MORE ON MORPHISMS

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1. **Introduction**

02GY In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [DG67].

2. **Thickenings**

04EW The following terminology may not be completely standard, but it is convenient.

04EX **Definition 2.1.** Thickenings.

1. We say a scheme $X'$ is a *thickening* of a scheme $X$ if $X$ is a closed subscheme of $X'$ and the underlying topological spaces are equal.

2. We say a scheme $X'$ is a *first order thickening* of a scheme $X$ if $X$ is a closed subscheme of $X'$ and the quasi-coherent sheaf of ideals $I \subset \mathcal{O}_{X'}$ defining $X$ has square zero.

3. Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f'(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme $Y$. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.

4. Let $S$ be a scheme. We similarly define *thickenings over $S$*, and *morphisms of thickenings over $S*.
schemes over $S$, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over $S$.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a finite order thickening if the ideal sheaf $\mathcal{I}$ is “globally” nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that $X'$ is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over $S$. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickenings are described as follows (see Modules, Lemma 25.11).

**Lemma 2.2.** Let $X$ be a scheme over a base $S$. Consider a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X$ where $\mathcal{A}$ is a sheaf of $\mathcal{f}^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $\mathcal{f}^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

1. $\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and
2. $\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over $S$. Moreover, any first order thickening over $S$ is of this form.

**Proof.** It is clear that $X'$ is a locally ringed space. Let $U = \text{Spec}(B)$ be an affine open of $X$. Set $A = \Gamma(U, \mathcal{A})$. Note that since $\mathcal{H}^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring $A$. By Schemes, Lemma 6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \to \text{Spec}(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$(U, \mathcal{O}_X|_U) \to \text{Spec}(B) \to \text{Spec}(A)$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$0 \to I_\mathfrak{p} \to A_\mathfrak{p} \to B_\mathfrak{p} \to 0$$

and

$$0 \to I_u \to A_u \to \mathcal{O}_{X,u} \to 0.$$
The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and $X'$ is a scheme as desired. □

**06AD Lemma 2.3.** Any thickening of an affine scheme is affine.

**Proof.** This is a special case of Limits, Proposition [11.2] □

**Proof for a finite order thickening.** Suppose that $X \subset X'$ is a finite order thickening with $X$ affine. Then we may use Serre’s criterion to prove $X'$ is affine.

More precisely, we will use Cohomology of Schemes, Lemma [3.1]. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. It suffices to show that $H^1(X', \mathcal{F}) = 0$. Denote $i : X \to X'$ the given closed immersion and denote $\mathcal{I} = \text{Ker}(i^*: \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$. By our discussion of finite order thickenings (following Definition [2.1]) there exists an $n \geq 0$ and a filtration

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

by quasi-coherent submodules such that $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$. Namely, we can take $\mathcal{F}_a = i^a\mathcal{F}$. Then $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}_a$, see Morphisms, Lemma [4.1]. We obtain

$$H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) = H^1(X', i_*\mathcal{G}_a) = H^1(X, \mathcal{G}_a) = 0$$

The second equality comes from Cohomology of Schemes, Lemma [2.4] and the last equality from Cohomology of Schemes, Lemma [2.2]. Thus $\mathcal{F}$ has a finite filtration whose successive quotients have vanishing first cohomology and it follows by a simple induction argument that $H^1(X', \mathcal{F}) = 0$. □

**09ZU Lemma 2.4.** Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.

**Proof.** Omitted. □

**0BPE Lemma 2.5.** If $S \subset S'$ and $S' \subset S''$ are thickenings, then so is $S \subset S''$.

**Proof.** Omitted. □

**0BPF Lemma 2.6.** The property of being a thickening is fpqc local. Similarly for first order thickenings.

**Proof.** The statement means the following: Let $X \to X'$ be a morphism of schemes and let $\{g_1 : X_1 \to X\}$ be an fpqc covering such that the base change $X_i \to X'$ is a thickening for all $i$. Then $X \to X'$ is a thickening. Since the morphisms $g_i$ are jointly surjective we conclude that $X \to X'$ is surjective. By Descent, Lemma [20.19] we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

**3. Morphisms of thickenings**

**0CF2** If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of schemes, then often properties of the morphism $f$ are inherited by $f'$. There are several variants.

**09ZV Lemma 3.1.** Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Then
(1) \( f \) is an affine morphism if and only if \( f' \) is an affine morphism,
(2) \( f \) is a surjective morphism if and only if \( f' \) is a surjective morphism,
(3) \( f \) is quasi-compact if and only if \( f' \) quasi-compact,
(4) \( f \) is universally closed if and only if \( f' \) is universally closed,
(5) \( f \) is integral if and only if \( f' \) is integral,
(6) \( f \) is (quasi-)separated if and only if \( f' \) is (quasi-)separated,
(7) \( f \) is universally injective if and only if \( f' \) is universally injective,
(8) \( f \) is universally open if and only if \( f' \) is universally open,
(9) \( f \) is quasi-affine if and only if \( f' \) is quasi-affine, and
(10) add more here.

**Proof.** Observe that \( S \to S' \) and \( X \to X' \) are universal homeomorphisms (see for example Morphisms, Lemma 43.6). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 2.3 which tells us that there is a 1-to-1 correspondence between affine opens of \( S \) and \( S' \) and between affine opens of \( X \) and \( X' \). Part (9) follows from Limits, Lemma 11.5 and the remark just made about affine opens of \( S \) and \( S' \). Part (5) follows from (1) and (4) by Morphisms, Lemma 42.7. Finally, note that 
\[
\begin{align*}
S \times_X S &= S \times_X S' \to S \times_X S' \to S' \times_{X'} S'
\end{align*}
\]
is a thickening (the two arrows are thickenings by Lemma 2.4). Hence applying (3) and (4) to the morphism \( (S \subset S') \to (S \times_X S \to S' \times_{X'} S') \) we obtain (6). 

**Lemma 3.2.** Let \((f, f') : (X \subset X') \to (S \subset S')\) be a morphism of thickenings. Let \( \mathcal{L}' \) be an invertible sheaf on \( X' \) and denote \( \mathcal{L} \) the restriction to \( X \). Then \( \mathcal{L}' \) is \( f' \)-ample if and only if \( \mathcal{L} \) is \( f \)-ample.

**Proof.** Recall that being relatively ample is a condition for each affine open in the base, see Morphisms, Definition 35.1. By Lemma 2.3 there is a 1-to-1 correspondence between affine opens of \( S \) and \( S' \). Thus we may assume \( S \) and \( S' \) are affine and we reduce to proving that \( \mathcal{L}' \) is ample if and only if \( \mathcal{L} \) is ample. This is Limits, Lemma 11.4.

**Lemma 3.3.** Let \((f, f') : (X \subset X') \to (S \subset S')\) be a morphism of thickenings such that \( X = S \times_{S'} X' \). If \( S \subset S' \) is a finite order thickening, then
(1) \( f \) is a closed immersion if and only if \( f' \) is a closed immersion,
(2) \( f \) is locally of finite type if and only if \( f' \) locally of finite type,
(3) \( f \) is locally quasi-finite if and only if \( f' \) locally quasi-finite,
(4) \( f \) is locally of finite type of relative dimension \( d \) if and only if \( f' \) locally of finite type of relative dimension \( d \),
(5) \( \Omega_{X/S} = 0 \) if and only if \( \Omega_{X'/S'} = 0 \),
(6) \( f \) is unramified if and only if \( f' \) is unramified,
(7) \( f \) is proper if and only if \( f' \) is proper,
(8) \( f \) is finite if and only if \( f' \) is finite,
(9) \( f \) is a monomorphism if and only if \( f' \) is a monomorphism,
(10) \( f \) is an immersion if and only if \( f' \) is an immersion, and
(11) add more here.

**Proof.** The properties \( P \) listed in the lemma are all stable under base change, hence if \( f' \) has property \( P \), then so does \( f \). See Schemes, Lemmas 18.2 and 23.5 and Morphisms, Lemmas 14.4, 19.13, 28.2, 31.10, 33.5, 39.5, and 42.6.
The interesting direction in each case is therefore to assume that \( f \) has the property and deduce that \( f' \) has it too. By induction on the order of the thickening we may assume that \( S \subset S' \) is a first order thickening, see discussion immediately following Definition \[2.1.\]

Most of the proofs will use a reduction to the affine case. Let \( U' \subset S' \) be an affine open and let \( V' \subset X' \) be an affine open lying over \( U' \). Let \( U' = \text{Spec}(A') \) and denote \( I \subset A' \) be the ideal defining the closed subscheme \( U' \cap S \). Say \( V' = \text{Spec}(B') \). Then \( V' \cap X = \text{Spec}(B'/IB') \). Setting \( A = A'/I \) and \( B = B'/IB' \) we get a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & IB' & \rightarrow & B & \rightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 & \rightarrow & IA' & \rightarrow & A' & \rightarrow & 0
\end{array}
\]

with exact rows and \( I^2 = 0 \).

The translation of (1) into algebra: If \( A \rightarrow B \) is surjective, then \( A' \rightarrow B' \) is surjective. This follows from Nakayama’s lemma (Algebra, Lemma \[19.1\]).

The translation of (2) into algebra: If \( A \rightarrow B \) is a finite type ring map, then \( A' \rightarrow B' \) is a finite type ring map. This follows from Nakayama’s lemma (Algebra, Lemma \[19.1\]) applied to a map \( A'[x_1, \ldots, x_n] \rightarrow B' \) such that \( A[x_1, \ldots, x_n] \rightarrow B \) is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma \[19.6\].

Proof of (4). Follows from (2) and that the additional property of “being of relative dimension \( d \)” can be checked on fibres (by definition, see Morphisms, Definition \[28.1\]).

The translation of (5) into algebra: If \( \Omega_{B/A} = 0 \), then \( \Omega_{B'/A'} = 0 \). By Algebra, Lemma \[130.12\] we have \( 0 = \Omega_{B/A} = \Omega_{B'/A'}/I\Omega_{B'/A'} \). Hence \( \Omega_{B'/A'} = 0 \) by Nakayama’s lemma (Algebra, Lemma \[19.1\]).

The translation of (6) into algebra: If \( A \rightarrow B \) is unramified map, then \( A' \rightarrow B' \) is unramified. Since \( A \rightarrow B \) is of finite type we see that \( A' \rightarrow B' \) is of finite type by (2) above. Since \( A \rightarrow B \) is unramified we have \( \Omega_{B/A} = 0 \). By part (5) we have \( \Omega_{B'/A'} = 0 \). Thus \( A' \rightarrow B' \) is unramified.

Proof of (7). Follows by combining (2) with results of Lemma \[3.1\] and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma \[3.1\] and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma \[42.4\]).

Proof of (9). As \( f \) is a monomorphism we have \( X = X \times_S X \). We may apply the results proved so far to the morphism of thickenings \( X \subset X' \) → \( X \times_S X' \subset X' \times_S X' \). We conclude \( X' \rightarrow X' \times_S X' \) is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion \( X' \rightarrow X' \times_S X' \) is contained in the pullback of the ideal \( I \subset \mathcal{O}_S \) cutting out \( S \) in \( S' \). Indeed, \( X = X \times_S X = (X' \times_S X') \times_S S \) is contained in \( X' \). Hence by Morphisms,
Lemma 31.7 it suffices to show that $\Omega_{X'/S'} = 0$ which follows from (5) and the corresponding statement for $X/S$.

Proof of (10). If $f : X \to S$ is an immersion, then it factors as $X \to U \to S$ where $U \to S$ is an open immersion and $X \to U$ is a closed immersion. Let $U' \subset S'$ be the open subscheme whose underlying topological space is the same as $U$. Then $X' \to S'$ factors through $U'$ and we conclude that $X' \to U'$ is a closed immersion by part (1). This finishes the proof. □

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from $f$ to $f'$), we instead take as given that each of $f$ and $f'$ is locally of finite type.

Lemma 3.4. Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings. Assume $f$ and $f'$ are locally of finite type and $X = Y \times_{Y'} X'$. Then

1. $f$ is locally quasi-finite if and only if $f'$ is locally quasi-finite,
2. $f$ is finite if and only if $f'$ is finite,
3. $f$ is a closed immersion if and only if $f'$ is a closed immersion,
4. $\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,
5. $f$ is unramified if and only if $f'$ is unramified,
6. $f$ is a monomorphism if and only if $f'$ is a monomorphism,
7. $f$ is an immersion if and only if $f'$ is an immersion,
8. $f$ is proper if and only if $f'$ is proper, and
9. add more here.

Proof. The properties $P$ listed in the lemma are all stable under base change, hence if $f'$ has property $P$, then so does $f$. See Schemes, Lemmas 18.2 and 23.5 and Morphisms, Lemmas 19.13, 28.2, 31.10, 33.5, 39.5, and 42.6. Hence in each case we need only to prove that if $f$ has the desired property, so does $f'$.

A morphism is locally quasi-finite if and only if it is locally of finite type and the scheme theoretic fibres are discrete spaces, see Morphisms, Lemma 19.8. Since the underlying topological space is unchanged by passing to a thickening, we see that $f'$ is locally quasi-finite if (and only if) $f$ is. This proves (1).

Case (2) follows from case (5) of Lemma 3.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms, Lemma 42.4).

Case (3). This follows immediately from Morphisms, Lemma 33.12.

Case (4) follows from the following algebra statement: Let $A$ be a ring and let $I \subset A$ be a locally nilpotent ideal. Let $B$ be a finite type $A$-algebra. If $\Omega_{(B/IB)/(A/I)} = 0$, then $\Omega_B/A = 0$. Namely, the assumption means that $I\Omega_B/A = 0$, see Algebra, Lemma 130.12. On the other hand $\Omega_B/A$ is a finite $B$-module, see Algebra, Lemma 130.16. Hence the vanishing of $\Omega_B/A$ follows from Nakayama’s lemma (Algebra, Lemma 19.1), and the fact that $IB$ is contained in the radical of $B$.

Case (5) follows immediately from (4) and Morphisms, Lemma 33.2.

Proof of (6). As $f$ is a monomorphism we have $X = X \times_{Y'} X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_{Y'} X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by
(3). In fact $\Delta_{X'/Y'}$ is a bijection on underlying sets, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms, Lemma \[20.12\]. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_Y X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_Y X'$ is zero by Morphisms, Lemma \[31.7\]. In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma \[20.5\].

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open immersion and $X \to V$ is a closed immersion. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is the same as $V$. Then $X' \to V'$ factors through $V'$ and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma \[3.1\] and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. □

4. Picard groups of thickenings

0C6Q Some material on Picard groups of thickenings.

0C6R Lemma 4.1. Let $X \subset X'$ be a first order thickening with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence

$$
0 \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X', \mathcal{O}_{X'}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow \\
\text{Pic}(X') \longrightarrow \text{Pic}(X) \longrightarrow \cdots
$$

of abelian groups.

Proof. This is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$
0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0
$$

where the first map sends a local section $f$ of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed space with the first cohomology group of the sheaf of invertible functions, see Cohomology, Lemma \[7.1\]. □

0C6S Lemma 4.2. Let $X \subset X'$ be a thickening. Let $n$ be an integer invertible in $\mathcal{O}_X$. Then the map $\text{Pic}(X')[n] \to \text{Pic}(X)[n]$ is bijective.

Proof for a finite order thickening. By the general principle explained following Definition \[2.1\] this reduces to the case of a first order thickening. Then may use Lemma \[2.1\] to see that it suffices to show that $H^1(X, \mathcal{I})[n]$, $H^1(X, \mathcal{I})/n$, and $H^2(X, \mathcal{I})[n]$ are zero. This follows as multiplication by $n$ on $\mathcal{I}$ is an isomorphism as it is an $\mathcal{O}_X$-module. □
Proof in general. Let \( I \subset \mathcal{O}_X \) be the quasi-coherent ideal sheaf cutting out \( X \). Then we have a short exact sequence of abelian groups

\[
0 \to (1 + I)^* \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0
\]

We obtain a long exact cohomology sequence as in the statement of Lemma 4.1 with \( H^i(X, I) \) replaced by \( H^i(X, (1 + I)^*) \). Thus it suffices to show that raising to the \( n \)th power is an isomorphism \( (1 + I)^* \to (1 + I)^* \). Taking sections over affine opens this follows from Algebra, Lemma 31.7.

5. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that \( i : Z \to X \) be an immersion of schemes. Choose an open subscheme \( U \subset X \) such that \( i \) identifies \( Z \) with a closed subscheme \( Z \subset U \). Let \( I \subset \mathcal{O}_U \) be the quasi-coherent sheaf of ideals defining \( Z \) in \( U \). Then we can consider the closed subscheme \( Z' \subset U \) defined by the quasi-coherent sheaf of ideals \( I^2 \).

Definition 5.1. Let \( i : Z \to X \) be an immersion of schemes. The first order infinitesimal neighbourhood of \( Z \) in \( X \) is the first order thickening \( Z \subset Z' \) over \( X \) described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open \( U \)).

Lemma 5.2. Let \( i : Z \to X \) be an immersion of schemes. The first order infinitesimal neighbourhood \( Z' \) of \( Z \) in \( X \) has the following universal property: Given any commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{a} & T \\
\downarrow{i} & & \downarrow{b} \\
X & \xleftarrow{a'} & T'
\end{array}
\]

where \( T \subset T' \) is a first order thickening over \( X \), there exists a unique morphism \((a', a) : (T \subset T') \to (Z \subset Z')\) of thickenings over \( X \).

Proof. Let \( U \subset X \) be the open used in the construction of \( Z' \), i.e., an open such that \( Z \) is identified with a closed subscheme of \( U \) cut out by the quasi-coherent sheaf of ideals \( I \). Since \( |T| = |T'| \) we see that \( b(T') \subset U \). Hence we can think of \( b \) as a morphism into \( U \). Let \( J \subset \mathcal{O}_T \) be the ideal cutting out \( T \). Since \( b(T) \subset Z \) by the diagram above we see that \( b^2(b^{-1}(T)) \subset J \). As \( T' \) is a first order thickening of \( T \) we see that \( J^2 = 0 \) hence \( b^2(b^{-1}(T^2)) = 0 \). By Schemes, Lemma 4.6 this implies that \( b \) factors through \( Z' \). Denote \( a' : T' \to Z' \) this factorization and everything is clear.

Lemma 5.3. Let \( i : Z \to X \) be an immersion of schemes. Let \( Z \subset Z' \) be the first order infinitesimal neighbourhood of \( Z \) in \( X \). Then the diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & Z' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

induces a map of conormal sheaves \( \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'} \) by Morphisms, Lemma 30.3. This map is an isomorphism.
Proof. This is clear from the construction of $Z'$ above. □

6. Formally unramified morphisms

Recall that a ring map $R \to A$ is called formally unramified (see Algebra, Definition 144.1) if for every commutative solid diagram

$$
\begin{array}{ccc}
A & \to & B/I \\
\downarrow & & \downarrow \\
R & \to & B
\end{array}
$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 6.1. Let $f : X \to S$ be a morphism of schemes. We say $f$ is formally unramified if given any solid commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & T \\
\downarrow & & \downarrow \\
S & \leftarrow & T'
\end{array}
$$

where $T \subset T'$ is a first order thickening of affine schemes over $S$ there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

Lemma 6.2. If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & T \\
\downarrow & & \downarrow \\
S & \leftarrow & T'
\end{array}
$$

where $T \subset T'$ is a first order thickening of schemes over $S$ there exists at most one dotted arrow making the diagram commute. In other words, in Definition 6.1 the condition that $T$ be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. □

Lemma 6.3. A composition of formally unramified morphisms is formally unramified.

Proof. This is formal. □

Lemma 6.4. A base change of a formally unramified morphism is formally unramified.

Proof. This is formal. □

Lemma 6.5. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If $f$ is formally unramified, so is $f|_U : U \to V$. 
Proof. Consider a solid diagram

\[ U \leftarrow T \]
\[ \downarrow f \]
\[ V \leftarrow T' \]

as in Definition 6.1. If \( f \) is formally ramified, then there exists at most one \( S \)-morphism \( a' : T' \to X \) such that \( a'|_T = a \). Hence clearly there exists at most one such morphism into \( U \).

\[ \square \]

Lemma 6.6. Let \( f : X \to S \) be a morphism of schemes. Assume \( X \) and \( S \) are affine. Then \( f \) is formally unramified if and only if \( \mathcal{O}_S(S) \to \mathcal{O}_X(X) \) is a formally unramified ring map.

Proof. This is immediate from the definitions (Definition 6.1 and Algebra, Definition 144.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. \( \square \)

Here is a characterization in terms of the sheaf of differentials.

Lemma 6.7. Let \( f : X \to S \) be a morphism of schemes. Then \( f \) is formally unramified if and only if \( \Omega_{X/S} = 0 \).

Proof. We give two proofs.

First proof. It suffices to show that \( \Omega_{X/S} \) is zero on the members of an affine open covering of \( X \). Choose an affine open \( U \subset X \) with \( f(U) \subset V \) where \( V \subset S \) is an affine open of \( S \). By Lemma 6.5, the restriction \( f_U : U \to V \) is formally unramified. By Morphisms, Lemma 31.5 we see that \( \Omega_{X/S}|_U \) is the quasi-coherent sheaf associated to the \( \mathcal{O}_X(U) \)-module \( \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)} \). By Lemma 6.6 we see that \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is a formally unramified ring map. Hence by Algebra, Lemma 144.2 we conclude that \( \Omega_{X/S}|_U = 0 \) as desired.

Second proof. We recall some of the arguments of the proof of Morphisms, Lemma 31.5. Let \( W \subset X \times_S X \) be an open such that \( \Delta : X \to X \times_S X \) induces a closed immersion into \( W \). Let \( \mathcal{J} \subset \mathcal{O}_W \) be the ideal sheaf of this closed immersion. Let \( X' \subset W \) be the closed subscheme defined by the quasi-coherent sheaf of ideals \( \mathcal{J}^2 \). Consider the two morphisms \( p_1, p_2 : X' \to X \) induced by the two projections \( X \times_S X \to X \). Note that \( p_1 \) and \( p_2 \) agree when composed with \( \Delta : X \to X' \) and that \( X \to X' \) is a closed immersion defined by an ideal whose square is zero. Moreover there is a short exact sequence

\[ 0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \]

and \( \Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 \). Moreover, \( \mathcal{J}/\mathcal{J}^2 \) is generated by the local sections \( p_1^*(f) - p_2^*(f) \) for \( f \) a local section of \( \mathcal{O}_X \).

Suppose that \( f : X \to S \) is formally unramified. By assumption this means that \( p_1 = p_2 \) when restricted to any affine open \( T' \subset X' \). Hence \( p_1 = p_2 \). By what was said above we conclude that \( \Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0 \).

Conversely, suppose that \( \Omega_{X/S} = 0 \). Then \( X' = X \). Take any pair of morphisms \( f'_1, f'_2 : T' \to X \) fitting as dotted arrows in the diagram of Definition 6.1. This gives a morphism \( (f'_1, f'_2) : T' \to X \times_S X \). Since \( f'_1|_T = f'_2|_T \) and \( |T| = |T'| \) we see that the image of \( T' \) under \( (f'_1, f'_2) \) is contained in the open \( W \) chosen above. Since
(f_1', f_2')(T) \subset \Delta(X)$ and since $T$ is defined by an ideal of square zero in $T'$ we see that $(f_1', f_2')$ factors through $X'$. As $X' = X$ we conclude $f_1' = f_2'$ as desired. \hfill \Box

**Lemma 6.8.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

1. The morphism $f$ is unramified (resp. $G$-unramified), and
2. the morphism $f$ is locally of finite type (resp. locally of finite presentation) and formally unramified.

**Proof.** Use Lemma 6.7 and Morphisms, Lemma 33.2. \hfill \Box

7. Universal first order thickenings

Let $h : Z \to X$ be a morphism of schemes. A *universal first order thickening* of $Z$ over $X$ is a first order thickening $Z \subset Z'$ over $X$ such that given any first order thickening $T \subset T'$ over $X$ and a solid commutative diagram

\[
\begin{array}{ccc}
Z & \xleftarrow{a} & T \\
\downarrow & & \downarrow \\
Z' & \xleftarrow{a'} & T' \\
\swarrow & & \searrow \\
X & \xleftarrow{b} & 
\end{array}
\]

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over $X$. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if $h$ is formally unramified then it does.

**Lemma 7.1.** Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of $Z$ over $X$.

**Proof.** During this proof we will say $Z \subset Z'$ is a universal first order thickening of $Z$ over $X$ if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over $X$ by gluing, starting with the affine case which is Algebra, Lemma 145.1. We begin with some general remarks.

If a universal first order thickening of $Z$ over $X$ exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of $Z$ over $X$. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of $V$ over $U$. Namely, suppose given any diagram

\[
\begin{array}{ccc}
V & \xleftarrow{a} & T \\
\downarrow & & \downarrow \\
U & \xleftarrow{b} & T' \\
\end{array}
\]

where $T \subset T'$ is a first order thickening over $U$. By the universal property of $Z'$ we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of $(a, a')$ as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over $U$. Uniqueness is
clear also. In a completely similar manner one proves that if \( h(Z) \subset U \) and \( Z \subset Z' \) is a universal first order thickening over \( U \), then \( Z \subset Z' \) is a universal first order thickening over \( X \).

Before we glue affine pieces let us show that the lemma holds if \( Z \) and \( X \) are affine. Say \( X = \text{Spec}(R) \) and \( Z = \text{Spec}(S) \). By Algebra, Lemma 145.1 there exists a first order thickening \( Z \subset Z' \) over \( X \) which has the universal property of the lemma for diagrams

\[
\begin{array}{c}
Z \\
\downarrow h \\
X \\
\downarrow b \\
T' \end{array} \quad \begin{array}{c}
\downarrow a \\
T \\
\end{array}
\]

where \( T, T' \) are affine. Given a general diagram we can choose an affine open covering \( T' = \bigcup T'_i \) and we obtain morphisms \( a'_i : T'_i \to Z' \) over \( X \) such that \( a''_i|_{T_i} = a|_{T_i} \). By uniqueness we see that \( a'_i \) and \( a'_j \) agree on any affine open of \( T'_i \cap T'_j \). Hence the morphisms \( a'_i \) glue to a global morphism \( a' : T' \to Z' \) over \( X \) as desired. Thus the lemma holds if \( X \) and \( Z \) are affine.

Choose an affine open covering \( Z = \bigcup Z_i \) such that each \( Z_i \) maps into an affine open \( U_i \) of \( X \). By Lemma 6.5 the morphisms \( Z_i \to U_i \) are formally unramified. Hence by the affine case we obtain universal first order thickenings \( Z_i \subset Z'_i \) over \( U_i \). By the general remarks above \( Z_i \subset Z'_i \) is also a universal first order thickening of \( Z_i \) over \( X \). Let \( Z'_{i,j} \subset Z'_i \) be the open subscheme such that \( Z_i \cap Z_j = Z'_{i,j} \cap Z_i \).

By the general remarks we see that both \( Z'_{i,j} \) and \( Z'_{j,i} \) are universal first order thickenings of \( Z_i \cap Z_j \) over \( X \). Thus, by the first of our general remarks, we see that there is a canonical isomorphism \( \varphi_{ij} : Z'_{i,j} \to Z'_{j,i} \) inducing the identity on \( Z_i \cap Z_j \). We claim that these morphisms satisfy the cocycle condition of Schemes, Section 14. (Verification omitted. Hint: Use that \( Z'_{i,j} \cap Z'_{i,k} \cap Z'_{j,k} \) is the universal first order thickening of \( Z_i \cap Z_j \cap Z_k \) which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 14 to get a first order thickening \( Z \subset Z' \) over \( X \) which the property that the open subscheme \( Z'_i \subset Z' \) with \( Z_i = Z'_i \cap Z \) is a universal first order thickening of \( Z_i \) over \( X \).

It turns out that this implies formally that \( Z' \) is a universal first order thickening of \( Z \) over \( X \). Namely, we have the universal property for any diagram

\[
\begin{array}{c}
Z \\
\downarrow h \\
X \\
\downarrow b \\
T' \end{array} \quad \begin{array}{c}
\downarrow a \\
T \\
\end{array}
\]

where \( a(T) \) is contained in some \( Z_i \). Given a general diagram we can choose an open covering \( T' = \bigcup T'_i \) such that \( a(T_i) \subset Z_i \). We obtain morphisms \( a'_i : T'_i \to Z' \) over \( X \) such that \( a'_i|_{T_i} = a|_{T_i} \). We see that \( a'_i \) and \( a'_j \) necessarily agree on \( T'_i \cap T'_j \) since both \( a'_i|_{T'_i \cap T'_i} \) and \( a'_j|_{T'_i \cap T'_i} \) are solutions of the problem of mapping into the universal first order thickening \( Z'_i \cap Z'_j \) of \( Z_i \cap Z_j \) over \( X \). Hence the morphisms \( a'_i \) glue to a global morphism \( a' : T' \to Z' \) over \( X \) as desired. This finishes the proof. \( \square \)

Definition 7.2. Let \( h : Z \to X \) be a formally unramified morphism of schemes.
(1) The universal first order thickening of $Z$ over $X$ is the thickening $Z \subset Z'$ constructed in Lemma 7.1.

(2) The conormal sheaf of $Z$ over $X$ is the conormal sheaf of $Z$ in its universal first order thickening $Z'$ over $X$.

We often denote the conormal sheaf $C_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to C_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z$. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

**Lemma 7.3.** Let $i : Z \to X$ be an immersion of schemes. Then

1. $i$ is formally unramified,
2. the universal first order thickening of $Z$ over $X$ is the first order infinitesimal neighbourhood of $Z$ in $X$ of Definition 5.1, and
3. the conormal sheaf of $i$ in the sense of Morphisms, Definition 30.1 agrees with the conormal sheaf of $i$ in the sense of Definition 7.2.

**Proof.** By Morphisms, Lemmas 33.7 and 33.8 an immersion is unramified, hence formally unramified by Lemma 6.8. The other assertions follow by combining Lemmas 5.2 and 5.3 and the definitions. □

**Lemma 7.4.** Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening $Z'$ is formally unramified over $X$.

**Proof.** There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 145.5. Then Lemma 6.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

$$
\begin{align*}
\begin{array}{c}
Z' \\
\uparrow^c
\end{array} & \xleftarrow{T} & \begin{array}{c}
T \\
\uparrow^a
\end{array} \\
\begin{array}{c}
X \\
\uparrow^b
\end{array} & \leftarrow \begin{array}{c}
T'
\end{array}
\end{align*}
$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since $Z'$ is a first order thickening of $Z$, we see that $T'$ is a first order thickening of $T'_a$. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As $T'$ is a first order thickening of $T$ it follows that $T'_a$ is a first order thickening of $T_0$. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of $T'_a$ into $Z'$ over $X$ which agree on $T_0$ as morphisms into $Z$. Hence by the universal property of $Z'$ we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus $a$ and $b$ are morphism from the first order thickening $T''$ of $T'_a$ whose restrictions to $T'_a$ agree as morphisms into $Z$. Thus using the universal property of $Z'$ once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □
Lemma 7.5. Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

with \( h \) and \( h' \) formally unramified. Let \( Z \subset Z' \) be the universal first order thickening of \( Z \) over \( X \). Let \( W \subset W' \) be the universal first order thickening of \( W \) over \( Y \). There exists a canonical morphism \((f, f') : (Z, Z') \to (W, W')\) of thickenings over \( Y \) which fits into the following commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{f'} & X' \\
\downarrow{f} & & \downarrow{g} \\
W' & \xrightarrow{h'} & Y
\end{array}
\]

In particular the morphism \((f, f')\) of thickenings induces a morphism of conormal sheaves \( f^*C_{W/Y} \to C_{Z/X} \).

Proof. The first assertion is clear from the universal property of \( W' \). The induced map on conormal sheaves is the map of Morphisms, Lemma 30.3 applied to \((Z \subset Z') \to (W \subset W')\). □

Lemma 7.6. Let

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

be a fibre product diagram in the category of schemes with \( h' \) formally unramified. Then \( h \) is formally unramified and if \( W \subset W' \) is the universal first order thickening of \( W \) over \( Y \), then \( Z = X \times_Y W \subset X \times_Y W' \) is the universal first order thickening of \( Z \) over \( X \). In particular the canonical map \( f^*C_{W/Y} \to C_{Z/X} \) of Lemma 7.5 is surjective.

Proof. The morphism \( h \) is formally unramified by Lemma 6.4. It is clear that \( X \times_Y W' \) is a first order thickening. It is straightforward to check that it has the universal property because \( W' \) has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 30.4 for why this implies that the map of conormal sheaves is surjective. □

Lemma 7.7. Let

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\downarrow{f} & & \downarrow{g} \\
W & \xrightarrow{h'} & Y
\end{array}
\]

be a fibre product diagram in the category of schemes with \( h' \) formally unramified and \( g \) flat. In this case the corresponding map \( Z' \to W' \) of universal first order thickenings is flat, and \( f^*C_{W/Y} \to C_{Z/X} \) is an isomorphism.
Proof. Flatness is preserved under base change, see Morphisms, Lemma \ref{lem-flatness-base-change}. Hence the first statement follows from the description of \( W' \) in Lemma \ref{lem-descr-W}. It is clear that \( X \times_Y W' \) is a first order thickening. It is straightforward to check that it has the universal property because \( W' \) has the universal property (by mapping properties of fibre products). See Morphisms, Lemma \ref{lem-vietoris-extension} for why this implies that the map of conormal sheaves is an isomorphism. \hfill \Box

**Lemma 7.8.** Taking the universal first order thickenings commutes with taking opens. More precisely, let \( h : Z \to X \) be a formally unramified morphism of schemes. Let \( V \subset Z, U \subset X \) be opens such that \( h(V) \subset U \). Let \( Z' \) be the universal first order thickening of \( Z \) over \( X \). Then \( h|_V : V \to U \) is formally unramified and the universal first order thickening of \( V \) over \( U \) is the open subscheme \( V' \subset Z' \) such that \( V = Z \cap V' \). In particular, \( C_{Z/X} |_V = C_{V/U} \).

**Proof.** The first statement is Lemma \ref{lem-first-order-neighborhood}. The compatibility of universal thickenings can be deduced from the proof of Lemma \ref{lem-thickening-unique}, or from Algebra, Lemma \ref{lem-thickening-unique} or deduced from Lemma \ref{lem-thickening-unique}.

**Lemma 7.9.** Let \( h : Z \to X \) be a formally unramified morphism of schemes over \( S \). Let \( Z \subset Z' \) be the universal first order thickening of \( Z \) over \( X \) with structure morphism \( h' : Z' \to X \). The canonical map

\[
c_{h'} : (h')^* \Omega_{X/S} \longrightarrow \Omega_{Z'/S}
\]

induces an isomorphism \( h^* \Omega_{X/S} \to \Omega_{Z'/S} \otimes O_Z \).

**Proof.** The map \( c_{h'} \) is the map defined in Morphisms, Lemma \ref{lem-first-order-extension}. If \( i : Z \to Z' \) is the given closed immersion, then \( i^* c_{h'} \) is a map \( h^* \Omega_{X/S} \to \Omega_{Z'/S} \otimes O_Z \). Checking that it is an isomorphism reduces to the affine case by localization, see Lemma \ref{lem-first-order-extension} and Morphisms, Lemma \ref{lem-first-order-extension}. In this case the result is Algebra, Lemma \ref{lem-first-order-extension}.

**Lemma 7.10.** Let \( h : Z \to X \) be a formally unramified morphism of schemes over \( S \). There is a canonical exact sequence

\[
C_{Z/X} \to h^* \Omega_{X/S} \to \Omega_{Z/S} \to 0.
\]

The first arrow is induced by \( d_{Z'/S} \) where \( Z' \) is the universal first order neighbourhood of \( Z \) over \( X \).

**Proof.** We know that there is a canonical exact sequence

\[
C_{Z'/X} \to \Omega_{Z'/S} \otimes O_Z \to \Omega_{Z/S} \to 0.
\]

see Morphisms, Lemma \ref{lem-first-order-extension}. Hence the result follows on applying Lemma \ref{lem-first-order-extension}.

**Lemma 7.11.** Let

\[
\begin{align*}
Z & \xrightarrow{i} X \\
& \downarrow j \\
Y & \xrightarrow{} Y
\end{align*}
\]

be a commutative diagram of schemes where \( i \) and \( j \) are formally unramified. Then there is a canonical exact sequence

\[
C_{Z/Y} \to C_{Z/X} \to i^* \Omega_{X/Y} \to 0
\]

where the first arrow comes from Lemma \ref{lem-first-order-extension} and the second from Lemma \ref{lem-first-order-extension}.

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**04FA** Lemma 7.8. Taking the universal first order thickenings commutes with taking opens. More precisely, let \( h : Z \to X \) be a formally unramified morphism of schemes. Let \( V \subset Z, U \subset X \) be opens such that \( h(V) \subset U \). Let \( Z' \) be the universal first order thickening of \( Z \) over \( X \). Then \( h|_V : V \to U \) is formally unramified and the universal first order thickening of \( V \) over \( U \) is the open subscheme \( V' \subset Z' \) such that \( V = Z \cap V' \). In particular, \( C_{Z/X} |_V = C_{V/U} \).

**Proof.** The first statement is Lemma \ref{lem-first-order-neighborhood}. The compatibility of universal thickenings can be deduced from the proof of Lemma \ref{lem-thickening-unique}, or from Algebra, Lemma \ref{lem-thickening-unique} or deduced from Lemma \ref{lem-thickening-unique}.

**04FB** Lemma 7.9. Let \( h : Z \to X \) be a formally unramified morphism of schemes over \( S \). Let \( Z \subset Z' \) be the universal first order thickening of \( Z \) over \( X \) with structure morphism \( h' : Z' \to X \). The canonical map

\[
c_{h'} : (h')^* \Omega_{X/S} \longrightarrow \Omega_{Z'/S}
\]

induces an isomorphism \( h^* \Omega_{X/S} \to \Omega_{Z'/S} \otimes O_Z \).

**Proof.** The map \( c_{h'} \) is the map defined in Morphisms, Lemma \ref{lem-first-order-extension}. If \( i : Z \to Z' \) is the given closed immersion, then \( i^* c_{h'} \) is a map \( h^* \Omega_{X/S} \to \Omega_{Z'/S} \otimes O_Z \). Checking that it is an isomorphism reduces to the affine case by localization, see Lemma \ref{lem-first-order-extension} and Morphisms, Lemma \ref{lem-first-order-extension}. In this case the result is Algebra, Lemma \ref{lem-first-order-extension}.

**04FC** Lemma 7.10. Let \( h : Z \to X \) be a formally unramified morphism of schemes over \( S \). There is a canonical exact sequence

\[
C_{Z/X} \to h^* \Omega_{X/S} \to \Omega_{Z/S} \to 0.
\]

The first arrow is induced by \( d_{Z'/S} \) where \( Z' \) is the universal first order neighbourhood of \( Z \) over \( X \).

**Proof.** We know that there is a canonical exact sequence

\[
C_{Z'/X} \to \Omega_{Z'/S} \otimes O_Z \to \Omega_{Z/S} \to 0.
\]

see Morphisms, Lemma \ref{lem-first-order-extension}. Hence the result follows on applying Lemma \ref{lem-first-order-extension}.

**067V** Lemma 7.11. Let

\[
\begin{align*}
Z & \xrightarrow{i} X \\
& \downarrow j \\
Y & \xrightarrow{} Y
\end{align*}
\]

be a commutative diagram of schemes where \( i \) and \( j \) are formally unramified. Then there is a canonical exact sequence

\[
C_{Z/Y} \to C_{Z/X} \to i^* \Omega_{X/Y} \to 0
\]

where the first arrow comes from Lemma \ref{lem-first-order-extension} and the second from Lemma \ref{lem-first-order-extension}.
Proof. Denote $Z \to Z'$ the universal first order thickening of $Z$ over $X$. Denote $Z \to Z''$ the universal first order thickening of $Z$ over $Y$. By Lemma 7.10 there is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$\begin{array}{ccc}
Z & \rightarrow & Z' \\
\downarrow & & \downarrow \\
& \rightarrow & X \\
Z'' & \rightarrow & Y
\end{array}$

Apply Morphisms, Lemma 31.18 to the left triangle to get an exact sequence

$C_{Z/Z''} \rightarrow C_{Z/Z'} \rightarrow (i')^*\Omega_{Z'/Z''} \rightarrow 0$

As $Z''$ is formally unramified over $Y$ (see Lemma 7.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 6.7 and Morphisms, Lemma 31.9). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 7.9. □

Lemma 7.12. Let $Z \to Y \to X$ be formally unramified morphisms of schemes.

