the base change $X_T \to T$ is not open for some morphism of schemes $g : T \to S$. Then we can find some affine opens $V \subset S$, $U \subset X$, $W \subset T$ with $f(U) \subset V$ and $g(W) \subset V$ such that $U \times_V W \to W$ is not open. If we can show that this implies $\mathbb{A}^n \times U \to \mathbb{A}^n \times V$ is not open, then $\mathbb{A}^n \times X \to \mathbb{A}^n \times S$ is not open and the proof is complete. This reduces us to the result proved in the next paragraph.

Let $A \to B$ be a ring map such that $A' \to B' = A' \otimes_A B$ does not induce an open map of spectra for some $A$-algebra $A'$. As the principal opens give a basis for the topology of $\text{Spec}(B')$ we conclude that the image of $D(g)$ in $\text{Spec}(A')$ is not open for some $g \in B'$. Write $g = \sum_{i=1}^n a'_i \otimes b_i$ for some $n$, $a'_i \in A'$, and $b_i \in B$. Consider the element $h = \sum_{i=1}^n x_i b_i$ in $B[x_1, \ldots, x_n]$. Assume that $D(h)$ maps to an open subset under the morphism

$$\text{Spec}(B[x_1, \ldots, x_n]) \to \text{Spec}(A[x_1, \ldots, x_n])$$

in order to get a contradiction. Then $D(h)$ would map surjectively onto a quasi-compact open $U \subset \text{Spec}(A[x_1, \ldots, x_n])$. Let $A[x_1, \ldots, x_n] \to A'$ be the $A$-algebra homomorphism sending $x_i$ to $a'_i$. This also induces a $B$-algebra homomorphism $B[x_1, \ldots, x_n] \to B'$ sending $h$ to $g$. Since

$$\text{Spec}(B[x_1, \ldots, x_n]) \leftarrow \text{Spec}(B')$$

$$\downarrow$$

$$\text{Spec}(A[x_1, \ldots, x_n]) \leftarrow \text{Spec}(A')$$

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of $U$ in $\text{Spec}(A')$ and hence open which is the desired contradiction. 

Lemma 66.2. Let $f : X \to Y$ be a morphism of schemes. If
(1) $f$ is locally quasi-finite,
(2) $Y$ is unibranch and locally Noetherian, and
(3) every irreducible component of $X$ dominates an irreducible component of $Y$,
then $f$ is universally open.

**Proof.** For any $n$ the scheme $\mathbb{A}^n \times Y$ is unibranch by Lemma [33.4]. Hence the hypotheses of the lemma hold for the morphisms $\mathbb{A}^n \times X \to \mathbb{A}^n \times Y$ for all $n$. By Lemma [66.1] it suffices to prove $f$ is open. By Morphisms, Lemma [22.2] it suffices to show that generalizations lift along $f$. Suppose that $y' \rightsquigarrow y$ is a specialization of points in $Y$ and $x \in X$ is a point mapping to $y$. As in Lemma [36.1] choose a diagram

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow \quad & & \downarrow \\
V & \longrightarrow & Y
\end{array}
\]

where $(V,v) \to (Y,y)$ is an étale neighbourhood, $U \to V$ is finite, $v$ is the unique point of $V$ mapping to $u$, $U \subset V \times Y$ is open, and $v \mapsto y$ and $u \mapsto x$. Let $E$ be an irreducible component of $U$ passing through $u$ (there is at least one of these). Since $U \to X$ is étale, $E$ maps to an irreducible component of $X$, which in turn dominates an irreducible component of $Y$ (by assumption). Since $U \to V$ is finite hence closed, we conclude that the image $E' \subset V$ of $E$ is an irreducible closed subset passing through $v$ which dominates an irreducible component of $Y$. Since $V \to Y$ is étale $E'$ must be an irreducible component of $V$ passing through $v$. Since $Y$ is unibranch we see that $E'$ is the unique irreducible component of $V$ passing through $v$ (Lemma [33.2]). Since $V$ is locally Noetherian we may after shrinking $V$ assume that $E' = V$ (equality of sets).

Since $V \to Y$ is étale we can find a specialization $v' \rightsquigarrow v$ whose image is $y' \rightsquigarrow y$. By the above we can find $u' \in U$ mapping to $v'$. Then $u' \rightsquigarrow u$ because $u$ is the only point of $U$ mapping to $v$ and $U \to V$ is closed. Then finally the image $x' \in X$ of $u'$ is a point specializing to $x$ and mapping to $y'$ and the proof is complete. \[\square\]

**0F33 Lemma 66.3.** Let $A \to B$ be a ring map. Say $B$ is generated as an $A$-module by $b_1, \ldots, b_d \in B$. Set $h = \sum x_i b_i \in B[x_1, \ldots, x_d]$. Then $\text{Spec}(B) \to \text{Spec}(A)$ is universally open if and only if the image of $D(h)$ in $\text{Spec}(A[x_1, \ldots, x_d])$ is open.

**Proof.** If $\text{Spec}(B) \to \text{Spec}(A)$ is universally open, then of course the image of $D(h)$ is open. Conversely, assume the image $U$ of $D(h)$ is open. Let $A \to A'$ be a ring map. It suffices to show that the image of any principal open $D(g) \subset \text{Spec}(A' \otimes_A B)$ in $\text{Spec}(A')$ is open. We may write $g = \sum_{i=1,\ldots,d} a_i' \otimes b_i$ for some $a_i' \in A'$. Let $A[x_1, \ldots, x_n] \to A'$ be the $A$-algebra homomorphism sending $x_i$ to $a_i'$. This also induces a $B$-algebra homomorphism $B[x_1, \ldots, x_n] \to A' \otimes_A B$ sending $h$ to $g$. Since

\[
\begin{array}{ccc}
\text{Spec}(B[x_1, \ldots, x_n]) & \phantom{\text{Spec}'} & \text{Spec}(B') \\
\downarrow & & \downarrow \\
\text{Spec}(A[x_1, \ldots, x_n]) & \phantom{\text{Spec}'} & \text{Spec}(A')
\end{array}
\]

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of $U$ in $\text{Spec}(A')$ and hence open. \[\square\]
\textbf{Lemma 66.4.} Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over $S_0$. Assume $S_0$, $X_0$, $Y_0$ are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of $f_0$ to $S_i$ and let $f : X \to Y$ be the base change of $f_0$ to $S$. If

\begin{enumerate}
\item $f$ is locally quasi-finite and universally open, and
\item $f_0$ is locally of finite presentation,
\end{enumerate}

then there exists an $i \geq 0$ such that $f_i$ is locally quasi-finite and universally open.

\textbf{Proof.} By Limits, Lemma 10.2 after increasing $0$ we may assume $f_0$ is locally quasi-finite. Let $x \in X$. By étale localization of quasi-finite morphisms we can find a diagram

$$
\begin{array}{ccc}
X & \xleftarrow{U} & Y \\
\downarrow & & \downarrow \\
Y & \xleftarrow{V} & V
\end{array}
$$

where $V \to Y$ is étale, $U \subset X_V$ is open, $U \to V$ is finite, and $x$ is in the image of $U \to X$, see Lemma 36.1. After shrinking $V$ we may assume $V$ and $U$ are affine. Since $X$ is quasi-compact, it follows, by taking a finite disjoint union of such $V$ and $U$, that we can make a diagram as above such that $U \to X$ is surjective. By Limits, Lemmas 10.1, 4.1, 8.1, 8.3, 8.10, and 4.13 after possibly increasing $0$ we may assume we have a diagram

$$
\begin{array}{ccc}
X_0 & \xleftarrow{U_0} & Y_0 \\
\downarrow & & \downarrow \\
Y_0 & \xleftarrow{V_0} & V_0
\end{array}
$$

where $V_0$ is affine, $V_0 \to Y_0$ is étale, $U_0 \subset (X_0)_{V_0}$ is open, $U_0 \to V_0$ is finite, and $U_0 \to X_0$ is surjective. Since $V_i \to Y_i$ is étale and hence universally open, follows that it suffices to prove that $U_i \to V_i$ is universally open for large enough $i$. This reduces us to the case discussed in the next paragraph.

Let $A = \colim A_i$ be a filtered colimit of rings. Let $A_0 \to B_0$ be a ring map. Set $B = A \otimes_{A_0} B_0$ and $B_i = A_i \otimes_{A_0} B_0$. Assume $A_0 \to B_0$ is finite, of finite presentation, and $A \to B$ is universally open. We have to show that $A_i \to B_i$ is universally open for $i$ large enough. Pick $b_{0,1}, \ldots, b_{0,d} \in B_0$ which generate $B_0$ as an $A_0$-module. Set $h_0 = \sum_{j=1}^{d} x_j b_{0,j}$ in $B_0[x_1, \ldots, x_d]$. Denote $h_i$ the image of $h_0$ in $B[x_1, \ldots, x_d]$, resp. $B_i[x_1, \ldots, x_d]$. The image $U$ of $D(h)$ in $\Spec(A[x_1, \ldots, x_d])$ is open as $A \to B$ is universally open. Of course $U$ is quasi-compact as the image of an affine scheme. For $i$ large enough there is a quasi-compact open $U_i \subset \Spec(A_i[x_1, \ldots, x_d])$ whose inverse image in $\Spec(A[x_1, \ldots, x_d])$ is $U$, see Limits, Lemma 4.11. After increasing $i$ we may assume that $D(h_i)$ maps into $U_i$; this follows from the same lemma by considering the pullback of $U_i$ in $D(h_i)$. Finally, for $i$ even larger the morphism of schemes $D(h_i) \to U_i$ will be surjective by an application of the already used Limits, Lemma 8.14. We conclude $A_i \to B_i$ is universally open by Lemma 66.3.

\textbf{Lemma 66.5.} Let $f : X \to Y$ be a locally quasi-finite morphism. Then

\begin{enumerate}
\item the functions $n_{X/Y}$ of Lemmas 25.3 and 26.3 agree,
\end{enumerate}
(2) If $X$ is quasi-compact, then $n_{X/Y}$ attains a maximum $d < \infty$.

**Proof.** Agreement of the functions is immediate from the fact that the (geometric) fibres of a locally quasi-finite morphism are discrete, see Morphisms, Lemma 19.8. Boundedness follows from Morphisms, Lemmas 54.2 and 54.10.

0F36 **Lemma 66.6.** Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 66.5. If $n_{X/Y}(y) \geq d$ for some $y \in Y$ and $d \geq 0$, then $n_{X/Y} \geq d$ in an open neighbourhood of $y$.

**Proof.** The question is local on $Y$ hence we may assume $Y$ affine. Let $K$ be an algebraic closure of the residue field $\kappa(y)$. Our assumption is that $(X_y)_K$ has $\geq d$ connected components. Then for a suitable quasi-compact open $X' \subset X$ the scheme $(X'_y)_K$ has $\geq d$ connected components; details omitted. After replacing $X$ by $X'$ we may assume $X$ is quasi-compact. Then $f$ is quasi-finite. Let $x_1, \ldots, x_n$ be the points of $X$ lying over $y$. Apply Lemma 36.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \prod_{i=1, \ldots, n} \prod_{j=1, \ldots, m_i} V_{i,j}$$

as in locus citatus. Observe that $n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since $f$ is universally open, we see that $V_{i,j} \to U$ is open for all $i, j$. Hence after shrinking $U$ we may assume $V_{i,j} \to U$ is surjective for all $i, j$. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = n_{X/Y}(y) \geq d$. Since the construction of $n_{X/Y}$ is compatible with base change we know that $n_{U \times_Y X/U} = d$. This means that $W$ has to be empty and we conclude that $U \times_Y X \to U$ is finite. By Descent, Lemma 20.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that $f$ is finite over an open neighbourhood of $y$ as desired. □

0F37 **Lemma 66.7.** Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 66.5. If $n_{X/Y}$ attains a maximum $d < \infty$, then the set

$$Y_d = \{ y \in Y \mid n_{X/Y}(y) = d \}$$

is open in $Y$ and the morphism $f^{-1}(Y_d) \to Y_d$ is finite.

**Proof.** The openness of $Y_d$ is immediate from Lemma 66.6. To prove finiteness over $Y_d$ we redo the argument of the proof of that lemma. Namely, let $y \in Y_d$. Then there are at most $d$ points of $X$ lying over $y$. Say $x_1, \ldots, x_n$ are the points of $X$ lying over $y$. Apply Lemma 36.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \prod_{i=1, \ldots, n} \prod_{j=1, \ldots, m_i} V_{i,j}$$

as in locus citatus. Observe that $d = n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since $f$ is universally open, we see that $V_{i,j} \to U$ is open for all $i, j$. Hence after shrinking $U$ we may assume $V_{i,j} \to U$ is surjective for all $i, j$ and we may assume $U$ maps into $W$. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = d$. Since the construction of $n_{X/Y}$ is compatible with base change we know that $n_{U \times_Y X/U} = d$. This means that $W$ has to be empty and we conclude that $U \times_Y X \to U$ is finite. By Descent, Lemma 20.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that $f$ is finite over an open neighbourhood of $y$ as desired. □
67. Weightings

The material in this section is taken from [AGV71, Exposee XVII, 6.2.4].