(1) If $Z \subset Z'$ is the universal first order thickening of $Z$ over $X$ and $Y \subset Y'$ is the universal first order thickening of $Y$ over $X$, then there is a morphism $Z' \to Y'$ and $Y \times_Y Z'$ is the universal first order thickening of $Z$ over $Y$.

(2) There is a canonical exact sequence

$i^*C_{Y/X} \rightarrow C_{Z/X} \rightarrow C_{Z/Y} \rightarrow 0$

where the maps come from Lemma 7.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 7.5. The assertion that $Y \times_Y Z'$ is the universal first order thickening of $Z$ over $Y$ is clear from the universal properties of $Z'$ and $Y'$. By Morphisms, Lemma 30.5 we have an exact sequence

$(i')^*C_{Y \times_Y Z'/Z'} \rightarrow C_{Z/Z'} \rightarrow C_{Z/Y \times_Y Z'} \rightarrow 0$

where $i' : Z \to Y \times_Y Z'$ is the given morphism. By Morphisms, Lemma 30.4 there exists a surjection $h^*C_{Y/Y'} \rightarrow C_{Y \times_Y Z'/Z'}$. Combined with the equalities $C_{Y/Y'} = C_{Y/X}, C_{Z/Z'} = C_{Z/X},$ and $C_{Z/Y \times_Y Z'} = C_{Z/Y}$ this proves the lemma. □

8. Formally étale morphisms

Recall that a ring map $R \to A$ is called formally étale (see Algebra, Definition 146.1) if for every commutative solid diagram

$\begin{array}{ccc}
A & \rightarrow & B/I \\
\downarrow & & \downarrow \\
R & \rightarrow & B
\end{array}$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.
Definition 8.1. Let \( f : X \to S \) be a morphism of schemes. We say \( f \) is \textit{formally étale} if given any solid commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{a} & T \\
\downarrow{f} & \swarrow{\iota} & \downarrow{i} \\
S & \xleftarrow{\alpha} & T'
\end{array}
\]

where \( T \subset T' \) is a first order thickening of affine schemes over \( S \) there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if \( f : X \to S \) is formally étale, then \( \Omega_{X/S} \) is zero, see Lemma 6.7.

Lemma 8.2. If \( f : X \to S \) is a formally étale morphism, then given any solid commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{a} & T \\
\downarrow{f} & \swarrow{\iota} & \downarrow{i} \\
S & \xleftarrow{\alpha} & T'
\end{array}
\]

where \( T \subset T' \) is a first order thickening of schemes over \( S \) there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 8.1 the condition that \( T \) be affine may be dropped.

Proof. Let \( T' = \bigcup T'_i \) be an affine open covering, and let \( T_i = T \cap T'_i \). Then we get morphisms \( a'_i : T'_i \to X \) fitting into the diagram. By uniqueness we see that \( a'_i \) and \( a'_j \) agree on any affine open subscheme of \( T'_i \cap T'_j \). Hence \( a'_i \) and \( a'_j \) agree on \( T'_i \cap T'_j \). Thus we see that the morphisms \( a'_i \) glue to a global morphism \( a' : T' \to X \). The uniqueness of \( a' \) we have seen in Lemma 6.2.

Lemma 8.3. A composition of formally étale morphisms is formally étale.

Proof. This is formal.

Lemma 8.4. A base change of a formally étale morphism is formally étale.

Proof. This is formal.

Lemma 8.5. Let \( f : X \to S \) be a morphism of schemes. Let \( U \subset X \) and \( V \subset S \) be open subschemes such that \( f(U) \subset V \). If \( f \) is formally étale, so is \( f|_U : U \to V \).

Proof. Consider a solid diagram

\[
\begin{array}{ccc}
U & \xleftarrow{a} & T \\
\downarrow{f|_U} & \swarrow{\iota} & \downarrow{i} \\
V & \xleftarrow{\alpha} & T'
\end{array}
\]

as in Definition 8.1. If \( f \) is formally ramified, then there exists exactly one \( S \)-morphism \( a' : T' \to X \) such that \( a'|_T = a \). Since \( |T'| = |T| \) we conclude that \( a'(T') \subset U \) which gives our unique morphism from \( T' \) into \( U \).

Lemma 8.6. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. \( f \) is formally étale,
(2) \( f \) is formally unramified and the universal first order thickening of \( X \) over \( S \) is equal to \( X \),

(3) \( f \) is formally unramified and \( \mathcal{C}_{X/S} = 0 \), and

(4) \( \Omega_{X/S} = 0 \) and \( \mathcal{C}_{X/S} = 0 \).

**Proof.** Actually, the last assertion only make sense because \( \Omega_{X/S} = 0 \) implies that \( \mathcal{C}_{X/S} \) is defined via Lemma 6.7 and Definition 7.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that \( f \) is formally unramified. Hence we may assume \( f \) is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening \( X' \) of \( X \) over \( S \) and the fact that \( X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0 \) since after all by definition \( \mathcal{C}_{X/S} = \mathcal{C}_{X/X'} \) is the ideal sheaf of \( X \) in \( X' \).

\( \square \)

**Lemma 8.7.** An unramified flat morphism is formally étale.

**Proof.** Say \( X \to S \) is unramified and flat. Then \( \Delta : X \to X \times_S X \) is an open immersion, see Morphisms, Lemma 33.13. We have to show that \( \mathcal{C}_{X/S} \) is zero. Consider the two projections \( p, q : X \times_S X \to X \). As \( f \) is formally unramified (see Lemma 6.8), \( q \) is formally unramified (see Lemma 6.4). As \( f \) is flat, \( p \) is flat, see Morphisms, Lemma 24.7. Hence \( p^* \mathcal{C}_{X/S} = \mathcal{C}_q \) by Lemma 7.7 where \( \mathcal{C}_q \) denotes the conormal sheaf of the formally unramified morphism \( q : X \times_S X \to X \). But \( \Delta(X) \subseteq X \times_S X \) is an open subscheme which maps isomorphically to \( X \) via \( q \). Hence by Lemma 7.8 we see that \( \mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0 \). In other words, the pullback of \( \mathcal{C}_{X/S} \) to \( X \) via the identity morphism is zero, i.e., \( \mathcal{C}_{X/S} = 0 \).

\( \square \)

**Lemma 8.8.** Let \( f : X \to S \) be a morphism of schemes. Assume \( X \) and \( S \) are affine. Then \( f \) is formally étale if and only if \( O_S(S) \to O_X(X) \) is a formally étale ring map.

**Proof.** This is immediate from the definitions (Definition 8.1 and Algebra, Definition 146.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5.

\( \square \)

**Lemma 8.9.** Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. The morphism \( f \) is étale, and
2. the morphism \( f \) is locally of finite presentation and formally étale.

**Proof.** Assume \( f \) is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 34. Hence \( f \) is locally of finite presentation and formally étale, see Lemma 8.7.

Conversely, suppose that \( f \) is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on \( X \) and \( S \), see Morphisms, Lemma 34.2. By Lemma 8.5 we can cover \( X \) by affine opens \( U \) which map into affine opens \( V \) such that \( U \to V \) is formally étale (and of finite presentation, see Morphisms, Lemma 20.2). By Lemma 8.8 we see that the ring maps \( O(V) \to O(U) \) are formally étale (and of finite presentation). We win by Algebra, Lemma 146.2. (We will give another proof of this implication when we discuss formally smooth morphisms.)
9. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening $X'$ of $X$ can be seen as a sheaf on $X$.

**Lemma 9.1.** Let $S$ be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over $S$. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over $S$. Assume that

1. $a = b$, and
2. the two maps $a^*C_{Y/Y'} \to C_{X/X'}$ (Morphisms, Lemma 30.3) are equal.

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$O_{Y'} \to O_Y \xrightarrow{D} a_*C_{X/X'} \to a_*O_{X'}$$

where $D$ is an $O_S$-derivation.

**Proof.** Instead of working on $Y$ we work on $X$. The advantage is that the pullback functor $a^*$ is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & C_{X/X'} & \rightarrow & O_{X'} & \rightarrow & O_X & \rightarrow & 0 \\
& & \downarrow{(a')^\sharp} & & \downarrow{(b')^\sharp} & & \\
0 & \rightarrow & a^{-1}C_{Y/Y'} & \rightarrow & a^{-1}O_{Y'} & \rightarrow & a^{-1}O_Y & \rightarrow & 0
\end{array}
$$

Now it is a general fact that in such a situation the difference of the $O_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $O_S$-derivation from $a^{-1}O_Y$ to $C_{X/X'}$. By adjointness of the functors $a^{-1}$ and $a_*$ this is the same thing as an $O_S$-derivation from $O_Y$ into $a_*C_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write $D$ as

$$D = d_{Y/S} \circ \theta$$

where $\theta$ is an $O_Y$-linear map $\theta : \Omega_{Y/S} \to a_*C_{X/X'}$. Of course, then by adjunction again we may view $\theta$ as an $O_X$-linear map $\theta : a^*\Omega_{Y/S} \to C_{X/X'}$.

**Lemma 9.2.** Let $S$ be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over $S$. Let

$$\theta : a^*\Omega_{Y/S} \to C_{X/X'}$$

be an $O_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 9.1 hold and the derivation $D$ and $\theta$ are related by Equation (9.1.1).

**Proof.** We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}O_Y \to O_X,$$

where $D$ is as in Equation (9.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $O_S$-algebras and that (1) and (2) of Lemma 9.1 hold. Equation (9.1.1) holds by construction. □
Remark 9.3. Assumptions and notation as in Lemma 9.2. The action of a local section \( \theta \) on \( a' \) is sometimes indicated by \( \theta \cdot a' \). Note that this means nothing else than the fact that \((a')^\sharp\) and \((\theta \cdot a')^\sharp\) differ by a derivation \( D \) which is related to \( \theta \) by Equation (9.1.1).

Lemma 9.4. Let \( S \) be a scheme. Let \( X \subseteq X' \) and \( Y \subseteq Y' \) be first order thickenings over \( S \). Assume given a morphism \( a : X \rightarrow Y \) and a map \( A : a^*C_{Y/Y'} \rightarrow C_{X/X'} \) of \( \mathcal{O}_X \)-modules. For an open subscheme \( U' \subseteq X' \) consider morphisms \( a' : U' \rightarrow Y' \) such that

1. \( a' \) is a morphism over \( S \),
2. \( a'|_U = a|_U \), and
3. the induced map \( a^*C_{Y/Y'}|_U \rightarrow C_{X/X'}|_U \) is the restriction of \( A \) to \( U \).

Here \( U = X \cap U' \). Then the rule

\[
(9.4.1) \quad U' \mapsto \{ a' : U' \rightarrow Y' \text{ such that (1), (2), (3) hold.} \}
\]
defines a sheaf of sets on \( X' \).

Proof. Denote \( \mathcal{F} \) the rule of the lemma. The restriction mapping \( \mathcal{F}(U') \rightarrow \mathcal{F}(V') \) for \( V' \subseteq U' \subseteq X' \) of \( \mathcal{F} \) is really the restriction map \( a' \mapsto a'|_{V'} \). With this definition in place it is clear that \( \mathcal{F} \) is a sheaf since morphisms are defined locally. \( \square \)

In the following lemma we identify sheaves on \( X \) and any thickening of \( X \).

Lemma 9.5. Same notation and assumptions as in Lemma 9.4. There is an action of the sheaf

\[
\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})
\]
on the sheaf \((9.4.1)\). Moreover, the action is simply transitive for any open \( U' \subseteq X' \) over which the sheaf \((9.4.1)\) has a section.

Proof. This is a combination of Lemmas 9.1, 9.2, and 9.4. \( \square \)

Remark 9.6. A special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where \( Y = Y' \). In this case the map \( A \) is always zero. The sheaf of Lemma 9.5 is just given by the rule

\[
U' \mapsto \{ a' : U' \rightarrow Y \text{ over } S \text{ with } a'|_U = a|_U \}
\]
and we act on this by the sheaf \( \mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'}) \).

Remark 9.7. Another special case of Lemmas 9.1, 9.2, 9.4, and 9.5 is where \( S \) itself is a thickening \( Z \subseteq Z' = S \) and \( Y = Z \times Z' Y' \). Picture

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(a',\cdot)} & (Y \subset Y') \\
\downarrow{(g,\cdot)} & & \downarrow{(h,\cdot)} \\
(Z \subset Z')
\end{array}
\]

In this case the map \( A : a^*C_{Y/Y'} \rightarrow C_{X/X'} \) is determined by \( a \): the map \( h^*C_{Z/Z'} \rightarrow C_{Y/Y'} \) is surjective (because we assumed \( Y = Z \times Z' Y' \)), hence the pullback \( g^*C_{Z/Z'} = a^*h^*C_{Z/Z'} \rightarrow a^*C_{Y/Y'} \) is surjective, and the composition \( g^*C_{Z/Z'} \rightarrow a^*C_{Y/Y'} \rightarrow C_{X/X'} \) has to be the canonical map induced by \( g' \). Thus the sheaf of Lemma 9.4 is just given by the rule

\[
U' \mapsto \{ a' : U' \rightarrow Y' \text{ over } Z' \text{ with } a'|_U = a|_U \}
\]
and we act on this by the sheaf \( \mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'}) \).
Let $S$ be a scheme. Let $X \subseteq X'$ be a first order thickening over $S$. Let $Y$ be a scheme over $S$. Let $a', b' : X' \rightarrow Y$ be two morphisms over $S$ with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \text{(b', a')} \\
Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y
\end{array}
\]

Since the horizontal arrows are immersions with conormal sheaves $C_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 30.3, we obtain a map $\theta : a^*\Omega_{Y/S} \rightarrow C_{X/X'}$. Then this $\theta$ and the derivation $D$ of Lemma 9.1 are related by Equation (9.1.1).

**Proof.** Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If $f$ is a local section of $O_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $C_{Y/(Y \times_S Y)}$ corresponding to $d_{Y/S}(f)$. It is mapped to the local section $(a')^!(f) - (b')^!(f) = D(f)$ of $C_{X/X'}$. In other words, $\theta(d_{Y/S}(f)) = D(f)$. \hfill \Box

For later purposes we need a result that roughly states that the construction of Lemma 9.2 is compatible with étale localization.

**Lemma 9.10.** Consider a commutative diagram of first order thickenings

\[
\begin{array}{ccc}
(T_2 \subset T'_2) & \xrightarrow{(a_2, a'_2)} & (X_2 \subset X'_2) \\
(h, h') & & \downarrow (f, f') \\
(T_1 \subset T'_1) & \xrightarrow{(a_1, a'_1)} & (X_1 \subset X'_1)
\end{array}
\]

with $X_2 \rightarrow X_1$ and $S_2 \rightarrow S_1$ étale. For any $O_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \rightarrow C_{T_1/T'_1}$ let $\theta_2$ be the composition

\[
a_2^*\Omega_{X_2/S_2} \xrightarrow{h^*a_1^*\Omega_{X_1/S_1}} h^*C_{T_1/T'_1} \xrightarrow{h^*\theta_1} C_{T_2/T'_2}
\]

(equality sign is explained in the proof). Then the diagram

\[
\begin{array}{ccc}
T'_2 & \xrightarrow{\theta_2 \cdot a'_2} & X'_2 \\
\downarrow & & \downarrow \\
T'_1 & \xrightarrow{\theta_1 \cdot a'_1} & X'_1
\end{array}
\]
Lemma 9.10 can be improved in the following way. Suppose $X \subset S$ are ideals of square zero and maps of pairs are ring maps sending ideals into ideals. Set $A_1 = A_i'/J_i$, $A_2 = A_j'/J_j$, $B_1 = B_i'/I_i$, and $B_2 = B_j'/I_j$. We are given that

$$A_2 \otimes_{A_1} \Omega_{A_1/I_1} \to \Omega_{A_2/J_2}$$

is an isomorphism. Then $\theta_1 : B_1 \otimes_{A_1} \Omega_{A_1/I_1} \to I_1$ is $B_1$-linear. This gives an $R_1$-derivation $D_1 = \theta_1 \circ d_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2 \otimes_{A_2} \Omega_{A_2/J_2} \to I_2$ gives rise to a $R_2$-derivation $D_2 = \theta_2 \circ d_{A_2/R_2} : A_2 \to I_2$. The construction of $\theta_2$ implies the following compatibility between $\theta_1$ and $\theta_2$: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f'(x))$$

as elements of $I_2$. We may view $D_1$ as a map $A_1' \to B_1'$ using $A_1' \to A_1 \xrightarrow{D_1} I_1 \to B_1$ similarly we may view $D_2$ as a map $A_2' \to B_2'$. Then the displayed equality holds for $x \in A_1'$. By the construction of the action in Lemma 9.2 and Remark 9.3 we know that $\theta_1 \cdot a'_1$ corresponds to the ring map $a'_1 + D_1 : A_1' \to B_1'$ and $\theta_2 \cdot a'_2$ corresponds to the ring map $a'_2 + D_2 : A_2' \to B_2'$. By the displayed equality we obtain that $h' \circ (a'_1 + D_1) = (a'_2 + D_2) \circ f'$ as desired.

**Remark 9.11.** Lemma 9.10 can be improved in the following way. Suppose that we have commutative diagrams as in Lemma 9.10 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a'_1 \Omega_{X_1/S_1} \to C_{T_1/T_1'}$ and $\theta_2 : a'_2 \Omega_{X_2/S_2} \to C_{T_2/T_2'}$ such that

$$\begin{array}{ccc}
\mathcal{O}_{X_1} & \xrightarrow{D_1} & a'_1 \cdot C_{T_1/T_1'} \\
f \downarrow & & \downarrow \text{induced by (h')}
\end{array}$$

$$\begin{array}{ccc}
f_* \mathcal{O}_{X_2} & \xrightarrow{f_*D_2} & f_*a'_2 \cdot C_{T_2/T_2'} \\
f' \downarrow & & \downarrow \text{induced by (h')}
\end{array}$$

is commutative where $D_i$ corresponds to $\theta_i$ as in Equation (9.1.1). Then we have the conclusion of Lemma 9.10. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a $\theta_2$ from $\theta_1$.

10. Infinitesimal deformations of schemes

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

**Lemma 10.1.** Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that $f$ is flat. Then the following are equivalent:

063X

063Y
(1) \( f' \) is flat and \( X = S \times_{S'} X' \), and
(2) the canonical map \( f^*C_{S/S'} \to C_{X/X'} \) is an isomorphism.

**Proof.** As the problem is local on \( X' \) we may assume that \( X, X', S, S' \) are affine schemes. Say \( S' = \text{Spec}(A') \), \( X' = \text{Spec}(B') \), \( S = \text{Spec}(A) \), \( X = \text{Spec}(B) \) with \( A = A'/I \) and \( B = B'/J \) for some square zero ideals. Then we obtain the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & J \\
\uparrow & & \uparrow \\
0 & \to & I
\end{array}
\begin{array}{ccc}
& \to & B' \\
& & \uparrow \\
& \to & B
\end{array}
\begin{array}{ccc}
& & 0 \\
& & \uparrow \\
& & A'
\end{array}
\begin{array}{ccc}
& \to & A \\
& & \uparrow \\
& \to & 0
\end{array}
\]

with exact rows. The canonical map of the lemma is the map

\[ I \otimes_A B = I \otimes_A' B' \to J. \]

The assumption that \( f \) is flat signifies that \( A \to B \) is flat. Assume (1). Then \( A' \to B' \) is flat and \( J = IB' \). Flatness implies \( \text{Tor}_1^{A'}(B', A) = 0 \) (see Algebra, Lemma 74.8). This means \( I \otimes_A B' \to B' \) is injective (see Algebra, Remark 74.9). Hence we see that \( I \otimes_A B \to J \) is an isomorphism.

Assume (2). Then it follows that \( J = IB' \), so that \( X = S \times_{S'} X' \). Moreover, we get \( \text{Tor}_1^{A'}(B', A'/I) = 0 \) by reversing the implications in the previous paragraph. Hence \( B' \) is flat over \( A' \) by Algebra, Lemma 98.8.

The following lemma is the “nilpotent” version of the “critère de platitude par fibres”, see Section 16.

**Lemma 10.2.** Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & & 
\end{array}
\]

of thickenings. Assume

1. \( X' \) is flat over \( S' \),
2. \( f \) is flat,
3. \( S \subset S' \) is a finite order thickening, and
4. \( X = S \times_{S'} X' \) and \( Y = S \times_{S'} Y' \).

Then \( f' \) is flat and \( Y' \) is flat over \( S' \) at all points in the image of \( f' \).

**Proof.** Immediate consequence of Algebra, Lemma 100.8.

Many properties of morphisms of schemes are preserved under flat deformations.

**Lemma 10.3.** Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & & 
\end{array}
\]
of thickenings. Assume $S \subset S'$ is a finite order thickening, $X'$ flat over $S'$, $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

06AH (1) $f$ is flat if and only if $f'$ is flat,
06AI (2) $f$ is an isomorphism if and only if $f'$ is an isomorphism,
06AJ (3) $f$ is an open immersion if and only if $f'$ is an open immersion,
06AK (4) $f$ is quasi-compact if and only if $f'$ is quasi-compact,
06AL (5) $f$ is universally closed if and only if $f'$ is universally closed,
06AM (6) $f$ is (quasi-)separated if and only if $f'$ is (quasi-)separated,
06AN (7) $f$ is a monomorphism if and only if $f'$ is a monomorphism,
06AP (8) $f$ is surjective if and only if $f'$ is surjective,
06AQ (9) $f$ is universally injective if and only if $f'$ is universally injective,
06AR (10) $f$ is affine if and only if $f'$ is affine,
06AS (11) $f$ is locally of finite type if and only if $f'$ is locally of finite type,
06AT (12) $f$ is locally quasi-finite if and only if $f'$ is locally quasi-finite,
06AU (13) $f$ is locally of finite presentation if and only if $f'$ is locally of finite presentation,
06AV (14) $f$ is locally of finite type of relative dimension $d$ if and only if $f'$ is locally of finite type of relative dimension $d$,
06AW (15) $f$ is universally open if and only if $f'$ is universally open,
06AX (16) $f$ is syntomic if and only if $f'$ is syntomic,
06AY (17) $f$ is smooth if and only if $f'$ is smooth,
06AZ (18) $f$ is unramified if and only if $f'$ is unramified,
06B0 (19) $f$ is étale if and only if $f'$ is étale,
06B1 (20) $f$ is proper if and only if $f'$ is proper,
06B2 (21) $f$ is integral if and only if $f'$ is integral,
06B3 (22) $f$ is finite if and only if $f'$ is finite,
06B4 (23) $f$ is finite locally free (of rank $d$) if and only if $f'$ is finite locally free (of rank $d$), and
(24) add more here.

**Proof.** The assumptions on $X$ and $Y$ mean that $f$ is the base change of $f'$ by $X \to X'$. The properties $P$ listed in (1) – (23) above are all stable under base change, hence if $f'$ has property $P$, then so does $f$. See Schemes, Lemmas 18.2 [19.3] and 21.13 [21.13] and Morphisms, Lemmas 9.4 [10.4] and 11.8 [11.8] and 14.4 [14.4] and 19.13 [19.13] and 20.4 [20.4] and 28.2 [28.2] and 29.4 [29.4] and 32.5 [32.5] and 33.5 [33.5] and 34.4 [34.4] and 39.5 [39.5] and 42.6 [42.6] and 45.4 [45.4].

The interesting direction in each case is therefore to assume that $f$ has the property and deduce that $f'$ has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over $W'$ with $f'(U') \subset V'$. Let $W' = \text{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \text{Spec}(B')$ and $V' = \text{Spec}(A')$. Then we get a commutative
with exact rows. Moreover $IB' \cong I \otimes_R B$, see proof of Lemma \ref{lemma}. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps $f$ and $f'$ (after any base change) is identical. (III) If $f$ is flat, then $f'$ is flat and $Y' \to S'$ is flat at every point in the image of $f'$, see Lemma \ref{lemma}.

Ad (1). This is general remark (III).

Ad (2). Assume $f$ is an isomorphism. By (III) we see that $Y' \to S'$ is flat. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_Y U'$ is affine. By Lemma \ref{lemma} we see that $U'$ is affine. Thus we have a diagram as in the general remark (I) and moreover $IA \cong I \otimes_R A$ because $R' \to A'$ is flat. Then $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus $f'$ is an isomorphism.

Ad (3). Assume $f$ is an open immersion. Then $f$ is an isomorphism of $X$ with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is $V$. Then $f'$ is a map from $X'$ to $V'$ which is an isomorphism by (2). Hence $f'$ is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma \ref{lemma} for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma \ref{lemma} for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_Y (X' \times_Y X')$ so that $X' \times_Y X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma \ref{lemma} for a more general statement.

Ad (7). Assume $f$ is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_Y X'$. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma \ref{lemma} for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma \ref{lemma} for a more general statement.

Ad (10). Assume $f$ is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_Y U'$ is affine. By Lemma \ref{lemma} we see that $U'$ is affine. Hence $f'$ is affine. See also Lemma \ref{lemma} for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in $B$ generate $B$ as an $A$-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma \ref{lemma} for a more general statement.
Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 19.6. See also Lemma 3.3 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal $K'$ by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As $B'$ is flat over $R'$ we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since $I$ is nilpotent we see that $f'_1, \ldots, f'_m$ generate $K'$ by Nakayama’s lemma, see Algebra, Lemma 19.1.

Ad (14). Follows from (11) and general remark (II). See also Lemma 3.3 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (16). Assume $f$ is syntomic. By (13) $f'$ is locally of finite presentation, by general remark (III) $f'$ is flat and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is syntomic by Morphisms, Lemma 29.11.

Ad (17). Assume $f$ is smooth. By (13) $f'$ is locally of finite presentation, by general remark (III) $f'$ is flat, and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is smooth by Morphisms, Lemma 32.3.

Ad (18). Assume $f$ unramified. By (11) $f'$ is locally of finite type and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is unramified by Morphisms, Lemma 33.12. See also Lemma 3.3 for a more general statement.

Ad (19). Assume $f$ étale. By (13) $f'$ is locally of finite presentation, by general remark (III) $f'$ is flat, and the fibres of $f'$ are the fibres of $f$. Hence $f'$ is étale by Morphisms, Lemma 34.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 3.3 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 42.7. See also Lemma 3.1 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 42.4. See also Lemma 3.3 for a more general statement.

Ad (23). Assume $f$ finite locally free. By (22) we see that $f'$ is finite, by general remark (III) $f'$ is flat, and by (13) $f'$ is locally of finite presentation. Hence $f'$ is finite locally free by Morphisms, Lemma 45.2.

The following lemma is the “locally nilpotent” version of the “critère de platitude par fibres”, see Section 16.
Lemma 10.4. Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & &
\end{array}
\]

of thickenings. Assume

1. \(Y' \to S'\) is locally of finite type,
2. \(X' \to S'\) is flat and locally of finite presentation,
3. \(f\) is flat, and
4. \(X = S \times_{S'} X'\) and \(Y = S \times_{S'} Y'\).

Then \(f'\) is flat and for all \(y' \in Y'\) in the image of \(f'\) the local ring \(\mathcal{O}_{Y',y'}\) is flat and essentially of finite presentation over \(\mathcal{O}_{S',s'}\).

Proof. Immediate consequence of Algebra, Lemma \(\textbf{127.10}\). \(\square\)

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

Lemma 10.5. Consider a commutative diagram

\[
\begin{array}{ccc}
(X \subset X') & \xrightarrow{(f,f')} & (Y \subset Y') \\
\downarrow & & \downarrow \\
(S \subset S') & &
\end{array}
\]

of thickenings. Assume \(Y' \to S'\) locally of finite type, \(X' \to S'\) flat and locally of finite presentation, \(X = S \times_{S'} X'\), and \(Y = S \times_{S'} Y'\). Then

1. \(f\) is flat if and only if \(f'\) is flat,
2. \(f\) is an isomorphism if and only if \(f'\) is an isomorphism,
3. \(f\) is an open immersion if and only if \(f'\) is an open immersion,
4. \(f\) is quasi-compact if and only if \(f'\) is quasi-compact,
5. \(f\) is universally closed if and only if \(f'\) is universally closed,
6. \(f\) is (quasi-)separated if and only if \(f'\) is (quasi-)separated,
7. \(f\) is a monomorphism if and only if \(f'\) is a monomorphism,
8. \(f\) is surjective if and only if \(f'\) is surjective,
9. \(f\) is universally injective if and only if \(f'\) is universally injective,
10. \(f\) is affine if and only if \(f'\) is affine,
11. \(f\) is locally quasi-finite if and only if \(f'\) is locally quasi-finite,
12. \(f\) is locally of finite type of relative dimension \(d\) if and only if \(f'\) is locally of finite type of relative dimension \(d\),
13. \(f\) is universally open if and only if \(f'\) is universally open,
14. \(f\) is syntomic if and only if \(f'\) is syntomic,
15. \(f\) is smooth if and only if \(f'\) is smooth,
16. \(f\) is unramified if and only if \(f'\) is unramified,
17. \(f\) is \(\acute{e}tale\) if and only if \(f'\) is \(\acute{e}tale\),
18. \(f\) is proper if and only if \(f'\) is proper,
19. \(f\) is finite if and only if \(f'\) is finite,
0CFP (20) \( f \) is finite locally free (of rank d) if and only if \( f' \) is finite locally free (of rank d), and

(21) add more here.

Proof. The assumptions on \( X \) and \( Y \) mean that \( f \) is the base change of \( f' \) by \( X \to X' \). The properties \( P \) listed in (1) – (20) above are all stable under base change, hence if \( f' \) has property \( P \), then so does \( f \). See Schemes, Lemmas 18.2 and 21.13 and 23.5 and Morphisms, Lemmas 9.4 and 10.4 and 11.8 and 19.13 and 28.2 and 29.4 and 32.5 and 34.4 and 39.5 and 42.6 and 45.4.

The interesting direction in each case is therefore to assume that \( f \) has the property and deduce that \( f' \) has it too. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let \( W' \subset S' \) be an affine open and let \( U' \subset X' \) and \( V' \subset Y' \) be affine opens lying over \( W' \) with \( f'(U') \subset V' \). Let \( W' = \text{Spec}(R') \) and denote \( I \subset R' \) be the ideal defining the closed subscheme \( W' \cap S \). Say \( U' = \text{Spec}(B') \) and \( V' = \text{Spec}(A') \). Then we get a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

with exact rows. (II) The morphisms \( X \to X' \) and \( Y \to Y' \) are universal homeomorphisms. Hence the topology of the maps \( f \) and \( f' \) (after any base change) is identical. (III) If \( f \) is flat, then \( f' \) is flat and \( Y' \to S' \) is flat at every point in the image of \( f' \), see Lemma 10.2.

Ad (1). This is general remark (III).

Ad (2). Assume \( f \) is an isomorphism. Choose an affine open \( V' \subset Y' \) and set \( U' = (f')^{-1}(V') \). Then \( V = Y \cap V' \) is affine which implies that \( V \cong f^{-1}(V) = U = Y \times Y'. U' \) is affine. By Lemma 2.3 we see that \( U' \) is affine. Thus we have a diagram as in the general remark (I). By Algebra, Lemma 25.10 we see that \( A' \to B' \) is an isomorphism, i.e., \( U' \cong V' \). Thus \( f' \) is an isomorphism.

Ad (3). Assume \( f \) is an open immersion. Then \( f \) is an isomorphism of \( X \) with an open subscheme \( V \subset Y \). Let \( V' \subset Y' \) be the open subscheme whose underlying topological space is \( V \). Then \( f' \) is a map from \( X' \) to \( V' \) which is an isomorphism by (2). Hence \( f' \) is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (6). Note that \( X \times Y X = Y \times Y' (X' \times Y'. X') \) so that \( X' \times Y', X' \) is a thickening of \( X \times Y X \). Hence the topology of the maps \( \Delta_{X/Y} \) and \( \Delta_{X'/Y'} \) matches and we win. See also Lemma 3.1 for a more general statement.

Ad (7). Assume \( f \) is a monomorphism. Consider the diagonal morphism \( \Delta_{X'/Y'} : X' \to X' \times Y' X' \). Observe that \( X' \times Y'. X' \to S' \) is locally of finite type. The base change of \( \Delta_{X'/Y'} \) by \( S \to S' \) is \( \Delta_{X/Y} \) which is an isomorphism by assumption. By (2) we conclude that \( \Delta_{X'/Y'} \) is an isomorphism.
Ad (8). This is clear. See also Lemma 3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 3.1 for a more general statement.

Ad (10). Assume $f$ is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_Y U'$ is affine. By Lemma 2.3 we see that $U'$ is affine. Hence $f'$ is affine. See also Lemma 3.1 for a more general statement.

Ad (11). Follows from the fact that $f'$ is locally of finite type (by Morphisms, Lemma 14.8) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 19.6.

Ad (12). Follows from general remark (II) and the fact that $f'$ is locally of finite type (Morphisms, Lemma 14.8).

Ad (13). Immediate from general remark (II). See also Lemma 3.1 for a more general statement.

Ad (14). Assume $f$ is syntomic. By Morphisms, Lemma 20.11 $f'$ is locally of finite presentation. By general remark (III) $f'$ is flat. The fibres of $f'$ are the fibres of $f$. Hence $f'$ is syntomic by Morphisms, Lemma 29.11.

Ad (15). Assume $f$ is smooth. By Morphisms, Lemma 20.11 $f'$ is locally of finite presentation. By general remark (III) $f'$ is flat. The fibres of $f'$ are the fibres of $f$. Hence $f'$ is smooth by Morphisms, Lemma 32.3.

Ad (16). Assume $f$ unramified. By Morphisms, Lemma 14.8 $f'$ is locally of finite type. The fibres of $f'$ are the fibres of $f$. Hence $f'$ is unramified by Morphisms, Lemma 33.12.

Ad (17). Assume $f$ étale. By Morphisms, Lemma 20.11 $f'$ is locally of finite presentation. By general remark (III) $f'$ is flat. The fibres of $f'$ are the fibres of $f$. Hence $f'$ is étale by Morphisms, Lemma 34.8.

Ad (18). This follows from a combination of (6), the fact that $f$ is locally of finite type (Morphisms, Lemma 14.8), (4), and (5).

Ad (19). Combine (5), (10), Morphisms, Lemma 42.7, the fact that $f$ is locally of finite type (Morphisms, Lemma 14.8), and Morphisms, Lemma 42.4.

Ad (20). Assume $f$ finite locally free. By (19) we see that $f'$ is finite. By general remark (III) $f'$ is flat. By Morphisms, Lemma 20.11 $f'$ is locally of finite presentation. Hence $f'$ is finite locally free by Morphisms, Lemma 45.2.

**Lemma 10.6** (Deformations of projective schemes). Let $f : X \to S$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $\mathcal{L}$ be $f$-ample. Assume $S$ is quasi-compact. There exists a $d_0 \geq 0$ such that for every cartesian diagram

\[
\begin{array}{ccc}
X & \overset{i'}{\longrightarrow} & X' \\
\downarrow f & & \downarrow f' \\
S & \overset{i}{\longrightarrow} & S'
\end{array}
\]

where $S \subset S'$ is a thickening and $f'$ is proper, flat, of finite presentation we have

1. $R^p(f')_* (\mathcal{L}') \otimes^d = 0$ for all $p > 0$ and $d \geq d_0$,
(2) \( \mathcal{A}'_d = (f')_*(\mathcal{L}')^\otimes d \) is finite locally free for \( d \geq d_0 \),
(3) \( \mathcal{A}' = \mathcal{O}_S \oplus \bigoplus_{d \geq d_0} \mathcal{A}'_d \) is a quasi-coherent \( \mathcal{O}_S \)-algebra of finite presentation,
(4) there is a canonical isomorphism \( r' : X' \to \text{Proj}_{S'}(\mathcal{A}') \), and
(5) there is a canonical isomorphism \( \theta' : (r')^*\mathcal{O}_{\text{Proj}_{S'}(\mathcal{A}')} \to \mathcal{L}' \).

The construction of \( \mathcal{A}' , r' , \theta' \) is functorial in the data \( (X', S' , i , i' , f' , \mathcal{L}') \).

**Proof.** We first describe the maps \( r' \) and \( \theta' \). Observe that \( \mathcal{L}' \) is \( f' \)-ample, see Lemma \( 3.2 \). There is a canonical map of quasi-coherent graded \( \mathcal{O}_S \)-algebras \( \mathcal{A}' \to \bigoplus_{d \geq 0} (f')_*(\mathcal{L}')^\otimes d \) which is an isomorphism in degrees \( \geq d_0 \). Hence this induces an isomorphism on relative \( \text{Proj} \) compatible with the Serre twists of the structure sheaf, see Constructions, Lemma \( 18.4 \). Hence we get the morphism \( r' \) by Morphisms, Lemma \( 35.4 \) (which in turn appeals to the construction given in Constructions, Lemma \( 19.1 \)) and it is an isomorphism by Morphisms, Lemma \( 41.17 \). We get the map \( \theta' \) from Constructions, Lemma \( 19.1 \). By Properties, Lemma \( 28.2 \) we find that \( \theta' \) is an isomorphism (this also uses that the morphism \( r' \) over affine opens of \( S' \) is the same as the morphism from Properties, Lemma \( 26.9 \) as is explained in the proof of Morphisms, Lemma \( 41.17 \)).

Assuming the vanishing and local freeness stated in parts (1) and (2), the functoriality of the construction can be seen as follows. Suppose that \( h : T \to S' \) is a morphism of schemes, denote \( f_T : X'_T \to T \) the base change of \( f' \) and \( \mathcal{L}_T \) the pullback of \( \mathcal{L} \) to \( X'_T \). By cohomology and base change (as formulated in Derived Categories of Schemes, Lemma \( 21.3 \) for example) we have the corresponding vanishing over \( T \) and moreover \( h^*\mathcal{A}'_d = f_T^*\mathcal{L}_T^\otimes d \) (and thus the local freeness of pushforwards as well as the finite generation of the corresponding graded \( \mathcal{O}_T \)-algebra \( \mathcal{A}_T \)). Hence the morphism \( r_T : X_T \to \text{Proj}_T(\bigoplus f_T^*\mathcal{L}_T^\otimes d) \) is simply the base change of \( r' \) to \( T \) and the pullback of \( \theta' \) is the map \( \theta_T \).

Having said all of the above, we see that it suffices to prove (1), (2), and (3). Pick \( d_0 \) such that \( R^pf_*\mathcal{L}^\otimes d = 0 \) for all \( d \geq d_0 \) and \( p > 0 \), see Cohomology of Schemes, Lemma \( 16.1 \). We claim that \( d_0 \) works.

By cohomology and base change (Derived Categories of Schemes, Lemma \( 26.4 \)) we see that \( E'_d = Rf_*f'_*(-\mathcal{L})^\otimes d \) is a perfect object of \( D(\mathcal{O}_S) \) and its formation commutes with arbitrary base change. In particular, \( E_d = \text{Li}^*E'_d = Rf_*\mathcal{L}^\otimes d \). By Derived Categories of Schemes, Lemma \( 28.4 \) we see that for \( d \geq d_0 \) the complex \( E_d \) is isomorphic to the finite locally free \( \mathcal{O}_S \)-module \( f_*\mathcal{L}^\otimes d \) placed in cohomological degree 0. Then by Derived Categories of Schemes, Lemma \( 27.3 \) we conclude that \( E'_d \) is isomorphic to a finite locally free module placed in cohomological degree 0. Of course this means that \( E'_d = \mathcal{A}'_d[0] \), that \( R^pf'_*(-\mathcal{L})^\otimes d = 0 \) for \( p > 0 \), and that \( \mathcal{A}'_d \) is finite locally free. This proves (1) and (2).