Let \( \pi : U \to V \) be a locally quasi-finite morphism of schemes with finite fibres. Given a function \( w : U \to \mathbb{Z} \) we define a function

\[
\int_\pi w : V \to \mathbb{Z}, \quad v \mapsto \sum_{u \in U, \pi(u) = v} w(u)[\kappa(u) : \kappa(v)]_s
\]

Note that the field extensions are finite (Morphisms, Lemma 19.5), \([\kappa' : \kappa]_s\) is the separable degree (Fields, Definition 14.7), and the sum is finite as the fibres of \( \pi \) are assumed finite. Another way to compute the value of \( \int_\pi w \) at a point \( v \in V \) is as follows. Choose an algebraically closed field \( k \) and a morphism \( \overline{v} : \text{Spec}(k) \to V \) whose image is \( v \). Then we have

\[
(f_\pi w)(v) = \sum_{\pi \in U_{\overline{v}}} w(\pi)
\]

where of course \( w(\pi) \) denotes the value of \( w \) at the image \( u \) of the point \( \pi \) under the morphism \( U_{\overline{v}} \to U \). Note that we may view \( U_{\overline{v}} \) as morphisms \( \pi : \text{Spec}(k) \to U \) such that \( \pi \circ \pi = \overline{v} \). Namely, since \( U \to V \) is locally quasi-finite with finite fibres, the scheme \( U_{\overline{v}} \) is the spectrum of a finite dimension algebra over \( k \) and all of whose prime ideals are maximal ideals with residue field \( k \). To see that the equality holds, note that the number of morphisms \( \pi \) lying over a given \( u \) is equal to \([\kappa(u) : \kappa(v)]_s\) by Fields, Lemma 14.8.

Lemma 67.1. Given a cartesian square

\[
\begin{array}{ccc}
U & \xrightarrow{h} & U' \\
\downarrow {\pi} & & \downarrow {\pi'} \\
V & \xrightarrow{g} & V'
\end{array}
\]

with \( \pi \) locally quasi-finite with finite fibres and a function \( w : U \to \mathbb{Z} \) we have \((\int_\pi w) \circ g = \int_{\pi'}(w \circ h)\).

Proof. This follows immediately from the second description of \( \int_\pi w \) given above. To prove it from the definition, you use that if \( E/F \) is a finite extension of fields and \( F'/F \) is another field extension, then writing \((E \otimes_F F')_{\text{red}} = \prod E'_i\) as a product of fields finite over \( F' \), we have

\[
[E : F]_s = \sum [E'_i : F']_s
\]

To prove this equality pick an algebraically closed field extension \( \Omega/F' \) and observe that

\[
[E : F]_s = |\text{Mor}_F(E, \Omega)| \\
= |\text{Mor}_{F'}(E \otimes_F F', \Omega)| \\
= |\text{Mor}_{F'}((E \otimes_F F')_{\text{red}}, \Omega)| \\
= \sum |\text{Mor}_{F'}(E'_i, \Omega)| \\
= \sum [E'_i : F']_s
\]

where we have used Fields, Lemma 14.8.
**Definition 67.2.** Let \( f : X \rightarrow Y \) be a locally quasi-finite morphism. A *weighting* or a *pondération* of \( f \) is a map \( w : X \rightarrow \mathbb{Z} \) such that for any diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & U \\
f \downarrow & & \downarrow \pi \\
Y & \xleftarrow{g} & V
\end{array}
\]

where \( V \rightarrow Y \) is étale, \( U \subset X_V \) is open, and \( U \rightarrow V \) finite, the function \( f_\pi(w \circ h) \) is locally constant.

Of course taking \( w = 0 \) we obtain a weighting of any locally quasi-finite morphism \( f \), albeit not a very interesting one. It will turn out that positive weightings, i.e., \( w : X \rightarrow \mathbb{Z}_{>0} \) are the most interesting ones for various purposes.

**Lemma 67.3.** Let \( f : X \rightarrow Y \) be a locally quasi-finite morphism. Let \( w : X \rightarrow \mathbb{Z} \) be a weighting. Let \( f' : X' \rightarrow Y' \) be the base change of \( f \) by a morphism \( Y' \rightarrow Y \). Then the composition \( w' : X' \rightarrow \mathbb{Z} \) of \( w \) and the projection \( X' \rightarrow X \) is a weighting of \( f' \).

**Proof.** Consider a diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{h'} & U' \\
f' \downarrow & & \downarrow \pi' \\
Y' & \xleftarrow{g'} & V'
\end{array}
\]

as in Definition 67.2 for the morphism \( f' \). For any \( v' \in V' \) we have to show that \( f'_{\pi'}(w' \circ h') \) is constant in an open neighbourhood of \( v' \). By Lemma 67.1 (and the fact that étale morphisms are open) we may replace \( V' \) by any étale neighbourhood of \( v' \). After replacing \( V' \) by an étale neighbourhood of \( v' \) we may assume that \( U' = U'_1 \amalg \ldots \amalg U'_n \) where each \( U'_i \) has a unique point \( u'_i \) lying over \( v' \) such that \( \kappa(u'_i)/\kappa(v') \) is purely inseparable, see Lemma 66.3. Clearly, it suffices to prove that \( f_{U'_i \rightarrow V'}(w'|_{U'_i}) \) is constant in a neighbourhood of \( v' \). This reduces us to the case discussed in the next paragraph.

We have \( v' \in V' \) and there is a unique point \( u' \) of \( U' \) lying over \( v' \) with \( \kappa(u')/\kappa(v') \) purely inseparable. Denote \( x \in X \) and \( y \in Y \) the image of \( u' \) and \( v' \). We can find an étale neighbourhood \( (V,v) \rightarrow (Y,y) \) and an open \( U \subset X_V \) such that \( \pi : U \rightarrow V \) is finite and such that there is a unique point \( u \in U \) lying over \( v \) which maps to \( x \in X \) via the projection \( h : U \rightarrow X \) such that moreover \( \kappa(u)/\kappa(v) \) is purely inseparable. This is possible by the lemma used above. Consider the morphism

\[
U'' = U \times_X U' \rightarrow V \times_Y V' = V''
\]

Since \( u \) and \( u' \) both map to \( x \in X \) there is a point \( u'' \in U'' \) mapping to \((u,u')\). Denote \( v'' \in V'' \) the image of \( u'' \). After replacing \( V' \), \( v' \) by \( V'', v'' \) we may assume that the composition \( V' \rightarrow Y' \rightarrow Y \) factors through a map of étale neighbourhoods \( (V',v') \rightarrow (V,v) \) such that the induced morphism \( X'_{V'} = X_V \rightarrow X_V \) sends \( u' \) to \( u \).