The last thing we have to show is finite presentation of \( \mathcal{A}' \) as a sheaf of \( \mathcal{O}_{S'} \)-algebras (this notion was introduced in Properties, Section \( 22 \)). Let \( U' = \text{Spec}(R') \subset S' \) be an affine open. Then \( \mathcal{A}' = \mathcal{A}'(U') \) is a graded \( R' \)-algebra whose graded parts are finite projective \( R' \)-modules. We have to show that \( \mathcal{A}' \) is a finitely presented \( R' \)-algebra. We will prove this by reduction to the Noetherian case. Namely, we can find a finite type \( \mathbf{Z} \)-subalgebra \( R'_0 \subset R' \) and a pair \( (X'_0 , \mathcal{L}'_0) \) over \( R'_0 ^{\text{an}} \)!
whose base change is \((X'_U, L'|_{X'_U})\), see Limits, Lemmas \ref{limits-lm-10.2} \ref{limits-lm-10.3} \ref{limits-lm-13.1} \ref{limits-lm-8.7}, and \ref{limits-lm-4.15}. Cohomology of Schemes, Lemma \ref{coh-lm-16.1} implies \(A'_0 = \bigoplus_{d \geq 0} H^d(X'_0, (L'_0)\otimes^d)\) is a finitely generated graded \(R'_0\)-algebra and implies there exists a \(d'_0\) such that \(H^p(X'_0, (L'_0)\otimes^d) = 0, p > 0\) for \(d \geq d'_0\). By the arguments given above applied to \(X'_0 \to \text{Spec}(R'_0)\) and \(L'_0\) we see that \((A'_0)_d\) is a finite projective \(R'_0\)-module and that
\[
A'_d = A'_d(U') = H^d(X'_U, (L')\otimes^d|_{X'_U}) = H^0(X'_0, (L'_0)\otimes^d) \otimes_{R'_0} R' = (A'_0)_d \otimes_{R'_0} R'
\]
for \(d \geq d'_0\). Now a small twist in the argument is that we don’t know that we can choose \(d'_0\) equal to \(d'_0^2\). To get around this we use the following sequence of arguments to finish the proof:

\(\text{(a)}\) The algebra \(B = R'_0 \oplus \bigoplus_{d \geq \max(d_0, d'_0)} (A'_0)_d\) is an \(R'_0\)-algebra of finite type: apply the Artin-Tate lemma to \(B \subset A'_0\), see Algebra, Lemma \ref{algebra-lm-50.7}.

\(\text{(b)}\) As \(R'_0\) is Noetherian we see that \(B\) is an \(R'_0\)-algebra of finite presentation.

\(\text{(c)}\) By right exactness of tensor product we see that \(B \otimes_{R'_0} R'\) is an \(R'\)-algebra of finite presentation.

\(\text{(d)}\) By the displayed equalities this exactly says that \(C = R' \oplus \bigoplus_{d \geq \max(d_0, d'_0)} A'_d\) is an \(R'\)-algebra of finite presentation.

\(\text{(e)}\) The quotient \(A'/C\) is the direct sum of the finite projective \(R'\)-modules \(A'_d\), \(d_0 \leq d \leq \max(d_0, d'_0)\), hence finitely presented as \(R'\)-module.

\(\text{(f)}\) The quotient \(A'/C\) is finitely presented as a \(C\)-module by Algebra, Lemma \ref{algebra-lm-6.4}.

\(\text{(g)}\) Thus \(A'\) is finitely presented as a \(C\)-module by Algebra, Lemma \ref{algebra-lm-7.3}.

\(\text{(h)}\) By Algebra, Lemma \ref{algebra-lm-7.4} this implies \(A'\) is finitely presented as \(C\)-algebra.

\(\text{(i)}\) Finally, by Algebra, Lemma \ref{algebra-lm-6.2} applied to \(R' \to C \to A'\) this implies \(A'\) is finitely presented as an \(R'\)-algebra.

This finishes the proof. \(\square\)

11. Formally smooth morphisms

Michael Artin’s position on differential criteria of smoothness (e.g., Morphisms, Lemma \ref{morph-lm-32.14}) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism \(X \to S\). Such a morphism is characterized by the property that \(T\)-valued points of \(X\) lift to infinitesimal thickenings of \(T\) provided \(T\) is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma \ref{morph-lm-11.7}.

It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map \(R \to A\) is called formally smooth (see Algebra, Definition \ref{algebra-lm-136.1}) if for every commutative solid diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B/I \\
\downarrow & & \downarrow \\
R & \longrightarrow & B
\end{array}
\]

where \(I \subset B\) is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.  

\(^2\)Actually, one can reduce to this case by doing more limit arguments.
Definition 11.1. Let \( f : X \to S \) be a morphism of schemes. We say \( f \) is formally smooth if given any solid commutative diagram

\[
\begin{array}{ccc}
  X & \leftarrow & T \\
  f \downarrow & & \downarrow i \\
  S & \leftarrow & T'
\end{array}
\]

where \( T \subset T' \) is a first order thickening of affine schemes over \( S \) there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that \( T' \) be affine could be dropped, see Lemmas 6.2 and 8.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on \( T' \).

In fact, analyzing the proof of Lemma 11.10 shows that this would be equivalent to the definition as it currently stands. In particular, the being formally smooth is Zariski local on the source (and in fact it is smooth local on the source, insert future reference here).

Lemma 11.2. A composition of formally smooth morphisms is formally smooth.

Proof. Omitted. \( \square \)

Lemma 11.3. A base change of a formally smooth morphism is formally smooth.

Proof. Omitted, but see Algebra, Lemma 136.2 for the algebraic version. \( \square \)

Lemma 11.4. Let \( f : X \to S \) be a morphism of schemes. Then \( f \) is formally étale if and only if \( f \) is formally smooth and formally unramified.

Proof. Omitted. \( \square \)

Lemma 11.5. Let \( f : X \to S \) be a morphism of schemes. Let \( U \subset X \) and \( V \subset S \) be open subschemes such that \( f(U) \subset V \). If \( f \) is formally smooth, so is \( f|_U : U \to V \).

Proof. Consider a solid diagram

\[
\begin{array}{ccc}
  U & \leftarrow & T \\
  f|_U \downarrow & & \downarrow i \\
  V & \leftarrow & T'
\end{array}
\]

as in Definition 11.1. If \( f \) is formally smooth, then there exists an \( S \)-morphism \( a' : T' \to X \) such that \( a'|_T = a \). Since the underlying sets of \( T \) and \( T' \) are the same we see that \( a' \) is a morphism into \( U \) (see Schemes, Section 3). And it clearly is a \( V \)-morphism as well. Hence the dotted arrow above as desired. \( \square \)

Lemma 11.6. Let \( f : X \to S \) be a morphism of schemes. Assume \( X \) and \( S \) are affine. Then \( f \) is formally smooth if and only if \( \mathcal{O}_S(S) \to \mathcal{O}_X(X) \) is a formally smooth ring map.

Proof. This is immediate from the definitions (Definition 11.1 and Algebra, Definition 136.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 6.5. \( \square \)
The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 6.1) that we can recognize whether a morphism \( f : X \to S \) is smooth in terms of “simple” properties of the functor \( h_X : \text{Sch}/S \to \text{Sets} \).

**Lemma 11.7** (Infinitesimal lifting criterion). Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. The morphism \( f \) is smooth, and
2. the morphism \( f \) is locally of finite presentation and formally smooth.

**Proof.** Assume \( f : X \to S \) is locally of finite presentation and formally smooth. Consider a pair of affine opens \( \text{Spec}(A) = U \subset X \) and \( \text{Spec}(R) = V \subset S \) such that \( f(U) \subset V \). By Lemma 11.5 we see that \( U \to V \) is formally smooth. By Lemma 11.6 we see that \( R \to A \) is formally smooth. By Morphisms, Lemma 20.2 we see that \( R \to A \) is of finite presentation. By Algebra, Proposition 136.13 we see that \( R \to A \) is smooth. Hence by the definition of a smooth morphism we see that \( X \to S \) is smooth.

Conversely, assume that \( f : X \to S \) is smooth. Consider a solid commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a} & T \\
\downarrow{f} & & \downarrow{i} \\
S & \xleftarrow{i} & T'
\end{array}
\]

as in Definition 11.1. We will show the dotted arrow exists thereby proving that \( f \) is formally smooth.

Let \( \mathcal{F} \) be the sheaf of sets on \( T' \) of Lemma 9.4 in the special case discussed in Remark 9.6. Let

\[\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})\]

be the sheaf of \( \mathcal{O}_T \)-modules with action \( \mathcal{H} \times \mathcal{F} \to \mathcal{F} \) as in Lemma 9.5. Our goal is simply to show that \( \mathcal{F}(T) \neq \emptyset \). In other words we are trying to show that \( \mathcal{F} \) is a trivial \( \mathcal{H} \)-torsor on \( T \) (see Cohomology, Section 5). There are two steps: (I) To show that \( \mathcal{F} \) is a torsor we have to show that \( \mathcal{F}_t \neq \emptyset \) for all \( t \in T \) (see Cohomology, Definition 5.1). (II) To show that \( \mathcal{F} \) is the trivial torsor it suffices to show that \( H^1(T, \mathcal{H}) = 0 \) (see Cohomology, Lemma 5.3— we may use either cohomology of \( \mathcal{H} \) as an abelian sheaf or as an \( \mathcal{O}_T \)-module, see Cohomology, Lemma 14.3).

First we prove (I). To see this, for every \( t \in T \) we can choose an affine open \( U \subset T \) neighbourhood of \( t \) such that \( a(U) \) is contained in an affine open \( \text{Spec}(A) = W \subset X \) which maps to an affine open \( \text{Spec}(R) = V \subset S \). By Morphisms, Lemma 32.2 the ring map \( R \to A \) is smooth. Hence by Algebra, Proposition 136.13 the ring map \( R \to A \) is formally smooth. Lemma 11.6 in turn implies that \( W \to V \) is formally smooth. Hence we can lift \( a|_U : U \to W \) to a \( V \)-morphism \( a' : U' \to W \subset X \) showing that \( \mathcal{F}(U) \neq \emptyset \).

Finally we prove (II). By Morphisms, Lemma 31.13 we see that \( \Omega_{X/S} \) is of finite presentation (it is even finite locally free by Morphisms, Lemma 32.12). Hence \( a^*\Omega_{X/S} \) is of finite presentation (see Modules, Lemma 11.4). Hence the sheaf \( \mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'}) \) is quasi-coherent by the discussion in Schemes, Section 24. Thus by Cohomology of Schemes, Lemma 2.2 we have \( H^1(T, \mathcal{H}) = 0 \) as desired. \( \square \)
Locally projective quasi-coherent modules are defined in Properties, Section 21.

Lemma 11.8. Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega^1_{X/Y}$ is locally projective on $X$.

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 11.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 11.6. Hence by Algebra, Lemma 136.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega^1_{\Gamma(U, \mathcal{O}_U)/\Gamma(V, \mathcal{O}_V)}$ is projective. Hence $\Omega^1_{U/V}$ is locally projective, see Properties, Section 21.

Lemma 11.9. Let $T$ be an affine scheme. Let $F, G$ be quasi-coherent $\mathcal{O}_T$-modules. Consider $H = \text{Hom}_{\mathcal{O}_T}(F, G)$. If $F$ is locally projective, then $H^1(T, H) = 0$.

Proof. By the definition of a locally projective sheaf on a scheme (see Properties, Definition 21.1) we see that $F$ is a direct summand of a free $\mathcal{O}_T$-module. Hence we may assume that $F = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $H = \prod_{i \in I} G$ is a product of quasi-coherent modules. By Cohomology, Lemma 12.12 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Cohomology of Schemes, Lemma 2.2.

Lemma 11.10. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. $f$ is formally smooth,
2. for every $x \in X$ there exist opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth,
3. for every pair of affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is formally smooth, and
4. there exists an affine open covering $Y = \bigcup V_j$ and for each $j$ an affine open covering $f^{-1}(V_j) = \bigcup U_i$ such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a formally smooth ring map for all $j$ and $i$.

Proof. The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), and (2) $\Rightarrow$ (4) follow from Lemma 11.5. The implication (3) $\Rightarrow$ (4) is immediate.

Assume (4). The proof that $f$ is formally smooth is the same as the second part of the proof of Lemma 11.7. Consider a solid commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a} & T \\
\downarrow{f} & & \downarrow{i} \\
Y & \xleftarrow{i} & T'
\end{array}
\]

as in Definition 11.1. We will show the dotted arrow exists thereby proving that $f$ is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $T'$ of Lemma 9.4 as in the special case discussed in Remark 9.6. Let

$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega^1_{X/Y}, \mathcal{C}_{T/T'})$

be the sheaf of $\mathcal{O}_T$-modules on $T$ with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 9.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology, Definition 5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ locally has a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$, see Cohomology, Lemma 5.3.
First we prove (I). To see this, for every $t \in T$ we can choose an affine open $W \subset T$ neighborhood of $t$ such that $a(W)$ is contained in $U_{ji}$ for some $i,j$. Let $W' \subset T'$ be the corresponding open subscheme. By assumption (4) we can lift $a|_W : W \to U_{ji}$ to a $V_j$-morphism $a' : W' \to U_{ji}$ showing that $\mathcal{F}(W')$ is nonempty.

Finally we prove (II). By Lemma 11.8 we see that $\Omega_{U_{ji}/V_j}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.2. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties, Lemma 21.3. Hence $H^1(T, H) = H^1(T, \mathcal{H}_{\text{Hom}}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$

by Lemma 11.9 as desired. □

**Lemma 11.11.** Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume $f$ is formally smooth. Then

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 31.7) is short exact.

**Proof.** The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 136.7 is exact. This is Algebra, Lemma 136.9. □

**Lemma 11.12.** Let $h : Z \to X$ be a formally unramified morphism of schemes over $S$. Assume that $Z$ is formally smooth over $S$. Then the canonical exact sequence

$$0 \to C_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 7.10 is short exact.

**Proof.** Let $Z \to Z'$ be the universal first order thickening of $Z$ over $X$. From the proof of Lemma 7.10 we see that our sequence is identified with the sequence

$$C_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$ 

Since $Z \to S$ is formally smooth we can locally on $Z'$ find a left inverse $Z' \to Z$ over $S$ to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 31.16. □

**Lemma 11.13.** Let

$$Z \xrightarrow{i} X \xrightarrow{f} Y$$

be a commutative diagram of schemes where $i$ and $j$ are formally unramified and $f$ is formally smooth. Then the canonical exact sequence

$$0 \to C_{Z/Y} \to C_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 7.11 is exact and locally split.
**Proof.** Denote \( Z \to Z' \) the universal first order thickening of \( Z \) over \( X \). Denote \( Z \to Z'' \) the universal first order thickening of \( Z \) over \( Y \). By Lemma 7.10 there is a canonical morphism \( Z' \to Z'' \) so that we have a commutative diagram

\[
Z \xrightarrow{i'} Z' \xrightarrow{a} X \quad \xrightarrow{f} \quad \xrightarrow{f} Z'' \xrightarrow{b} Y
\]

In the proof of Lemma 7.11 we identified the sequence above with the sequence

\[
C_{Z/Z''} \to C_{Z/Z'} \to (i')^* \Omega_{Z'/Z''} \to 0
\]

Let \( U'' \subset Z'' \) be an affine open. Denote \( U \subset Z \) and \( U' \subset Z' \) the corresponding affine open subschemes. As \( f \) is formally smooth there exists a morphism \( h : U'' \to X \) which agrees with \( i \) on \( U \) and such that \( f \circ h \) equals \( b|_{U''} \). Since \( Z' \) is the universal first order thickening we obtain a unique morphism \( g : U'' \to Z' \) such that \( g = a \circ h \). The universal property of \( Z'' \) implies that \( k \circ g \) is the inclusion map \( U'' \to Z'' \). Hence \( g \) is a left inverse to \( k \). Picture

\[
\begin{array}{c}
U'' \xrightarrow{g} Z' \\
\downarrow \quad \downarrow k
\end{array}
\quad \xrightarrow{U'' \xrightarrow{g} Z''}
\]

Thus \( g \) induces a map \( C_{Z/Z'}|_U \to C_{Z/Z''}|_U \) which is a left inverse to the map \( C_{Z/Z''} \to C_{Z/Z'} \) over \( U \). □

### 12. Smoothness over a Noetherian base

02HW It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

02HX **Lemma 12.1.** Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \). Assume that \( S \) is locally Noetherian and \( f \) locally of finite type. The following are equivalent:

1. \( f \) is smooth at \( x \),
2. for every solid commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & \text{Spec}(B) \\
\downarrow f & & \downarrow i \\
S & \xleftarrow{\beta} & \text{Spec}(B')
\end{array}
\]

where \( B' \to B \) is a surjection of local rings with \( \text{Ker}(B' \to B) \) of square zero, and \( \alpha \) mapping the closed point of \( \text{Spec}(B) \) to \( x \) there exists a dotted arrow making the diagram commute,
3. same as in (2) but with \( B' \to B \) ranging over small extensions (see Algebra, Definition 139.1), and
4. same as in (2) but with \( B' \to B \) ranging over small extensions such that \( \alpha \) induces an isomorphism \( \kappa(x) \to \kappa(m) \) where \( m \subset B \) is the maximal ideal.
**Proof.** Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of $x$ such that $f(U) \subset V$. For any “test” diagram as in (2) the morphism $\alpha$ will map $\text{Spec}(B)$ into $U$ and the morphism $\beta$ will map $\text{Spec}(B')$ into $V$ (see Schemes, Section 13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, $V$ is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 5.2 and Morphisms, Lemma 14.2.) In this affine case the lemma is identical to Algebra, Lemma 139.2. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions “centered” at points of finite type (see Morphisms, Section 15).

**Lemma 12.2.** Let $f : X \to S$ be a morphism of schemes. Assume that $S$ is locally Noetherian and $f$ locally of finite type. The following are equivalent:

1. $f$ is smooth,
2. for every solid commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & \text{Spec}(B) \\
\downarrow f & \nearrow i & \downarrow \\
S & \leftarrow & \text{Spec}(B') \\
\end{array}
$$

where $B' \to B$ is a small extension of Artinian local rings and $\beta$ of finite type (!) there exists a dotted arrow making the diagram commute.

**Proof.** If $f$ is smooth, then the infinitesimal lifting criterion (Lemma 11.7) says $f$ is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where $f$ is not smooth forms a closed subset $T$ of $X$. By the discussion in Morphisms, Section 15 if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$
\text{Spec}(\kappa(x)) \to X \to S
$$

is of finite type (namely, pick any point $x$ of $T$ which is closed in an affine open of $X$). By Morphisms, Lemma 15.2 given any local Artinian ring $B'$ with residue field $\kappa(x)$ then any morphism $\beta : \text{Spec}(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 12.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a $x$ a contradiction. □

Here is a useful application.

**Lemma 12.3.** Let $f : X \to S$ be a finite type morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with $n$th infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$.

1. If $X_n \to Z_n$ is smooth for all $n$, then $f$ is smooth at every point of $f^{-1}(Z)$.
2. If $X_n \to Z_n$ is étale for all $n$, then $f$ is étale at every point of $f^{-1}(Z)$.

**Proof.** Assume $X_n \to Z_n$ is smooth for all $n$. Let $x \in X$ be a point lying over a point of $Z$. Given a small extension $B' \to B$ and morphisms $\alpha$, $\beta$ as in Lemma 12.1 part (3) the maximal ideal of $B'$ is nilpotent (as $B'$ is Artinian) and hence the morphism $\beta$ factors through $Z_n$ and $\alpha$ factors through $X_n$ for a suitable $n$. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. □
Lemma 12.4. Let $f : X \to S$ be a morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with $n$th infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$. If $X_n \to Z_n$ is flat for all $n$, then $f$ is flat at every point of $f^{-1}(Z)$.

Proof. This is a translation of Algebra, Lemma 98.11 into the language of schemes. □

13. The naive cotangent complex

Definition 13.1. Let $f : X \to Y$ be a morphism of schemes. The naive cotangent complex of $f$ is the complex defined in Modules, Definition 26.6. Notation: $NL_f$ or $NL_{X/Y}$.

Lemma 13.2. Let $f : X \to Y$ be a morphism of schemes. Let $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ be affine opens with $f(U) \subset V$. There is a canonical map

$$\hat{NL}_{A/R} \to NL_{X/Y}|_U$$

of complexes which is an isomorphism in $D(O_U)$.

Proof. From the construction of $NL_{X/Y}$ in Modules, Section 26 we see there is a canonical map of complexes $NL_{O_X(U)/f^{-1}O_Y(U)} \to NL_{X/Y}(U)$ of $A = O_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $R \to f^{-1}O_Y(U)$ we obtain a canonical map $NL_{A/R} \to NL_{O_X(U)/f^{-1}O_Y(U)}$ of complexes of $A$-modules. Using the universal property of the $\hat{-}$ functor (see Schemes, Lemma 7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows from Algebra, Lemma 132.11 and 132.13 and Modules, Lemma 26.4 (and the description of the stalks of $O_X$ and $f^{-1}O_Y$ at a point $p \in \text{Spec}(A)$ as $A_p$ and $R_q$ where $q = R \cap p$; references used are Schemes, Lemma 5.4 and Sheaves, Lemma 21.5). □

Lemma 13.3. Let $f : X \to Y$ be a morphism of schemes. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees $-1, 0$ and equal to $\Omega_{X/Y}$ in degree 0.

Proof. By construction of the naive cotangent complex in Modules, Section 26 we have that $NL_{X/Y}$ is a complex sitting in degrees $-1, 0$ and that its cohomology in degree 0 is $\Omega_{X/Y}$. The sheaf of differentials is quasi-coherent (by Morphisms, Lemma 31.7). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking over affines using Lemma 132.2. □

Lemma 13.4. Let $f : X \to Y$ be a morphism of schemes. If $f$ is locally of finite presentation, then $NL_{X/Y}$ is locally on $X$ quasi-isomorphic to a complex

$$\ldots \to 0 \to F^{-1} \to F^0 \to 0 \to \ldots$$

of quasi-coherent $O_X$-modules with $F^0$ of finite presentation and $F^{-1}$ of finite type.
Proof. By Lemma 13.2 it suffices to show that $NL_{A/R}$ has this shape if $R \to A$ is a finitely presented ring map. Write $A = R[x_1, \ldots, x_n]/I$ with $I$ finitely generated. Then $I/I^2$ is a finite $A$-module and $NL_{A/R}$ is quasi-isomorphic to

$$\ldots \to 0 \to I/I^2 \to \bigoplus_{i=1}^{n} \text{Ad}x_i \to 0 \to \ldots$$

by Algebra, Section 13.2 and in particular Algebra, Lemma 13.2. □

**Lemma 13.5.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

1. $f$ is formally smooth,
2. $H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.

Proof. This follows from Algebra, Proposition 13.6 and Lemma 11.10. □

**Lemma 13.6.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

1. $f$ is formally étale,
2. $H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = 0$.

Proof. A formally étale morphism is formally smooth and hence we have $H^{-1}(NL_{X/Y}) = 0$ by Lemma 13.5. On the other hand, we have $\Omega_{X/Y} = 0$ by Lemma 8.6. Conversely, if (2) holds, then $f$ is formally smooth by Lemma 13.5 and formally unramified by Lemma 6.7 and hence formally étale by Lemma 11.4. □

**Lemma 13.7.** Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

1. $f$ is smooth, and
2. $f$ is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.

Proof. This follows from the definition of a smooth ring homomorphism (Algebra, Definition 13.1), Lemma 13.2, and the definition of a smooth morphism of schemes (Morphisms, Definition 32.1). We also use that finite locally free is the same as finite projective for modules over rings (Algebra, Lemma 77.2). □

**Lemma 13.8.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a canonical six term exact sequence

$$H^{-1}(f^*NL_{Y/Z}) \to H^{-1}(NL_{X/Y}) \to H^{-1}(NL_{X/Z}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of cohomology sheaves.

Proof. Special case of Modules, Lemma 26.3. □

14. Pushouts in the category of schemes

In this section we collect some results on pushouts in the category of schemes. Please see the detailed discussion in [Fer03]. See Categories, Section 4 for a general discussion of pushouts in any category.
Lemma 14.1. Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow j & & \downarrow a \\
Y & \xrightarrow{b} & W
\end{array}
\]

and set \( c = a \circ i = b \circ j \). If there exists an fpqc covering \( \{ W_i \to W \} \) such that for all \( i \) and \( i' \) the diagrams

\[
\begin{array}{ccc}
Z \times_{c,W} W_i & \to & X \times_{a,W} W_i \\
\downarrow & & \downarrow \\
Y \times_{b,W} W_i & \to & W_i
\end{array}
\]

\[
\begin{array}{ccc}
Z \times_{c,W} (W_i \times_W W_{i'}) & \to & X \times_{a,W} (W_i \times_W W_{i'}) \\
\downarrow & & \downarrow \\
Y \times_{b,W} (W_i \times_W W_{i'}) & \to & (W_i \times_W W_{i'})
\end{array}
\]

are cocartesian, then so is the original diagram.

Proof. Namely, for a scheme \( T \) a morphism \( W \to T \) is the same thing as a collection of morphism \( W_i \to T \) which agree on overlaps, see Descent, Lemma 10.3. \( \square \)

A suitably general case is where one morphism is a closed immersion and the second is integral, however a further condition is needed to ensure the pushout exists.

Proposition 14.2. Let \( i : Z \to X \) and \( j : Z \to Y \) be morphisms of schemes. Assume

1. \( i \) is a closed immersion,
2. \( j \) is an integral morphism of schemes,
3. for \( y \in Y \) there exists an affine open \( U \subset X \) with \( j^{-1}(\{ y \}) \subset i^{-1}(U) \).

Then the pushout \( Y \amalg_Z X \) exists in the category of schemes. Picture

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow j & & \downarrow a \\
Y & \xrightarrow{b} & Y \amalg_Z X
\end{array}
\]

The diagram is a fibre square, the morphism \( a \) is integral, the morphism \( b \) is a closed immersion, and there is a short exact sequence

\[ 0 \to \mathcal{O}_W \to a_* \mathcal{O}_X \oplus b_* \mathcal{O}_Y \to c_* \mathcal{O}_Z \to 0 \]

where \( c = a \circ i = b \circ j \).

Proof. As a topological space we set \( Y \amalg_Z X \) equal to the pushout of the diagram in the category of topological spaces (Topology, Section 29). This is just the pushout of the underlying sets (Topology, Lemma 29.1) endowed with the quotient topology. On \( Y \amalg_Z X \) we have the maps of sheaves of rings

\[ b_* \mathcal{O}_Y \to c_* \mathcal{O}_Z \leftarrow a_* \mathcal{O}_X \]

and we can define

\[ \mathcal{O}_{Y \amalg_Z X} = b_* \mathcal{O}_Y \times_{c_* \mathcal{O}_Z} a_* \mathcal{O}_X \]

as the fibre product in the category of sheaves of rings. To prove that we obtain a scheme we have to show that every point has an affine open neighbourhood. This is clear for points not in the image of \( c \) as the image of \( c \) is a closed subset whose complement is isomorphic to a ringed space over \( (Y \setminus j(\{ z \})) \amalg (X \setminus i(\{ z \})) \).
A point in the image of \( c \) corresponds to a unique \( y \in Y \) in the image of \( j \). Choose an affine open \( U \subset X \) such that \( j^{-1}(\{y\}) \subset i^{-1}(U) \). Choose an affine open \( V \subset Y \) neighbourhood of \( y \) such that \( j^{-1}(V) \subset i^{-1}(U) \). This is possible because \( j : Z \to Y \) is a closed morphism (Morphisms, Lemma 42.7) and \( i^{-1}(U) \) contains the fibre over \( y \). Since \( j \) is integral, the scheme theoretic fibre \( Z_y \) is the spectrum of an algebra integral over a field. By Limits, Lemma 11.6 we can find an \( \mathcal{F} \in \Gamma(i^{-1}(U), \mathcal{O}_{i^{-1}(U)}) \) such that \( Z_y \subset D(\mathcal{F}) \subset j^{-1}(V) \). Since \( \iota_i^{-1}(U) : i^{-1}(U) \to U \) is a closed immersion of affines, we can choose an \( f \in \Gamma(U, \mathcal{O}_U) \) whose restriction to \( i^{-1}(U) \) is \( \mathcal{F} \). After replacing \( U \) by the principal open \( D(f) \subset U \) we find affine opens \( y \in V \subset Y \) and \( U \subset X \) with
\[
j^{-1}(\{y\}) \subset i^{-1}(U) \subset j^{-1}(V)
\]
Now we (in some sense) repeat the argument. Namely, we choose \( g \in \Gamma(V, \mathcal{O}_V) \) such that \( y \in D(g) \) and \( j^{-1}(D(g)) \subset i^{-1}(U) \) (possible by the same argument as above). Then we can pick \( f \in \Gamma(U, \mathcal{O}_U) \) whose restriction to \( i^{-1}(U) \) is the pullback of \( g \) by \( i^{-1}(U) \to V \) (again possible by the same reason as above). Then we finally have affine opens \( y \in V' = D(g) \subset V \subset Y \) and \( U' = D(f) \subset U \subset X \) with \( j^{-1}(V') = i^{-1}(V') \). Since the construction of the first paragraph is clearly compatible with restriction to compatible open subschemes, to prove that it produces a scheme we may assume \( X, Y, \) and \( Z \) are affine.

If \( X = \text{Spec}(A), Y = \text{Spec}(B) \), and \( Z = \text{Spec}(C) \) are affine, then More on Algebra, Lemma 6.2 shows that \( Y \amalg_Z X = \text{Spec}(B \times_C A) \) as topological spaces. To finish the proof that \( Y \times_Z X \) is a scheme, it suffices to show that on \( \text{Spec}(B \times_C A) \) the structure sheaf is the fibre product of the pushforwards. This follows by applying More on Algebra, Lemma 6.3 to principal affine opens of \( \text{Spec}(B \times_C A) \).

The discussion above shows the scheme \( Y \amalg_Z X \) has an affine open covering \( Y \amalg_Z X = \bigcup \mathcal{W}_i \) such that \( U_i = a^{-1}(\mathcal{W}_i), V_i = b^{-1}(\mathcal{W}_i), \) and \( \Omega_i = c^{-1}(\mathcal{W}_i) \) are affine open in \( X, Y, \) and \( Z \). Thus \( a \) and \( b \) are affine. Moreover, if \( A_i, B_i, C_i \) are the rings corresponding to \( U_i, V_i, \Omega_i \), then \( A_i \to C_i \) is surjective and \( W_i \) corresponds to \( A_i \times_C B_i \) which surjects onto \( B_i \). Hence \( b \) is a closed immersion. The ring map \( A_i \times_C B_i \to A_i \) is integral by More on Algebra, Lemma 6.3 hence \( a \) is integral. The short exact sequence comes from the short exact sequence
\[
0 \to A_i \times_C B_i \to A_i \times B_i \to C_i \to 0
\]
The diagram is cartesian because
\[
C_i \cong B_i \otimes_{B_i \times_C A_i} A_i
\]
This follows as \( B_i \times_C A_i \to B_i \) and \( A_i \to C_i \) are surjective maps whose kernels are the same.

We finish the proof by showing our construction gives a pushout in the category of schemes. Let \( f : X \to T \) and \( g : Y \to T \) be morphisms of schemes with \( f \circ i = g \circ j \).

Then we obtain a map of topological spaces \( (g, f) : Y \amalg_Z X \to T \). We can use the maps \( f^2 : f^{-1}\mathcal{O}_T \to \mathcal{O}_X \) and \( g^2 : g^{-1}\mathcal{O}_T \to \mathcal{O}_Y \) the equalities \( f = (g, f) \circ a \), \( g = (g, f) \circ b \) and adjunction of the pairs \( (a^{-1}, a_*), \) and \( (b^{-1}, b_*) \) to get maps
\[
(g, f)^{-1}\mathcal{O}_T \to a_*\mathcal{O}_X \text{ and } (g, f)^{-1}\mathcal{O}_T \to b_*\mathcal{O}_Y
\]
A computation (omitted) shows that the compositions into \( c_* \mathcal{O}_Z \) are equal. Thus we get a morphism of ringed spaces

\[
h : Y \amalg \mathbb{Z} X \longrightarrow T
\]

by our choice of the structure sheaf of \( Y \amalg \mathbb{Z} X \) as the fibre product of \( a_* \mathcal{O}_X \) and \( b_* \mathcal{O}_Y \) over \( c_* \mathcal{O}_Z \). To show that \( h \) is a morphism of locally ringed spaces (and hence a morphism of schemes), let \( s \in Y \amalg \mathbb{Z} X \) be a point mapping to \( t \in T \). Then either \( s \) is the image of a \( y \in Y \) or the image of an \( x \in X \). Consider

\[
\mathcal{O}_{T,t} \rightarrow \mathcal{O}_{Y \amalg \mathbb{Z} X, s} \rightarrow \mathcal{O}_{Y, y} \quad \text{or} \quad \mathcal{O}_{T,t} \rightarrow \mathcal{O}_{Y \amalg \mathbb{Z} X, s} \rightarrow \mathcal{O}_{X, x}
\]

Since the composition and the second map are local ring homomorphisms, we conclude. \( \square \)

**Lemma 14.3.** Let \( S \) be a scheme. Let \( i : Z \rightarrow X \) and \( j : Z \rightarrow Y \) be morphisms of schemes over \( S \). Assume

1. \( i \) is a closed immersion,
2. \( j \) is an integral morphism,
3. for \( y \in Y \) there exists an affine open \( U \subset X \) with \( j^{-1}(\{y\}) \subset i^{-1}(U) \).

If \( X \) and \( Y \) are separated, then the pushout \( Y \amalg \mathbb{Z} X \) (Proposition 14.2) is separated. Same with “separated over \( S \), “quasi-separated”, and “quasi-separated over \( S \”).

**Proof.** The morphism \( Y \amalg X \rightarrow Y \amalg \mathbb{Z} X \) is surjective and universally closed. Thus we may apply Morphisms, Lemma 39.11. \( \square \)

**Lemma 14.4.** Let \( S \) be a locally Noetherian scheme. Let \( i : Z \rightarrow X \) and \( j : Z \rightarrow Y \) be morphisms of schemes locally of finite type over \( S \). Assume

1. \( i \) is a closed immersion,
2. \( j \) is a finite morphism,
3. for \( y \in Y \) there exists an affine open \( U \subset X \) with \( j^{-1}(\{y\}) \subset i^{-1}(U) \).

Then the pushout \( Y \amalg \mathbb{Z} X \) (Proposition 14.2) is locally of finite type over \( S \).

**Proof.** Looking on affine opens we recover the result of More on Algebra, Lemma 5.1. \( \square \)

Next, we discuss existence in the case where both morphisms are closed immersions.

**Lemma 14.5.** Let \( i : Z \rightarrow X \) and \( j : Z \rightarrow Y \) be closed immersions of schemes. Then the pushout \( Y \amalg \mathbb{Z} X \) exists in the category of schemes. Picture

\[
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow \quad j & & \quad \downarrow a \\
Y & \longrightarrow & Y \amalg \mathbb{Z} X
\end{array}
\]

The diagram is a fibre square, the morphisms \( a \) and \( b \) are closed immersions, and there is a short exact sequence

\[
0 \rightarrow \mathcal{O}_{Y \amalg \mathbb{Z} X} \rightarrow a_* \mathcal{O}_X \oplus b_* \mathcal{O}_Y \rightarrow c_* \mathcal{O}_Z \rightarrow 0
\]

where \( c = a \circ i = b \circ j \).

**Proof.** This is a special case of Proposition 14.2. Observe that condition (3) in the proposition is immediate be cause the fibres of \( j \) are singletons. Finally, reverse the roles of the arrows to conclude that both \( a \) and \( b \) are closed immersions. \( \square \)
Lemma 14.6. Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes and let $\varphi : X' \times_{Z,j} Z \to Y' \times_{Y,j} Z$ be an isomorphism of schemes over $Z$. Consider the morphism
\[ h : X' \amalg_{X' \times_{X,i} Z,\varphi} Y' \to X \amalg_{Z} Y \]
Then we have
1. $h$ is locally of finite type if and only if $f$ and $g$ are locally of finite type,
2. $h$ is flat if and only if $f$ and $g$ are flat,
3. $h$ is flat and locally of finite presentation if and only if $f$ and $g$ are flat and locally of finite presentation,
4. $h$ is smooth if and only if $f$ and $g$ are smooth,
5. $h$ is étale if and only if $f$ and $g$ are étale, and
6. add more here as needed.

Proof. We know that the pushouts exist by Lemma 14.5. In particular we get the morphism $h$. Hence we may replace all schemes in sight by affine schemes. In this case the assertions of the lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7. □

An important case for us will be the case where one of the morphisms is a thickening and the other is an affine morphism.

Lemma 14.7. Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Then there exists a pushout
\[ X \to X' \quad \quad Y \to Y \amalg_{X} X' \]
in the category of schemes. Moreover $Y' = Y \amalg_{X} X'$ is a thickening of $Y$ and
\[ \mathcal{O}_{Y'} = \mathcal{O}_{Y} \times_{f, \mathcal{O}_{X}} f'_* \mathcal{O}_{X'} \]
as sheaves on $|Y| = |Y'|$.

Proof. We first construct $Y'$ as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals $f$. As structure sheaf $\mathcal{O}_{Y'}$, we take the right hand side of the equation of the lemma. To see that $Y'$ is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume $Y$ is affine. Then $X$ is affine (as $f$ is affine) and $X'$ is affine as well (see Lemma 2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_{A} A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\text{Spec}(B') = \text{Spec}(B)$ (as topological spaces). We claim that $Y' = \text{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_g$. Namely, by More on Algebra, Lemma 5.3 we see that
\[ (B')_{g'} = B_g \times_{A_h} A'_{h'} \]
where $h \in A, h' \in A'$ are the images of $g'$. Since $B_g$, resp. $A_h$, resp. $A'_{h'}$ is equal to $\mathcal{O}_{Y}(D(g))$, resp. $f_* \mathcal{O}_{X}(D(g))$, resp. $f'_* \mathcal{O}_{X'}(D(g))$ the claim follows.
Finally, we prove the universal property of the pushout holds for $Y'$ and the morphisms $Y \to Y'$ and $X' \to Y'$. Namely, let $S$ be a scheme and let $b : Y \to S$ and $a' : X' \to S$ be morphisms such that

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & S \\
\end{array}
$$

commutes. Note that $a' = b \circ f'$ on underlying topological spaces. Denote also $(a')^\sharp : b^{-1}\mathcal{O}_S \to f'_*\mathcal{O}_{X'}$, the map which is adjoint to $(a')^\sharp : (a')^{-1}\mathcal{O}_S = (f')^{-1}b^{-1}\mathcal{O}_S \to \mathcal{O}_{X'}$. Then we get a map

$$b^{-1}\mathcal{O}_S \xrightarrow{(b^\sharp,(a')^\sharp)} \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'} = \mathcal{O}_Y.$$

which defines a morphism of ringed spaces $b' : Y' \to S$ compatible with $a'$ and $b$. Since $Y \subset Y'$ is a thickening it follows that $b'$ is a morphism of locally ringed spaces, i.e., a morphism of schemes. This finishes the proof. 

In the following lemma we use the fibre product of categories as defined in Categories, Example [30.3]

**Lemma 14.8.** Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \times_X X'$ be the pushout (see Lemma 14.7).

Base change gives a functor

$$F : (\text{Sch}/Y') \longrightarrow (\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$$

given by $V' \mapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which has a left adjoint

$$G : (\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X') \longrightarrow (\text{Sch}/Y')$$

which sends the triple $(V,U',\varphi)$ to the pushout $V \amalg_{(V \times_Y X)} U'$. Finally, $F \circ G$ is isomorphic to the identity functor.

**Proof.** Let $(V,U',\varphi)$ be an object of the fibre product category. Set $U = U' \times_X X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_X X$, $X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies $U$ with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 11.8. Hence we can apply Lemma 14.7 to get a pushout $V' = V \amalg_{U} U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that $V'$ is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on $U$ as morphisms into $Y'$. Setting $G(V,U',\varphi) = V'$ gives the functor $G$.

Let us prove that $G$ is a left adjoint to $F$. Let $Z$ be a scheme over $Y'$. We have to show that

$$\text{Mor}(V',Z) = \text{Mor}((V,U',\varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of $g'$ with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then $(g,f')$ is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g,f') : (V,U',\varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of $g$, resp. $f$ by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from $U$ to $Z$. By the universal
property of pushout, we obtain a morphism \( g' : V' \to Z \), i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that \( F \circ G \) is isomorphic to the identity we have to show that the adjunction mapping \( (V, U', \varphi) \to F(G(V, U', \varphi)) \) is an isomorphism. To do this we may work affine locally. Say \( X = \text{Spec}(A) \), \( X' = \text{Spec}(A') \), and \( Y = \text{Spec}(B) \). Then \( A' \to A \) and \( B \to A \) are ring maps as in More on Algebra, Lemma 6.4 and \( Y' = \text{Spec}(B') \) with \( B' = B \times_A A' \).