Inside the base change \( X'_{V'} = X_V \) we have two open subschemes, namely \( U' \) and the inverse image \( U_{V'} \) of \( U \subset X_V \). By construction both contain a unique point lying over \( v' \), namely \( u' \) for both of them. Thus after shrinking \( V' \) we may assume these open subsets are the same; namely, \( U' \setminus (U' \cap U_{V'}) \) and \( U_{V'} \setminus (U' \cap U_{V'}) \) have a closed image in \( V' \) and these images do not contain \( v' \). Thus \( U' = U_{V'} \) and we
find a cartesian diagram as in Lemma \[67.1\]. Since \( \int_x w \circ h \) is locally constant by assumption we conclude. □

**Lemma 67.4.** Let \( f : X \to Y \) be a locally quasi-finite morphism. Let \( w : X \to \mathbb{Z} \) be a weighting. If \( w(x) > 0 \) for all \( x \in X \), then \( f \) is universally open.

**Proof.** Since the property is preserved by base change, see Lemma \[67.3\], it suffices to prove that \( f \) is open. Since we may also replace \( X \) by any open of \( X \), it suffices to prove that \( f(X) \) is open. Let \( y \in f(X) \). Choose \( x \in X \) with \( f(x) = y \). It suffices to prove that \( f(X) \) contains an open neighbourhood of \( y \) and it suffices to do so after replacing \( Y \) by an étale neighbourhood of \( y \). By étale localization of quasi-finite morphisms, see Section 36, we may assume there is an open neighbourhood \( U \subset X \) of \( x \) such that \( \pi = f|_U : U \to Y \) is finite. Then \( \int_x w|_U \) is locally constant and has positive value at \( y \). Hence \( \pi(U) \) contains an open neighbourhood of \( y \) and the proof is complete. □

**Lemma 67.5.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( f \) is locally quasi-finite, locally of finite presentation, and flat. Then there is a positive weighting \( w : X \to \mathbb{Z}_{>0} \) of \( f \) given by the rule that sends \( x \in X \) lying over \( y \in Y \) to

\[
(w(x) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]_i)
\]

where \([\kappa' : \kappa]\) is the inseparable degree (Fields, Definition 14.7).

**Proof.** Consider a diagram as in Definition \[67.2\]. Let \( u \in U \) with images \( x, y, v \) in \( X, Y, V \). Then we claim that

\[
\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x}) = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/m_y \mathcal{O}_{U,u})
\]

and

\[
[\kappa(x) : \kappa(y)]_i = [\kappa(u) : \kappa(v)]_i
\]

The first equality follows as \( \mathcal{O}_{X,x} \to \mathcal{O}_{U,u} \) is a flat local homomorphism such that \( m_y \mathcal{O}_{U,u} = m_u \mathcal{O}_{U,u} \) and \( m_y \mathcal{O}_{U,u} = m_u \) (because \( \mathcal{O}_{Y,y} \to \mathcal{O}_{V,v} \) and \( \mathcal{O}_{X,x} \to \mathcal{O}_{U,u} \) are unramified) and hence the equality by Algebra, Lemma 51.13. The second equality follows because \( \kappa(y) / \kappa(v) \) is a finite separable extension and \( \kappa(u) \) is a factor of \( \kappa(x) \otimes_{\kappa(y)} \kappa(v) \) and hence the inseparable degree is unchanged. Having said this, we see that formation of the function in the lemma commutes with étale base change. This reduces the problem to the discussion of the next paragraph.

Assume that \( f \) is a finite, flat morphism of finite presentation. We have to show that \( \int_x w \) is locally constant on \( Y \). In fact, \( f \) is finite locally free (Morphisms, Lemma \[46.2\]) and we will show that \( \int_x w \) is equal to the degree of \( f \) (which is a locally constant function on \( Y \)). Namely, for \( y \in Y \) we see that

\[
(\int_x w)(y) = \sum_{f(x) = y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]_i[\kappa(x) : \kappa(y)]_s
\]

\[
= \sum_{f(x) = y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/m_y \mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]
\]

\[
= \text{length}_{\mathcal{O}_{V,v}}((f_* \mathcal{O}_X)_y/m_y (f_* \mathcal{O}_X)_y)
\]

Last equality by Algebra, Lemma 51.12. The final number is the rank of \( f_* \mathcal{O}_X \) at \( y \) as desired. □

**Lemma 67.6.** Let \( f : X \to Y \) be a morphism of schemes. Assume (1) \( f \) is locally quasi-finite, and
(2) $Y$ is unibranch and locally Noetherian. Then there is a weighting $w : X \to \mathbb{Z}_{\geq 0}$ given by the rule that sends $x \in X$ lying over $y \in Y$ to the “generic separable degree” of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,y}$.

**Proof.** It follows from Algebra, Lemmas [151.15] and [149.3] that $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is finite. Since $Y$ is unibranch there is a unique minimal prime $p$ in $\mathcal{O}^{sh}_{Y,y}$. Write

$$(\kappa(p) \otimes_{\mathcal{O}^{sh}_{Y,y}} \mathcal{O}^{sh}_{X,x})_{\text{red}} = \prod K_i$$

as a finite product of fields. We set $w(x) = \sum [K_i : \kappa(p)]_s$.

Since this definition is clearly insensitive to étale localization, in order to show that $w$ is a weighting we reduce to showing that if $w$ is a finite morphism, then $\int_f w$ is locally constant. Observe that the value of $\int_f w$ in a generic point $y$ of $Y$ is just the number of points of the geometric fibre $X_{\eta}$ of $X \to Y$ over $\eta$. Moreover, since $Y$ is unibranch a point $y$ of $Y$ is the specialization of a unique generic point $\eta$. Hence it suffices to show that $(\int_f w)(y)$ is equal to the number of points of $X_{\eta}$.

After passing to an affine neighbourhood of $y$ we may assume $X \to Y$ is given by a finite ring map $A \to B$. Suppose $\mathcal{O}^{sh}_{Y,y}$ is constructed using a map $\kappa(y) \to k$ into an algebraically closed field $k$. Then

$$\mathcal{O}^{sh}_{Y,y} \otimes_A B = \prod_{f(x)=y} \prod_{p \in \text{Mor}_{\kappa(y)}(\kappa(x),k)} \mathcal{O}^{sh}_{X,x}$$

by the references on strict henselization given above. Observe that the minimal prime $p$ of $\mathcal{O}^{sh}_{Y,y}$ maps to the prime of $A$ corresponding to $\eta$. Hence we see that the desired equality holds because the number of points of a geometric fibre is unchanged by a field extension. \hfill \Box

### 68. More on weightings

0F3F We prove a few more basic properties of weightings. Although at first it appears that weightings can be very wild, it actually turns out the condition imposed in Definition [67.2] is rather strong.