Next, suppose that \( V = \text{Spec}(D) \), \( U' = \text{Spec}(C') \) and \( \varphi \) is given by an \( A \)-algebra isomorphism \( D \otimes_B A \to C' \otimes_{A'} A = C'/IC' \). Set \( D' = D \times_{C'/IC'} C' \).

In this case the statement we have to prove is that \( D' \otimes_{B'} B \cong D \) and \( D' \otimes_{B'} A' \cong C' \). This is a special case of More on Algebra, Lemma 6.4. 

\[ \square \]

**08KU Lemma 14.9.** Let \( X \to X' \) be a thickening of schemes and let \( X \to Y \) be an affine morphism of schemes. Let \( Y' = Y \amalg_X X' \) be the pushout (see Lemma 14.7).

Let \( V' \to V \) be a morphism of schemes. Set \( V = Y \times_Y V' \), \( U' = X' \times_Y V' \), and \( U = X \times_Y V' \). There is an equivalence of categories between

1. quasi-coherent \( \mathcal{O}_{V'} \)-modules flat over \( V' \), and
2. the category of triples \((\mathcal{G}, \mathcal{F}', \varphi)\) where
   - \( \mathcal{G} \) is a quasi-coherent \( \mathcal{O}_V \)-module flat over \( V \),
   - \( \mathcal{F}' \) is a quasi-coherent \( \mathcal{O}_{V'} \)-module flat over \( V' \), and
   - \( \varphi : (U \to V)'^* \mathcal{G} \to (U' \to V')^* \mathcal{F}' \) is an isomorphism of \( \mathcal{O}_U \)-modules.

The equivalence maps \( \mathcal{G}' \) to \( ((V \to V')^* \mathcal{G}', (U' \to V')^* \mathcal{F}', \text{can}) \). Suppose \( \mathcal{G}' \) corresponds to the triple \((\mathcal{G}, \mathcal{F}', \varphi)\). Then

1. \( \mathcal{G}' \) is a finite type \( \mathcal{O}_{V'} \)-module if and only if \( \mathcal{G} \) and \( \mathcal{F}' \) are finite type \( \mathcal{O}_V \) and \( \mathcal{O}_{V'} \)-modules.
2. if \( V' \to Y' \) is locally of finite presentation, then \( \mathcal{G}' \) is an \( \mathcal{O}_{V'} \)-module of finite presentation if and only if \( \mathcal{G} \) and \( \mathcal{F}' \) are \( \mathcal{O}_V \) and \( \mathcal{O}_{V'} \)-modules of finite presentation.

**Proof.** A quasi-inverse functor assigns to the triple \((\mathcal{G}, \mathcal{F}', \varphi)\) the fibre product

\[ (V \to V'), \mathcal{G} \times_{(U \to V')} \mathcal{F} \to (U' \to V'), \mathcal{F}' \]

where \( \mathcal{F} = (U \to U')^* \mathcal{F}' \). This works, because on affines we recover the equivalence of More on Algebra, Lemma 7.5. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 7.4 and 7.6. \[ \square \]

**07RX Lemma 14.10.** In the situation of Lemma 14.9 If \( V' = G(V, U', \varphi) \) for some triple \((V, U', \varphi)\), then

1. \( V' \to Y' \) is locally of finite type if and only if \( V \to Y \) and \( U' \to X' \) are locally of finite type,
2. \( V' \to Y' \) is flat if and only if \( V \to Y \) and \( U' \to X' \) are flat,
3. \( V' \to Y' \) is flat and locally of finite presentation if and only if \( V \to Y \) and \( U' \to X' \) are flat and locally of finite presentation,
4. \( V' \to Y' \) is smooth if and only if \( V \to Y \) and \( U' \to X' \) are smooth,
5. \( V' \to Y' \) is étale if and only if \( V \to Y \) and \( U' \to X' \) are étale, and
6. add more here as needed.

If \( W' \) is flat over \( Y' \), then the adjunction mapping \( G(F(W')) \) to \( W' \) is an isomorphism. Hence \( F \) and \( G \) define mutually quasi-inverse functors between the category of schemes flat over \( Y' \) and the category of triples \((V, U', \varphi)\) with \( V \to Y \) and \( U' \to X' \) flat.
Proof. Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 7.7.

15. Openness of the flat locus

This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

Theorem 15.1. Let $S$ be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $F$ be a quasi-coherent $O_X$-module which is locally of finite presentation. Then

$$U = \{ x \in X \mid F \text{ is flat over } S \text{ at } x \}$$

is open in $X$.

Proof. We may test for openness locally on $X$ hence we may assume that $f$ is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 128.4.

Lemma 15.2. Let $S$ be a scheme. Let $X'$\(\to\)\(\to\)\(\downarrow\)\(\downarrow\) $Y$\(\rightarrow\)\(\rightarrow\)\(\downarrow\)\(\downarrow\) $S$ be a cartesian diagram of schemes. Let $F$ be a quasi-coherent $O_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = g'(s')$.

1. If $F$ is flat over $S$ at $x$, then $(g')^*F$ is flat over $S'$ at $x'$.
2. If $g$ is flat at $s'$ and $(g')^*F$ is flat over $S'$ at $x'$, then $F$ is flat over $S$ at $x$.

In particular, if $g$ is flat, $f$ is locally of finite presentation, and $F$ is locally of finite presentation, then formation of the open subset of Theorem 15.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$O_{X',x'} \xleftarrow{g'} O_{X,s} \xrightarrow{f} O_{S,s}$$

Note that $O_{X',x'}$ is a localization of $O_{X,s} \otimes_{O_S,s} O_{S',s'}$, and that $((g')^*F)_{x'}$ is equal to $F_x \otimes_{O_{X,s}} O_{X',x'}$. Hence the lemma follows from Algebra, Lemma 99.1.

16. Critère de platitude par fibres

Consider a commutative diagram of schemes (left hand diagram)

and a quasi-coherent $O_X$-module $F$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $F$ flat over $Y$ at $x$? If $F$ is flat over $S$
at $x$, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^* \mathcal{F}$$

is flat over $Y_s$ at $x$. Below you will find a “Noetherian” version, a “finitely presented” version, and earlier we treated a “nilpotent” version, see Lemma 10.2.

**Theorem 16.1.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of $x$ in $S$. Assume $S, X, Y$ locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:

1. $\mathcal{F}$ is flat over $S$ at $x$, and $\mathcal{F}_s$ is flat over $Y_s$ at $x$, and
2. $Y$ is flat over $S$ at $y$ and $\mathcal{F}$ is flat over $Y$ at $x$.

**Proof.** Consider the ring maps

$$\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at $x$ is the module $\mathcal{F}_x/m_s\mathcal{F}_x$ and the local ring of $Y_s$ at $y$ is $\mathcal{O}_{Y,y}/m_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 98.15. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 38.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/m_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_{Y,y}$, hence flat by Algebra, Lemma 38.7. □

Here is the non-Noetherian version.

**Theorem 16.2.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

1. $X$ is locally of finite presentation over $S$,
2. $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and
3. $Y$ is locally of finite type over $S$.

Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be the image of $x$ in $S$. If $\mathcal{F}_x \neq 0$, then the following are equivalent:

1. $\mathcal{F}$ is flat over $S$ at $x$, and $\mathcal{F}_s$ is flat over $Y_s$ at $x$, and
2. $Y$ is flat over $S$ at $y$ and $\mathcal{F}$ is flat over $Y$ at $x$.

Moreover, the set of points $x$ where (1) and (2) hold is open in $\text{Supp}(\mathcal{F})$.

**Proof.** Consider the ring maps

$$\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at $x$ is the module $\mathcal{F}_x/m_s\mathcal{F}_x$ and the local ring of $Y_s$ at $y$ is $\mathcal{O}_{Y,y}/m_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 127.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 38.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/m_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/m_{Y,y}$, hence flat by Algebra, Lemma 38.7. □
This set is open in $X$ by Theorem 15.1. Note that if $x \in U$, then $\mathcal{F}_x$ is flat at $x$ over $Y$, as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 24.6. Hence at every point of $U \cap \text{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \text{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \text{Supp}(\mathcal{F})$. □

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

**Lemma 16.3.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Assume

1. $S$, $X$, $Y$ are locally Noetherian,
2. $X$ is flat over $S$,
3. for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat.

Then $f$ is flat. If $f$ is also surjective, then $Y$ is flat over $S$.

**Proof.** This is a special case of Theorem 16.1 □

**Lemma 16.4.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Assume

1. $X$ is locally of finite presentation over $S$,
2. $X$ is flat over $S$,
3. for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat, and
4. $Y$ is locally of finite type over $S$.

Then $f$ is flat. If $f$ is also surjective, then $Y$ is flat over $S$.

**Proof.** This is a special case of Theorem 16.2 □

**Lemma 16.5.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

1. $X$ is locally of finite presentation over $S$,
2. $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,
3. $\mathcal{F}$ is flat over $S$, and
4. $Y$ is locally of finite type over $S$.

Then the set

$$U = \{ x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y \}.$$

is open in $X$ and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and $U'$ is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

**Proof.** By Morphisms, Lemma 20.11 the morphism $f$ is locally of finite presentation. Hence $U$ is open by Theorem 15.1. Because we have assumed that $\mathcal{F}$ is flat over $S$ we see that Theorem 16.2 implies

$$U = \{ x \in X \mid \mathcal{F}_x \text{ flat at } x \text{ over } Y_s \}.$$

where $s$ always denotes the image of $x$ in $S$. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence $U'$ has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{ x' \in X_{s'}' \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y_{s'} \}.$$
is the inverse image of the corresponding locus in $X_s$. This is true by Lemma 15.2 because in the cartesian diagram

\[
\begin{array}{ccc}
X'_{s'} & \longrightarrow & X_s \\
\downarrow & & \downarrow \\
Y'_{s'} & \longrightarrow & Y_s
\end{array}
\]

the horizontal morphisms are flat as they are base changes by the flat morphism $\text{Spec}(\kappa(s')) \to \text{Spec}(\kappa(s))$. □

**Lemma 16.6.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Assume

1. $X$ is locally of finite presentation over $S$,
2. $X$ is flat over $S$, and
3. $Y$ is locally of finite type over $S$.

Then the set

\[
U = \{ x \in X \mid X \text{ flat at } x \text{ over } Y \}
\]

is open in $X$ and its formation commutes with arbitrary base change.

**Proof.** This is a special case of Lemma 16.5. □

The following lemma is a variant of Algebra, Lemma 98.4. Note that the hypothesis that $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module means that $(\mathcal{F}_s)_x$ is a free $\mathcal{O}_{X_s,x}$-module which is always the case if $x \in X_s$ is a generic point of an irreducible component of $X_s$ and $X_s$ is reduced (namely, in this case $\mathcal{O}_{X_s,x}$ is a field, see Algebra, Lemma 24.1).

**Lemma 16.7.** Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over $S$. Let $x \in X$ with image $s \in S$. If $\mathcal{F}$ is flat at $x$ over $S$ and $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module, then $\mathcal{F}$ is finite free in a neighbourhood of $x$.

**Proof.** If $\mathcal{F}_x \otimes \kappa(x)$ is zero, then $\mathcal{F}_x = 0$ by Nakayama’s lemma (Algebra, Lemma 19.1) and hence $\mathcal{F}$ is zero in a neighbourhood of $x$ (Modules, Lemma 9.5) and the lemma holds. Thus we may assume $\mathcal{F}_x \otimes \kappa(x)$ is not zero and we see that Theorem 16.2 applies with $f = \text{id} : X \to X$. We conclude that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{F}_x$ is free, see Algebra, Lemma 77.4 for example. Choose an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ which map to a basis in $\mathcal{F}_x$. The corresponding map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective after shrinking $U$ (Modules, Lemma 9.5). Then $\text{Ker}(\psi)$ is of finite type (see Modules, Lemma 11.3) and $\text{Ker}(\psi)_x = 0$. Whence after shrinking $U$ once more $\psi$ is an isomorphism. □

**Lemma 16.8.** Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over $S$. Then the set

\[
\{ x \in X : \mathcal{F} \text{ free in a neighbourhood of } x \}
\]

is open in $X$ and its formation commutes with arbitrary base change $S' \to S$.

**Proof.** Openness holds trivially. Let $x \in X$ mapping to $s \in S$. By Lemma 16.7 we see that $x$ is in our set if and only if $\mathcal{F}_{|_{X_s}}$ is flat at $x$ over $X_s$. Clearly this is also equivalent to $\mathcal{F}$ being flat at $x$ over $X$ (because this statement is implied by freeness of $\mathcal{F}_x$ and implies flatness of $\mathcal{F}_{|_{X_s}}$ at $x$ over $X_s$). Thus the base change statement follows from Lemma 16.5 applied to $\text{id} : X \to X$ over $S$. □
17. Normalization revisited

081J Normalization commutes with smooth base change.

081K **Lemma 17.1.** Let $f : Y \to X$ be a smooth morphism of schemes. Let $A$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*A$ is equal to $f^*A'$ where $A' \subset A$ is the integral closure of $\mathcal{O}_X$ in $A$.

**Proof.** This is a translation of Algebra, Lemma 143.4 into the language of schemes. Details omitted. □

03GV **Lemma 17.2** (Normalization commutes with smooth base change). Let

\[
\begin{array}{ccc}
Y_2 & \longrightarrow & Y_1 \\
\downarrow f_2 & & \downarrow f_1 \\
X_2 & \longrightarrow & X_1
\end{array}
\]

be a fibre square in the category of schemes. Assume $f_1$ is quasi-compact and quasi-separated, and $\varphi$ is smooth. Let $Y_i \to X'_i \to X_i$ be the normalization of $X_i$ in $Y_i$. Then $X'_2 \cong X_2 \times_{X_1} X'_1$.

**Proof.** The base change of the factorization $Y_1 \to X'_1 \to X_1$ to $X_2$ is a factorization $Y_2 \to X_2 \times_{X_1} X'_1 \to X_2$ and $X_2 \times_{X_1} X'_1 \to X_2$ is integral (Morphisms, Lemma 42.6). Hence we get a morphism $h : X'_2 \to X_2 \times_{X_1} X'_1$ by the universal property of integral closures, Lemma 50.4. Observe that $X'_2$ is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2*}\mathcal{O}_{Y_2}$. If $A' \subset f_{1*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X'_1$ is the relative spectrum of $\varphi^*A'$, see Constructions, Lemma 4.6. By Cohomology of Schemes, Lemma 5.2 we know that $f_{2*}\mathcal{O}_{Y_2} = \varphi^*f_{1*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 17.1. □

07TD **Lemma 17.3** (Normalization and smooth morphisms). Let $X \to Y$ be a smooth morphism of schemes. Assume every quasi-compact open of $Y$ has finitely many irreducible components. Then the same is true for $X$ and there is a canonical isomorphism $X^v = X \times_Y Y^v$.

**Proof.** By Descent, Lemma 13.3 every quasi-compact open of $X$ has finitely many irreducible components. Note that $X_{\text{red}} = X \times_Y Y_{\text{red}}$ as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 15.1. Hence we may assume that $X$ and $Y$ are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that $X' = X \times_Y Y^v$ is a normal scheme by Descent, Lemma 15.2. The morphism $X' \to Y^v$ is smooth (hence flat) thus the generic points of irreducible components of $X'$ lie over generic points of irreducible components of $Y^v$. Since $Y^v \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y^v$ induces an isomorphism on fibres over generic points of $Y$). We conclude that there exists a factorization $X' \to X' \to X$, see Morphisms, Lemma 51.5 which is an isomorphism as $X'$ is normal and integral over $X$. □

0CBM **Lemma 17.4** (Normalization and henselization). Let $X$ be a locally Noetherian scheme. Let $\nu : X^v \to X$ be the normalization morphism. Then for any point $x \in X$ the base change

$$X^v \times_X \text{Spec}(\mathcal{O}_{X,x}^h) \to \text{Spec}(\mathcal{O}_{X,x}^h), \quad \text{resp.} \quad X^v \times_X \text{Spec}(\mathcal{O}_{X,x}^{sh}) \to \text{Spec}(\mathcal{O}_{X,x}^{sh})$$

is an isomorphism.
is the normalization of \( \text{Spec}(O_{X,x}^h) \), resp. \( \text{Spec}(O_{X,x}^{sh}) \).

**Proof.** Let \( \eta_1, \ldots, \eta_r \) be the generic points of the irreducible components of \( X \) passing through \( x \). The base change of the normalization to \( \text{Spec}(O_{X,x}) \) is the spectrum of the integral closure of \( O_{X,x} \) in \( \prod \kappa(\eta_i) \). This follows from our construction of the normalization of \( X \) in Morphisms, Definition 51.1 and Morphisms, Lemma 50.1; you can also use the description of the normalization in Morphisms, Lemma 51.3. Thus we reduce to the following algebra problem. Let \( A \) be a Noetherian local ring; recall that this implies the henselization \( A^h \) and strict henselization \( A^{sh} \) are Noetherian too (More on Algebra, Lemma 42.3). Let \( p_1, \ldots, p_r \) be its minimal primes. Let \( A' \) be the integral closure of \( A \) in \( \prod \kappa(p_i) \). Problem: show that \( A' \otimes_A A^h \), resp. \( A' \otimes_A A^{sh} \) is constructed from the Noetherian local ring \( A^h \), resp. \( A^{sh} \) in the same manner.

Since \( A^h \), resp. \( A^{sh} \) are colimits of étale \( A \)-algebras, we see that the minimal primes of \( A \) and \( A^{sh} \) are exactly the primes of \( A^h \), resp. \( A^{sh} \) lying over the minimal primes of \( A \) (by going down, see Algebra, Lemmas 38.18 and 29.7). Thus More on Algebra, Lemma 42.13 tells us that \( A^h \otimes_A \prod \kappa(p_i) \), resp. \( A^{sh} \otimes_A \prod \kappa(p_i) \) is the product of the residue fields at the minimal primes of \( A^h \), resp. \( A^{sh} \). We know that taking the integral closure in an overring commutes with étale base change, see Algebra, Lemma 143.2. Writing \( A^h \) and \( A^{sh} \) as a limit of étale \( A \)-algebras we see that the same thing is true for the base change to \( A^h \) and \( A^{sh} \) (you can also use the more general Algebra, Lemma 143.5).

\[ \square \]

18. Normal Morphisms

In the article [DM69] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

**Definition 18.1.** Let \( f : X \to Y \) be a morphism of schemes. Assume that all the fibres \( X_y \) are locally Noetherian schemes.

(1) Let \( x \in X \), and \( y = f(x) \). We say that \( f \) is normal at \( x \) if \( f \) is flat at \( x \), and the scheme \( X_y \) is geometrically normal at \( x \) over \( \kappa(y) \) (see Varieties, Definition 10.1).

(2) We say \( f \) is a normal morphism if \( f \) is normal at every point of \( X \).

So the condition that the morphism \( X \to Y \) is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

**Lemma 18.2.** Let \( f : X \to Y \) be a morphism of schemes. Assume all fibres of \( f \) are locally Noetherian. The following are equivalent

(1) \( f \) is normal, and
(2) \( f \) is flat and its fibres are geometrically normal schemes.

**Proof.** This follows directly from the definitions. \[ \square \]

**Lemma 18.3.** A smooth morphism is normal.

---

3 The other types are coprof \( \leq k \), Cohen-Macaulay, \((S_k)\), regular, \((R_k)\), and reduced. See [DG67, IV Definition 6.8.1.]. Gorenstein morphisms will be defined in Duality for Schemes, Section 25.
Proof. Let $f : X \to Y$ be a smooth morphism. As $f$ is locally of finite presentation, see Morphisms, Lemma \[32.8\] the fibres $X_y$ are locally of finite type over a field, hence locally Noetherian. Moreover, $f$ is flat, see Morphisms, Lemma \[32.9\]. Finally, the fibres $X_y$ are smooth over a field (by Morphisms, Lemma \[32.5\]) and hence geometrically normal by Varieties, Lemma \[25.4\]. Thus $f$ is normal by Lemma \[18.2\]. \[\square\]

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

**Lemma 18.4.** The property $\mathcal{P}(f) =$ “the fibres of $f$ are locally Noetherian” is local in the fppf topology on the source and the target.

**Proof.** Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$  

Moreover, as $\varphi_i$ is of finite presentation the field extension $\kappa(y) \subset \kappa(y_i)$ is finitely generated. Hence in this situation we have that $X_y$ is locally Noetherian if and only if $X_{i,y_i}$ is locally Noetherian, see Varieties, Lemma \[11.1\]. This fact implies locality on the target.

Let $\{X_i \to X\}$ be an fppf covering of $X$. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma \[13.1\]. \[\square\]

**Lemma 18.5.** The property $\mathcal{P}(f) =$ “the fibres of $f$ are locally Noetherian and $f$ is normal” is local in the fppf topology on the target and local in the smooth topology on the source.

**Proof.** We have $\mathcal{P}(f) = \mathcal{P}_1(f) \land \mathcal{P}_2(f) \land \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$ “the fibres of $f$ are locally Noetherian”, $\mathcal{P}_2(f) =$ “$f$ is flat”, and $\mathcal{P}_3(f) =$ “the fibres of $f$ are geometrically normal”. We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma \[18.4\] and Descent, Lemmas \[20.15\] and \[24.1\]. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$  

Hence in this situation we have that $X_y$ is geometrically normal if and only if $X_{i,y_i}$ is geometrically normal, see Varieties, Lemma \[10.4\]. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of $X$. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma \[15.2\]. Combining the above the lemma follows. \[\square\]
19. Regular morphisms

**Definition 19.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres $X_y$ are locally Noetherian schemes.

1. Let $x \in X$, and $y = f(x)$. We say that $f$ is regular at $x$ if $f$ is flat at $x$, and the scheme $X_y$ is geometrically regular at $x$ over $\kappa(y)$ (see Varieties, Definition 12.1).
2. We say $f$ is a regular morphism if $f$ is regular at every point of $X$.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

**Lemma 19.2.** Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of $f$ are locally Noetherian. The following are equivalent

1. $f$ is regular,
2. $f$ is flat and its fibres are geometrically regular schemes,
3. for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is regular,
4. there exists an open covering $Y = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and
5. there exists an affine open covering $Y = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(U_i)$ are regular.

**Proof.** The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition $f$ is flat at $x$ if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and $X_y$ is geometrically regular at $x$ over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence whether or not $f$ is regular at $x$ depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 38.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(p)$ are Noetherian and geometrically regular for all primes $p \subset A$. By Varieties, Lemma 12.3, this is equivalent to $\text{Spec}(B \otimes_A \kappa(p))$ being a geometrically regular scheme over $\kappa(p)$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that $f$ is flat (see Morphisms, Lemma 24.3). Moreover, if $y \in Y$, then $y \in V_j$ for some $j$ and we see that $X_y = \bigcup_{i \in I_j} \hat{U}_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 12.3. Another application of Varieties, Lemma 12.3 shows that $X_y$ is geometrically regular. Hence (2) holds and the proof of the lemma is finished.

**Lemma 19.3.** A smooth morphism is regular.

**Proof.** Let $f : X \to Y$ be a smooth morphism. As $f$ is locally of finite presentation, see Morphisms, Lemma 32.8, the fibres $X_y$ are locally of finite type over a field, hence locally Noetherian. Moreover, $f$ is flat, see Morphisms, Lemma 32.9. Finally, the fibres $X_y$ are smooth over a field (by Morphisms, Lemma 32.5) and hence geometrically regular by Varieties, Lemma 25.4. Thus $f$ is regular by Lemma 19.2.

□
Lemma 19.4. The property $P(f) = \text{“the fibres of } f \text{ are locally Noetherian and } f \text{ is regular”}$ is local in the fppf topology on the target and local in the smooth topology on the source.

Proof. We have $P(f) = P_1(f) \land P_2(f) \land P_3(f)$ where $P_1(f) = \text{“the fibres of } f \text{ are locally Noetherian”}$, $P_2(f) = \text{“ } f \text{ is flat”, and } P_3(f) = \text{“the fibres of } f \text{ are geometrically regular”}$. We have already seen that $P_1$ and $P_2$ are local in the fppf topology on the source and the target, see Lemma 18.4 and Descent, Lemmas 20.15 and 24.1. Thus we have to deal with $P_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{ \varphi_i : Y_i \to Y \}_{i \in I}$ be an fpqc covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y.$$ 

Hence in this situation we have that $X_y$ is geometrically regular if and only if $X_{i,y_i}$ is geometrically regular, see Varieties, Lemma 12.4. This fact implies $P_3$ is fpqc local on the target.

Let $\{ X_i \to X \}$ be a smooth covering of $X$. Let $y \in Y$. In this case $\{ X_{i,y} \to Y \}$ is a smooth covering of the fibre. Hence the locality of $P_3$ for the smooth topology on the source follows from Descent, Lemma 15.4. Combining the above the lemma follows. □

20. Cohen-Macaulay morphisms

Lemma 20.2. Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of $f$ are locally Noetherian. The following are equivalent

1. $f$ is Cohen-Macaulay,
2. $f$ is flat and its fibres are Cohen-Macaulay schemes.

Proof. This follows directly from the definitions. □

Lemma 20.3. Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is locally of finite type and Cohen-Macaulay. For every point $x$ in $X$ with image $y$ in $Y$,

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where $X_y$ denotes the fiber over $y$.
Proof. After replacing $X$ by an open neighborhood of $x$, there is a natural number $d$ such that all fibers of $X \to Y$ have dimension $d$ at every point, see Morphisms, Lemma [28.4]. Then $f$ is flat, locally of finite type and of relative dimension $d$. Hence the result follows from Morphisms, Lemma [28.6]. □

0C0W Lemma 20.4. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the fibers of $f$, $g$, and $g \circ f$ are locally Noetherian. Let $x \in X$ with images $y \in Y$ and $z \in Z$.

1. If $f$ is Cohen-Macaulay at $x$ and $g$ is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at $x$.
2. If $f$ and $g$ are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.
3. If $g \circ f$ is Cohen-Macaulay at $x$ and $f$ is flat at $x$, then $f$ is Cohen-Macaulay at $x$ and $g$ is Cohen-Macaulay at $f(x)$.
4. If $f \circ g$ is Cohen-Macaulay and $f$ is flat, then $f$ is Cohen-Macaulay and $g$ is Cohen-Macaulay at every point in the image of $f$.

Proof. Consider the map of Noetherian local rings $\mathcal{O}_{Y, y} \to \mathcal{O}_{X, x}$ and observe that its fibre is $\mathcal{O}_{X, x}/\mathfrak{m}_{Y, y}\mathcal{O}_{X, x} = \mathcal{O}_{X, y}$.

Thus the lemma this follows from Algebra, Lemma [157.3]. □

0C0X Lemma 20.5. Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If $X$ is Cohen-Macaulay, then $f$ is Cohen-Macaulay and $\mathcal{O}_{Y, f(x)}$ is Cohen-Macaulay for all $x \in X$.

Proof. After translating into algebra this follows from Algebra, Lemma [157.3]. □

045T Lemma 20.6. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibers $X_y$ are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of $f$. Let $x' \in X'$ be a point with image $x \in X$.

1. If $f$ is Cohen-Macaulay at $x$, then $f' : X' \to Y'$ is Cohen-Macaulay at $x'$.
2. If $f$ is flat at $x$ and $f'$ is Cohen-Macaulay at $x'$, then $f$ is Cohen-Macaulay at $x$.
3. If $Y' \to Y$ is flat at $f'(x')$ and $f'$ is Cohen-Macaulay at $x'$, then $f$ is Cohen-Macaulay at $x$.

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $k(y) \subset k(y')$ is finitely generated. Hence also all the fibres $X'_y = (X_y)_{k(y')}$ are locally Noetherian, see Varieties, Lemma [11.1]. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$
\begin{array}{ccc}
\mathcal{O}_{X', x'} & \xrightarrow{\mathcal{O}_{X, x}} & \mathcal{O}_{X, x} \\
\mathcal{O}_{Y', y'} & \xleftarrow{\mathcal{O}_{Y, y}} & \mathcal{O}_{Y, y}
\end{array}
$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.
Assume $f$ is Cohen-Macaulay at $x$. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma [99.1]. The fact that $\mathcal{O}_{X,x}/m_x\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/m_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay, see Varieties, Lemma [13.1]. Hence we see that $f'$ is Cohen-Macaulay at $x'$.

Assume $f$ is flat at $x$ and $f'$ is Cohen-Macaulay at $x'$. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/m_x\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma [13.1]. Hence we see that $f$ is Cohen-Macaulay at $x$.

Assume $Y' \to Y$ is flat at $y'$ and $f'$ is Cohen-Macaulay at $x'$. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma [99.1]. The fact that $\mathcal{O}_{X,x}/m_x\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/m_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay, see Varieties, Lemma [13.1]. Hence we see that $f$ is Cohen-Macaulay at $x$.

0BUU Lemma 20.7. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let

$$W = \{ x \in X \mid f \text{ is Cohen-Macaulay at } x \}$$

Then

1. $W = \{ x \in X \mid \mathcal{O}_{X,f(x),x} \text{ is Cohen-Macaulay} \}$,
2. $W$ is open in $X$,
3. $W$ dense in every fibre of $X \to S$,
4. the formation of $W$ commutes with arbitrary base change of $f$: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of $f$ and the projection $g' : X' \to X$. Then the corresponding set $W'$ for the morphism $f'$ is equal to $W' = (g')^{-1}(W)$.

Proof. As $f$ is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma [129.5] Part (4) follows either from Algebra, Lemma [129.7] or Varieties, Lemma [13.1].

0BUU Lemma 20.8. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent

1. $f$ is Cohen-Macaulay at $x$,
2. there exists an open neighbourhood $U \subset X$ of $x$ and a locally quasi-finite morphism $U \to \mathbb{A}^d_S$ over $S$ which is flat at $x$,
3. there exists an open neighbourhood $U \subset X$ of $x$ and a locally quasi-finite flat morphism $U \to \mathbb{A}^d_S$ over $S$,
4. for any $S$-morphism $g : U \to \mathbb{A}^d_S$ of an open neighbourhood $U \subset X$ of $x$ we have: $g$ is quasi-finite at $x \Rightarrow g$ is flat at $x$.

Proof. Openness of flatness shows (2) and (3) are equivalent, see Theorem [15.1]. Choose affine open $U = \text{Spec}(A) \subset X$ with $x \in U$ and $V = \text{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $p \subset A$ be the prime ideal corresponding to $x$. After replacing $A$ by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma [124.2]. Thus there exists at least one pair $(U, g)$ consisting of an open neighbourhood $U \subset X$ of $x$ and a locally quasi-finite morphism $g : U \to \mathbb{A}^d_S$.

4If $S$ is quasi-separated, then $g$ will be quasi-finite.
The results in this section eventually lead to the assertion that the fppf topology is the same as the “finitely presented, flat, quasi-finite” topology. The following lemma is very closely related to Divisors, Lemma 18.9.

21. Slicing Cohen-Macaulay morphisms

Theorem [20.9]. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties:

1. $W = \bigcup_{d \geq 0} U_d$ is dense in every fibre of $f$, and
2. $U_d \to S$ is of relative dimension $d$ (see Morphisms, Definition 28.1).

Proof. This follows by combining Lemma 20.7 with Morphisms, Lemma 28.4.

Lemma 20.10. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Suppose $x' \to x$ is a specialization of points of $X$ with image $s' \to s$ in $S$. If $x$ is a generic point of an irreducible component of $X_\eta$ then $\dim_{x'}(X_{x'}) = \dim_x(X_\eta)$.

Proof. The point $x$ is contained in $U_d$ for some $d$, where $U_d$ as in Lemma 20.9.

Lemma 20.11. The property $P(f) =$ “the fibres of $f$ are locally Noetherian and $f$ is Cohen-Macaulay” is local in the fppf topology on the target and local in the syntomic topology on the source.

Proof. We have $P(f) = P_1(f) \land P_2(f)$ where $P_1(f) =$ “$f$ is flat”, and $P_2(f) =$ “the fibres of $f$ are locally Noetherian and Cohen-Macaulay”. We know that $P_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 20.15 and 24.1. Thus we have to deal with $P_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of $Y$. Denote $f_i : X_i \to Y_i$ the base change of $f$ by $\varphi_i$. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that $X_{i,y_i} = \text{Spec}(\kappa(y_i)) \times_{\text{Spec}(\kappa(y))} X_y$, and that $\kappa(y) \subset \kappa(y_i)$ is a finitely generated field extension. Hence if $X_y$ is locally Noetherian, then $X_{i,y_i}$ is locally Noetherian, see Varieties, Lemma 11.1. And if in addition $X_y$ is Cohen-Macaulay, then $X_{i,y_i}$ is Cohen-Macaulay, see Varieties, Lemma 13.1. Thus $P_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of $X$. Let $y \in Y$. In this case $X_{i,y} \to X_y$ is a syntomic covering of the fibre. Hence the locality of $P_2$ for the syntomic topology on the source follows from Descent, Lemma 14.2. Combining the above the lemma follows.

The property $P(f) =$ “the fibres of $f$ are locally Noetherian and $f$ is Cohen-Macaulay” is local in the fppf topology on the target and local in the syntomic topology on the source.
Lemma 21.1. Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Let \( h \in m_x \subseteq \mathcal{O}_{X,x} \). Assume

1. \( f \) is locally of finite presentation,
2. \( f \) is flat at \( x \), and
3. the image \( \overline{h} \) of \( h \) in \( \mathcal{O}_{X,x} = \mathcal{O}_{X,x}/m_x \mathcal{O}_{X,x} \) is a nonzerodivisor.

Then there exists an affine open neighbourhood \( U \subseteq X \) of \( x \) such that \( h \) comes from \( h \in \Gamma(U, \mathcal{O}_U) \) and such that \( D = V(h) \) is an effective Cartier divisor in \( U \) with \( x \in D \) and \( D \to S \) flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 15.1) we may assume, after replacing \( X \) by an open neighbourhood of \( x \), that \( X \to S \) is flat. We may also assume that \( X \) and \( S \) are affine. After possible shrinking \( X \) a bit we may assume that there exists an \( h \in \Gamma(X, \mathcal{O}_X) \) which maps to our given \( h \).

We may write \( S = \text{Spec}(A) \) and we may write \( A = \text{colim}_i A_i \) as a directed colimit of finite type \( \mathbb{Z} \) algebras. Then by Algebra, Lemma 162.1 or Limits, Lemmas 10.1 8.2 and 10.1 we can find a cartesian diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X_0 \\
\downarrow f & & \downarrow f_0 \\
S & \longrightarrow & S_0
\end{array}
\]

with \( f_0 \) flat and of finite presentation, \( X_0 \) affine, and \( S_0 \) affine and Noetherian. Let \( x_0 \in X_0 \), resp. \( s_0 \in S_0 \) be the image of \( x \), resp. \( s \). We may also assume there exists an element \( h_0 \in \Gamma(X_0, \mathcal{O}_{X_0}) \) which restricts to \( h \) on \( X \). (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 10.1 by thinking of \( h \) as a morphism \( X \to \mathbb{A}^1_S \).) Note that \( \mathcal{O}_{X_0,x} \) is a localization of \( \mathcal{O}_{(X_0,x_0),x_0} \otimes_{\kappa(s_0)} \kappa(s) \), so that \( \mathcal{O}_{(X_0,x_0),x_0} \to \mathcal{O}_{X_0,x} \) is a flat local ring map, in particular faithfully flat.

Hence the image \( \overline{h_0} \in \mathcal{O}_{(X_0,x_0),x_0} \) is contained in \( m_{(X_0,x_0),x_0} \) and is a nonzerodivisor. We claim that after replacing \( X_0 \) by a principal open neighbourhood of \( x_0 \) the element \( h_0 \) is a nonzerodivisor in \( B_0 = \Gamma(X_0, \mathcal{O}_{X_0}) \) such that \( B_0/h_0B_0 \) is flat over \( A_0 = \Gamma(S_0, \mathcal{O}_{S_0}) \). If so then

\[
0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0
\]

is a short exact sequence of flat \( A_0 \)-modules. Hence this remains exact on tensoring with \( A \) (by Algebra, Lemma 38.12) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript \( 0 \) here): Let \( A \to B \) be a flat, finite type ring map of Noetherian rings. Let \( \mathfrak{q} \subseteq B \) be a prime lying over \( \mathfrak{p} \subseteq A \). Assume \( h \in \mathfrak{q} \) maps to a nonzerodivisor in \( B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} \). Goal: show that after possible replacing \( B \) by \( B_g \) for some \( g \in B, g \notin \mathfrak{q} \) the element \( h \) becomes a nonzerodivisor and \( B/hB \) becomes flat over \( A \). By Algebra, Lemma 98.2 we see that \( h \) is a nonzerodivisor in \( B_\mathfrak{q} \) and that \( B_\mathfrak{q}/hB_\mathfrak{q} \) is flat over \( A \). By openness of flatness, see Algebra, Theorem 128.4 or Theorem 15.1 we see that \( B/hB \) is flat over \( A \) after replacing \( B \) by \( B_g \) for some \( g \in B, g \notin \mathfrak{q} \). Finally, let \( I = \{ b \in B \mid hb = 0 \} \) be the annihilator of \( h \). Then \( IB_\mathfrak{q} = 0 \) as \( h \) is a nonzerodivisor in \( B_\mathfrak{q} \). Also \( I \) is finitely generated as \( B \) is
Noetherian. Hence there exists a \( g \in B, g \not\in q \) such that \( IB_q = 0 \). After replacing \( B \) by \( B_g \) we see that \( h \) is a nonzerodivisor.

**Lemma 21.2.** Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Let \( h_1, \ldots, h_r \in \mathcal{O}_{X,x} \). Assume

1. \( f \) is locally of finite presentation,
2. \( f \) is flat at \( x \), and
3. the images of \( h_1, \ldots, h_r \) in \( \mathcal{O}_{X,x} = \mathcal{O}_{X,x}/\mathfrak{m}_x\mathcal{O}_{X,x} \) form a regular sequence.

Then there exists an affine open neighbourhood \( U \subset X \) of \( x \) such that \( h_1, \ldots, h_r \) come from \( h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U) \) and such that \( Z = V(h_1, \ldots, h_r) \to U \) is a regular immersion with \( x \in Z \) and \( Z \to S \) flat and locally of finite presentation. Moreover, the base change \( Z_{S'} \to U_{S'} \) is a regular immersion for any scheme \( S' \) over \( S \).

**Proof.** (Our conventions on regular sequences imply that \( h_i \in \mathfrak{m}_x \) for each \( i \).) The case \( r = 1 \) follows from Lemma [21.1](#) combined with Divisors, Lemma [18.1](#) to see that \( V(h_1) \) remains an effective Cartier divisor after base change. The case \( r > 1 \) follows from a straightforward induction on \( r \) (applying the result for \( r = 1 \) exactly \( r \) times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section [22](#). Namely, first by openness of flatness (see Theorem [15.1](#)) we may assume, after replacing \( X \) by an open neighbourhood of \( x \), that \( X \to S \) is flat. We may also assume that \( X \) and \( S \) are affine. After possible shrinking \( X \) a bit we may assume that we have \( h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X) \). Set \( Z = V(h_1, \ldots, h_r) \). Note that \( X_s \) is a Noetherian scheme (because it is an algebraic \( \kappa(s) \)-scheme, see Varieties, Section [20](#)) and that the topology on \( X_s \) is induced from the topology on \( X \) (see Schemes, Lemma [18.3](#)). Hence after shrinking \( X \) a bit more we may assume that \( Z_s \subset X_s \) is a regular immersion cut out by the \( r \) elements \( h_i|_{X_s} \), see Divisors, Lemma [20.8](#) and its proof. It is also clear that \( r = \dim_x(X_s) - \dim_x(Z_s) \) because

\[
\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)),
\]

\[
\dim_x(Z_s) = \dim(\mathcal{O}_{Z_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)),
\]

\[
\dim(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{Z,x}) + r
\]

the first two equalities by Algebra, Lemma [115.3](#) and the second by \( r \) times applying Algebra, Lemma [59.12](#) Hence Divisors, Lemma [22.6](#) part (3) applies to show that (after Zariski shrinking \( X \)) the morphism \( Z \to X \) is a regular immersion to which Divisors, Lemma [22.4](#) applies (which gives the flatness and the statement on base change).