0F3G **Lemma 68.1.** Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbb{Z}$ be a weighting of $f$. Then the level sets of the function $w$ are locally constructible in $X$.

**Proof.** The question is Zariski local on $X$ and $Y$, hence we may assume $X$ and $Y$ are affine. (Observe that the restriction of a weighting to an open is a weighting of the restriction of $f$ to that open.) By Lemma [67.3] after base change we still have a weighting. If we can find a surjective morphism $Y' \to Y$ of finite presentation such that the level sets of $w$ pull back to locally constructible subsets of $X' = Y' \times_Y X$, then we conclude by Morphisms, Theorem [21.3]. We may choose an immersion $X \to T$ where $T \to Y$ is finite, see Lemma [38.3] by Morphisms, Lemma [16.6] after replacing $Y$ by $Y'$ surjective finite locally free over $Y$, replacing $X$ by $Y' \times_Y X$ and $T$ by a scheme finite locally free over $Y'$ containing $Y' \times_Y T$ as a closed subscheme, we may assume $T$ is finite locally free over $Y$, contains closed subschemes $T_i$ mapping isomorphically to $Y$ such that $T = \bigcup_{i=1,\ldots,n} T_i$ (set theoretically). Observe that for $I \subset \{1,\ldots,n\}$ the intersection $\bigcap_{i \in I} T_i$ is a constructible closed subset of $T$ (and hence maps to a constructible closed subset of $Y$).
For a disjoint union decomposition \( \{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r \) with nonempty parts consider the subset \( Y_{I_1, \ldots, I_r} \subset Y \) consisting of points \( y \in Y \) such that \( T_y = \{x_1, \ldots, x_r\} \) consists of exactly \( r \) points with \( x_j \in T_i \Leftrightarrow i \in I_j \). This is a constructible partition of \( Y \), hence we may assume that \( Y = Y_{I_1, \ldots, I_r} \) for some disjoint union decomposition \( \{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r \). (You can base change by an affine scheme \( Y' \) of finite presentation over \( Y \) such that the image of \( Y' \to Y \) is exactly \( Y_{I_1, \ldots, I_r} \), see Algebra, Lemma [28.3].) Thus \( T = T(1) \amalg \ldots \amalg T(r) \) with \( T(j) = \bigcap_{i \in I_j} T_i \) is a decomposition of \( T \) into disjoint closed (and hence open) subsets. Intersecting with the locally closed subscheme \( X \) we obtain an analogous decomposition \( X = X(1) \amalg \ldots \amalg X(r) \) into open and closed parts. The morphism \( X(j) \to Y \) an immersion. Since \( w \) is a weighting, it follows that \( w|_{X(j)} \) is locally constant\(^{13}\) and we conclude. \( \square \)

**Lemma 68.2.** Let \( f : X \to Y \) be a locally quasi-finite morphism of finite presentation. Let \( w : X \to \mathbb{Z} \) be a weighting of \( f \). Then the level sets of the function \( \int f w \) are locally constructible in \( Y \).

**Proof.** By Lemma 67.1 formation of the function \( \int f w \) commutes with arbitrary base change and by Lemma 67.3 after base change we still have a weighting. This means that if we can find \( Y' \to Y \) surjective and of finite presentation, then it suffices to prove the result after base change to \( Y' \), see Morphisms, Theorem 21.3.

The question is local on \( Y \) hence we may assume \( Y \) is affine. Then \( X \) is quasi-compact and quasi-separated (as \( f \) is of finite presentation). Suppose that \( X = U \cup V \) are quasi-compact open. Then we have

\[
\int f w = \int f|_U w|_U + \int f|_V w|_V - \int f|_{U \cap V} w|_{U \cap V}
\]

Thus if we know the result for \( w|_U, w|_V, w|_{U \cap V} \) then we know the result for \( w \). By the induction principle (Cohomology of Schemes, Lemma 4.1) it suffices to prove the lemma when \( X \) is affine.

Assume \( X \) and \( Y \) are affine. We may choose an open immersion \( X \to T \) where \( T \to Y \) is finite, see Lemma 38.3. Because we may still base change with a suitable \( Y' \to Y \) we can use Morphisms, Lemma 46.6 to reduce to the case where all residue field extensions induced by the morphism \( T \to Y \) (and a fortiori induced by \( X \to Y \)) are trivial. In this situation \( \int f w \) is just taking the sums of the values of \( w \) in fibres. The level sets of \( w \) are locally constructible in \( X \) (Lemma 68.1). The function \( w \) only takes a finite number of values by Properties, Lemma 2.7. Hence we conclude by Morphisms, Theorem 21.3 and some elementary arguments on sums of integers. \( \square \)

**Lemma 68.3.** Let \( f : X \to Y \) be a locally quasi-finite morphism. Let \( w : X \to \mathbb{Z}_{>0} \) be a positive weighting of \( f \). Then \( w \) is upper semi-continuous.

**Proof.** Let \( x \in X \) with image \( y \in Y \). Choose an étale neighbourhood \( (V, v) \to (Y, y) \) and an open \( U \subset X_V \) such that \( \pi : U \to V \) is finite and there is a unique point \( u \in U \) mapping to \( v \) with \( k(u)/k(v) \) purely inseparable. See Lemma 36.3. Then \( (\int w|_U)(v) = w(u) \). It follows from Definition 67.2 that after replacing \( V \) by a neighbourhood of \( v \) we we have \( w|_U(u') \leq w|_U(u) = w(x) \) for all \( u' \in U \). Namely,

\(^{13}\)In fact, if \( f : X \to Y \) is an immersion and \( w \) is a weighting of \( f \), then \( f \) restricts to an open map on the locus where \( w \) is nonzero.
Lemma 68.4. Let \( f : X \rightarrow Y \) be a separated, locally quasi-finite morphism with finite fibres. Let \( w : X \rightarrow \mathbb{Z}_{\geq 0} \) be a positive weighting of \( f \). Then \( \int_f w \) is lower semi-continuous.

Proof. Let \( y \in Y \). Let \( x_1, \ldots, x_r \in X \) be the points lying over \( y \). Apply Lemma 36.5 to get an étale neighbourhood \((U, u) \rightarrow (Y, y)\) and a decomposition

\[
U \times_Y X = W \amalg \coprod_{i=1}^{n} \coprod_{j=1}^{m_i} V_{i,j}
\]

as in locus citatus. Observe that \((\int_f w)(y) = \sum w(v_{i,j})\) where \(w(v_{i,j}) = w(x_i)\). Since \(\int_{V_{i,j}} \rightarrow U \wedge \wedge V_{i,j}\) is locally constant by definition, we may after shrinking \(U\) assume that these functions are constant with \(w(v_{i,j})\). We conclude that

\[
\int_{U \times_Y X \rightarrow U \times U \wedge \wedge X} w|_{U \times U \wedge \wedge X} \wedge \wedge w + \sum \int_{V_{i,j} \rightarrow U \wedge \wedge V_{i,j}} w|_{V_{i,j}} = \int_{U \rightarrow U} w|_{U} + (\int_f w)(y)
\]

This is \(\geq (\int_f w)(y)\) and we conclude because \(U \rightarrow Y\) is open and formation of the integral commutes with base change (Lemma 67.1). \(\square\)

Lemma 68.5. Let \( f : X \rightarrow Y \) be a locally quasi-finite morphism with \( X \) quasi-compact. Let \( w : X \rightarrow \mathbb{Z} \) be a weighting of \( f \). Then \( \int_f w \) attains its maximum.

Proof. It follows from Lemma 68.1 and Properties, Lemma 2.7 that \(w \) only takes a finite number of values on \(X\). It follows from Morphisms, Lemma 54.10 that \(X \rightarrow Y\) has bounded geometric fibres. This shows that \(\int_f w \) is bounded. \(\square\)

Lemma 68.6. Let \( f : X \rightarrow Y \) be a separated, locally quasi-finite morphism. Let \( w : X \rightarrow \mathbb{Z}_{\geq 0} \) be a positive weighting of \( f \). Assume \(\int_w f\) attains its maximum \(d\) and let \(Y_d \subset Y\) be the open set of points \( y \) with \((\int_f w)(y) = d\). Then the morphism \(f^{-1}(Y_d) \rightarrow Y_d\) is finite.

Proof. Observe that \(Y_d\) is open by Lemma 68.4. Let \( y \in Y_d\). Say \(x_1, \ldots, x_n\) are the points of \(X\) lying over \(y\). Apply Lemma 36.5 to get an étale neighbourhood \((U, u) \rightarrow (Y, y)\) and a decomposition

\[
U \times_Y X = W \amalg \coprod_{i=1}^{n} \coprod_{j=1}^{m_i} V_{i,j}
\]

as in locus citatus. Observe that \(d = \sum w(v_{i,j})\) where \(w(v_{i,j}) = w(x_i)\). Since \(\int_{V_{i,j} \rightarrow U \wedge \wedge V_{i,j}} w|_{V_{i,j}}\) is locally constant by definition, we may after shrinking \(U\) assume that these functions are constant with \(w(v_{i,j})\). We conclude that

\[
\int_{U \times_Y X \rightarrow U \times U \wedge \wedge X} w|_{U \times U \wedge \wedge X} \wedge \wedge w + \sum \int_{V_{i,j} \rightarrow U \wedge \wedge V_{i,j}} w|_{V_{i,j}} = \int_{U \rightarrow U} w|_{U} + (\int_f w)(y)
\]

This is \(\geq (\int_f w)(y) = d\) and we conclude that \(W\) must be the emptyset. Thus \(U \times_Y X \rightarrow U\) is finite. By Descent, Lemma 20.23 this implies that \(X \rightarrow Y\) is finite over the image of the open morphism \(U \rightarrow Y\). In other words, we see that \(f\) is finite over an open neighbourhood of \(y\) as desired. \(\square\)

Lemma 68.7. Let \( A \rightarrow B \) be a ring map which is finite and of finite presentation. There exists a finitely presented ring map \( A \rightarrow A_{univ} \) and an idempotent \( e_{univ} \in B \otimes_A A_{univ} \) such that for any ring map \( A \rightarrow A' \) and idempotent \( e \in B \otimes_A A' \) there is a ring map \( A_{univ} \rightarrow A' \) mapping \( e_{univ} \) to \( e \).
Proof. Choose \( b_1, \ldots, b_n \in B \) generating \( B \) as an \( A \)-module. For each \( i \) choose a monic \( P_i \in A[x] \) such that \( P_i(b_i) = 0 \) in \( B \), see Algebra, Lemma \[35.3\] Thus \( B \) is a quotient of the finite \( A \)-algebra \( B' = A[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n)) \). Let \( J \subset B' \) be the kernel of the surjection \( B' \to B \). Then \( J = (f_1, \ldots, f_m) \) is finitely generated as \( B \) is a finitely generated \( A \)-algebra, see Algebra, Lemma \[6.2\]. Choose an \( A \)-basis \( b_1', \ldots, b_N' \) of \( B' \). Consider the algebra
\[
A_{univ} = A[z_1, \ldots, z_n, y_1, \ldots, y_m]/I
\]
where \( I \) is the ideal generated by the coefficients in \( A[z_1, \ldots, z_n, y_1, \ldots, y_m] \) of the basis elements \( b_1', \ldots, b_N' \) of the expression
\[
(\sum z_j b_j')^2 - \sum z_j b_j' + \sum y_k f_k
\]
in \( B'[z_1, \ldots, z_n, y_1, \ldots, y_m] \). By construction the element \( \sum z_j b_j' \) maps to an idempotent \( e_{univ} \) in the algebra \( B \otimes_A A_{univ} \). Moreover, if \( e \in B \otimes_A A' \) is an idempotent, then we can lift \( e \) to an element of the form \( \sum b_j' \otimes a_j' \) in \( B' \otimes_A A' \) and we can find \( a_k'' \in A' \) such that
\[
(\sum b_j' \otimes a_j')^2 - \sum b_j' \otimes a_j' + \sum f_k \otimes a_k''
\]
is zero in \( B' \otimes_A A' \). Hence we get an \( A \)-algebra map \( A_{univ} \to A \) sending \( z_j \) to \( a_j' \) and \( y_k \) to \( a_k'' \) mapping \( e_{univ} \) to \( e \). This finishes the proof. \( \square \)

Lemma \[68.8\]. Let \( X \to Y \) be a morphism of affine schemes which is quasi-finite and of finite presentation. There exists a morphism \( Y_{univ} \to Y \) of finite presentation and an open subscheme \( U_{univ} \subset Y_{univ} \times_Y X \) such that \( U_{univ} \to Y_{univ} \) is finite with the following property: given any morphism \( Y' \to Y \) of affine schemes and an open subscheme \( U' \subset Y' \times_Y X \) such that \( U' \to Y' \) is finite, there exists a morphism \( Y' \to Y_{univ} \) such that the inverse image of \( U_{univ} \) is \( U' \).