**Lemma 21.3.** Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Assume

1. \( f \) is locally of finite presentation,
2. \( f \) is flat at \( x \), and
3. \( \mathcal{O}_{X,x} \) has depth \( \geq 1 \).

Then there exists an affine open neighbourhood \( U \subset X \) of \( x \) and an effective Cartier divisor \( D \subset U \) containing \( x \) such that \( D \to S \) is flat and of finite presentation.

**Proof.** Pick any \( h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x} \) which maps to a nonzerodivisor in \( \mathcal{O}_{X,x} \) and apply Lemma [21.1](#)
Lemma 21.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is locally of finite presentation,
2. $f$ is Cohen-Macaulay at $x$, and
3. $x$ is a closed point of $X_s$.

Then there exists a regular immersion $Z \to X$ containing $x$ such that

1. $Z \to S$ is flat and locally of finite presentation,
2. $Z \to S$ is locally quasi-finite, and
3. $Z_s = \{x\}$ set theoretically.

Proof. We may and do replace $S$ by an affine open neighbourhood of $s$. We will prove the lemma for affine $S$ by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take $Z$ to be an open neighbourhood of $x$. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at $x$, see Morphisms, Lemma 28.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 52.2. Thus after replacing $X$ by $U$ we see that the fibre $X_s$ is a finite discrete set. Hence after replacing $X$ by a further affine open neighbourhood of $X$ we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on $X_s$ is induced from the topology on $X$, see Schemes, Lemma 18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because $x$ is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 19.3). Thus we see

$$\text{depth} (\mathcal{O}_{X_s,x}) = \dim (\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 27.1. Thus we may apply Lemma 21.3 to find a diagram

$$\begin{array}{ccc}
D & \longrightarrow & U \\
\downarrow & & \downarrow \\
S & \longrightarrow & X \\
\end{array}$$

with $x \in D$. Note that $\mathcal{O}_{D_{s,x}} = \mathcal{O}_{X_{s,x}}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma 18.1. Hence $\mathcal{O}_{D_{s,x}}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X_{s,x}}$, see Algebra, Lemma 103.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at $x$ with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 21.7. This finishes the proof. \qed

Lemma 21.5. Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $s \in S$ be a point in the image of $f$. Then there exists a commutative diagram

$$\begin{array}{ccc}
S' & \longrightarrow & X \\
\downarrow & & \downarrow \\
S & \longrightarrow & f \\
\end{array}$$
where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

**Proof.** The fibre $X_s$ is not empty by assumption. Hence there exists a closed point $x \in X_s$ where $f$ is Cohen-Macaulay, see Lemma 20.7. Apply Lemma 21.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the “quasi-finite, flat, finite presentation” topology.

**Lemma 21.6.** Let $S$ be a scheme. Let $U = \{S_i \to S\}_{i \in I}$ be an fppf covering of $S$, see Topologies, Definition 7.4. Then there exists an fppf covering $V = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 8.2) $U$ such that each $T_j \to S$ is locally quasi-finite.

**Proof.** For every $s \in S$ there exists an $i \in I$ such that $s$ is in the image of $S_i \to S$. By Lemma 21.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that $g_s$ factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of $S$ that refines $U$. □

## 22. Generic fibres

Some results on the relationship between generic fibres and nearby fibres.

**Lemma 22.1.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.

**Proof.** Follows immediately from the more general Morphisms, Lemma 8.4 □

**Lemma 22.2.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.

**Proof.** This follows, upon taking affine opens, from Algebra, Lemma 29.2 (Of course it also follows from generic flatness.) □

**Lemma 22.3.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. If $Z \subset X$ is a closed subset with $Z_\eta$ nowhere dense in $X_\eta$, then there exists a nonempty open $V \subset Y$ such that $Z_V$ is nowhere dense in $X_y$ for all $y \in V$.

**Proof.** Let $Y' \subset Y$ be the reduction of $Y$. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 43.6 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that $Y$ is integral, see Properties, Lemma 3.4. By Morphisms, Proposition 26.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to Z$ are flat and of finite presentation. We claim that $V$ works. Pick $y \in V$. If $Z_y$ has a nonempty interior, then $Z_y$ contains a generic point $\xi$ of an irreducible component of $X_y$. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 24.8. By Lemma 20.10 we see that

\[ \dim_R(Z_\eta) = \dim_R(Z_y) = \dim_R(X_y) = \dim_R(X_\eta). \]

Hence some irreducible component of $Z_\eta$ passing through $\xi'$ has dimension $\dim_R(X_\eta)$ which contradicts the assumption that $Z_\eta$ is nowhere dense in $X_\eta$ and we win. □
Lemma 22.4. Let \( f : X \to Y \) be a finite type morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \). Let \( U \subset X \) be an open subscheme such that \( U_\eta \) is scheme theoretically dense in \( X_\eta \). Then there exists a nonempty open \( V \subset Y \) such that \( U_y \) is scheme theoretically dense in \( X_y \) for all \( y \in V \).

Proof. Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' = Y' \times_Y X \) and \( U' = Y' \times_Y U \). As \( Y' \to Y \) induces a bijection on points, and as \( U' \to U \) and \( X' \to X \) induce isomorphisms of scheme theoretic fibres, we may replace \( Y \) by \( Y' \) and \( X \) by \( X' \). Thus we may assume that \( Y \) is integral, see Properties, Lemma 3.4. We may also replace \( Y \) by a nonempty affine open. In other words we may assume that \( Y = \text{Spec}(A) \) where \( A \) is a domain with fraction field \( K \).

As \( f \) is of finite type we see that \( X \) is quasi-compact. Write \( X = X_1 \cup \ldots \cup X_n \) for some affine opens \( X_i \). By Morphisms, Definition 7.1 we see that \( U_i = X_i \cap U \) is an open subscheme of \( X_i \) such that \( U_i,\eta \) is scheme theoretically dense in \( X_i,\eta \). Thus it suffices to prove the result for the pairs \((X_i,U_i)\), in other words we may assume that \( X \) is affine.

Write \( X = \text{Spec}(B) \). Note that \( B_K \) is Noetherian as it is a finite type \( K \)-algebra. Hence \( U_\eta \) is quasi-compact. Thus we can find finitely many \( g_1,\ldots,g_m \in B \) such that \( D(g_j) \subset U \) and such that \( U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta \). The fact that \( U_\eta \) is scheme theoretically dense in \( X_\eta \) means that \( B_K \to \bigoplus_j (B_K)_{g_j} \) is injective, see Morphisms, Example 7.4. By Algebra, Lemma 22.4 this is equivalent to the injectivity of \( B_K \to \bigoplus_{j=1,\ldots,m} B_K, b \mapsto (g_1 b,\ldots,g_m b) \). Let \( M \) be the cokernel of this map over \( A \), i.e., such that we have an exact sequence

\[
0 \to I \to B \to \bigoplus_{j=1,\ldots,m} B \to M \to 0
\]

After replacing \( A \) by \( A_h \) for some nonzero \( h \) we may assume that \( B \) is a flat, finitely presented \( A \)-algebra, and that \( M \) is flat over \( A \), see Algebra, Lemma 17.3. The flatness of \( B \) over \( A \) implies that \( B \) is torsion free as an \( A \)-module, see More on Algebra, Lemma 20.9. Hence \( B \subset B_K \). By assumption \( I_K = 0 \) which implies that \( I = 0 \) (as \( I \subset B \subset B_K \) is a subset of \( I_K \)). Hence now we have a short exact sequence

\[
0 \to B \to \bigoplus_{j=1,\ldots,m} B \to M \to 0
\]

with \( M \) flat over \( A \). Hence for every homomorphism \( A \to \kappa \) where \( \kappa \) is a field, we obtain a short exact sequence

\[
0 \to B \otimes_A \kappa \to \bigoplus_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0
\]

see Algebra, Lemma 38.12. Reversing the arguments above this means that \( \bigcup D(g_j) \otimes 1 \) is scheme theoretically dense in \( \text{Spec}(B \otimes_A \kappa) \). As \( \bigcup D(g_j) \otimes 1 = \bigcup D(g_j)_\kappa \subset U_\kappa \) we obtain that \( U_\kappa \) is scheme theoretically dense in \( X_\kappa \) which is what we wanted to prove. \( \square \)

Suppose given a morphism of schemes \( f : X \to Y \) and a point \( y \in Y \). Recall that the fibre \( X_y \) is homeomorphic to the subset \( f^{-1}({\{ y \}}) \) of \( X \) with induced topology, see Schemes, Lemma 38.5. Suppose given a closed subset \( T(y) \subset X_y \). Let \( T \) be the closure of \( T(y) \) in \( X \). Endow \( T \) with the induced reduced scheme structure. Then \( T \) is a closed subscheme of \( X \) with the property that \( T_y = T(y) \) set-theoretically. In fact \( T \) is the smallest closed subscheme of \( X \) with this property. Thus it is
“harmless” to denote a closed subset of $X_y$ by $T_y$ if we so desire. In the following lemma we apply this to the generic fibre of $f$.

**Lemma 22.5.** Let $f : X \to Y$ be a finite type morphism of schemes. Assume $Y$ irreducible with generic point $\eta$. Let $X_\eta = Z_1,\eta \cup \ldots \cup Z_n,\eta$ be a covering of the generic fibre by closed subsets of $X_\eta$. Let $Z_i$ be the closure of $Z_i,\eta$ in $X$ (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.

**Proof.** If $Y$ is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over $Y$ and we can directly apply Lemma 22.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace $Y$ by its reduction. Thus $Y$ is integral, see Properties, Lemma 3.4. After shrinking $Y$ we may assume that $X \to Y$ is flat, see Morphisms, Proposition 26.1.

In this case every point $x$ in $X_y$ is a specialization of a point $x' \in X_y$ by Morphisms, Lemma 22.7. As the $Z_i$ are closed in $X$ and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired.

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

**Lemma 22.6.** Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of $Y$. Then $(X_\eta)_{\text{red}} = (X_{\text{red}})_{\eta}$.

**Proof.** Choose an affine neighbourhood $\text{Spec}(A) \subset Y$ of $\eta$. Choose an affine open $\text{Spec}(B) \subset X$ mapping into $\text{Spec}(A)$ via the morphism $f$. Let $p \subset A$ be the minimal prime corresponding to $\eta$. Let $B_{\text{red}}$ be the quotient of $B$ by $\sqrt{(0)}$. The algebraic content of the lemma is that $B_{\text{red}} \otimes_A \kappa(p)$ is reduced. To prove this, suppose that $x \in B_{\text{red}} \otimes_A \kappa(p)$ is nilpotent. Say $x^n = 0$ for some $n > 0$. Pick an $f \in A$, $f \not\in p$ such that $fx$ is the image of $y \in B_{\text{red}}$. Then $gy^n \in pB_{\text{red}}$ for some $g \in A$, $g \not\in p$. By Algebra, Lemma 24.1 we see that $pA_\eta$ is locally nilpotent. By Algebra, Lemma 31.2 we see that $p(B_{\text{red}})_p$ is locally nilpotent. Hence we conclude that $gy^n$ is nilpotent in $(B_{\text{red}})_p$. Thus there exists a $h \in A$, $h \not\in p$ and an $m > 0$ such that $h(gy^n)^m = 0$ in $B_{\text{red}}$. This implies that $hgy$ is nilpotent in $B_{\text{red}}$, i.e., that $hgy = 0$. Of course this means that $x = 0$ as desired.

**Lemma 22.7.** Let $f : X \to Y$ be a morphism of schemes. Assume that $Y$ is irreducible and $f$ is of finite type. There exists a diagram

\[
\begin{array}{cccc}
X' & \longrightarrow & X_V & \longrightarrow & X \\
n' & & f & & \\
Y' & \longrightarrow & V & \longrightarrow & Y
\end{array}
\]

where

1. $V$ is a nonempty open of $Y$,
2. $X_V = V \times_Y X$,
3. $g : Y' \to V$ is a finite universal homeomorphism,
4. $X' = (Y' \times_Y X)_{\text{red}} = (Y' \times_Y X_V)_{\text{red}}$,
5. $g'$ is a finite universal homeomorphism,
6. $Y'$ is an integral affine scheme,
7. $f'$ is flat and of finite presentation, and
(8) the generic fibre of \( f' \) is geometrically reduced.

**Proof.** Let \( V = \text{Spec}(A) \) be a nonempty affine open of \( Y \). By assumption the radical of \( A \) is a prime ideal \( p \). Let \( K = \kappa(p) \). Let \( p \) be the characteristic of \( K \) if positive and \( 1 \) if the characteristic is zero. By Varieties, Lemma 8.11 there exists a finite purely inseparable field extension \( K \subset K' \) such that \( X_{K'} \) is geometrically reduced over \( K' \). Choose elements \( x_1, \ldots, x_n \in K' \) which generate \( K' \) over \( K \) and such that some \( p \)-power of \( x_i \) is in \( A/p \). Let \( A' \subset K' \) be the finite \( A \)-subalgebra of \( K' \) generated by \( x_1, \ldots, x_n \). Note that \( A' \) is a domain with fraction field \( K' \).

By Algebra, Lemma 45.6 we see that \( A \rightarrow A' \) is a universal homeomorphism. Set \( Y' = \text{Spec}(A') \). Set \( X' = (Y' \times_Y X)_{\text{red}} \). The generic fibre of \( X' \rightarrow Y' \) is \((X_K)_{\text{red}} \) by Lemma 22.7 which is geometrically reduced by construction. Note that \( X' \rightarrow X_V \) is a finite universal homeomorphism as the composition of the reduction morphism \( X' \rightarrow Y' \times_Y X \) (see Morphisms, Lemma 43.6) and the base change of \( g \). At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking \( Y' \) and hence \( V \), see Morphisms, Proposition 26.1 \( \square \)

**Lemma 22.8.** Let \( f : X \rightarrow Y \) be a morphism of schemes. Assume that \( Y \) is irreducible and \( f \) is of finite type. There exists a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X_V \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

where

1. \( V \) is a nonempty open of \( Y \),
2. \( X_V = V \times_Y X \),
3. \( g : Y' \rightarrow V \) is surjective finite étale,
4. \( X' = Y' \times_Y X = Y' \times_X X_V \),
5. \( g' \) is surjective finite étale,
6. \( Y' \) is an irreducible affine scheme, and
7. all irreducible components of the generic fibre of \( f' \) are geometrically irreducible.

**Proof.** Let \( V = \text{Spec}(A) \) be a nonempty affine open of \( Y \). By assumption the radical of \( A \) is a prime ideal \( p \). Let \( K = \kappa(p) \). By Varieties, Lemma 8.11 there exists a finite separable field extension \( K \subset K' \) such that all irreducible components of \( X_{K'} \) are geometrically irreducible over \( K' \). Choose an element \( \alpha \in K' \) which generates \( K' \) over \( K \), see Fields, Lemma 19.1. Let \( P(T) \in K[T] \) be the minimal polynomial for \( \alpha \) over \( K \). After replacing \( \alpha \) by \( f\alpha \) for some \( f \in A \), \( f \not\in p \) we may assume that there exists a monic polynomial \( T^d + a_1 T^{d-1} + \ldots + a_d \in A[T] \) which maps to \( P(T) \in K' \) under the map \( A[T] \rightarrow K[T] \). Set \( A' = A[T]/(P) \). Then \( A \rightarrow A' \) is a finite free ring map such that there exists a unique prime \( q \) lying over \( p \), such that \( K = \kappa(p) \subset \kappa(q) = K' \) is finite separable, and such that \( p A'_q \) is the maximal ideal of \( A'_q \). Hence \( g : Y' = \text{Spec}(A') \rightarrow V = \text{Spec}(A) \) is étale at \( q \), see Algebra, Lemma 141.7. This means that there exists an open \( W \subset \text{Spec}(A') \) such that \( g|_W : W \rightarrow \text{Spec}(A) \) is étale. Since \( g \) is finite and since \( q \) is the only point lying over \( p \) we see that \( Z = g(Y' \setminus W) \) is a closed subset of \( V \) not containing \( p \). Hence after replacing \( V \) by a principal affine open of \( V \) which does not meet \( Z \) we obtain that \( g \) is finite étale. \( \square \)
23. Relative assassins

**Lemma 23.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in \text{Ass}_{X/S}(\mathcal{F})$ and set $Z = \{\xi\} \subset X$. If $f$ is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of $V_s$ are elements of $\text{Ass}_{X/S}(\mathcal{F})$.

**Proof.** We may replace $S$ by an affine open neighbourhood of $f(\xi)$ and $X$ by an affine open neighbourhood of $\xi$. Hence we may assume $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ and that $f$ is given by the finite type ring map $A \to B$, see Morphisms, Lemma [14.2.1]. Moreover, we may write $\mathcal{F} = M$ for some finite $B$-module $M$, see Properties, Lemma [16.1]. Let $q \subset B$ be the prime corresponding to $\xi$ and let $p \subset A$ be the corresponding prime of $A$. By assumption $q \in \text{Ass}_B(M \otimes_A \kappa(p))$, see Algebra, Remark [64.6.1] and Divisors, Lemma [2.2]. With this notation $Z = V(q) \subset \text{Spec}(B)$.

In particular $f(Z) \subset V(p)$. Hence clearly it suffices to prove the lemma after replacing $A$, $B$, and $M$ by $A/pA$, $B/pB$, and $M/pM$. In other words we may assume that $A$ is a domain with fraction field $K$ and $q \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma [117.3.1] there exists a nonzero $g \in A$ such that $M_g$ is flat as an $A_g$-module. After replacing $A$ by $A_g$ we may assume that $M$ is flat as an $A$-module.

In this case, by Algebra, Lemma [64.4.1] we see that $q$ is also an associated prime of $M$. Hence we obtain an injective $B$-module map $B/q \to M$. Let $Q$ be the cokernel so that we obtain a short exact sequence

$$0 \to B/q \to M \to Q \to 0$$

of finite $B$-modules. After applying generic flatness Algebra, Lemma [117.3.1] once more, this time to the $B$-module $Q$, we may assume that $Q$ is a flat $A$-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma [38.12]. In this situation $(B/q) \otimes_A \kappa(p') \subset M \otimes_A \kappa(p')$ for any prime $p'$ of $A$. The lemma follows as a minimal prime $q'$ of the support of $(B/q) \otimes_A \kappa(p')$ is an associated prime of $(B/q) \otimes_A \kappa(p')$ by Divisors, Lemma [2.9].

**Lemma 23.2.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume

(1) $f$ is of finite type,
(2) $\mathcal{F}$ is of finite type,
(3) $Y$ is irreducible with generic point $\eta$, and
(4) $\text{Ass}_{X/\eta}(\mathcal{F}_\eta)$ is not contained in $U_\eta$.

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $\text{Ass}_{X/\eta}(\mathcal{F}_y)$ is not contained in $U_y$.

**Proof.** Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition [5.5.1]. Then $Z_\eta$ is the scheme theoretic support of $\mathcal{F}_\eta$ (Morphisms, Lemma [24.13.3]). Hence the generic points of irreducible components of $Z_\eta$ are contained in $\text{Ass}_{X/\eta}(\mathcal{F}_\eta)$ by Divisors, Lemma [2.9.7]. Hence we see that $Z_\eta \cap U_\eta = \emptyset$. Thus $T = Z \setminus U$ is a closed subset of $Z$ with $T_\eta = \emptyset$. If we endow $T$ with the induced
Let $T \to Y$ be a morphism of finite type. By Lemma 22.1 there is a nonempty open $V \subset Y$ with $T_V = \emptyset$. Then $V$ works. □

Lemma 23.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume

1. $f$ is of finite type,
2. $\mathcal{F}$ is of finite type,
3. $Y$ is irreducible with generic point $\eta$, and
4. $\text{Ass}_{X,\eta}(\mathcal{F}_\eta) \subset U_\eta$.

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $\text{Ass}_{X,y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 22.4.) We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace $Y$ by its reduction. Hence we may assume that $Y$ is integral, see Properties, Lemma 3.4. We may also assume that $Y = \text{Spec}(A)$ is affine. Then $A$ is a domain with fraction field $K$.

As $f$ is of finite type we see that $X$ is quasi-compact. Write $X = \bigcup X_i$ for some affine opens $X_i$ and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of $U_i = X_i \cap U$ contains $\text{Ass}_{X_i,\eta}(\mathcal{F}_i,\eta)$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that $X$ is affine.

Write $X = \text{Spec}(B)$. Let $N$ be a finite $B$-module such that $\mathcal{F} = \mathcal{F}\mathbb{N}$. Note that $B_K$ is Noetherian as it is a finite type $K$-algebra. Hence $U_\eta$ is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1) \cup \ldots \cup D(g_m)$. Since $\text{Ass}_{X,\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K = \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 22.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_{K,\eta}$, $n \mapsto (g_1n, \ldots, g_mn)$. Let $I$ and $M$ be the kernel and cokernel of this map over $A$, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1, \ldots, g_m)} \bigoplus_{j=1,\ldots,m} N \to M \to 0$$

After replacing $A$ by $A_h$ for some nonzero $h$ we may assume that $B$ is a flat, finitely presented $A$-algebra and that both $M$ and $N$ are flat over $A$, see Algebra, Lemma 117.3. The flatness of $N$ over $A$ implies that $N$ is torsion free as an $A$-module, see More on Algebra, Lemma 20.9. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of $I_K$). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1, \ldots, g_m)} \bigoplus_{j=1,\ldots,m} N \to M \to 0$$

with $M$ flat over $A$. Hence for every homomorphism $A \to \kappa$ where $\kappa$ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1 \otimes 1, \ldots, g_m \otimes 1)} \bigoplus_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 38.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\text{Ass}_{B \otimes_A \kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j) \subset U_\kappa$ we obtain that $U_\kappa$ contains $\text{Ass}_{X \otimes \kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. □

Lemma 23.4. Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme.
Let \( g : S' \to S \) be a morphism of schemes, let \( f' : X' = X_{S'} \to S' \) be the base change of \( f \), let \( g' : X' \to X \) be the projection, set \( F' = (g')^* F \), and set \( U' = (g')^{-1}(U) \). Finally, let \( s' \in S' \) with image \( s = g(s') \). In this case

\[
\text{Ass}_{X_s}(F_s) \subset U_s \iff \text{Ass}_{X'_{s'}}(F'_{s'}) \subset U'_{s'}.
\]

**Proof.** This follows immediately from Divisors, Lemma 7.3. See also Divisors, Remark 7.4. \( \square \)

**Lemma 23.5.** Let \( f : X \to Y \) be a morphism of finite presentation. Let \( F \) be a quasi-coherent \( \mathcal{O}_X \)-module of finite presentation. Let \( U \subset X \) be an open subscheme such that \( U \to Y \) is quasi-compact. Then the set

\[
E = \{ y \in Y \mid \text{Ass}_{X_y}(F_y) \subset U_y \}
\]

is locally constructible in \( Y \).

**Proof.** Let \( y \in Y \). We have to show that there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( E \cap V \) is constructible in \( V \). Thus we may assume that \( Y \) is affine. Write \( Y = \text{Spec}(A) \) and \( A = \text{colim} A_i \) as a directed limit of finite type \( \mathbf{Z} \)-algebras. By Limits, Lemma 10.1 we can find an \( i \) and a morphism \( f_i : X_i \to \text{Spec}(A_i) \) of finite presentation whose base change to \( Y \) recovers \( f \). After possibly increasing \( i \) we may assume there exists a quasi-coherent \( \mathcal{O}_{X_i} \)-module \( F_i \) of finite presentation whose pullback to \( X \) is isomorphic to \( F \), see Limits, Lemma 10.2. After possibly increasing \( i \) one more time we may assume there exists an open subscheme \( U_i \subset X_i \) whose inverse image in \( X \) is \( U \), see Limits, Lemma 4.11. By Lemma 23.4 it suffices to prove the lemma for \( f_i \). Thus we reduce to the case where \( Y \) is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that \( E \) is constructible in case \( Y \) is a Noetherian scheme. To see this let \( Z \subset Y \) be an irreducible closed subscheme. We have to show that \( E \cap Z \) either contains a nonempty open subset or is not dense in \( Z \). This follows from Lemmas 23.2 and 23.3 applied to the base change \( (X,F_i,U) \times_Y Z \) over \( Z \). \( \square \)

### 24. Reduced fibres

**Lemma 24.1.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) is nonreduced, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) is nonreduced.

**Proof.** Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' \to Y' \) be the base change of \( f \). Note that \( Y' \to Y \) induces a bijection on points and that \( X' \to Y' \) identifies fibres. Hence we may assume that \( Y' \) is reduced, i.e., integral, see Properties, Lemma 3.4. We may also replace \( Y \) by an affine open. Hence we may assume that \( Y = \text{Spec}(A) \) with \( A \) a domain. Denote \( K \) the fraction field of \( A \). Pick an affine open \( \text{Spec}(B) = U \subset X \) and a section \( h_\eta \in \Gamma(U_\eta, \mathcal{O}_U_\eta) = B_K \) which is nonzero and nilpotent. After shrinking \( Y \) we may assume that \( h \) comes from \( h \in \Gamma(U, \mathcal{O}_U) = B \). After shrinking \( Y \) a bit more we may assume that \( h \) is nilpotent. Let \( I = \{ b \in B \mid bh = 0 \} \) be the annihilator of \( h \). Then \( C = B/I \) is a finite type \( A \)-algebra whose generic fiber \( (B/I)_K \) is nonzero (as \( h_\eta \neq 0 \)). We apply generic flatness to \( A \to C \) and \( A \to B/hB \), see Algebra, Lemma 117.3 and we obtain
a \, g \in A, \, g \neq 0 \text{ such that } C_g \text{ is free as an } A_g\text{-module and } (B/hB)_g \text{ is flat as an } A_g\text{-module. Replace } Y \text{ by } D(g) \subseteq Y. \text{ Now we have the short exact sequence}

\[ 0 \to C \to B \to B/hB \to 0. \]

with \( B/hB \) flat over \( A \) and with \( C \) nonzero free as an \( A \)-module. It follows that for any homomorphism \( A \to \kappa \) to a field the ring \( C \otimes_A \kappa \) is nonzero and the sequence

\[ 0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0 \]

is exact, see Algebra, Lemma 38.12. Note that \( B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa) \) by right exactness of tensor product. Thus we conclude that multiplication by \( h \) is not zero on \( B \otimes_A \kappa \). This clearly means that for any point \( y \in Y \) the element \( h \) restricts to a nonzero element of \( U_y \), whence \( X_y \) is nonreduced. □

\textbf{Lemma 24.2.} Let \( f : X \to Y \) be a morphism of schemes. Let \( g : Y' \to Y \) be any morphism, and denote \( f' : X' \to Y' \) the base change of \( f \). Then

\[
\{ y' \in Y' \mid X'_{y'} \text{ is geometrically reduced} \} = g^{-1}(\{ y \in Y \mid X_y \text{ is geometrically reduced} \}).
\]

\textbf{Proof.} This comes down to the statement that for \( y' \in Y' \) with image \( y \in Y \) the fibre \( X'_{y'} = X_y \times_Y y' \) is geometrically reduced over \( \kappa(y') \) if and only if \( X_y \) is geometrically reduced over \( \kappa(y) \). This follows from Varieties, Lemma 6.6. □

\textbf{Lemma 24.3.} Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) is not geometrically reduced, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) is not geometrically reduced.

\textbf{Proof.} Apply Lemma 22.7 to get

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X_V \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & V \\
\end{array}
\]

with all the properties mentioned in that lemma. Let \( \eta' \) be the generic point of \( Y' \). Consider the morphism \( X' \to X_{Y'} \) (which is the reduction morphism) and the resulting morphism of generic fibres \( X'_{\eta'} \to X_{\eta'} \). Since \( X'_{\eta'} \) is geometrically reduced, and \( X_{\eta} \) is not this cannot be an isomorphism, see Varieties, Lemma 6.6. Hence \( X_{Y'} \) is nonreduced. Hence by Lemma 24.1 the fibres of \( X_{Y'} \to Y' \) are nonreduced at all points \( y' \in V' \) of a nonempty open \( V' \subset Y' \). Since \( g : Y' \to V \) is a homeomorphism Lemma 24.2 proves that \( g(V') \) is the open we are looking for. □

\textbf{Lemma 24.4.} Let \( f : X \to Y \) be a morphism of schemes. Assume

(1) \( Y \) is irreducible with generic point \( \eta \),
(2) \( X_\eta \) is geometrically reduced, and
(3) \( f \) is of finite type.

Then there exists a nonempty open subscheme \( V \subset Y \) such that \( X_V \to V \) has geometrically reduced fibres.
Proof. Let $Y' \subset Y$ be the reduction of $Y$. Let $X' \to Y'$ be the base change of $f$. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that $Y'$ is reduced, i.e., integral, see Properties, Lemma 3.4. We may also replace $Y$ by an affine open. Hence we may assume that $Y = \text{Spec}(A)$ with $A$ a domain. Denote $K$ the fraction field of $A$. After shrinking $Y$ a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 26.1.

As $X_n$ is geometrically reduced there exists an open dense subset $V \subset X_n$ such that $V \to \text{Spec}(K)$ is smooth, see Varieties, Lemma 32.15. Let $U \subset X$ be the set of points where $f$ is smooth. By Morphisms, Lemma 32.15 we see that $V \subset U_n$. Thus the generic fibre of $U$ is dense in the generic fibre of $X$. Since $X_n$ is reduced, it follows that $U_n$ is scheme theoretically dense in $X_n$, see Morphisms, Lemma 7.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically reduced. Thus it suffices to show that, after shrinking $Y$, for all $y \in Y$ the scheme $U_y$ is scheme theoretically dense in $X_y$, see Morphisms, Lemma 7.9. This follows from Lemma 22.3. \qed

Lemma 24.5. Let $f : X \to Y$ be a morphism which is quasi-compact and locally of finite presentation. Then the set

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in $Y$.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Then $X$ is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. Then the fibres of $U_i \to Y$ at $y$ form an affine open covering of the fibre of $X \to Y$ at $y$. Hence we may assume $X$ is affine as well. Write $Y = \text{Spec}(A)$. Write $A = \text{colim} A_i$ as a directed limit of finite type $\mathbb{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 24.2 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in $Z$. If $X_\xi$ is geometrically reduced, then Lemma 24.4 (applied to the morphism $X_\xi \to Z$) implies that all fibres $X_y$ are geometrically reduced for a nonempty open $V \subset Z$. If $X_\xi$ is not geometrically reduced, then Lemma 24.3 (applied to the morphism $X_\xi \to Z$) implies that all fibres $X_y$ are geometrically reduced for a nonempty open $V \subset Z$. Thus we win. \qed

Lemma 24.6. Let $X \to \text{Spec}(R)$ be a proper flat morphism where $R$ is a discrete valuation ring. If the special fibre is reduced, then both $X$ and the generic fibre $X_\eta$ are reduced.

Proof. Let $x \in X$ be a point in the generic fibre $X_\eta$ such that $\mathcal{O}_{X_x}$ is nonreduced. Then $\mathcal{O}_{X,x}$ is nonreduced. Let $x \leadsto x'$ be a specialization with $x'$ in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Let $\pi \in R$ be a uniformizer. If $a \in A$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \not\in \pi A$. This follows from Krull
intersection theorem (Algebra, Lemma 50.4). If \( a \) is nilpotent, so is \( a' \), because \( \pi \) is a nonzerodivisor by flatness of \( A \) over \( \mathbf{R} \). But \( a' \) maps to a nonzero element of the reduced ring \( A/\pi A = \mathcal{O}_{X, x'} \). This is a contradiction unless \( A \) is reduced, which is what we wanted to show. \( \square \)

**Lemma 24.7.** Let \( f : X \to Y \) be a flat proper morphism of finite presentation. Then the set \( \{ y \in Y \mid X_y \text{ is geometrically reduced} \} \) is open in \( Y \).

**Proof.** We may assume \( Y \) is affine. Then \( Y \) is a cofiltered limit of affine schemes of finite type over \( \mathbb{Z} \). Hence we can assume \( X \to Y \) is the base change of \( X_0 \to Y_0 \) where \( Y_0 \) is the spectrum of a finite type \( \mathbb{Z} \)-algebra and \( X_0 \to Y_0 \) is flat and proper. See Limits, Lemma 10.1, 8.7, and 13.1. Since the formation of the set of points where the fibres are geometrically reduced commutes with base change (Lemma 24.2), we may assume the base is Noetherian.

Assume \( Y \) is Noetherian. The set is constructible by Lemma 24.5. Hence it suffices to show the set is stable under generalization (Topology, Lemma 19.9). By Properties, Lemma 7.10 we reduce to the case where \( Y = \text{Spec}(R) \), \( R \) is a discrete valuation ring, and the closed fibre \( X_y \) is geometrically reduced. To show: the generic fibre \( X_\eta \) is geometrically reduced.

If not then there exists a finite extension \( L \) of the fraction field of \( R \) such that \( X_L \) is not reduced, see Varieties, Lemma 6.4. There exists a discrete valuation ring \( R' \subset L \) with fraction field \( L \) dominating \( R \), see Algebra, Lemma 119.18. After replacing \( R \) by \( R' \) we reduce to Lemma 24.6. \( \square \)

## 25. Irreducible components of fibres

**Lemma 25.1.** Let \( f : X \to Y \) be a morphism of schemes. Assume \( Y \) irreducible with generic point \( \eta \) and \( f \) of finite type. If \( X_\eta \) has \( n \) irreducible components, then there exists a nonempty open \( V \subset Y \) such that for all \( y \in V \) the fibre \( X_y \) has at least \( n \) irreducible components.

**Proof.** As the question is purely topological we may replace \( X \) and \( Y \) by their reductions. In particular this implies that \( Y \) is integral, see Properties, Lemma 3.4. Let \( X_\eta = X_{1, \eta} \cup \ldots \cup X_{n, \eta} \) be the decomposition of \( X_\eta \) into irreducible components. Let \( X_i \subset X \) be the reduced closed subscheme whose generic fibre is \( X_{i, \eta} \). Note that \( Z_{i, j} = X_i \cap X_j \) is a closed subset of \( X_i \) whose generic fibre \( Z_{i, j, \eta} \) is nowhere dense in \( X_{i, \eta} \). Hence after shrinking \( Y \) we may assume that \( Z_{i, j, y} \) is nowhere dense in \( X_{i, y} \) for every \( y \in Y \), see Lemma 22.3. After shrinking \( Y \) some more we may assume that \( X_y = \bigcup X_{i, y} \) for \( y \in Y \), see Lemma 22.5. Moreover, after shrinking \( Y \) we may assume that each \( X_i \to Y \) is flat and of finite presentation, see Morphisms, Proposition 26.1. The morphisms \( X_i \to Y \) are open, see Morphisms, Lemma 24.9. Thus there exists an open neighbourhood \( V \) of \( \eta \) which is contained in \( f(X_i) \) for each \( i \). For each \( y \in V \) the schemes \( X_{i, y} \) are nonempty closed subsets of \( X_y \), we have \( X_y = \bigcup X_{i, y} \) and the intersections \( Z_{i, j, y} = X_{i, y} \cap X_{j, y} \) are not dense in \( X_{i, y} \). Clearly this implies that \( X_y \) has at least \( n \) irreducible components. \( \square \)
Lemma 25.2. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of $f$. Then
\[
\{ y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible} \} = g^{-1}( \{ y \in Y \mid X_y \text{ is geometrically irreducible} \} ).
\]

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if $X_y$ is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 8.2.

Lemma 25.3. Let $f : X \to Y$ be a morphism of schemes. Let
\[ n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty \} \]
be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where $K$ is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then
\[ n_{X'/Y'} = n_{X/Y} \circ g \]
where $X' \to Y'$ is the base change of $f$.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram
\[
\begin{array}{ccc}
K & \longrightarrow & K'' \\
\downarrow & & \downarrow \\
\kappa(y) & \longrightarrow & \kappa(y')
\end{array}
\]
of fields. The result follows as the morphisms of schemes
\[
(X_{y'})_{K'} \leftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K
\]
induce bijections between irreducible components, see Varieties, Lemma 8.7.

Lemma 25.4. Let $A$ be a domain with fraction field $K$. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of $K$. Assume $P$ is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.

Proof. There exists an automorphism $\Psi$ of $A[x_1, \ldots, x_n]$ over $A$ such that $\Psi(P) = ax^d_1 + \text{ lower order terms in } x_n$ with $a \neq 0$, see Algebra, Lemma 14.2. We may replace $P$ by $\Psi(P)$ and we may replace $A$ by $A_a$. Thus we may assume that $P$ is monic in $x_n$ of degree $d > 0$. For $i = 1, \ldots, n-1$ let $d_i$ be the degree of $P$ in $x_i$. Note that this implies that $P^\varphi$ is monic of degree $d$ in $x_n$ and has degree $\leq d_i$ in $x_i$ for every homomorphism $\varphi : A \to \kappa$ where $\kappa$ is a field. Thus if $P^\varphi$ is reducible, then we can write
\[ P^\varphi = Q_1 Q_2 \]
with $Q_1, Q_2$ monic of degree $e_1, e_2 \geq 0$ in $x_n$ with $e_1 + e_2 = d$ and having degree $\leq d_i$ in $x_i$ for $i = 1, \ldots, n-1$. In other words we can write
\[ Q_j = x_n^{e_j} + \sum_{0 \leq l \leq e_j} \left( \sum_{L \in L} a_{j,l,L} x_L^l \right) x_n^{e_j-l} \]

(25.4.1)
where the sum is over the set $\mathcal{L}$ of multi-indices $L$ of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the $A$-algebra

$$B_{e_1, e_2} = A[\{a_{1, l, m} \mid 0 \leq l d, m \leq e_2, L \in \mathcal{L}\}]/(relations)$$

where the (relations) is the ideal generated by the coefficients of the polynomial $P - Q_1 Q_2 \in A[\{a_{1, l, m} \mid 0 \leq l \leq e_1, L \in \mathcal{L}\}, \{a_{2, l, m} \mid 0 \leq l \leq e_2, L \in \mathcal{L}\}][x_1, \ldots, x_n]$ with $Q_1$ and $Q_2$ defined as in (25.4.1). OK, and the assumption that $P$ is irreducible over $\mathcal{K}$ implies that there does not exist any $A$-algebra homomorphism $B_{e_1, e_2} \to \mathcal{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 33.1, this means that $B_{e_1, e_2} \otimes_A K = 0$. As $B_{e_1, e_2}$ is a finitely generated $A$-algebra this signifies that we can find an $f_{e_1, e_2} \in A$ such that $(B_{e_1, e_2})_{f_{e_1, e_2}} = 0$. By construction this means that if $\varphi : B_{f_{e_1, e_2}} \to \kappa$ is a homomorphism to a field, then $P \varphi$ does not have a factorization $P \varphi = Q_1 Q_2$ with $Q_1$ of degree $e_1$ in $x_n$ and $Q_2$ of degree $e_2$ in $x_n$. Thus taking $f = \prod_{e_1, e_2 \geq 0, e_1 + e_2 = d} f_{e_1, e_2}$ we win. \hfill \Box

**Lemma 25.5.** Let $f : X \to Y$ be a morphism of schemes. Assume

1. $Y$ is irreducible with generic point $\eta$,
2. $X_\eta$ is geometrically irreducible, and
3. $f$ is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

**First proof of Lemma 25.5.** We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X_V & \longrightarrow & X \\
f' \downarrow & & \downarrow f & & \downarrow f \\
Y' & \longrightarrow & V & \longrightarrow & Y
\end{array}
$$

as in Lemma 22.7. Note that the generic fibre of $f'$ is the reduction of the geometric fibre of $f$ (see Lemma 22.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism $f'$. Then after shrinking $V$ all the fibres of $f'$ are geometrically irreducible. As $X' = (Y' \times_Y X_V)_{\text{red}}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 25.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \text{Spec}(A)$ with $A$ a domain.

Let $x \in X_\eta$ be the generic point. As $X_\eta$ is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 6.2 and 8.6. In particular the field extension $K \subset L$ is separable, see Algebra, Lemma 43.1. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate $L$ over $K$ and such that $x_1, \ldots, x_r$ is a transcendence basis for $L$ over $K$, see Algebra, Lemma 11.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for $x_{r+1}$. Clearing denominators we may assume that $P$ has coefficients in $A[x_1, \ldots, x_r]$. Note that as $L$ is geometrically reduced and geometrically irreducible over $K$, the polynomial $P$ is irreducible.
Lemma 25.4. and set \( X' = \text{Spec}(B') \). By construction the fraction field of \( B' \) is isomorphic to \( L = \kappa(x) \) as \( K \)-extensions. Hence there exists an open \( U \subset X \), and open \( U' \subset X' \) and a \( Y \)-isomorphism \( U \to U' \), see Morphisms, Lemma [47.7]. Here is a diagram:

\[
\begin{array}{ccc}
X & \leftarrow & U \\
\downarrow & & \downarrow \\
Y & \longrightarrow & U' \rightarrow X' \longrightarrow \text{Spec}(B')
\end{array}
\]

Note that \( U_y \subset X_y \) and \( U'_y \subset X'_y \) are dense opens. Thus after shrinking \( Y \) by applying Lemma [22.3] we obtain that \( U_y \) is dense in \( X_y \) and \( U'_y \) is dense in \( X'_y \) for all \( y \in Y \). Thus it suffices to prove the lemma for \( X' \to Y \) which is the content of Lemma [25.3].

Second proof of Lemma [25.5] Let \( Y' \subset Y \) be the reduction of \( Y \). Let \( X' \to X \) be the reduction of \( X \). Note that \( X' \to X \to Y \) factors through \( Y' \), see Schemes, Lemma [12.6] As \( Y' \to Y \) and \( X' \to X \) are universal homeomorphisms by Morphisms, Lemma [43.6] we see that it suffices to prove the lemma for \( X' \to Y' \). Thus we may assume that \( X \) and \( Y \) are reduced. In particular \( Y \) is integral, see Properties, Lemma [3.4]. Thus by Morphisms, Proposition [26.1] there exists a nonempty affine open \( V \subset Y \) such that \( X_V \to V \) is flat and of finite presentation. After replacing \( Y \) by \( V \) we may assume, in addition to (1), (2), (3) that \( Y \) is integral affine, \( X \) is reduced, and \( f \) is flat and of finite presentation. In particular \( f \) is universally open, see Morphisms, Lemma [24.9].

Pick a nonempty affine open \( U \subset X \). Then \( U \to Y \) is flat and of finite presentation with geometrically irreducible generic fibre. The complement \( X_y \setminus U_y \) is nowhere dense. Thus after shrinking \( Y \) we may assume \( U_y \subset X_y \) is open dense for all \( y \in Y \), see Lemma [22.3]. Thus we may replace \( X \) by \( U \) and we reduce to the case where \( Y \) is integral affine and \( X \) is reduced affine, flat and of finite presentation over \( Y \) with geometrically irreducible generic fibre \( X_y \).

Write \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \). Then \( A \) is a domain, \( B \) is reduced, \( A \to B \) is flat of finite presentation, and \( B_K \) is geometrically irreducible over the fraction field \( K \) of \( A \). In particular we see that \( B_K \) is a domain. Let \( L \) be the fraction field of \( B_K \). Note that \( L \) is a finitely generated field extension of \( K \) as \( B \) is an \( A \)-algebra of finite presentation. Let \( K \subset K' \) be a finite purely inseparable extension such that \( (L \otimes_K K')_{\text{red}} \) is a separably generated field extension, see Algebra, Lemma [44.3]. Choose \( x_1, \ldots, x_n \in K' \) which generate the field extension \( K' \) over \( K \), and such that \( x_i^{p_i} \in A \) for some prime power \( q_i \) (proof existence \( x_i \) omitted). Let \( A' \) be the \( A \)-subalgebra of \( K' \) generated by \( x_1, \ldots, x_n \). Then \( A' \) is a finite \( A \)-subalgebra \( A' \subset K' \) whose fraction field is \( K' \). Note that \( \text{Spec}(A') \to \text{Spec}(A) \) is a universal homeomorphism, see Algebra, Lemma [45.6]. Hence it suffices to prove the result after base changing to \( \text{Spec}(A') \). We are going to replace \( A \) by \( A' \) and \( B \) by \( (B \otimes_A A')_{\text{red}} \) to arrive at the situation where \( L \) is a separably generated field extension of \( K \). Of course it may happen that \( (B \otimes_A A')_{\text{red}} \) is no longer flat, or of finite presentation over \( A' \), but this can be remedied by replacing \( A' \) by \( A'_f \) for a suitable \( f \in A' \), see Algebra, Lemma [117.3].
At this point we know that $A$ is a domain, $B$ is reduced, $A \to B$ is flat and of finite presentation, $B_K$ is a domain whose fraction field $L$ is a separably generated field extension of the fraction field $K$ of $A$. By Algebra, Lemma [11.3] we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over $K$, and $x_{r+1}$ is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of $x_{r+1}$ over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since $L/K$ is separable and since $L$ is geometrically irreducible over $K$, the polynomial $P$ is irreducible over the algebraic closure $\overline{K}$ of $K$. Denote $B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$. By construction the fraction fields of $B$ and $B'$ are isomorphic as $K$-extensions. Hence there exists an isomorphism of $A$-algebras $B_h \cong B'_h$, for suitable $h \in B$ and $h' \in B'$, see Morphisms, Lemma [47.7]. In other words $X$ and $X' = \text{Spec}(B')$ have a common affine open $U$. Here is a diagram:

$$
\begin{array}{ccc}
  X = \text{Spec}(B) & \leftarrow & U \rightarrow \text{Spec}(B') = X' \\
  \downarrow & & \downarrow \\
  Y = \text{Spec}(A) & & 
\end{array}
$$

After shrinking $Y$ once more (by applying Lemma [22.3] to $Z = X \setminus U$ in $X$ and $Z' = X' \setminus U$ in $X'$) we see that $U_y$ is dense in $X_y$ and $U_{y'}$ is dense in $X'_{y'}$ for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma [25.4].

055A Lemma 25.6. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ counting the numbers of geometrically irreducible components of fibres of $f$ introduced in Lemma [25.5]. Assume $f$ of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \{y\}$ such that $n_{X/Y}|_V$ is constant.

Proof. Let $Z$ be the reduced induced scheme structure on $\{y\}$. Let $f_Z : X_Z \to Z$ be the base change of $f$. Clearly it suffices to prove the lemma for $f_Z$ and the generic point of $Z$. Hence we may assume that $Y$ is an integral scheme, see Properties, Lemma [3.4]. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma [22.8] to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma [34.13]), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma [25.3]. Thus we see that we may assume that all irreducible components of the generic fibre $X_\eta$ are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let $X_i$ be the closure of $X_{i,\eta}$ in $X$. After shrinking $Y$ we may assume that $X = \bigcup X_i$, see Lemma [22.5]. After shrinking $Y$ some more we see that each fibre of $f$ has at least $n$ irreducible components, see Lemma [25.1]. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking $Y$ some more we obtain that $X_{i,y}$ is geometrically irreducible for each $i$ and all $y \in Y$, see Lemma [25.5]. Since $X_\eta = \bigcup X_{i,\eta}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □
Lemma 25.7. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ counting the numbers of geometrically irreducible components of fibres of $f$ introduced in Lemma 25.3. Assume $f$ is of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in $Y$.

Proof. Fix $n$. Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E_n \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim} A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 25.3 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E_n$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in $Z$. Let $\xi \in Z$ be the generic point. Then Lemma 25.6 shows that $n_{X/Y}$ is constant in a neighbourhood of $\xi$ in $Z$. This clearly implies what we want. \qed

26. Connected components of fibres

Lemma 26.1. Let $f : X \to Y$ be a morphism of schemes. Assume $Y$ irreducible with generic point $\eta$ and $f$ of finite presentation. If $X_{\eta}$ has $n$ connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre $X_y$ has at least $n$ connected components.

Proof. As the question is purely topological we may replace $X$ and $Y$ by their reductions. In particular this implies that $Y$ is integral, see Properties, Lemma 3.4. Let $X_{\eta} = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of $X_{\eta}$ into connected components. Let $X_1 \subset X$ be the reduced closed subscheme whose generic fibre is $X_{1,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of $X$ whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking $Y$ we may assume that $Z_{i,j} = \emptyset$, see Lemma 22.1. After shrinking $Y$ some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 22.5. Moreover, after shrinking $Y$ we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 26.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 24.9. Thus there exists an open neighbourhood $V$ of $\eta$ which is contained in $f(X_i)$ for each $i$. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of $X_y$, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that $X_y$ has at least $n$ connected components. \qed

Lemma 26.2. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of $f$. Then

$$\{y' \in Y' \mid X_{y'}' \text{ is geometrically connected}\} \subseteq g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}).$$
Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_y = X_y \times_Y y'$ is geometrically connected over $\kappa(y')$ if and only if $X_y$ is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 07.3. □

**Lemma 26.3.** Let $f : X \to Y$ be a morphism of schemes. Let
\[ n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\} \]
be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where $K$ is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then
\[ n_{X'/Y'} = n_{X/Y} \circ g \]
where $X' \to Y'$ is the base change of $f$.

**Proof.** Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram
\[
\begin{array}{ccc}
K & \longrightarrow & K' \\
\uparrow & & \uparrow \\
\kappa(y) & \longrightarrow & \kappa(y')
\end{array}
\]
of fields. The result follows as the morphisms of schemes
\[ (X'_y)_K' \leftarrow (X'_y)_K'' = (X_y)_K'' \to (X_y)_K \]
induce bijections between connected components, see Varieties, Lemma 07.6. □

**Lemma 26.4.** Let $f : X \to Y$ be a morphism of schemes. Assume
\begin{enumerate}
\item $Y$ is irreducible with generic point $\eta$,
\item $X_\eta$ is geometrically connected, and
\item $f$ is of finite type.
\end{enumerate}
Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

**Proof.** Choose a diagram
\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \to & Y
\end{array}
\]
as in Lemma 07.8. Note that the generic fibre of $f'$ is geometrically connected (for example by Lemma 26.3). Suppose that the lemma holds for the morphism $f'$. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over $W$ is geometrically connected. Then, as $g$ is an open morphism by Morphisms, Lemma 34.13 all the fibres of $f$ at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 26.3. In this way we see that we may assume that the irreducible components of the generic fibre $X_\eta$ are geometrically irreducible.

Let $Y'$ be the reduction of $Y$, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 26.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$
Let \( \eta \) be the decomposition of the generic fibre into (geometrically) irreducible components. Let \( X_i \) be the closure of \( X_{i,\eta} \) in \( X \). After shrinking \( Y \) we may assume that \( X = \bigcup X_i \), see Lemma 22.5. Let \( Z_{i,j} = X_i \cap X_j \). Let
\[
\{1,\ldots,n\} \times \{1,\ldots,n\} = I \amalg J
\]
where \( (i,j) \in I \) if \( Z_{i,j,\eta} = \emptyset \) and \( (i,j) \in J \) if \( Z_{i,j,\eta} \neq \emptyset \). After shrinking \( Y \) we may assume that \( Z_{i,j} = \emptyset \) for all \( (i,j) \in I \), see Lemma 22.1. After shrinking \( Y \) we obtain that \( X_{i,\eta} \) is geometrically irreducible for each \( i \) and all \( y \in Y \), see Lemma 25.5. After shrinking \( Y \) some more we achieve the situation where each \( Z_{i,j} \to Y \) is flat and of finite presentation for all \( (i,j) \in J \), see Morphisms, Proposition 26.1. This means that \( f(Z_{i,j}) \subset Y \) is open, see Morphisms, Lemma 24.9. We claim that
\[
V = \bigcap_{(i,j) \in J} f(Z_{i,j})
\]
works, i.e., that \( X_y \) is geometrically connected for each \( y \in V \). Namely, the fact that \( X_{i,\eta} \) is connected implies that the equivalence relation generated by the pairs in \( J \) has only one equivalence class. Now if \( y \in V \) and \( K \supseteq \kappa(y) \) is a separably closed extension, then the irreducible components of \((X_{i,y})_K \) are the fibres \((X_{i,y})_K \). Moreover, we see by construction and \( y \in V \) that \((X_{i,y})_K \) meets \((X_{j,y})_K \) if and only \((i,j) \in J \). Hence the remark on equivalence classes shows that \((X_y)_K \) is connected and we win.

**Lemma 26.5.** Let \( f : X \to Y \) be a morphism of schemes. Let \( n_{X/Y} \) be the function on \( Y \) counting the numbers of geometrically connected components of fibres of \( f \) introduced in Lemma 26.3. Assume \( f \) of finite type. Let \( y \in Y \) be a point.

Then there exists a nonempty open \( V \subset \{y\} \) such that \( n_{X/Y}|_V \) is constant.

**Proof.** Let \( Z \) be the reduced induced scheme structure on \( \{y\} \). Let \( f_Z : X_Z \to Z \) be the base change of \( f \). Clearly it suffices to prove the lemma for \( f_Z \) and the generic point of \( Z \). Hence we may assume that \( Y \) is an integral scheme, see Properties, Lemma 3.4. Our goal in this case is to produce a nonempty open \( V \subset Y \) such that \( n_{X/Y}|_V \) is constant.

We apply Lemma 22.8 to \( f : X \to Y \) and we get \( g : Y' \to V \subset Y \). As \( g : Y' \to V \) is surjective finite étale, in particular open (see Morphisms, Lemma 34.19), it suffices to prove that there exists an open \( V' \subset Y' \) such that \( n_{X'/Y'}|_{V'} \) is constant, see Lemma 25.3. Thus we see that we may assume that all irreducible components of the generic fibre \( X_\eta \) are geometrically irreducible over \( \kappa(\eta) \). By Varieties, Lemma 8.15 this implies that also the connected components of \( X_\eta \) are geometrically connected.

At this point suppose that \( X_\eta = X_{1,\eta} \cup \cdots \cup X_{n,\eta} \) is the decomposition of the generic fibre into (geometrically) connected components. In particular \( n_{X/Y}(\eta) = n \). Let \( X_i \) be the closure of \( X_{i,\eta} \) in \( X \). After shrinking \( Y \) we may assume that \( X = \bigcup X_i \), see Lemma 22.5. After shrinking \( Y \) some more we see that each fibre of \( f \) has at least \( n \) connected components, see Lemma 26.1. Hence \( n_{X/Y}(y) \geq n \) for all \( y \in Y \). After shrinking \( Y \) some more we obtain that \( X_{i,y} \) is geometrically irreducible.
connected for each $i$ and all $y \in Y$, see Lemma 26.4. Since $X_y = \bigcup_i X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

**Lemma 26.6.** Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ counting the numbers of geometric connected components of fibres of $f$ introduced in Lemma 26.3. Assume $f$ of finite presentation. Then the level sets

$E_n = \{ y \in Y \mid n_{X/Y}(y) = n \}$

of $n_{X/Y}$ are locally constructible in $Y$.

**Proof.** Fix $n$. Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E_n \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \colim A_i$ as a directed limit of finite type $\mathcal{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 26.3 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 16.3 to prove that $E_n$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in $Z$. Let $\xi \in Z$ be the generic point. Then Lemma 26.5 shows that $n_{X/Y}$ is constant in a neighbourhood of $\xi$ in $Z$. This clearly implies what we want. □

**Lemma 26.7.** Let $f : X \to S$ be a morphism of schemes. Assume that

1. $S$ is the spectrum of a discrete valuation ring,
2. $f$ is flat,
3. $X$ is connected,
4. the closed fibre $X_s$ is reduced.

Then the generic fibre $X_\eta$ is connected.

**Proof.** Write $Y = \text{Spec}(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that $X_\eta$ is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = \text{Spec}(A)$ be any affine open in $X$. Note that $\pi$ is a nonzerodivisor on $A$ as $A$ is flat over $R$, see More on Algebra, Lemma 20.9 for example. Then $e|_U$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \mod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of $n$ implies $n = 0$. Thus we conclude that $e \in A!$. In other words $e \in \Gamma(X, \mathcal{O}_X)$. As $X$ is connected it follows that $e$ is a trivial idempotent which is a contradiction. □

### 27. Connected components meeting a section

**Situation 27.1.** Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of $f$. For every $y \in Y$ we denote $X^0_y$ the connected component of $X_y$ containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X^0_y$. 
055M Lemma 27.2. Let \( f : X \to Y \), \( s : Y \to X \) be as in Situation 27.1. If \( g : Y' \to Y \) is any morphism, consider the base change diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow_{s'} & \nearrow_{f} & \nearrow_{f} \\
Y' & \longrightarrow & Y
\end{array}
\]

so that we obtain \((X')^0 \subset X'\). Then \((X')^0 = (g')^{-1}(X^0)\).

**Proof.** Let \( y' \in Y' \) with image \( y \in Y \). We may think of \( X^0_y \) as a closed subscheme of \( X_y \), see for example Morphisms, Definition 25.3. As \( s(y) \in X^0_y \) we conclude from Varieties, Lemma 7.14 that \( X^0_y \) is a geometrically connected scheme over \( \kappa(y) \). Hence \( X^0_y \times_y y' \to X_{y'} \) is a connected closed subscheme which contains \( s'(y') \). Thus \( X^0_y \times_y y' \subset (X')^0 \). The other inclusion \( X^0_y \times_y y' \supset (X')^0 \) is clear as the image of \((X')^0 \) in \( X_y \) is a connected subset of \( X_y \) which contains \( s(y) \). \( \square \)

055N Lemma 27.3. Let \( f : X \to Y \), \( s : Y \to X \) be as in Situation 27.1. Assume \( f \) of finite type. Let \( y \in Y \) be a point. Then there exists a nonempty open \( V \subset \{y\} \) such that the inverse image of \( X^0 \) in the base change \( X_V \) is open and closed in \( X_V \).

**Proof.** Let \( Z \subset Y \) be the induced reduced closed subscheme structure on \( \{y\} \). Let \( f_Z : X_Z \to Z \) and \( s_Z : Z \to X_Z \) be the base changes of \( f \) and \( s \). By Lemma 27.2 we have \((X_Z)^0 = (X^0)_Z \). Hence it suffices to prove the lemma for the morphism \( X_Z \to Z \) and the point \( x \in X_Z \) which maps to the generic point of \( Z \). In other words we have reduced the problem to the case where \( Y \) is an integral scheme (see Properties, Lemma 3.4) with generic point \( \eta \). Our goal is to show that after shrinking \( Y \) the subset \( X^0 \) becomes an open and closed subset of \( X \).

Note that the scheme \( X_\eta \) is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write \( X_\eta = X^0_\eta \cup T_\eta \) for some open and closed subset \( T_\eta \) of \( X_\eta \). Next, let \( T \subset X \) be the closure of \( T_\eta \) and let \( X^{\text{open}} \subset X \) be the closure of \( X^0_\eta \). Note that \( T_\eta \), resp. \( X^0_\eta \) is the generic fibre of \( T \), resp. \( X^{\text{open}} \), see discussion preceding Lemma 22.5. Moreover, that lemma implies that after shrinking \( Y \) we may assume that \( X = X^{\text{open}} \cup T \) (set theoretically). Note that \((T \cap X^{\text{open}})_\eta = T_\eta \cap X^0_\eta = \emptyset \). Hence after shrinking \( Y \) we may assume that \( T \cap X^{\text{open}} = \emptyset \), see Lemma 22.1. In particular \( X^{\text{open}} \) is open in \( X \). Note that \( X^0_\eta \) is connected and has a rational point, namely \( s(\eta) \), hence it is geometrically connected, see Varieties, Lemma 7.14. Thus after shrinking \( Y \) we may assume that all fibres of \( X^{\text{open}} \to Y \) are geometrically connected, see Lemma 26.4. At this point it follows that the fibres \( X^{y}_0 \) are open, closed, and connected subsets of \( X_y \) containing \( \sigma(y) \). It follows that \( X^0 = X^{\text{open}} \) and we win. \( \square \)

055P Lemma 27.4. Let \( f : X \to Y \), \( s : Y \to X \) be as in Situation 27.1. If \( f \) is of finite presentation then \( X^0 \) is locally constructible in \( X \).

**Proof.** Let \( x \in X \). We have to show that there exists an open neighbourhood \( U \) of \( x \) such that \( X^0 \cap U \) is constructible in \( U \). This reduces us to the case where \( Y \) is affine. Write \( Y = \text{Spec}(A) \) and \( A = \text{colim} A_i \) as a directed limit of finite type \( \mathbf{Z} \)-algebras. By Limits, Lemma 10.11 we can find an \( i \) and a morphism \( f_i : X_i \to \text{Spec}(A_i) \) of finite presentation, endowed with a section \( s_i : \text{Spec}(A_i) \to X_i \) whose base change to \( Y \) recovers \( f \) and the section \( s \). By Lemma 27.2 it suffices to prove
the lemma for $f_i, s_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

Assume $Y$ is a Noetherian affine scheme. Since $f$ is of finite presentation, i.e., of finite type, we see that $X$ is a Noetherian scheme too, see Morphisms, Lemma 14.6

In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of $Z$ or is not dense in $Z$, see Topology, Lemma 16.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 27.3 there exists a nonempty open subset $V \subset \{y\}$ such that $X^0 \cap X_V$ is open and closed in $X_V$. Since $f(Z) \subset \{y\}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of $Z$. It follows that $X^0 \cap W$ is open and closed in $W$. Since $W$ is irreducible we see that $X^0 \cap W$ is either empty or equal to $W$. This proves the lemma.

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**Lemma 27.5.** Let $f : X \to Y, s : Y \to X$ be as in Situation 27.1. Let $y \in Y$ be a point. Assume

1. $f$ is of finite presentation and flat, and
2. the fibre $X_y$ is geometrically reduced.

Then $X^0$ is a neighbourhood of $X^0_y$ in $X$.

**Proof.** We may replace $Y$ with an affine open neighbourhood of $y$. Write $Y = \text{Spec}(A)$ and $A = \colim A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \text{Spec}(A_i) \to X_i$ whose base change to $Y$ recovers $f$ and the section $s$. After possibly increasing $i$ we may also assume that $f_i$ is flat, see Limits, Lemma 8.7. Let $y_i$ be the image of $y$ in $Y_i$. Note that $X_y = (X_{i,y_i})_{\text{Spa}} (y)$. Hence $X_{i,y_i}$ is geometrically reduced, see Varieties, Lemma 6.6. By Lemma 27.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.

Assume $Y$ is the spectrum of a Noetherian ring. Since $f$ is of finite presentation, i.e., of finite type, we see that $X$ is a Noetherian scheme too, see Morphisms, Lemma 14.6. Let $x \in X^0$ be a point lying over $y$. By Topology, Lemma 16.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through $x$ the intersection $X^0 \cap Z$ is dense in $Z$. In particular it suffices to prove that the generic point $x' \in Z$ is in $X^0$. By Properties, Lemma 5.10 we can find a discrete valuation ring $R$ and a morphism $\text{Spec}(R) \to X$ which maps the special point to $x$ and the generic point to $x'$. We are going to think of $\text{Spec}(R)$ as a scheme over $Y$ via the composition $\text{Spec}(R) \to X \to Y$. By Lemma 27.2 we have that $(X_R)^0$ is the inverse image of $X^0$. By construction we have a second section $t : \text{Spec}(R) \to X_R$ (besides the base change $s_R$ of $s$) of the structure morphism $X_R \to \text{Spec}(R)$ such that $t(\eta_R)$ is a point of $X_R$ which maps to $x'$ and $t(0_R)$ is a point of $X_R$ which maps to $x$. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \sim t(0_R)$. Thus it suffices to prove that this implies that $t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where $Y$ is the spectrum of a discrete valuation ring and $y$ its closed point.

Assume $Y$ is the spectrum of a discrete valuation ring and $y$ is its closed point. Our goal is to prove that $X^0$ is a neighbourhood of $X^0_y$. Note that $X^0_y$ is open and closed in $X_y$, as $X_y$ has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in $X$. Thus $U = X \setminus C$ is an open neighbourhood of $X^0_y$ and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other
words, we may assume that $X_y$ is connected. Suppose that $X$ is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the $X_i$. Say $s(Y) \subseteq X_1$. Then $X^0 \subseteq X_1$. Hence we may replace $X$ by $X_1$ and assume that $X$ is connected. At this point Lemma 26.7 implies that $X_y$ is connected, i.e., $X^0 = X$ and we win.

Lemma 27.6. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 27.1 Assume

1. $f$ is of finite presentation and flat, and
2. all fibres of $f$ are geometrically reduced.

Then $X^0$ is open in $X$.

Proof. This is an immediate consequence of Lemma 27.5. □

28. Dimension of fibres

Lemma 28.1. Let $f : X \to Y$ be a morphism of schemes. Assume $Y$ irreducible with generic point $\eta$ and $f$ of finite type. If $X_\eta$ has dimension $n$, then there exists a nonempty open $V \subseteq Y$ such that for all $y \in V$ the fibre $X_y$ has dimension $n$.

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 27.4 this is a closed subset of $X$. By assumption $Z_\eta = \emptyset$. Hence by Lemma 22.1 we may shrink $Y$ and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n - 1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of $X$. By assumption we have $Z'_\eta \neq \emptyset$. Hence after shrinking $Y$ we may assume that $Z' \to Y$ is surjective, see Lemma 22.2. Hence we win. □

Lemma 28.2. Let $f : X \to Y$ be a morphism of finite type. Let $n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$ be the function which associates to $y \in Y$ the dimension of $X_y$. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of $f$.

Proof. This follows from Morphisms, Lemma 27.3. □

Lemma 28.3. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Assume $f$ of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in $Y$.

Proof. Fix $n$. Let $y \in Y$. We have to show that there exists an open neighbourhood $V$ of $y$ in $Y$ such that $E_n \cap V$ is constructible in $V$. Thus we may assume that $Y$ is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim} A_i$ as a directed limit of finite type $\mathbb{Z}$-algebras. By Limits, Lemma 10.1 we can find an $i$ and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to $Y$ recovers $f$. By Lemma 28.2 it suffices to prove the lemma for $f_i$. Thus we reduce to the case where $Y$ is the spectrum of a Noetherian ring.
We will use the criterion of Topology, Lemma 16.3 to prove that $E_n$ is constructible in case $Y$ is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in $Z$. Let $\xi \in Z$ be the generic point. Then Lemma 28.1 shows that $n_{X/Y}$ is constant in a neighbourhood of $\xi$ in $Z$. This implies what we want. □

**Lemma 28.4.** Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Then $n_{X/Y}$ is lower semi-continuous.

**Proof.** Let $W \subset X$, $W = \coprod_{d \geq 0} U_d$ be the open constructed in Lemmas 20.7 and 20.9. Let $y \in Y$ be a point. If $n_{X/Y}(y) = \dim(X_y) = n$, then $y$ is in the image of $U_n \to Y$. By Morphisms, Lemma 24.9 we see that $f(U_n)$ is open in $Y$. Hence there is an open neighbourhood of $y$ where $n_{X/Y}$ is $\geq n$. □

**Lemma 28.5.** Let $f : X \to Y$ be a proper morphism of schemes. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.4. Then $n_{X/Y}$ is upper semi-continuous.

**Proof.** Let $Z_d = \{x \in X \mid \dim_r(X_{f(x)}) > d\}$. Then $Z_d$ is a closed subset of $X$ by Morphisms, Lemma 27.4. Since $f$ is proper $f(Z_d)$ is closed. Since $y \in f(Z_d)$ if and only if $n_{X/Y}(y) = d$ we see that the lemma is true. □

**Lemma 28.6.** Let $f : X \to Y$ be a proper, flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on $Y$ giving the dimension of fibres of $f$ introduced in Lemma 28.2. Then $n_{X/Y}$ is locally constant.

**Proof.** Immediate consequence of Lemmas 28.4 and 28.5. □

### 29. Theorem of the cube

**Lemma 29.1.** Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For a morphism $g : T \to S$ consider the base change diagram

$$
\begin{array}{ccc}
X_T & \longrightarrow & X \\
\downarrow p & & \downarrow f \\
T & \longrightarrow & S
\end{array}
$$

Assume $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism for all $g : T \to S$. Then there is a locally closed subscheme $Z \subset S$ such that a morphism $g : T \to S$ factors through $Z$ if and only if there exists an invertible $\mathcal{O}_T$-module $N$ with $p^*N \cong q^*\mathcal{L}$.

**Proof.** By cohomology and base change (more precisely by Derived Categories of Schemes, Lemma 26.4) we see that $E = Rf_*\mathcal{L}$ is a perfect object of the derived category of $S$ and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{L}^{\otimes -1}$. Since there is never any cohomology in degrees $< 0$, we see that $E$ and $E'$ have (locally) tor-amplitude in $[0, b]$ for some $b$. Observe that for any $g : T \to S$ we have $p_*(q^*\mathcal{L}) = H^0(Lg^*E)$ and $p_*(q^*\mathcal{L}^{\otimes -1}) = H^0(Lg^*E')$. Let $j : Z \to S$ and $j' : Z' \to S$ be the locally closed immersions constructed in Derived Categories of Schemes, Lemma 27.4 for $E$ and $E'$ with $a = 0$; these
are characterized by the property that \( H^0(L_j^*E) \) and \( H^0((j')^*E') \) are invertible modules compatible with pullback.

Let \( g : T \to S \) be a morphism. If there exists an \( \mathcal{N} \) as in the lemma, then, using the projection formula Cohomology, Lemma \([15.2]\) we see that the modules

\[
p_*(q^*\mathcal{L}) \cong p_*(p^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_T} p_*\mathcal{O}_T \cong \mathcal{N}
\]

and similarly \( p_*(q^*\mathcal{L}^{-1}) \cong \mathcal{N}^{-1} \) are invertible and remain invertible after any further base change \( T' \to T \). Hence in this case \( T \to S \) factors through \( j \) and through \( j' \). Thus we may replace \( S \) by \( Z \times_S Z' \) and assume that \( f_!\mathcal{L} \) and \( f_!\mathcal{L}^{-1} \) are invertible \( \mathcal{O}_S \)-modules whose formation commutes with arbitrary change of base.

In this situation if \( g : T \to S \) be a morphism and there exists an \( \mathcal{N} \) as in the lemma, then the map (cup product in degree 0)

\[
p_*(q^*\mathcal{L}) \otimes_{\mathcal{O}_T} p_*(q^*\mathcal{L}^{-1}) \to \mathcal{O}_T
\]

is an isomorphism. Conversely, if this cup product map is an isomorphism, then we see that locally on \( T \) we have sections \( \sigma \) in \( p_*(q^*\mathcal{L}) \) and \( \sigma' \) in \( p_*(q^*\mathcal{L}^{-1}) \) whose product is 1. Thinking of \( \sigma \) as a section of \( q^*\mathcal{L} \) on \( X_T \) and \( \sigma' \) as a section of \( q^*\mathcal{L}^{-1} \) on \( X_T \) with \( \sigma \cdot \sigma' = 1 \), we conclude that \( \mathcal{O}_{X_T} \to q^*\mathcal{L} \) is an isomorphism. In other words, we see that \( p^!p_*q^*\mathcal{L} \cong q^*\mathcal{L} \). But the condition that the cup product is nonzero picks out an open subscheme and the proof is complete.

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**Lemma 29.2.** Let \( f : X \to S \) and \( \mathcal{L} \) be as in Lemma 29.1 If moreover the geometric fibres of \( f \) are integral, then \( Z \) is closed in \( S \).

**Proof.** We first do a standard argument to reduce to the Noetherian case. Namely, the question is local on \( S \), hence we may assume that \( S = \text{Spec}(R) \) is affine. Then we write \( R = \colim_{i \in I} R_i \) as a filtered colimit with \( R_i \) of finite type over \( \mathbf{Z} \). Set \( S_i = \text{Spec}(R_i) \). For some \( i \) there exists a flat proper morphism \( f_i : X_i \to S_i \) and an invertible \( \mathcal{O}_{X_i} \)-module \( \mathcal{L}_i \) whose base change to \( S \) gives back \( f : X \to S \) and \( \mathcal{L} \). See Limits, Lemmas 10.1 [8.7] 13.1 and 10.3 Pick \( i \in I \). By Lemmas 24.5 and 24.7 the set \( E \subseteq S_i \) of points where the fibres of \( f_i \) are geometrically integral is constructible. Since \( S \to S_i \) maps into \( E \) by assumption (and Lemmas 24.2 and 25.2) after increasing \( i \) we may assume the fibres of \( f_i \) are geometrically irreducible, see Limits, Lemma 4.10 By Derived Categories of Schemes, Lemma 26.4 \( \mathcal{Rf}_i^*\mathcal{O}_X \) is a perfect object of \( \mathcal{D}(\mathcal{O}_{S_i}) \) whose formation commutes with arbitrary base change. Let \( T \subseteq S_i \) be the locally closed subscheme of \( S_i \) constructed in Derived Categories of Schemes, Lemma 27.4 for \( \mathcal{Rf}_i^*\mathcal{O}_X \), with \( a = 0 \). By our assumption that \( f_*\mathcal{O}_X = \mathcal{O}_S \) universally we see that \( S \to S_i \) factors through \( T \). Set \( Y = X_i \times_{S_i} T \to T \) and \( \mathcal{M} = \mathcal{L}_i|_Y \). By construction the morphism \( g : Y \to T \) satisfies \( g_*\mathcal{O}_Y = \mathcal{O}_T \) universally and we have a cartesian diagram

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{f} & \mathcal{M} \\
\downarrow & & \downarrow g \\
S & \xrightarrow{g} & T
\end{array}
\]

Thus if we can prove the lemma for \( g \) and \( \mathcal{M} \), then it follows for \( f \) and \( \mathcal{L} \). Since \( T \) is Noetherian, we have reduced to the Noetherian case.

Assume \( S \) is Noetherian. Since \( Z \) is a locally closed subscheme of a Noetherian scheme it suffices to show that \( Z \) is closed under specialization in order to prove
that it is closed. By Properties, Lemma 5.10 and base change we see that it suffices to prove the lemma in case $S$ is the spectrum of a dvr $A$. In other words, suppose we have a flat proper morphism $X \to \text{Spec}(A)$ with integral scheme theoretic fibres $X_\eta$ (generic), $X_0$ (closed) and an invertible $\mathcal{O}_X$-module $\mathcal{L}$ whose restriction to $X_\eta$ is trivial. Goal: show that $\mathcal{L}$ is trivial. This follows from Divisors, Lemma 26.1.

However, we can prove this special case directly as follows: take a trivializing section $s \in \Gamma(X_\eta, \mathcal{L}_\eta)$. After replacing $s$ by $\pi n s$ if necessary ($\pi \in A$ a uniformizer) we can assume that $s \in \Gamma(X, \mathcal{L})$. If $s|_{X_0} = 0$, then we see that $s$ is divisible by $\pi$ (because $X_0$ is the scheme theoretic fibre and $X$ is flat over $A$). Thus we may assume that $s|_{X_0}$ is nonzero. Then the zero locus $Z(s)$ of $s$ is contained in $X_0$ but does not contain the generic point of $X_0$ (because $X_0$ is integral). This means that the $Z(s)$ has codimension $\geq 2$ in $X$ which contradicts Divisors, Lemma 15.3.

□

**Lemma 29.3.** Consider a commutative diagram of schemes

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{S} & \nearrow{f} & \\
S & & 
\end{array}
$$

with $f' : X' \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 29.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}'$ be the pullback to $X'$. Let $Z \subset S$, resp. $Z' \subset S$ be the locally closed subscheme constructed in Lemma 29.1 for $(f, \mathcal{L})$, resp. $(f', \mathcal{L}')$ so that $Z \subset Z'$. If $s \in Z$ and $H^1(X_s, \mathcal{O}) \longrightarrow H^1(X'_s, \mathcal{O})$ is injective, then $Z \cap U = Z' \cap U$ for some open neighbourhood $U$ of $s$.

**Proof.** We may replace $S$ by $Z'$. After shrinking $S$ to an affine open neighbourhood of $s$ we may assume that $\mathcal{L}' = \mathcal{O}_{X'}$. Let $E = Rf_* \mathcal{L}$ and $E' = Rf'_* \mathcal{L}' = Rf'_* \mathcal{O}_{X'}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 26.4). In particular we see that

$$E \otimes_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_i = \dim_{\kappa(s)} H^i(X_s, \mathcal{O}_{X_s})$. After shrinking $S$ we can represent $E$ by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 68.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 9.7). Similarly, we may assume $E'$ is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h'_1} \to \mathcal{O}_S^{\oplus h'_2} \to \ldots$$

where $h'_i = \dim_{\kappa(s)} H^i(X'_s, \mathcal{O}_{X'_s})$. By functoriality of cohomology we have a map

$$E \longrightarrow E'$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing $E$ is a finite complex of finite free modules and since $S$ is affine, we
can choose a map of complexes

\[
\begin{array}{ccc}
O_S & \xrightarrow{d} & O_S^\oplus b_1 \\
& a & \\
& b & c \\
O_S & \xrightarrow{d'} & O_S^\oplus b'_1 \\
\end{array}
\]

representing the given map \( E \to E' \). Since \( s \in Z \) we see that the trivializing section of \( L_s \) pulls back to a trivializing section of \( L'_s = \mathcal{O}_{X'} \). Thus \( a \otimes \kappa(s) \) is an isomorphism, hence after shrinking \( S \) we see that \( a \) is an isomorphism. Finally, we use the hypothesis that \( H^1(X_s, \mathcal{O}) \to H^1(X'_s, \mathcal{O}) \) is injective, to see that there exists a \( h_1 \times h_1 \) minor of the matrix defining \( b \) which maps to a nonzero element in \( \kappa(s) \). Hence after shrinking \( S \) we may assume that \( b \) is injective. However, since \( L' = \mathcal{O}_{X'} \) we see that \( d' = 0 \). It follows that \( d = 0 \). In this way we see that the trivializing section of \( L_s \) lifts to a section of \( L \) over \( X \). A straightforward topological argument (omitted) shows that this means that \( L \) is trivial after possibly shrinking \( S \) a bit further.

**Lemma 29.4.** Consider \( n \) commutative diagrams of schemes

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_i} & X \\
& S \\
& f \\
\end{array}
\]

with \( f_i : X_i \to S \) and \( f : X \to S \) satisfying the hypotheses of Lemma 29.3. Let \( L \) be an invertible \( \mathcal{O}_X \)-module and let \( L_i \) be the pullback to \( X_i \). Let \( Z \subset S \), resp. \( Z_i \subset S \) be the locally closed subscheme constructed in Lemma 29.1 for \( (f, L) \), resp. \( (f_i, L_i) \) so that \( Z \subset \bigcap_{i=1,\ldots,n} Z_i \). If \( s \in Z \) and

\[
H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})
\]

is injective, then \( Z \cap U = (\bigcap_{i=1,\ldots,n} Z_i) \cap U \) (scheme theoretic intersection) for some open neighbourhood \( U \) of \( s \).

**Proof.** This lemma is a variant of Lemma 29.3 and we strongly urge the reader to read that proof first; this proof is basically a copy of that proof with minor modifications. It follows from the description of (scheme valued) points of \( Z \) and the \( Z_i \) that \( Z \subset \bigcap_{i=1,\ldots,n} Z_i \) where we take the scheme theoretic intersection. Thus we may replace \( S \) by the scheme theoretic intersection \( \bigcap_{i=1,\ldots,n} Z_i \). After shrinking \( S \) to an affine open neighbourhood of \( s \) we may assume that \( L_i = \mathcal{O}_{X_i} \) for \( i = 1, \ldots, n \).