Proof. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma \[28.5\] By Lemma \[30.9\] applied with \( d = 0 \) we reduce to the case where \( X \) and \( Y \) are Noetherian. We may choose an open immersion \( X \to X' \) such that \( X' \to Y \) is finite, see Algebra, Lemma \[122.14\] Note that if we have \( Y' \to Y \) and \( U' \) as in (2), then
\[
U' \to Y' \times_Y X \to Y' \times_Y X'
\]
is open immersion between schemes finite over \( Y' \) and hence is closed as well. We conclude that \( U' \) corresponds to an idempotent in
\[
\Gamma(Y', \mathcal{O}_{Y'}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})
\]
whose corresponding open and closed subset is contained in the open \( Y' \times_Y X \). Let \( Y'_{univ} \to Y \) and idempotent
\[
e_{univ}' \in \Gamma(Y_{univ}, \mathcal{O}_{Y_{univ}}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})
\]
be the pair constructed in Lemma \[68.7\] for the ring map \( \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X', \mathcal{O}_{X'}) \) (here we use that \( Y \) is Noetherian to see that \( X' \) is of finite presentation over \( Y \)). Let \( U_{univ}' \subset Y_{univ} \times_Y X' \) be the corresponding open and closed subscheme. Then we see that
\[
U_{univ}' \setminus Y_{univ} \times_Y X
\]
In this section we give a bound for the affine stratification number of a scheme. Let $Y = \lim_{i \to \infty} Y_i$ be a directed limit of affine schemes. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $f : X \to Y$ and $f_i : X_i \to Y_i$ for $i \geq 0$ be the base changes of $f_0$. If $w : X \to \mathbb{Z}$ is a weighting of $f$, then for sufficiently large $i$ there exists a weighting $w_i : X_i \to \mathbb{Z}$ of $f_i$ whose pullback to $X$ is $w$.

**Proof.** By Lemma 68.4 the level sets of $w$ are constructible subsets $E_k$ of $X$. This implies the function $w$ only takes a finite number of values by Properties, Lemma 27. Thus there exists an $i$ such that $E_k$ descends to a constructible subset $E_{i,k}$ in $X_i$ for all $i$; moreover, we may assume $X_i = \coprod E_{i,k}$. This follows as the topological space of $X$ is the limit in the category of topological spaces of the spectral spaces $X_i$ along a directed system with spectral transition maps. See Limits, Section 4 and Topology, Section 24. We define $w_i : X_i \to \mathbb{Z}$ such that its level sets are the constructible sets $E_{i,k}$.

Choose $Y_{i,\text{univ}} \to Y_i$ and $U_{i,\text{univ}} \subset Y_{i,\text{univ}} \times Y_i X_i$ as in Lemma 68.8. By the universal property of the construction, in order to show that $w_i$ is a weighting, it would suffice to show that

$$\tau_i = \int_{U_{i,\text{univ}}} w_i|U_{i,\text{univ}}$$

is locally constant on $Y_{i,\text{univ}}$. By Lemma 68.2 this function has constructible level sets but it may not (yet) be locally constant. Set $Y_{\text{univ}} = Y_{i,\text{univ}} \times Y_i Y$ and let $U_{\text{univ}} \subset Y_{\text{univ}} \times Y X$ be the inverse image of $U_{i,\text{univ}}$. Then, since the pullback of $w$ to $Y_{\text{univ}} \times Y X$ is a weighting for $Y_{\text{univ}} \times Y X \to Y_{\text{univ}}$ (Lemma 67.3) we do have that

$$\tau = \int_{U_{\text{univ}}} w_i|U_{\text{univ}}$$

is locally constant on $Y_{\text{univ}}$. Thus the level sets of $\tau$ are open and closed. Finally, we have $Y_{\text{univ}} = \lim_{i \to \infty} Y_{i,\text{univ}}$ and the level sets of $\tau$ are the inverse limits of the level sets of $\tau_i$ (similarly defined). Hence the references above imply that for sufficiently large $i'$ the level sets of $\tau_i'$ are open as well. For such an index $i'$ we conclude that $w_{i'}$ is a weighting of $f_{i'}$ as desired.

### 69. Weightings and affine stratification numbers

**Lemma 69.1.** Let $f : X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $w : X \to \mathbb{Z}_{\geq 0}$ be a positive weighting of $f$. Let $d < \infty$ be the maximum value of $\int_f w$. The open

$$Y_d = \{ y \in Y \mid (\int_f w)(y) = d \}$$

of $Y$ is affine.

**Proof.** Observe that $\int_f w$ attains its maximum by Lemma 68.5. The set $Y_d$ is open by Lemma 68.4. Thus the statement of the lemma makes sense.
Reduction to the Noetherian case; please skip this paragraph. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 28.5. By Lemma 30.9 applied with $d = 0$ we can find a quasi-finite morphism $f_0 : X_0 \to Y_0$ of affine Noetherian schemes and a morphism $Y \to Y_0$ such that $f$ is the base change of $f_0$. Then we can write $Y = \lim Y_i$ as a directed limit of affine schemes of finite type over $Y_0$, see Algebra, Lemma 126.2. By Lemma 68.9 we can find an $i$ such that our weighting $w$ descends to a weighting $w_i$ of the base change $f_i : X_i \to Y_i$ of $f_0$. Now if the lemma holds for $f_i, w_i$, then it implies the lemma for $f$ as formation of $\int_f w$ commutes with base change, see Lemma 67.1.

Assume $X$ and $Y$ Noetherian. Let $X' \to Y'$ be the base change of $f$ by a morphism $g : Y' \to Y$. The formation of $\int_f w$ and hence the open $Y_d$ commute with base change. If $g$ is finite and surjective, then $Y'_d \to Y_d$ is finite and surjective. In this case proving that $Y_d$ is affine is equivalent to showing that $Y'_d$ is affine, see Cohomology of Schemes, Lemma 13.3.

We may choose an immersion $X \to T$ with $T$ finite over $Y$, see Lemma 38.3. We are going to apply Morphisms, Lemma 46.4 to the finite morphism $T \to Y$. This lemma tells us that there is a finite surjective morphism $Y' \to Y$ such that $Y' \times_Y T$ is a closed subscheme of a scheme $T'$ finite over $Y'$ which has a special form. By the discussion in the first paragraph, we may replace $Y$ by $Y'$, $T$ by $T'$, and $X$ by $Y' \times_Y X$. Thus we may assume there is an immersion $X \to T$ (not necessarily open or closed) and closed subschemes $T_i \subset T$, $i = 1, \ldots, n$ where

1. $T \to Y$ is finite (and locally free),
2. $T_i \to Y$ is an isomorphism, and
3. $T = \bigcup_{i=1,..,n} T_i$ set theoretically.