Let \( E = Rf_* \mathcal{L} \) and \( E_i = Rf_{i,*} \mathcal{L}_i = Rf_{i,*} \mathcal{O}_{X_i} \). These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 26.4). In particular we see that

\[
E \otimes^L_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})
\]

The second equality because \( s \in Z \). Set \( h_j = \dim_{\kappa(s)} H^j(X_s, \mathcal{O}_{X_s}) \). After shrinking \( S \) we can represent \( E \) by a complex

\[
O_S \to O_S^\oplus h_1 \to O_S^\oplus h_2 \to \ldots
\]
see More on Algebra, Lemma 08.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 3.5 and 9.7). Similarly, we may assume $E_i$ is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_{i,1}} \to \mathcal{O}_S^{\oplus h_{i,2}} \to \ldots$$

where $h_{i,j} = \dim_{\kappa(s)} H^j(X_{i,s}, \mathcal{O}_{X_{i,s}})$. By functoriality of cohomology we have a map

$$E \to E_i$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing $E$ is a finite complex of finite free modules and since $S$ is affine, we can choose a map of complexes

$$\begin{array}{cccc}
\mathcal{O}_S & \to & \mathcal{O}_S^{\oplus h_1} & \to & \mathcal{O}_S^{\oplus h_2} & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_S & \to & \mathcal{O}_S^{\oplus h_{i,1}} & \to & \mathcal{O}_S^{\oplus h_{i,2}} & \to & \ldots
\end{array}$$

representing the given map $E \to E_i$. Since $s \in \mathbb{Z}$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}_{i,s} = \mathcal{O}_{X_{i,s}}$. Thus $a_i \otimes \kappa(s)$ is an isomorphism, hence after shrinking $S$ we see that $a_i$ is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1, \ldots, n} H^1(X_{i,s}, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining $\oplus b_i$ which maps to a nonzero element in $\kappa(s)$. Hence after shrinking $S$ we may assume that $(b_1, \ldots, b_n) : \mathcal{O}_S^{h_1} \to \bigoplus_{i=1, \ldots, n} \mathcal{O}_S^{h_{i,1}}$ is injective. However, since $\mathcal{L}_i = \mathcal{O}_{X_i}$ we see that $d_i = 0$ for $i = 1, \ldots, n$. It follows that $d = 0$ because $(b_1, \ldots, b_n) \circ d = (\oplus d_i) \circ (a_1, \ldots, a_n)$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over $X$. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking $S$ a bit further.

\[ \begin{proof} \] By K"unneth (Varieties, Lemma 29.1) the map

$$H^1(X_s \times \text{Spec}(\kappa(s)), \mathcal{O}) \to H^1(X_s, \mathcal{O}) \oplus H^1(Y_s, \mathcal{O})$$

is injective. Thus we may apply Lemma 29.3 to the two morphisms

$$1 \times \tau : X \to X \times S Y \quad \text{and} \quad \sigma \times 1 : Y \to X \times S Y$$

to conclude. \[ \]
Proof. Since $X$ and $Y$ are geometrically integral and proper over $k$ the product $X \times_k Y$ is geometrically integral and proper over $k$. This implies that $H^0(X \times Y, \mathcal{O}_{X \times Y}) = k$ and that the same remains true after any base change. Thus we may apply Lemma 29.1 to the morphism
\[ p : X \times Y \times Z \to Z \]
and the invertible module $\mathcal{L}$ to get a locally closed subscheme $Z' \subset Z$ such that $\mathcal{L}|_{X \times Y \times Z'}$ is the pullback of an invertible module $\mathcal{N}$ on $Z'$. By Lemma 29.2 we see that $Z' \subset Z$ is a closed subscheme. Hence if $Z'$ contains an open neighbourhood of $z$, then $Z' = Z$ and we see that $\mathcal{N} \cong \mathcal{O}_Z$ and $\mathcal{L}$ is trivial. To get the desired open neighbourhood of $z$ apply Lemma 29.5 to the morphism $p$, the point $z$, and the sections $\sigma : Z \to X \times Z$ and $\tau : Z \to Y \times Z$ given by $x$ and $y$. □

30. Limit arguments

05FA Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

05FB Lemma 30.1. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram
\[
\begin{array}{ccc}
X_0 & \xleftarrow{g} & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \xleftarrow{g} & S
\end{array}
\]
such that
(1) $X_0$, $S_0$ are affine schemes,
(2) $S_0$ of finite type over $\mathbb{Z}$,
(3) $f_0$ is finite of finite type.

Proof. Write $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$. As $f$ is of finite presentation we see that $B$ is of finite presentation as an $A$-algebra, see Morphisms, Lemma 20.2. Thus the lemma follows from Algebra, Lemma 126.18. □

05FC Lemma 30.2. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 30.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \overline{M}$. As $f$ is of finite presentation we see that $B$ is of finite presentation as an $A$-algebra, see Morphisms, Lemma 20.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that $M$ is of finite presentation as a $B$-module, see Properties, Lemma 16.2. Thus the lemma follows from Algebra, Lemma 126.18. □

05FD Lemma 30.3. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over $S$. Then we may choose a diagram as in Lemma 30.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over $S_0$. 
**Proof.** Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As $f$ is of finite presentation we see that $B$ is of finite presentation as an $A$-algebra, see Morphisms, Lemma 20.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that $M$ is of finite presentation as a $B$-module, see Properties, Lemma 16.2. As $\mathcal{F}$ is flat over $S$ we see that $M$ is flat over $A$, see Morphisms, Lemma 24.2. Thus the lemma follows from Algebra, Lemma 162.1. □

**Lemma 30.4.** Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ is flat.

**Proof.** This is a special case of Lemma 30.3. □

**Lemma 30.5.** Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ is smooth.

**Proof.** Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and as $f$ is smooth we see that $B$ is smooth as an $A$-algebra, see Morphisms, Lemma 32.2. Hence the lemma follows from Algebra, Lemma 136.14. □

**Lemma 30.6.** Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically reduced fibres.

**Proof.** Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{g} & X \\
\downarrow f_0 & & \downarrow f \\
S_0 & \leftarrow h & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbb{Z}$. By Lemma 24.5 the set $E \subset S_0$ of points where the fibre of $f_0$ is geometrically reduced is a constructible subset. By Lemma 24.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that Spec($A_i$) $\to S_0$ has image contained in $E$ for some $i$. After replacing $S_0$ by Spec($A_i$) and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ are geometrically reduced. □

**Lemma 30.7.** Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically irreducible fibres.

**Proof.** Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{g} & X \\
\downarrow f_0 & & \downarrow f \\
S_0 & \leftarrow h & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbb{Z}$. By Lemma 25.7 the set $E \subset S_0$ of points where the fibre of $f_0$ is geometrically irreducible is a constructible subset. By Lemma 25.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that Spec($A_i$) $\to S_0$ has...
image contained in $E$ for some $i$. After replacing $S_0$ by $\text{Spec}(A_i)$ and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ are geometrically irreducible. □

Lemma 30.8. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ has geometrically connected fibres.

Proof. Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \xleftarrow{g} & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \xleftarrow{h} & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbb{Z}$. By Lemma 26.6 the set $E \subset S_0$ of points where the fibre of $f_0$ is geometrically connected is a constructible subset. By Lemma 26.3 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in $E$ for some $i$. After replacing $S_0$ by $\text{Spec}(A_i)$ and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ are geometrically connected. □

Lemma 30.9. Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension $d$. Then there exists a diagram as in Lemma 30.1 such that in addition all fibres of $f_0$ have dimension $d$.

Proof. Apply Lemma 30.1 to get a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \xleftarrow{g} & X \\
\downarrow{f_0} & & \downarrow{f} \\
S_0 & \xleftarrow{h} & S
\end{array}
$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbb{Z}$. By Lemma 28.3 the set $E \subset S_0$ of points where the fibre of $f_0$ has dimension $d$ is a constructible subset. By Lemma 28.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type $A_0$-algebras. By Limits, Lemma 4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in $E$ for some $i$. After replacing $S_0$ by $\text{Spec}(A_i)$ and $X_0$ by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of $f_0$ have dimension $d$. □

Lemma 30.10. Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 29.1). Then there exists a diagram as in Lemma 30.1 such that in addition $f_0$ is standard syntomic.

Proof. This lemma is a copy of Algebra, Lemma 134.12. □

Lemma 30.11. (Noetherian approximation and combining properties.) Let $P, Q$ be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can
find cartesian diagrams

\[ \begin{array}{ccc}
X_1 & \xleftarrow{f_1} & X \\
\downarrow & & \downarrow \\
S_1 & \xleftarrow{f} & S
\end{array} \quad \text{and} \quad \begin{array}{ccc}
X_2 & \xleftarrow{f_2} & X \\
\downarrow & & \downarrow \\
S_2 & \xleftarrow{f} & S
\end{array} \]

of affine schemes, with \( S_1, S_2 \) of finite type over \( \mathbb{Z} \) and \( f_1, f_2 \) of finite type such that \( f_1 \) has property \( P \) and \( f_2 \) has property \( Q \). Then we can find a cartesian diagram

\[ \begin{array}{ccc}
X_0 & \xleftarrow{f_0} & X \\
\downarrow & & \downarrow \\
S_0 & \xleftarrow{f} & S
\end{array} \]

of affine schemes with \( S_0 \) of finite type over \( \mathbb{Z} \) and \( f_0 \) of finite type such that \( f_0 \) has both property \( P \) and property \( Q \).

**Proof.** The given pair of diagrams correspond to cocartesian diagrams of rings

\[ \begin{array}{ccc}
B_1 & \longrightarrow & B \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & A
\end{array} \quad \text{and} \quad \begin{array}{ccc}
B_2 & \longrightarrow & B \\
\downarrow & & \downarrow \\
A_2 & \longrightarrow & A
\end{array} \]

Let \( A_0 \subset A \) be a finite type \( \mathbb{Z} \)-subalgebra of \( A \) containing the image of both \( A_1 \to A \) and \( A_2 \to A \). Such a subalgebra exists because by assumption both \( A_1 \) and \( A_2 \) are of finite type over \( \mathbb{Z} \). Note that the rings \( B_{0,1} = B_1 \otimes_{A_1} A_0 \) and \( B_{0,2} = B_2 \otimes_{A_2} A_0 \) are finite type \( A_0 \)-algebras with the property that \( B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A \) as \( A \)-algebras. As \( A \) is the directed colimit of its finite type \( A_0 \)-subalgebras, by Limits, Lemma \[10.1\] we may assume after enlarging \( A_0 \) that there exists an isomorphism \( B_{0,1} \cong B_{0,2} \) as \( A_0 \)-algebras. Since properties \( P \) and \( Q \) are assumed stable under base change we conclude that setting \( S_0 = \text{Spec}(A_0) \) and

\[ X_0 = X_1 \times_{S_1} S_0 = \text{Spec}(B_{0,1}) \cong \text{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0 \]

works. \( \square \)

### 31. Étale neighbourhoods

**02LD** It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

**02LE Definition** 31.1. Let \( S \) be a scheme. Let \( s \in S \) be a point.

1. An *étale neighbourhood of* \((S, s)\) is a pair \((U, u)\) together with an étale morphism of schemes \( \varphi : U \to S \) such that \( \varphi(u) = s \).
2. A *morphism of étale neighbourhoods* \( f : (V, v) \to (U, u) \) of \((S, s)\) is simply a morphism of \( S \)-schemes \( f : V \to U \) such that \( f(v) = u \).
3. An *elementary étale neighbourhood* is an étale neighbourhood \( \varphi : (U, u) \to (S, s) \) such that \( \kappa(s) = \kappa(u) \).

If \( f : (V, v) \to (U, u) \) is a morphism of étale neighbourhoods, then \( f \) is automatically étale, see Morphisms, Lemma \[34.18\]. Hence it turns \((V, v)\) into an étale neighbourhood of \((U, u)\). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma \[34.3\]) we see that conversely any étale neighbourhood
$(V,v)$ of $(U,u)$ is an étale neighbourhood of $(S,s)$ as well. We also remark that if $U \subset S$ is an open neighbourhood of $s$, then $(U,s) \to (S,s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 34.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(s) \subset \kappa(u)$ is a finite separable extension if $(U,u) \to (S,s)$ is an étale neighbourhood, see Morphisms, Lemma 34.15.

**Lemma 31.2.** Let $S$ be a scheme. Let $s \in S$. Let $\kappa(s) \subset k$ be a finite separable field extension. Then there exists an étale neighbourhood $(U,u) \to (S,s)$ such that the field extension $\kappa(s) \subset \kappa(u)$ is isomorphic to $\kappa(s) \subset k$.

**Proof.** We may assume $S$ is affine. In this case the lemma follows from Algebra, Lemma 141.15. \hfill \square

**Lemma 31.3.** Let $S$ be a scheme, and let $s$ be a point of $S$. The category of étale neighborhoods has the following properties:

1. Let $(U_i,u_i)_{i=1,2}$ be two étale neighborhoods of $s$ in $S$. Then there exists a third étale neighborhood $(U,u)$ and morphisms $(U,u) \to (U_i,u_i)$, $i = 1, 2$.
2. Let $h_1, h_2 : (U,u) \to (U',u')$ be two morphisms between étale neighborhoods of $s$. Assume $h_1, h_2$ induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood $(U'',u'')$ and a morphism $h : (U'',u'') \to (U,u)$ which equalizes $h_1$ and $h_2$, i.e., such that $h_1 \circ h = h_2 \circ h$.

**Proof.** For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both $U_1$ and $U_2$ because étale morphisms are preserved under base change, see Morphisms, Lemma 34.4. There is a point of $U$ mapping to both $u_1$ and $u_2$ for example by the description of points of a fibre product in Schemes, Lemma 17.3.

For part (2), define $U''$ as the fibre product

$$
\begin{array}{ccc}
U'' & \longrightarrow & U \\
\downarrow & & \downarrow \langle h_1, h_2 \rangle \\
U' & \longrightarrow & U' \times_S U'.
\end{array}
$$

Since $h_1$ and $h_2$ induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over $u'$ with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since $U'$ is étale over $S$, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 34.4 and 34.3). Hence the vertical arrow $(h_1, h_2)$ is étale by Morphisms, Lemma 34.18. Therefore $U''$ is étale over $U'$ by base change, and hence also étale over $S$ (because compositions of étale morphisms are étale). Thus $(U'',u'')$ is a solution to the problem. \hfill \square

**Lemma 31.4.** Let $S$ be a scheme, and let $s$ be a point of $S$. The category of elementary étale neighborhoods of $(S,s)$ is cofiltered (see Categories, Definition 20.1).

**Proof.** This is immediate from the definitions and Lemma 31.3. \hfill \square

**Lemma 31.5.** Let $S$ be a scheme. Let $s \in S$. Then we have

$$
\mathcal{O}_{S,s}^h = \text{colim}_{(U,u)} \mathcal{O}(U)
$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods $(U,u)$ of $(S,s)$. 


Let \( \text{Spec}(A) \subset S \) be an affine neighbourhood of \( s \). Let \( p \subset A \) be the prime ideal corresponding to \( s \). With these choices we have canonical isomorphisms \( \mathcal{O}_{S,s} = A_p \) and \( \kappa(s) = \kappa(p) \). A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods \((U, u)\) such that \( U \) is affine and \( U \to S \) factors through \( \text{Spec}(A) \). In other words, we see that the right hand side is equal to \( \text{colim} (B, q) \) where the colimit is over étale \( A \)-algebras \( B \) endowed with a prime \( q \) lying over \( p \) with \( \kappa(p) = \kappa(q) \). Thus the lemma follows from Algebra, Lemma \[\text{Lemma 32.1.}\]

We can lift étale neighbourhoods of points on fibres to the total space.

**Lemma 31.6.** Let \( X \to S \) be a morphism of schemes. Let \( x \in X \) with image \( s \in S \). Let \((V, v) \to (X, x)\) be an étale neighbourhood. Then there exists an étale neighbourhood \((U, u) \to (X, x)\) such that there exists a morphism \((U_s, u) \to (V, v)\) of étale neighbourhoods of \((X, x)\) which is an open immersion.

**Proof.** We may assume \( X, V, \) and \( S \) affine. Say the morphism \( X \to S \) is given by \( A \to B \) the point \( x \) by a prime \( q \subset B \), the point \( s \) by \( p = A \cap q \), and the morphism \( V \to X_s \) by \( B \otimes_A \kappa(p) \to C \). Since \( \kappa(p) \) is a localization of \( A/p \) there exists an \( f \in A \), \( f \not\in p \) and an étale ring map \( B \otimes_A (A/p)_f \to D \) such that

\[
C = (B \otimes_A \kappa(p)) \otimes B \otimes_A (A/p),
\]

See Algebra, Lemma \[\text{Lemma 141.3} \text{ part (9).}\] After replacing \( A \) by \( A_f \) and \( B \) by \( B_f \) we may assume \( D \) is étale over \( B \otimes_A A/p = B/pB \). Then we can apply Algebra, Lemma \[\text{Lemma 141.10}.\] This proves the lemma.

### 32. Étale neighbourhoods and Artin approximation

In this section we prove results of the form: if two pointed schemes have isomorphic complete local rings, then they have isomorphic étale neighbourhoods. We will rely on Popescu’s theorem, see Smoothing Ring Maps, Theorem \[\text{Theorem 12.1}.\]

**Lemma 32.1.** Let \( S \) be a locally Noetherian scheme. Let \( X, Y \) be schemes locally of finite type over \( S \). Let \( x \in X \) and \( y \in Y \) be points lying over the same point \( s \in S \). Assume \( \mathcal{O}_{S,s} \) is a \( G \)-ring. Assume further we are given a local \( \mathcal{O}_{S,s} \)-algebra map \( \varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}^\wedge \) between the complete local rings. For every \( N \geq 1 \) there exists an elementary étale neighbourhood \((U, u) \to (X, x)\) and an \( S \)-morphism \( f : U \to Y \) mapping \( u \) to \( y \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X,x}^\wedge & \to & \mathcal{O}_{U,u}^\wedge \\
\varphi \uparrow & & \uparrow \\
\mathcal{O}_{Y,y} & \to & \mathcal{O}_{U,u}
\end{array}
\]

commutes modulo \( m_y^N \).

**Proof.** The question is local on \( X \) hence we may assume \( X, Y, S \) are affine. Say \( S = \text{Spec}(R) \), \( X = \text{Spec}(A) \), \( Y = \text{Spec}(B) \). Write \( B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \). Let \( p \subset A \) be the prime ideal corresponding to \( x \). The local ring \( \mathcal{O}_{X,x} = A_p \) is a \( G \)-ring by More on Algebra, Proposition \[\text{Proposition 47.10}.\] The map \( \varphi \) is a map

\[
B_q^\wedge \to A_p^\wedge
\]
where $q \subset B$ is the prime corresponding to $y$. Let $a_1, \ldots, a_n \in A_\eta^\wedge$ be the images of $x_1, \ldots, x_n$ via $R[x_1, \ldots, x_n] \to B \to B_q^\wedge \to A_\eta^\wedge$. Then we can apply Smoothing Ring Maps, Lemma 13.4 to get an étale ring map $A \to A'$ and a prime ideal $p' \subset A'$ and $b_1, \ldots, b_n \in A'$ such that $\kappa(p) = \kappa(p')$, $a_i - b_i \in (p')^N(A'_\eta)^\wedge$, and $f_j(b_1, \ldots, b_n) = 0$ for $j = 1, \ldots, n$. This determines an $R$-algebra map $B \to A'$ by sending the class of $x_i$ to $b_i \in A'$. This finishes the proof by taking $U = \text{Spec}(A') \to \text{Spec}(B)$ as the morphism $f$ and $u = p'$.

**Lemma 32.2.** Let $S$ be a locally Noetherian scheme. Let $X, Y$ be schemes locally of finite type over $S$. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $O_{S,s}$ is a G-ring. Assume we have an $O_{S,s}$-algebra isomorphism $\varphi : O_{Y,y}^\wedge \to O_{X,x}^\wedge$ between the complete local rings. Then for every $N \geq 1$ there exists morphisms

$$(X, x) \leftarrow (U, u) \to (Y, y)$$

of pointed schemes over $S$ such that both arrows define elementary étale neighbourhoods and such that the diagram

$$
\begin{array}{ccc}
O_{U,u}^\wedge & \rightarrow & \varphi \downarrow \\
\downarrow & & \downarrow \\
\varphi \downarrow & & \downarrow \\
O_{Y,y}^\wedge & \rightarrow & O_{X,x}^\wedge
\end{array}
$$

commutes modulo $m^N_u$.

**Proof.** We may assume $N \geq 2$. Apply Lemma 32.1 to get $(U, u) \to (X, x)$ and $f : (U, u) \to (Y, y)$. We claim that $f$ is étale at $u$ which will finish the proof. In fact, we will show that the induced map $O_{Y,y}^\wedge \to O_{U,u}^\wedge$ is an isomorphism. Having proved this, Lemma 12.1 will show that $f$ is smooth at $u$ and of course $f$ is unramified at $u$ as well, so Morphisms, Lemma 34.5 tells us $f$ is étale at $u$. For a local ring $(R, m)$ we set $\text{Gr}_m(R) = \bigoplus_{n \geq 0} m^n/m^{n+1}$. To prove the claim we look at the induced diagram of graded rings

$$
\begin{array}{ccc}
\text{Gr}_m(O_{U,u}) & \rightarrow & \varphi \\
\downarrow & & \downarrow \\
\text{Gr}_m(O_{Y,y}) & \rightarrow & \text{Gr}_m(O_{X,x})
\end{array}
$$

Since $N \geq 2$ this diagram is actually commutative as the displayed graded algebras are generated in degree 1! By assumption the lower arrow is an isomorphism. By More on Algebra, Lemma 40.8 (for example) the map $O_{X,x}^\wedge \to O_{U,u}^\wedge$ is an isomorphism and hence the north-west arrow in the diagram is an isomorphism. We conclude that $f$ induces an isomorphism $\text{Gr}_m(O_{X,x}) \to \text{Gr}_m(O_{U,u})$. Using induction and the short exact sequences

$$0 \to \text{Gr}_m^n(R) \to R/m^{n+1} \to R/m^n \to 0$$

for both local rings we conclude (from the snake lemma) that $f$ induces isomorphisms $O_{Y,y}/m^n_y \to O_{U,u}/m^n_u$ for all $n$ which is what we wanted to show. $\square
Lemma 32.3. Consider a diagram

\[
\begin{array}{ccc}
X & \searrow & Y \\
\downarrow & & \downarrow \\
S & \leftarrow & T
\end{array}
\]

with points

\[
\begin{array}{ccc}
x & \searrow & y \\
\downarrow & & \downarrow \\
s & \leftarrow & t
\end{array}
\]

where \( S \) be a locally Noetherian scheme and the morphisms are locally of finite type. Assume \( \mathcal{O}_{S,s} \) is a \( G \)-ring. Assume further we are given a local \( \mathcal{O}_{S,s} \)-algebra map

\[
\sigma : \mathcal{O}_{T,t} \rightarrow \mathcal{O}_{S,s}^\wedge
\]

and a local \( \mathcal{O}_{S,s} \)-algebra map

\[
\varphi : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,\sigma y}^\wedge
\]

where \( Y_\sigma = Y \times_{T,\sigma} \text{Spec}(\mathcal{O}_{S,s}^\wedge) \) and \( y_\sigma \) is the unique point of \( Y_\sigma \) lying over \( y \). For every \( N \geq 1 \) there exists a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & X \times_S V \\
\downarrow & & \downarrow \\
S & \leftarrow & V
\end{array}
\]

of schemes over \( S \) and points \( w \in W, v \in V \) such that

1. \( v \mapsto s, \tau(v) = t, f(w) = (x, v), \) and \( w \mapsto (y, v) \),
2. \( (V, v) \rightarrow (S, s) \) is an elementary étale neighbourhood,
3. the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{S,s}^\wedge & \rightarrow & \mathcal{O}_{V,v}^\wedge \\
\downarrow \sigma & & \downarrow \\
\mathcal{O}_{T,t} \tau^t_v & \rightarrow & \mathcal{O}_{V,v}
\end{array}
\]

commutes module \( m_N^v \),
4. \( (W, w) \rightarrow (Y \times_{T,\tau} V, (y, v)) \) is an elementary étale neighbourhood,
5. the diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X,x} & \rightarrow & \mathcal{O}_{Y_\sigma, y_\sigma}^\wedge \\
\downarrow \varphi & & \downarrow \\
\mathcal{O}_{X, x V, (x,v)} & \rightarrow & \mathcal{O}_{Y_\sigma, y_\sigma} / m_{y_\sigma}^N = \mathcal{O}_{Y, y} / m_{y}^N \cong \mathcal{O}_{X, x V, (x,v)} f_w^T \\
\downarrow & & \downarrow \\
\mathcal{O}_{X, x V, (x,v)} & \rightarrow & \mathcal{O}_{W, w} / m_w^N
\end{array}
\]

commutes. The equality comes from the fact that \( Y_\sigma \) and \( Y \times_{T,\tau} V \) are canonically isomorphic over \( \mathcal{O}_{V,v} / m_{v}^N = \mathcal{O}_{S,s} / m_{s}^N \) by parts (2) and (3).

Proof. After replacing \( X, S, T, Y \) by affine open subschemes we may assume the diagram in the statement of the lemma comes from applying Spec to a diagram

\[
\begin{array}{ccc}
A & \leftarrow & B \\
\uparrow & & \uparrow \\
R & \leftarrow & C
\end{array}
\]

with primes

\[
\begin{array}{ccc}
p_A & \leftarrow & p_B \\
\uparrow & & \uparrow \\
p_R & \leftarrow & p_C
\end{array}
\]

of Noetherian rings and finite type ring maps. In this proof every ring \( E \) will be a Noetherian \( R \)-algebra endowed with a prime ideal \( p_E \) lying over \( p_R \) and all ring
maps will be $R$-algebra maps compatible with the given primes. Moreover, if we write $E^\wedge$ we mean the completion of the localization of $E$ at $p_E$. We will also use without further mention that an étale ring map $E_1 \to E_2$ such that $\kappa(p_{E_1}) = \kappa(p_{E_2})$ induces an isomorphism $E_1^\wedge = E_2^\wedge$ by More on Algebra, Lemma 40.8.

With this notation $\sigma$ and $\varphi$ correspond to ring maps

$$\sigma : C \to R^\wedge \quad \text{and} \quad \varphi : A \to (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

Here is a picture:

![Diagram](image)

Observe that $R^\wedge$ is a G-ring by More on Algebra, Proposition 47.6. Thus $B \otimes_{C,\sigma} R^\wedge$ is a G-ring by More on Algebra, Proposition 47.10. By Lemma 12.1 (translated into algebra) there exists an étale ring map $B \otimes_{C,\sigma} R^\wedge \to B'$ inducing an isomorphism $\kappa(p_{B \otimes_{C,\sigma} R^\wedge}) \to \kappa(p_B)$ and an $R$-algebra map $A \to B'$ such that the composition

$$A \to B' \to (B')^\wedge = (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

is the same as $\varphi$ modulo $(p_{B \otimes_{C,\sigma} R^\wedge})^N$. Thus we may replace $\varphi$ by this composition because the only way $\varphi$ enters the conclusion is via the commutativity requirement in part (5) of the statement of the lemma. Picture:

![Diagram](image)

Next, we use that $R^\wedge$ is a filtered colimit of smooth $R$-algebras (Smoothing Ring Maps, Theorem 12.1) because $R_{p_{R'}}$ is a G-ring by assumption. Since $C$ is of finite presentation over $R$ we get a factorization

$$C \to R' \to R^\wedge$$

for some $R \to R'$ smooth, see Algebra, Lemma 126.3. After increasing $R'$ we may assume there exists an étale $B \otimes_{C} R'$-algebra $B''$ whose base change to $B \otimes_{C,\sigma} R^\wedge$ is $B'$, see Algebra, Lemma 141.3. Then $B'$ is the filtered colimit of these $B''$ and we conclude that after increasing $R'$ we may assume there is an $R$-algebra map $A \to B''$ such that $A \to B'' \to B'$ is the previously constructed map (same
reference as above). Picture

\[
\begin{array}{c}
B'' \ar[r] & B' \ar[r] & (B')^\wedge \\
A \ar[u] \ar[r] & B \otimes_C R' \ar[r] & B \otimes_{C, \sigma} R^\wedge \ar[r] & (B \otimes_{C, \sigma} R^\wedge)^\wedge \\
R \ar[u] \ar[r] & C \ar[r] & R' \ar[r] & R^\wedge \ar[u]
\end{array}
\]

and

\[B' = B'' \otimes_{(B \otimes_C R')^\wedge} (B \otimes_{C, \sigma} R^\wedge)\]

This means that we may replace \(C\) by \(R', \sigma : C \to R^\wedge\) by \(R' \to R^\wedge\), and \(B\) by \(B''\) so that we simplify to the diagram

\[
\begin{array}{c}
A \ar[r] & B \ar[r] & B \otimes_{C, \sigma} R^\wedge \\
R \ar[u] \ar[r] & C \ar[r] & \sigma \ar[r] & R^\wedge \ar[u]
\end{array}
\]

with \(\varphi\) equal to the composition of the horizontal arrows followed by the canonical map from \(B \otimes_{C, \sigma} R^\wedge\) to its completion. The final step in the proof is to apply Lemma 32.1 (or its proof) one more time to \(\text{Spec}(C)\) and \(\text{Spec}(R)\) over \(\text{Spec}(R)\) and the map \(C \to R^\wedge\). The lemma produces a ring map \(C \to D\) such that \(R \to D\) is étale, such that \(\kappa(p_R) = \kappa(p_D)\), and such that

\[C \to D \to D^\wedge = R^\wedge\]

is equal to \(\sigma : C \to R^\wedge\) modulo \((p_{R^\wedge})^N\). Then we can take

\[V = \text{Spec}(D)\quad\text{and}\quad W = \text{Spec}(B \otimes_C D)\]

as our solution to the problem posed by the lemma. Namely the diagram

\[
\begin{array}{c}
A \ar[r] & B \otimes_{C, \sigma} R^\wedge \ar[r] & B \otimes_{C, \sigma} R^\wedge / (p_{R^\wedge})^N \ar[r] & B \otimes_C D / (p_D)^N \\
A \ar@{=}[r] & A \otimes_R D \ar[r] & B \otimes_R D \ar[r] & B \otimes_C D / (p_D)^N
\end{array}
\]

commutes because \(C \to D \to D^\wedge = R^\wedge\) is equal to \(\sigma\) modulo \((p_{R^\wedge})^N\). This proves part (5) and the other properties are immediate from the construction. \(\square\)

**Lemma 32.4.** Let \(T \to S\) be finite type morphisms of Noetherian schemes. Let \(t \in T\) map to \(s \in S\) and let \(\sigma : \mathcal{O}_{T, t} \to \mathcal{O}_{S, s}^\wedge\) be a local \(\mathcal{O}_{S, s}\)-algebra map. For every \(N \geq 1\) there exists a finite type morphism \((T', t') \to (T, t)\) such that \(\sigma\) factors through \(\mathcal{O}_{T, t} \to \mathcal{O}_{T', t'}\) and such that for every local \(\mathcal{O}_{S, s}\)-algebra map \(\sigma' : \mathcal{O}_{T, t} \to \mathcal{O}_{S, s}^\wedge\) which factors through \(\mathcal{O}_{T, t} \to \mathcal{O}_{T', t'}\) the maps \(\sigma\) and \(\sigma'\) agree modulo \(m_s^N\).

**Proof.** We may assume \(S\) and \(T\) are affine. Say \(S = \text{Spec}(R)\) and \(T = \text{Spec}(C)\). Let \(c_1, \ldots, c_n \in C\) be generators of \(C\) as an \(R\)-algebra. Let \(p \subset R\) be the prime ideal corresponding to \(s\). Say \(p = (f_1, \ldots, f_m)\). After replacing \(R\) by a principal
Let which factors through \( O \) with localization (to clear denominators in \( R \)) we may assume there exist \( r_1, \ldots, r_n \in R \) and \( a_{i,j} \in O^{\wedge}_{S,s} \) where \( I = (i_1, \ldots, i_m) \) with \( \sum i_j = N \) such that
\[
\sigma(c_i) = r_i + \sum t_i a_{i,1}f_1^{i_1} \cdots f_m^{i_m}
\]
in \( O^{\wedge}_{S,s} \). Then we consider
\[
C' = C[t_i,1]/\left( c_i - r_i - \sum t_i a_{i,1}f_1^{i_1} \cdots f_m^{i_m} \right)
\]
with \( p' = pC' + (t_i,1) \) and factorization of \( \sigma : C \to O^{\wedge}_{S,s} \) through \( C' \) given by sending \( t_{i,j} \) to \( a_{i,j} \). Taking \( T' = \text{Spec}(C') \) works because any \( \sigma' \) as in the statement of the lemma will send \( c_i \) to \( r_i \) modulo the maximal ideal to the power \( N \). \( \square \)

**Lemma 32.5.** Let \( Y \to T \to S \) be finite type morphisms of Noetherian schemes. Let \( t \in T \) map to \( s \in S \) and let \( \sigma : O_{T,t} \to O^{\wedge}_{S,s} \) be a local \( O_{S,s} \)-algebra map. There exists a finite type morphism \( (T',t') \to (T,t) \) such that \( \sigma \) factors through \( O_{T,t} \to O_{T',t'} \) and such that for every local \( O_{S,s} \)-algebra map \( \sigma' : O_{T,t} \to O^{\wedge}_{S,s} \) which factors through \( O_{T,t} \to O_{T',t'} \) the closed immersions
\[
Y \times_{T,\sigma} \text{Spec}(O^{\wedge}_{S,s}) = Y_s \leftarrow Y_t \to Y_{s'} = Y \times_{T,\sigma'} \text{Spec}(O^{\wedge}_{S,s})
\]
have isomorphic conormal algebras.

**Proof.** A useful observation is that \( \kappa(s) = \kappa(t) \) by the existence of \( \sigma \). Observe that the statement makes sense as the fibres of \( Y_s \) and \( Y_{s'} \) over \( s \in \text{Spec}(O^{\wedge}_{S,s}) \) are both canonically isomorphic to \( Y_t \). We will think of the property “\( \sigma' \) factors through \( O_{T,t} \to O_{T',t'} \)” as a constraint on \( \sigma' \). If we have several such constraints, say given by \( (T_i',t_i') \to (T,t), i = 1, \ldots, n \) then we can combined them by considering \( (T_1' \times_T \cdots \times_T T_n', (t_1', \ldots, t_n')) \to (T,t) \). We will use this without further mention in the following.

By Lemma 32.4 we can assume that any \( \sigma' \) as in the statement of the lemma is the same as \( \sigma \) modulo \( m_s^2 \). Note that the conormal algebra of \( Y_t \) in \( Y_s \) is just the quasi-coherent graded \( O_{Y_t} \)-algebra
\[
\bigoplus_{n \geq 0} m_s^n O_{Y_t}/m_s^{n+1} O_{Y_s}
\]
and similarly for \( Y_{s'} \). Since \( \sigma \) and \( \sigma' \) agree modulo \( m_s^2 \) we see that these two algebras are the same in degrees 0 and 1. On the other hand, these conormal algebras are generated in degree 1 over degree 0. Hence if there is an isomorphism extending the isomorphism just constructed in degrees 0 and 1, then it is unique.

We may assume \( S \) and \( T \) are affine. Let \( Y = Y_1 \cup \cdots \cup Y_n \) be an affine open covering. If we can construct \( (T_i',t_i') \to (T,t) \) as in the lemma such that the desired isomorphism (see previous paragraph) exists for \( Y_i \to T \to S \) and \( \sigma \), then these glue by uniqueness to prove the result for \( Y \to T \). Thus we may assume \( Y \) is affine.

Write \( S = \text{Spec}(R), T = \text{Spec}(C), \) and \( Y = \text{Spec}(B) \). Choose a presentation \( B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \). Denote \( R^{\wedge} = O^{\wedge}_{S,s} \). Let \( a_{k,j} \in R^{\wedge}[x_1, \ldots, x_n] \) be polynomials such that
\[
\sum_{j=1}^m a_{k,j} \sigma(f_j) = 0, \quad \text{for } k = 1, \ldots, K
\]
is a set of generators for the module of relations among the \( \sigma(f_j) \in R^\wedge[x_1, \ldots, x_n] \). Thus we have an exact sequence

\[(32.5.1) \quad R^\wedge[x_1, \ldots, x_n]^\oplus K \to R^\wedge[x_1, \ldots, x_n]^\oplus m \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C, \sigma} R^\wedge \to 0\]

Let \( c \) be an integer which works in the Artin-Rees lemma for both the first and the second map in this sequence and the ideal \( \mathfrak{m}_{R^\wedge} R^\wedge[x_1, \ldots, x_n] \) as defined in More on Algebra, Section \[4\].

Write

\[ a_{kj} = \sum_{l \in \Omega} a_{kj,l} x^l \quad \text{and} \quad f_j = \sum_{l \in \Omega} f_j,l x^l \]

in multiindex notation where \( a_{kj,l} \in R^\wedge \), \( f_j,l \in C \), and \( \Omega \) a finite set of multiindices. Then we see that

\[ \sum_{j=1, \ldots, m} \sum_{l \in \Omega, i+l'=l''} a_{kj,l} \sigma(f_{j,l''}) = 0, \quad l'' \text{ a multiindex} \]

in \( R^\wedge \). Thus we take

\[ C' = C[t_{jk,l}] / \left( \sum_{j=1, \ldots, m} \sum_{l \in \Omega, i+l'=l''} t_{kj,l} f_{j,l''}, \quad l'' \text{ a multiindex} \right) \]

Then \( \sigma \) factors through a map \( \tilde{\sigma} : C' \to R^\wedge \) sending \( t_{kj,l} \) to \( a_{kj,l} \). Thus \( T' = \text{Spec}(C') \) comes with a point \( t' \in T' \) such that \( \sigma \) factors through \( O_{T,t} \to O_{T',t'} \).

Let \( t_{kj} = \sum t_{kj,l} x^l \) in \( C'[x_1, \ldots, x_n] \). Then we see that we have a complex

\[(32.5.2) \quad C'[x_1, \ldots, x_n]^\oplus K \to C'[x_1, \ldots, x_n]^\oplus m \to C'[x_1, \ldots, x_n] \to B \otimes_{C} C' \to 0\]

which is exact at \( C'[x_1, \ldots, x_n] \) and whose base change by \( \tilde{\sigma} \) gives \((32.5.1)\).

By Lemma \([32.4]\) we can find a further morphism \((T'', t'') \to (T', t')\) such that \( \tilde{\sigma} \) factors through \( O_{T'', t''} \to O_{T', t'} \) and such that if \( \sigma' : C \to R^\wedge \) factors through \( O_{T'', t''} \), then the induced map \( \tilde{\sigma}' : C' \to R^\wedge \) agrees modulo \( \mathfrak{m}_s^{n+1} \) with \( \tilde{\sigma} \). Thus if \( \sigma' \) is such a map, then we obtain a complex

\[ R^\wedge[x_1, \ldots, x_n]^\oplus K \to R^\wedge[x_1, \ldots, x_n]^\oplus m \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C, \sigma'} R^\wedge \to 0 \]

over \( R^\wedge[x_1, \ldots, x_n] \) by applying \( \tilde{\sigma}' \) to the polynomials \( t_{kj} \) and \( f_j \). In other words, this is the base change of the complex \((32.5.2)\) by \( \tilde{\sigma}' \). The matrices defining this complex are congruent modulo \( \mathfrak{m}_s^{n+1} \) to the matrices defining the complex \((32.5.1)\) because \( \tilde{\sigma} \) and \( \tilde{\sigma}' \) are congruent modulo \( \mathfrak{m}_s^{n+1} \). Since \((32.5.1)\) is exact, we can apply More on Algebra, Lemma \([4.2]\) to conclude that

\[ \text{Gr}_{\mathfrak{m}_s}(B \otimes_{C, \sigma'} R^\wedge) \cong \text{Gr}_{\mathfrak{m}_s}(B \otimes_{C, \sigma} R^\wedge) \]

as desired. \( \square \)

**Lemma 32.6.** With notation as in Lemma \([32.3]\) assume that \( \varphi \) induces an isomorphism on completions. Then we can choose our diagram such that \( f \) is étale.

**Proof.** We may assume \( N \geq 2 \) and we may replace \((T, t)\) with \((T', t')\) as in Lemma \([32.5]\). Since \((V, v) \to (S, s)\) is an elementary étale neighbourhood, so is \((X \times_S V, (x, v)) \to (X, x)\). Thus \( \mathcal{O}_{X,x} \to \mathcal{O}_{X \times_S V, (x, v)} \) induces an isomorphism on completions by More on Algebra, Lemma \([40.8]\). We claim \( \mathcal{O}_{X,x} \to \mathcal{O}_{W,w} \) induces an isomorphism on completions. Having proved this, Lemma \([12.1]\) will show that \( f \) is smooth at \( w \) and of course \( f \) is unramified at \( u \) as well, so Morphisms, Lemma \([34.3]\) tells us \( f \) is étale at \( w \).
First we use the commutativity in part (5) of Lemma \[32.3\] to see that for \(i \leq N\) there is a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_m^i(O_{X,x}) & \overset{\varphi}{\longrightarrow} & \text{Gr}_m^i(O_{Y^T,T_v,(y,v)}) \\
\approx & & \approx \\
\text{Gr}_m^i(O_{X,x}) & \overset{\simeq}{\longrightarrow} & \text{Gr}_m^i(O_{X,T_v,\tilde{v}}) \\
\end{array}
\]

This implies that \(f_w^T\) defines an isomorphism \(\kappa(x) \rightarrow \kappa(w)\) on residue fields and an isomorphism \(m_x/m_x^2 \rightarrow m_w/m_w^2\) on cotangent spaces. Hence \(f_w^T\) defines a surjection \(O_{X,x}^\wedge \rightarrow O_{W,w}^\wedge\) on complete local rings.