Let $Y' = \coprod Y_k$ be the disjoint union of the irreducible components of $Y$ (viewed as integral closed subschemes of $Y$). Then we may base change once more by $Y' \to Y$; here we are using that $Y$ is Noetherian. Thus we may in addition assume $Y$ is integral and Noetherian.

We also may and do assume that $T_i \neq T_j$ if $i \neq j$ by removing repeats. Since $Y$ and hence all $T_i$ are integral, this means that if $T_i$ and $T_j$ intersect, then they intersect in a closed subset which maps to a proper closed subset of $Y$.

Observe that $V_i = X \cap T_i$ is a locally closed subset which is in addition a closed subscheme of $X$ hence affine. Let $\eta \in Y$ and $\eta_i \in T_i$ be the generic points. If $\eta \not\in Y_d$, then $Y_d = \emptyset$ and we're done. Assume $\eta \in Y_d$. Denote $I = \{1, \ldots, n\}$ the subset of indices $i$ such that $\eta_i \in V_i$. For $i \in I$ the locally closed subset $V_i \subset T_i$ contains the generic point of the irreducible space $T_i$ and hence is open. On the other hand, since $f$ is open (Lemma 67.4), for any $x \in X$ we can find an $i \in I$ and a specialization $\eta_i \leadsto x$. It follows that $x \in T_i$ and hence $x \in V_i$. In other words, we see that $X = \bigcup_{i \in I} V_i$ set theoretically. We claim that $Y_d = \bigcap_{i \in I} \text{Im}(V_i \to Y)$; this will finish the proof as the intersection of affine opens $\text{Im}(V_i \to Y)$ of $Y$ is affine.

For $y \in Y$ let $f^{-1}\{y\} = \{x_1, \ldots, x_r\} \subset X$. For each $i \in I$ there is at most one $j(i) \in \{1, \ldots, r\}$ such that $\eta_i \leadsto x_{j(i)}$. In fact, $j(i)$ exists and is equal to $j$ if and only if $x_j \in V_i$. If $i \in I$ is such that $j = j(i)$ exists, then $V_i \to Y$ is an isomorphism in a neighbourhood of $x_j \mapsto y$. Hence $\bigcup_{i \in I, j(i) = j} V_i \to Y$ is finite after replacing source and target by neighbourhoods of $x_j \mapsto y$. Thus the definition of a weighting
MORE ON MORPHISMS 212

0F3S Proposition 69.2. Let \( f : X \rightarrow Y \) be a surjective quasi-finite morphism of schemes. Let \( w : X \rightarrow \mathbb{Z}_{>0} \) be a positive weighting of \( f \). Assume \( X \) affine and \( Y \) separated. Then the affine stratification number of \( Y \) is at most the number of distinct values of \( \int f w \).

Proof. Note that since \( Y \) is separated, the morphism \( X \rightarrow Y \) is affine (Morphisms, Lemma 11.11). The function \( \int f w \) attains its maximum \( d \) by Lemma 68.5. We will use induction on \( d \). Consider the open subscheme \( Y_d = \{ y \in Y \mid (\int f w)(y) = d \} \) of \( Y \) and recall that \( f^{-1}(Y_d) \rightarrow Y_d \) is finite, see Lemma 68.6. By Lemma 69.1 for every affine open \( W \subset Y \) we have that \( Y_d \cap W \) is affine (this uses that \( W \times_Y X \) is affine, being affine over \( X \)). Hence \( Y_d \rightarrow Y \) is an affine morphism of schemes. We conclude that \( f^{-1}(Y_d) = Y_d \times_Y X \) is an affine scheme being affine over \( X \). Then \( f^{-1}(Y_d) \rightarrow Y_d \) is surjective and hence \( Y_d \) is affine by Limits, Lemma 11.1. Set \( X' = X \setminus f^{-1}(Y_d) \) and \( Y' = Y \setminus Y_d \) viewed as closed subschemes of \( X \) and \( Y \). Since \( X' \) is closed in \( X \) it is affine. Since \( Y' \) is closed in \( Y \) it is separated. The morphism \( f' : X' \rightarrow Y' \) is surjective and \( w \) induces a weighting \( w' \) of \( f' \), see Lemma 67.3. By induction \( Y' \) has an affine stratification of length \( \leq \) the number of distinct values of \( \int f' w' \) and the proof is complete. □

70. Other chapters

Preliminaries

<table>
<thead>
<tr>
<th>Preliminaries</th>
<th>Schemes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Introduction</td>
<td>(21) Cohomology on Sites</td>
</tr>
<tr>
<td>(2) Conventions</td>
<td>(22) Differential Graded Algebra</td>
</tr>
<tr>
<td>(3) Set Theory</td>
<td>(23) Divided Power Algebra</td>
</tr>
<tr>
<td>(4) Categories</td>
<td>(24) Differential Graded Sheaves</td>
</tr>
<tr>
<td>(5) Topology</td>
<td>(25) Hypercoverings</td>
</tr>
<tr>
<td>(6) Sheaves on Spaces</td>
<td>(25) Schemes</td>
</tr>
<tr>
<td>(7) Sites and Sheaves</td>
<td>(26) Constructions of Schemes</td>
</tr>
<tr>
<td>(8) Stacks</td>
<td>(27) Properties of Schemes</td>
</tr>
<tr>
<td>(9) Fields</td>
<td>(28) Morphisms of Schemes</td>
</tr>
<tr>
<td>(10) Commutative Algebra</td>
<td>(29) Cohomology of Schemes</td>
</tr>
<tr>
<td>(11) Brauer Groups</td>
<td>(30) Divisors</td>
</tr>
<tr>
<td>(12) Homological Algebra</td>
<td>(31) Limits of Schemes</td>
</tr>
<tr>
<td>(13) Derived Categories</td>
<td>(32) Varieties</td>
</tr>
<tr>
<td>(14) Simplicial Methods</td>
<td>(33) Topologies on Schemes</td>
</tr>
<tr>
<td>(15) More on Algebra</td>
<td>(34) Descent</td>
</tr>
<tr>
<td>(16) Smoothing Ring Maps</td>
<td>(35) Derived Categories of Schemes</td>
</tr>
<tr>
<td>(17) Sheaves of Modules</td>
<td>(36) More on Morphisms</td>
</tr>
<tr>
<td>(18) Modules on Sites</td>
<td>(37) More on Flatness</td>
</tr>
<tr>
<td>(19) Injectives</td>
<td>(38) Groupoid Schemes</td>
</tr>
<tr>
<td>(20) Cohomology of Sheaves</td>
<td>(39) More on Groupoid Schemes</td>
</tr>
</tbody>
</table>

14It suffices if the diagonal of \( Y \) is affine.