By Lemma \[32.5\] there is an isomorphism of \(\text{Gr}_m^i(O_{Y^T,T_v,(y,v)})\) with \(\text{Gr}_m^i(O_{X,T_v,\tilde{v}})\). This follows by taking stalks of the isomorphism of conormal sheaves at the point \(y\). Since our local rings are Noetherian taking associated graded with respect to \(m_s\) commutes with completion because completion with respect to an ideal is an exact functor on finite modules over Noetherian rings. This produces the right vertical isomorphism in the diagram of graded rings

\[
\begin{array}{ccc}
\text{Gr}_m^i(O_{W,w}^\wedge) & \overset{\varphi}{\longrightarrow} & \text{Gr}_m^i(O_{Y^T,T_v,(y,v)}) \\
\approx & & \approx \\
\text{Gr}_m^i(O_{X,x}^\wedge) & \overset{\varphi}{\longrightarrow} & \text{Gr}_m^i(O_{Y^T,T_v,(y,v)}) \\
\end{array}
\]

We do not claim the diagram commutes. By the result of the previous paragraph the left arrow is surjective. The other three arrows are isomorphisms. It follows that the left arrow is a surjective map between isomorphic Noetherian rings. Hence it is an isomorphism by Algebra, Lemma \[30.10\] (you can argue this directly using Hilbert functions as well). In particular \(O_{X,x}^\wedge \rightarrow O_{W,w}^\wedge\) must be injective as well as surjective which finishes the proof. \(\square\)

### 33. Étale neighbourhoods and branches

The number of (geometric) branches of a scheme at a point was defined in Properties, Section \[15\]. In Varieties, Section \[39\] we related this to fibres of the normalization morphism. In this section we discuss a characterization of this number in terms of étale neighbourhoods.

**Lemma 33.1.** Let \(R = \text{colim} R_i\) be colimit of a directed system of rings whose transition maps are faithfully flat. Then the number of minimal primes of \(R\) taken as an element of \(\{0, 1, 2, \ldots, \infty\}\) is the supremum of the numbers of minimal primes of the \(R_i\).

**Proof.** If \(A \rightarrow B\) is a flat ring map, then \(\text{Spec}(B) \rightarrow \text{Spec}(A)\) maps minimal primes to minimal primes by going down (Algebra, Lemma \[38.18\]). If \(A \rightarrow B\) is faithfully flat, then every minimal prime is the image of a minimal prime (by Algebra, Lemma \[38.16\] and \[29.7\]). Hence the number of minimal primes of \(R_i\) is \(\geq\) the number of minimal primes of \(R_{i'}\) if \(i \leq i'\). By Algebra, Lemma \[38.20\] each of the maps \(R_i \rightarrow R\) is faithfully flat and we also see that the number of minimal primes of \(R\) is \(\geq\) the number of minimal primes of \(R_i\). Finally, suppose that \(q_1, \ldots, q_n\) are pairwise distinct minimal primes of \(R\). Then we can find an
Let \( R_i \cap q_1, \ldots, R_i \cap q_n \) are pairwise distinct (as sets and hence as prime ideals). This implies the lemma.

**Lemma 33.2.** Let \( X \) be a scheme and \( x \in X \) a point. Then

1. the number of branches of \( X \) at \( x \) is equal to the supremum of the number of irreducible components of \( U \) passing through \( u \) taken over elementary étale neighbourhoods \((U, u) \to (X, x)\),
2. the number of geometric branches of \( X \) at \( x \) is equal to the supremum of the number of irreducible components of \( U \) passing through \( u \) taken over étale neighbourhoods \((U, u) \to (X, x)\),
3. \( X \) is unibranch at \( x \) if and only if for every elementary étale neighbourhood \((U, u) \to (X, x)\) there is exactly one irreducible component of \( U \) passing through \( u \), and
4. \( X \) is geometrically unibranch at \( x \) if and only if for every étale neighbourhood \((U, u) \to (X, x)\) there is exactly one irreducible component of \( U \) passing through \( u \).

**Proof.** Parts (3) and (4) follow from parts (1) and (2) via Properties, Lemma 15.6

Proof of (1). Let \( \text{Spec}(A) \) be an affine open neighbourhood of \( x \) and let \( p \subset A \) be the prime ideal corresponding to \( x \). We may replace \( X \) by \( \text{Spec}(A) \) and it suffices to consider affine elementary étale neighbourhoods \((U, u)\) in the supremum as they form a cofinal subsystem. Recall that the henselization \( A_p^{h} \) is the colimit of the rings \( B_q \) over the category of pairs \((B, q)\) where \( B \) is an étale \( A \)-algebra and \( q \) is a prime lying over \( p \) with \( \kappa(q) = \kappa(p) \), see Algebra, Lemma 150.7 These pairs \((B, q)\) correspond exactly to the affine elementary étale neighbourhoods \((U, u)\) by the correspondence between rings and affine schemes. Observe that irreducible components of \( \text{Spec}(B) \) passing through \( q \) are exactly the minimal prime ideals of \( B_q \). The number of minimal primes of \( A_p^{h} \) is the number of branches of \( X \) at \( x \) by Properties, Definition 15.4. Observe that the transition maps \( B_q \to B'_q \) in the system are all flat. Since a flat local ring map is faithfully flat (Algebra, Lemma 38.17) we see that the lemma follows from Lemma 33.1

Proof of (2). The proof is the same as the proof of (1), except that we use Algebra, Lemma 150.13. There is a tiny difference: given a separable algebraic closure \( \kappa^{sep} \) of \( \kappa(x) \) for every étale neighbourhood \((U, u)\) we can choose a \( \kappa(x) \)-embedding \( \phi: \kappa(u) \to \kappa^{sep} \) because \( \kappa(u)/\kappa(x) \) is finite separable (Morphisms, Lemma 34.15). Hence we can look at the supremum over all triples \((U, u, \phi)\) where \((U, u) \to (X, x)\) is an affine étale neighbourhood and \( \phi: \kappa(u) \to \kappa^{sep} \) is a \( \kappa(x) \)-embedding. These triples correspond exactly to the triples in Algebra, Lemma 150.13 and the rest of the proof is exactly the same.

We will need a relative variant of the lemma above.

**Lemma 33.3.** Let \( X \to S \) be a morphism of schemes and \( x \in X \) a point with image \( s \). Then

1. the number of branches of the fibre \( X_s \) at \( x \) is equal to the supremum of the number of irreducible components of the fibre \( U_s \) passing through \( u \) taken over elementary étale neighbourhoods \((U, u) \to (X, x)\),
2. the number of geometric branches of the fibre \( X_s \) at \( x \) is equal to the supremum of the number of irreducible components of the fibre \( U_s \) passing through \( u \) taken over étale neighbourhoods \((U, u) \to (X, x)\).
the fibre $X_s$ is unibranch at $x$ if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of the fibre $U_s$ passing through $u$, and

(4) $X$ is geometrically unibranch at $x$ if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of $U_s$ passing through $u$.

Proof. Combine Lemmas 33.2 and 31.6.

Lemma 33.4. Let $X \to S$ be a smooth morphism of schemes. Let $x \in X$ with image $s \in S$. Then

(1) The number of geometric branches of $X$ at $x$ is equal to the number of geometric branches of $S$ at $s$.

(2) If $\kappa(x)/\kappa(s)$ is a purely inseparable extension of fields, then number of branches of $X$ at $x$ is equal to the number of branches of $S$ at $s$.

Proof. Follows immediately from More on Algebra, Lemma 89.8 and the definitions.

34. Slicing smooth morphisms

In this section we explain a result that roughly states that smooth coverings of a scheme $S$ can be refined by étale coverings. The technique to prove this relies on a slicing argument.

Lemma 34.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume

(1) $f$ is smooth at $x$, and

(2) the image $\overline{dh}$ of $dh$ in

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ such that $h$ comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in $U$ with $x \in D$ and $D \to S$ smooth.

Proof. As $f$ is smooth at $x$ we may assume, after replacing $X$ by an open neighbourhood of $x$ that $f$ is smooth. In particular we see that $f$ is flat and locally of finite presentation. By Lemma 21.1 we already know there exists an open neighbourhood $U \subset X$ of $x$ such that $h$ comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in $U$ with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 31.15 we have a short exact sequence

$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$

where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of $D$ in $U$. As $D$ is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps

\[ \text{In fact, it would suffice if } \kappa(x) \text{ is geometrically irreducible over } \kappa(s). \] If we ever need this we will add a detailed proof.
the free generator \( h \) to the section \( dh|_D \) of \( i^*\Omega_{U/S} \) which has nonzero value in the fibre \( \Omega_{U/S,x} \otimes \kappa(x) \) by assumption. By right exactness of \( \otimes \kappa(x) \) we conclude that
\[
\dim_{\kappa(x)}(\Omega_{D/S,x} \otimes \kappa(x)) = \dim_{\kappa(x)}(\Omega_{U/S,x} \otimes \kappa(x)) - 1.
\]

By Morphisms, Lemma \[32.14\] we see that \( \Omega_{U/S,x} \otimes \kappa(x) \) can be generated by at most \( \dim_x(U_s) \) elements. By the displayed formula we see that \( \Omega_{D/S,x} \otimes \kappa(x) \) can be generated by at most \( \dim_x(U_s) - 1 \) elements. Note that \( \dim_x(D_s) = \dim_x(U_s) - 1 \) for example because \( \dim(\mathcal{O}_{D,x}) = \dim(\mathcal{O}_{U,x}) - 1 \) by Algebra, Lemma \[59.12\] (also \( D_s \subset U_s \) is effective Cartier, see Divisors, Lemma \[18.1\]) and then using Morphisms, Lemma \[27.1\]. Thus we conclude that \( \Omega_{D/S,x} \otimes \kappa(x) \) can be generated by at most \( \dim_x(D_s) \) elements and we conclude that \( D \to S \) is smooth at \( x \) by Morphisms, Lemma \[32.14\] again. After shrinking \( U \) we get that \( D \to S \) is smooth and we win. \( \square \)

\textbf{Lemma 34.2.} Let \( f : X \to S \) be a morphism of schemes. Let \( x \in X \) be a point with image \( s \in S \). Assume

1. \( f \) is smooth at \( x \), and
2. the map
\[
\Omega_{X,x/s} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}
\]
has a nonzero kernel.

Then there exists an affine open neighbourhood \( U \subset X \) of \( x \) and an effective Cartier divisor \( D \subset U \) containing \( x \) such that \( D \to S \) is smooth.

\textbf{Proof.} Write \( k = \kappa(s) \) and \( R = \mathcal{O}_{X,x} \). Denote \( m \) the maximal ideal of \( R \) and \( \kappa = R/m \) so that \( \kappa = \kappa(x) \). As formation of modules of differentials commutes with localization (see Algebra, Lemma \[130.8\]) we have \( \Omega_{X,x/s} = \Omega_R/k \). By Algebra, Lemma \[27.1\], there is an exact sequence
\[
m/m^2 \to \Omega_{R/k} \otimes \kappa \to \Omega_{\kappa/k} \to 0.
\]

Hence if (2) holds, there exists an element \( \overline{h} \in m \) such that \( d\overline{h} \) is nonzero. Choose a lift \( h \in \mathcal{O}_{X,x} \) of \( \overline{h} \) and apply Lemma \[34.1\]. \( \square \)

\textbf{Remark 34.3.} The second condition in Lemma \[34.2\] is necessary even if \( x \) is a closed point of a positive dimensional fibre. An example is the following: Let \( k \) be a field of characteristic \( p \neq 0 \) which is imperfect. Let \( a \in k \) be an element which is not a \( p \)th power. Let \( m = (x, y^p - a) \subset k[x, y] \). This corresponds to a closed point \( w \) of \( X = \mathbb{A}^2_k \). Set \( S = \mathbb{A}^1_k \) and let \( f : X \to S \) be the morphism corresponding to \( k[x] \to k[x, y] \). Then there does not exist any commutative diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & X \\
\uparrow & \downarrow & \downarrow \text{f} \\
S & \downarrow \text{g} & \\
& & \\
\end{array}
\]

with \( g \) étale and \( w \) in the image of \( h \). This is clear as the residue field extension \( \kappa(f(w)) \subset \kappa(w) \) is purely inseparable, but for any \( s' \in S' \) with \( g(s') = f(w) \) the extension \( \kappa(f(w)) \subset \kappa(s') \) would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark \[34.3\] does not happen. Here is the precise result.
Lemma 34.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is smooth at $x$,
2. the residue field extension $\kappa(s) \subset \kappa(x)$ is separable, and
3. $x$ is not a generic point of $X_s$.

Then there exists an affine open neighbourhood $U \subset X$ of $x$ and an effective Cartier divisor $D \subset U$ containing $x$ such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X,s}$. Denote $m$ the maximal ideal of $R$ and $\kappa = R/m$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 130.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 138.4 the map

$$d : m/m^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa$$

is injective. Assumption (3) implies that $m/m^2 \neq 0$. Thus there exists an element $\tilde{h} \in m$ such that $d\tilde{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\tilde{h}$ and apply Lemma 34.1. $\square$

The subscheme $Z$ constructed in the following lemma is really a complete intersection in an affine open neighbourhood of $x$. If we ever need this we will explicitly formulate a separate lemma stating this fact.

Lemma 34.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

1. $f$ is smooth at $x$, and
2. $x$ is a closed point of $X_s$ and $\kappa(s) \subset \kappa(x)$ is separable.

Then there exists an immersion $Z \to X$ containing $x$ such that

1. $Z \to S$ is étale, and
2. $Z_s = \{x\}$ set theoretically.

Proof. We may and do replace $S$ by an affine open neighbourhood of $s$. We may and do replace $X$ by an affine open neighbourhood of $x$ such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take $Z$ to be an open neighbourhood of $x$. Namely, if $d = 0$, then $X \to S$ is quasi-finite at $x$, see Morphisms, Lemma 28.3. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 52.2. Thus after replacing $X$ by $U$ we see that $X$ is quasi-finite and smooth over $S$, hence smooth of relative dimension $0$ over $S$, hence étale over $S$. Moreover, the fibre $X_s$ is a finite discrete set. Hence after replacing $X$ by a further affine open neighbourhood of $X$ we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on $X_s$ is induced from the topology on $X$, see Schemes, Lemma 18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because $x$ is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 19.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions.
Thus we may apply Lemma 34.2 to find a diagram

\[
\begin{array}{ccc}
D & \xrightarrow{u} & U \\
& & \downarrow \pi \\
S & \xrightarrow{f} & X
\end{array}
\]

with \(x \in D\). Note that \(\dim_x(D_s) = \dim_x(X_s) - 1\) for example because \(\dim(O_{D_s,x}) = \dim(O_{X_s,x}) - 1\) by Algebra, Lemma 39.12 (also \(D_s \subset X_s\) is effective Cartier, see Divisors, Lemma 18.1) and then using Morphisms, Lemma 27.1. Thus the morphism \(D \to S\) is smooth with \(\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1\). By induction hypothesis we can find an immersion \(Z \to D\) as desired, which finishes the proof. \(\square\)

**Lemma 34.6.** Let \(f : X \to S\) be a smooth morphism of schemes. Let \(s \in S\) be a point in the image of \(f\). Then there exists an étale neighbourhood \((S', s') \to (S, s)\) and a \(S\)-morphism \(S' \to X\).

**First proof of Lemma 34.6** By assumption \(X_s \neq \emptyset\). By Varieties, Lemma 25.6 there exists a closed point \(x \in X_s\) such that \(\kappa(x)\) is a finite separable field extension of \(\kappa(s)\). Hence by Lemma 34.5 there exists an immersion \(Z \to S\) such that \(Z \to S\) is étale and such that \(x \in Z\). Take \((S', s') = (Z, x)\). \(\square\)

**Second proof of Lemma 34.6** Pick a point \(x \in X\) with \(f(x) = s\). Choose a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & A^d_s \\
\downarrow & & \downarrow \\
Y & \xleftarrow{v} & V
\end{array}
\]

with \(\pi\) étale, \(x \in U\) and \(V = \text{Spec}(R)\) affine, see Morphisms, Lemma 34.20. In particular \(s \in V\). The morphism \(\pi : U \to A^d_s\) is open, see Morphisms, Lemma 34.13. Thus \(W = \pi(V) \cap A^d_s\) is a nonempty open subset of \(A^d_s\). Let \(w \in W\) be a point with \(\kappa(s) \subseteq \kappa(w)\) finite separable, see Varieties, Lemma 25.5. By Algebra, Lemma 113.1 there exist \(d\) elements \(f_1, \ldots, f_d \in \kappa(s)[x_1, \ldots, x_d]\) which generate the maximal ideal corresponding to \(w\) in \(\kappa(s)[x_1, \ldots, x_d]\). After replacing \(R\) by a principal localization we may assume there are \(f_1, \ldots, f_d \in R[x_1, \ldots, x_d]\) which map to \(f_1, \ldots, f_d \in \kappa(s)[x_1, \ldots, x_d]\). Consider the \(R\)-algebra

\[
R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)
\]

and set \(S' = \text{Spec}(R')\). By construction we have a closed immersion \(j : S' \to A^d_s\) over \(V\). By construction the fibre of \(S' \to V\) over \(s\) is a single point \(s'\) whose residue field is finite separable over \(\kappa(s)\). Let \(q' \subset R'\) be the corresponding prime. By Algebra, Lemma 134.11 we see that \((R'_{q'})\) is a relative global complete intersection over \(R\) for some \(g \in R', g \notin q\). Thus \(S' \to V\) is flat and of finite presentation in a neighbourhood of \(s'\), see Algebra, Lemma 134.14. By construction the scheme theoretic fibre of \(S' \to V\) over \(s\) is \(\text{Spec}(\kappa(s'))\). Hence it follows from Morphisms, Lemma 34.15 that \(S' \to S\) is étale at \(s'\). Set

\[
S'' = U \times_{\pi, A^d_s, j} S'.
\]

By construction there exists a point \(s'' \in S'' \) which maps to \(s'\) via the projection \(p : S'' \to S'\). Note that \(p\) is étale as the base change of the étale morphism \(\pi\), see Morphisms, Lemma 34.4. Choose a small affine neighbourhood \(S''' \subset S''\) of \(s''\)
which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. □

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

**Lemma 34.7.** Let $S$ be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be a smooth covering of $S$, see Topologies, Definition 5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ (see Topologies, Definition 4.1) which refines (see Sites, Definition 8.1) $\mathcal{U}$.

**Proof.** For every $s \in S$ there exists an $i \in I$ such that $s$ is in the image of $S_i \to S$. By Lemma 34.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ and such that $g_s$ factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of $S$ that refines $\mathcal{U}$. □

### 35. Finite free locally dominates étale

**Lemma 35.1.** Let $S$ be a scheme. Let $s \in S$. Let $f : (U, u) \to (S, s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subseteq S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subseteq T$ and a commutative diagram

$$
\begin{array}{c}
T \\
\downarrow \pi \\
W_t \\
\downarrow h_t \\
V \\
\downarrow \\
S
\end{array}
$$

with $h_t(t) = u$.

**Proof.** The problem is local on $S$ hence we may replace $S$ by any open neighbourhood of $s$. We may also replace $U$ by an open neighbourhood of $u$. Hence, by Morphisms, Lemma 34.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 141.17. □

**Lemma 35.2.** Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:

$$
\begin{array}{c}
\bigsqcup T_i \\
\downarrow \pi \\
T \\
\downarrow f \\
S
\end{array}
$$

where the south-west arrow is a Zariski-covering.

**Proof.** This is a restatement of Algebra, Lemma 141.18. □
Remark 35.3. In terms of topologies Lemmas 35.1 and 35.2 mean the following. Let $S$ be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of $S$. There exists a Zariski open covering $S = \bigcup V_j$, for each $j$ a finite locally free, surjective morphism $W_j \to V_j$, and for each $j$ a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with $\pi$ finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\text{Br}(X) = \text{Br}'(X)$ for an affine scheme $X$, see [Hoo82].

36. Étale localization of quasi-finite morphisms

04HF Now we come to a series of lemmas around the theme “quasi-finite morphisms become finite after étale localization”. The general idea is the following. Suppose given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of $s$ in $S$. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

$$
\begin{array}{ccc}
V & \longrightarrow & X_U \\
\downarrow & & \downarrow f \\
U & \varphi & \longrightarrow S
\end{array}
$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens $V$? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

Lemma 36.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that

(1) $f$ is locally of finite type, and
(2) $x \in X_s$ is isolated.

Then there exist

(a) an elementary étale neighbourhood $(U, u) \to (S, s)$,
(b) an open subscheme $V \subset X_U$ (see 36.0.1)

such that

(i) $V \to U$ is a finite morphism,
(ii) there is a unique point $v$ of $V$ mapping to $u$ in $U$, and
(iii) the point $v$ maps to $x$ under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$, the triple $(U', u', V')$ satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that $x$ is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for $f$. Hence we reduce to the case where $f$ is a morphism of affine schemes. This case is Algebra, Lemma 141.21. □

---

6In the presence of (1) this means that $f$ is quasi-finite at $x$, see Morphisms, Lemma 19.6
In the preceding and following lemma we do not assume that the morphism $f$ is separated. This means that the opens $V$, $V_i$ created in them are not necessarily closed in $X_U$. Moreover, if we choose the neighbourhood $U$ to be affine, then each $V_i$ is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

02LL **Lemma 36.2.** Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image $s$ in $S$. Assume that

1. $f$ is locally of finite type, and
2. $x_i \in X_s$ is isolated for $i = 1, \ldots, n$.

Then there exist

(a) an elementary étale neighbourhood $(U, u) \to (S, s)$,

(b) for each $i$ an open subscheme $V_i \subset X_U$,

such that for each $i$ we have

(i) $V_i \to U$ is a finite morphism,

(ii) there is a unique point $v_i$ of $V_i$ mapping to $u$ in $U$, and

(iii) the point $v_i$ maps to $x_i$ in $X$ and $\kappa(x_i) = \kappa(v_i)$.

**Proof.** We will use induction on $n$. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also $x'_n$ is isolated in $X_u$. Hence by Lemma 36.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for $x'_n$ and hence for $x_n$. By the final assertion of Lemma 36.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. □

If we allow a nontrivial field extension $\kappa(s) \subset \kappa(u)$, i.e., general étale neighbourhoods, then we can split the points as follows.

02LM **Lemma 36.3.** Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image $s$ in $S$. Assume that

1. $f$ is locally of finite type, and
2. $x_i \in X_s$ is isolated for $i = 1, \ldots, n$.

Then there exist

(a) an étale neighbourhood $(U, u) \to (S, s)$,

(b) for each $i$ an integer $m_i$ and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$

such that we have

(i) each $V_{i,j} \to U$ is a finite morphism,

(ii) there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to $u$ in $U$ with $\kappa(u) \subset \kappa(v_{i,j})$

finite purely inseparable,

(iv) if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and

(iii) the points $v_{i,j}$ map to $x_i$ in $X$ and no other points of $(X_U)_u$ map to $x_i$.

**Proof.** This proof is a variant of the proof of Algebra 141.23 in the language of schemes. By Morphisms, Lemma 19.6 the morphism $f$ is quasi-finite at each of the points $x_i$. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each $i$ (Morphisms, Lemma 19.5). For each $i$, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension $\kappa(s) \subset L$
such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 31.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of $X_U$ lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of $L$ (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair $i, j$. Also, by Morphisms, Lemma 19.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all $i, j$.

Apply Lemma 36.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u') = \kappa(u)$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. \qed

\begin{Lemma} \label{lemma-02LN}
Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

\begin{enumerate}
\item $f$ is locally of finite type,
\item $f$ is separated, and
\item $x_1, \ldots, x_n$ are pairwise distinct isolated points of $X_s$.
\end{enumerate}

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V_1 \amalg \ldots \amalg V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over $u$ are singletons $\{v_i\}$, each $v_i$ maps to $x_i$ with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over $u$ contains no points mapping to any of the $x_i$.

\end{Lemma}

\begin{proof}
Choose $(U, u) \to (S, s)$ and $V_i \subset X_{U'}$ as in Lemma 36.2 Since $X_U \to U$ is separated (Schemes, Lemma 21.13) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 42.10) we see that $V_i \subset X_{U'}$ is closed by Morphisms, Lemma 39.7. Hence $V_i \cap V_j$ is a closed subset of $V_i$ which does not contain $v_i$. Hence the image of $V_i \cap V_j$ in $U$ is a closed set (because $V_i \to U$ proper) not containing $u$. After shrinking $U$ we may therefore assume that $V_i \cap V_j = \emptyset$ for all $i, j$. This gives the decomposition as in the lemma. \qed

Here is the variant where we reduce to purely inseparable field extensions.

\begin{Lemma} \label{lemma-02LO}
Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

\begin{enumerate}
\item $f$ is locally of finite type,
\item $f$ is separated, and
\item $x_1, \ldots, x_n$ are pairwise distinct isolated points of $X_s$.
\end{enumerate}

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg \coprod_{i=1}^n \coprod_{j=1}^{m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over $u$ are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to $x_i$, $\kappa(u) \subset \kappa(v_{i,j})$
Lemma 36.6. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

1. $f$ is locally of finite type,
2. $f$ is separated, and
3. $X_s$ has at most finitely many isolated points.

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre $W_u$ of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 36.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of $X_s$ and setting $V = \bigcup V_i$. 

37. Étale localization of integral morphisms

Some variants of the results of Section 36 for the case of integral morphisms.
$R' \to B'$ is finite (as $R \to S'$ is finite) we can after localizing $R'$ at some $g' \in R'$, $g' \not\in p'$ assume that $B' = 0$. Via the map $S' \otimes_R R' \to S \otimes_R R'$ we get the corresponding decomposition for $S$.

\begin{proof}
This proof uses strict henselization. First, assume $R$ is strictly henselization with maximal ideal $p$. Then $S/pS$ has finitely many primes corresponding to $q_1, \ldots, q_n$, each maximal, each with purely inseparable residue field over $\kappa(p)$. Hence $S/pS$ is equal to $\prod (S/pS)_{p_i}$. By More on Algebra, Lemma 10.8 we can lift this product decomposition to a product composition of $S$ as in the statement.

In the general case, let $R^{sh}$ be the strict henselization of $R$. Then we can apply the result of the first paragraph to $R^{sh} \to S \otimes_R R^{sh}$.

Consider the $m$ mutually orthogonal idempotents in $S \otimes_R R^{sh}$ corresponding to the product decomposition. Since $R^{sh}$ is a filtered colimit of étale ring maps $(R, p) \to (R', p')$ by Algebra, Lemma 150.13 we see that these idempotents descend to some $R'$ as desired.
\end{proof}

### 38. Zariski’s Main Theorem

In this section we prove Zariski’s main theorem as reformulated by Grothendieck.

Often when we say “Zariski’s main theorem” in this content we mean either of Lemma 38.1, Lemma 38.2, or Lemma 38.3. In most texts people refer to the last of these as Zariski’s main theorem. These lemmas have many consequences some of which the reader can find in this section.

We have already proved the algebraic version in Algebra, Theorem 122.13 and we have already restated this algebraic version in the language of schemes, see Morphisms, Theorem 52.1. The version in this section is more subtle; to get the full result we use the étale localization techniques of Section 36 to reduce to the algebraic case.

\begin{lemma}
Let $f : X \to S$ be a morphism of schemes. Assume $f$ is of finite type and separated. Let $S'$ be the normalization of $S$ in $X$, see Morphisms, Definition 50.3. Picture:

\[
\begin{tikzcd}
X \ar{dr}{f'} \ar{rr}{f} & & S' \\
& S \ar{ur}{\nu}
\end{tikzcd}
\]

Then there exists an open subscheme $U' \subset S'$ such that

1. $(f')^{-1}(U') \to U'$ is an isomorphism, and
2. $(f')^{-1}(U') \subset X$ is the set of points at which $f$ is quasi-finite.

\end{lemma}

\begin{proof}
By Morphisms, Lemma 52.2 the subset $U \subset X$ of points where $f$ is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over $S$ (as an open subscheme of $X$) and for $v \in V$ the residue field extension $\kappa(v) \supset \kappa(\nu(v))$
is algebraic (as $V \subset S'$ and $S'$ is integral over $S$). Hence the fibres of $V \to S$ are discrete (Morphisms, Lemma 19.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 19.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since $x$ was arbitrary we see that (a), (b), and (c) are true.

Let $s = f(x)$. Let $(T, t) \to (S, s)$ be an elementary étale neighbourhood. Denote by a subscript $\tau$ the base change to $T$. Let $y = (x, t) \in X_T$ be the unique point in the fibre $X_t$ lying over $x$. Note that $U_T \subset X_T$ is the set of points where $f_T$ is quasi-finite, see Morphisms, Lemma 19.13. Note that $X_T \xrightarrow{f'_T} S'_T \xrightarrow{\nu_T} T$ is the normalization of $T$ in $X_T$, see Lemma 17.2. Suppose that the claim holds for $y \in U_T \subset X_T \to S'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(y) \in V' \subset S'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $S'_T \to S'$ is étale hence the image $V \subset S'$ of $V'$ is open. Observe that $f'(x) \in V$ as $f'_T(y) \in V'$. Observe that $(f'_T)^{-1}V' \xrightarrow{(f')^{-1}(V)} V'$ is a fibre square (as $S'_T \times_{S'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent, Lemma 20.17. In other words, the claim holds for $x \in U \subset X \to S'$.

By the result of the previous paragraph we may replace $S$ by an elementary étale neighbourhood of $s = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subschemes where $V \to S$ is finite and $x \in V$, see Lemma 36.4. Since $X$ is a disjoint union of $V$ and $W$ over $S$ and since $V \to S$ is finite we see that the normalization of $S$ in $X$ is the morphism

$$X = V \amalg W \to V \amalg W' \to S$$

where $W'$ is the normalization of $S$ in $W$, see Morphisms, Lemmas 50.10, 52.4, and 50.12. The claim follows and we win. 

\begin{lemma}
Let $f : X \to S$ be a morphism of schemes. Assume $f$ is quasi-finite and separated. Let $S'$ be the normalization of $S$ in $X$, see Morphisms, Definition 50.3. Picture:

$$
\begin{tikzcd}
X \ar{rr}{f'} \ar{dr}[swap]{f} & & S' \\
& S \ar{ur}{\nu} &
\end{tikzcd}
$$

Then $f'$ is a quasi-compact open immersion and $\nu$ is integral. In particular $f$ is quasi-affine.
\end{lemma}
Proof. This follows from Lemma 38.1. Namely, by that lemma there exists an open subscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, $f'$ is an open immersion. Note that $f'$ is quasi-compact as $f$ is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 21.15). It follows that $f$ is quasi-affine by Morphisms, Lemma 12.3.

\begin{lemma}[Zariski's Main Theorem]
Let $f : X \to S$ be a morphism of schemes. Assume $f$ is quasi-finite and separated and assume that $S$ is quasi-compact and quasi-separated. Then there exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f} & T \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow{\pi} & S 
\end{array}
$$

where $j$ is a quasi-compact open immersion and $\pi$ is finite.
\end{lemma}

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 38.2. By Properties, Lemma 22.13 we can write $\nu_*\mathcal{O}_{S'} = \colim_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{S'}$. Then $\pi_i : T_i = \Spec(\mathcal{A}_i) \to S$ is a finite morphism for each $i$. Note that the transition morphisms $T_i \to T_i$ are affine and that $S' = \lim T_i$.

By Limits, Lemma 4.11 there exists an $i$ and a quasi-compact open $U_i \subset T_i$ whose inverse image in $S'$ equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of $U_i$ in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits, Lemma 2.2. By Limits, Lemma 4.16 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large $i'$ but we don’t need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms, Lemma 3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win.

\begin{lemma}
Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

\begin{enumerate}
\item $f$ is finite,
\item $f$ is proper with finite fibres,
\item $f$ is proper and locally quasi-finite,
\item $f$ is universally closed, separated, locally of finite type and has finite fibres.
\end{enumerate}
\end{lemma}

Proof. We have (1) implies (2) by Morphisms, Lemmas 42.10 and 19.10. We have (2) implies (3) by Morphisms, Lemma 19.7. We have (3) implies (4) by the definition of proper morphisms and Morphisms, Lemmas 19.9 and 19.10.

Assume (3). Pick $s \in S$. By Morphisms, Lemma 19.7 we see that all the finitely many points of $X_s$ are isolated in $X_s$. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \amalg W$ as in Lemma 36.6. Note that $W_u = \emptyset$ because all points of $X_s$ are isolated. Since $f$ is universally closed we see that the image of $W$ in $U$ is a closed set not containing $u$. After shrinking $U$ we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over $U$. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover $S$ such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 4.11. Hence it is an fpqc covering, see Topologies, Lemma 9.6. Hence we conclude $f$ is finite by Descent, Lemma 20.23.
As a consequence we have the following useful results.

**Lemma 38.5.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that $f$ is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of $s$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

**Proof.** The morphism $f$ is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 19.7. By Morphisms, Lemma 52.2 the set of points at which $f$ is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since $f$ is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 19.9), hence has finite fibres (Morphisms, Lemma 19.10), hence is finite by Lemma 38.4.

**Lemma 38.6.** Consider a commutative diagram of schemes

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
f \downarrow & & \downarrow g \\
S & \xrightarrow{j} & 
\end{array}
$$

Let $s \in S$. Assume

1. $X \to S$ is a proper morphism,
2. $Y \to S$ is separated and locally of finite type, and
3. the image of $X_s \to Y_s$ is finite.

Then there is an open subspace $U \subset S$ containing $s$ such that $X_U \to Y_U$ factors through a closed subscheme $Z \subset Y_U$ finite over $U$.

**Proof.** Let $Z \subset Y$ be the scheme theoretic image of $h$, see Morphisms, Section 6. By Morphisms, Lemma 39.9 the morphism $X \to Z$ is surjective and $Z \to S$ is proper. Thus $X_s \to Z_s$ is surjective. We see that either (3) implies $Z_s$ is finite. Hence $Z \to S$ is finite in an open neighbourhood of $s$ by Lemma 38.5.

**Lemma 38.7.** Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.

**Proof.** If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 42.3. Thus the problem is local on $X$ and we may assume that $X$ is affine.

Assume $X$ is affine. Write $Y = \bigcup_{j=1, \ldots, m} V_j$ with $V_j$ affine. This is possible since $f$ is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of $X$. We see from Morphisms, Lemmas 19.10 and 48.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose $U_\eta$ such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of $X$ is dense in $X$. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace $X$ by an affine open subscheme of $W$ and assume that each $V_j \to X$ is finite.
Assume $X$ is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with $V_j$ affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set
\[
\Delta_{ij} = \left( V_i \cap V_j \setminus V_i \cap V_j \right) \cap V_j.
\]
This is a nowhere dense closed subset of $V_j$ because it is the boundary of the open subset $V_i \cap V_j$ in $V_j$. By Morphisms, Lemma \[45.7\] the image $f(\Delta_{ij})$ is a nowhere dense closed subset of $X$. By Topology, Lemma \[21.2\] the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of $X$. Thus $U = X \setminus T$ is a dense open subset of $X$. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V_j' = Y' \cap V_j = U' \times_X V_j$. Consider the restriction
\[
f' = f|_{Y'} : Y' \to U'
\]
of $f$. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V_j'$ is an affine open covering, each $V_j' \to U'$ is finite, and $V_j' \cap V_j'$ is (open and) closed both in $V_j'$ and $V_j'$. Hence $V_j' \cap V_j'$ is affine, and the map
\[
\mathcal{O}(V_j') \otimes \mathcal{O}(V_j') \longrightarrow \mathcal{O}(V_j' \cap V_j')
\]
is surjective. This implies that $Y'$ is separated, see Schemes, Lemma \[21.8\]. Finally, consider the commutative diagram
\[
\begin{array}{ccc}
\bigcup_{j=1,\ldots,m} V_j' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
U' & \longrightarrow & Y'
\end{array}
\]
The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma \[39.8\] we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma \[38.3\] and we win. \qed

**Lemma 38.8.** Let $f : X \to S$ be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets
\[
\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S
\]
such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \bigsqcup_{r=0,\ldots,n} S_r$ is characterized by the following universal property: Given $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree $r$ if and only if $g(T) \subset S_r$ (set theoretically).

**Proof.** Let $n$ be an integer bounding the degree of the fibres of $X \to S$. By Morphisms, Lemma \[33.6\] we see that any base change has degrees of fibres bounded by $n$ also. In particular, all the integers $r$ that occur in the statement of the lemma will be $\leq n$. We will prove the lemma by induction on $n$. The base case is $n = 0$ which is obvious.

We claim the set of points $s \in S$ with $\deg_{S_n(s)}(X_s) = n$ is an open subset $S_n \subset S$ and that $X \times_S S_n \to S_n$ is finite locally free of degree $n$. Namely, suppose that $s \in S$ is such a point. Choose an elementary étale morphism $(U, u) \to (S, s)$ and a decomposition $U \times_S X = W \amalg V$ as in Lemma \[36.0\]. Since $V \to U$ is finite, flat, and locally of finite presentation, we see that $V \to U$ is finite locally free, see Morphisms, Lemma \[45.2\]. After shrinking $U$ to a smaller neighbourhood of $u$ we may assume $V \to U$ is finite locally free of some degree $d$, see Morphisms, Lemma
As \( u \mapsto s \) and \( W_u = \emptyset \) we see that \( d = n \). Since \( n \) is the maximum degree of a fibre we see that \( W = \emptyset \). Thus \( U \times_S X \to U \) is finite locally free of degree \( n \).

By Descent, Lemma \( \ref{descent-flat-desc-finite-deg} \) we conclude that \( X \to S \) is finite locally free of degree \( n \) over \( \text{Im}(U \to S) \) which is an open neighbourhood of \( s \) (Morphisms, Lemma \( \ref{morphisms-flat-desc-finite-deg} \)). This proves the claim.

Let \( S' = S \setminus S_n \) endowed with the reduced induced scheme structure and set \( X' = X \times_S S' \). Note that the degrees of fibres of \( X' \to S' \) are universally bounded by \( n - 1 \). By induction we find a stratification \( S' = S_0 \amalg \ldots \amalg S_{n-1} \) adapted to the morphism \( X' \to S' \). We claim that \( S = \coprod_{r=0, \ldots, n} S_r \) works for the morphism \( X \to S \). Let \( g : T \to S \) be a morphism of schemes and assume that \( X \times_S T \to T \) is finite locally free of degree \( r \). As remarked above this implies that \( r \leq n \). If \( r = n \), then it is clear that \( T \to S \) factors through \( S_n \). If \( r < n \), then \( g(T) \subset S' = S \setminus S_d \) (set theoretically) hence \( T_{\text{red}} \to S \) factors through \( S' \), see Schemes, Lemma \( \ref{schemes-flat-desc-finite-deg} \).

Note that \( X \times_S T_{\text{red}} \to T_{\text{red}} \) is also finite locally free of degree \( r \) as a base change. By the universal property of the stratification \( S' = \coprod_{i=0, \ldots, n-1} S_r \), we see that \( g(T) = g(T_{\text{red}}) \) is contained in \( S_r \). Conversely, suppose that we have \( g : T \to S \) such that \( g(T) \subset S_r \) (set theoretically). If \( r = n \), then \( g \) factors through \( S_n \) and it is clear that \( X \times_S T \to T \) is finite locally free of degree \( n \) as a base change. If \( r < n \), then \( X \times_S T \to T \) is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to \( T_{\text{red}} \) is finite locally free of degree \( r \). Since \( T_{\text{red}} \to T \) is a universal homeomorphism, we conclude that \( X \times_S T_{\text{red}} \to \coprod X \times_S T_{\text{red}} \) is a universal homeomorphism too and hence \( X \times_S T \to T \) is universally closed (as this is true for the finite morphism \( X \times_S T_{\text{red}} \to T_{\text{red}} \)). It follows that \( X \times_S T \to T \) is finite, for example by Lemma \( \ref{morphisms-flat-desc-finite-deg} \). Then we can use Morphisms, Lemma \( \ref{morphisms-flat-desc-finite-deg} \) to see that \( X \times_S T \to T \) is finite locally free. Finally, the degree is \( r \) as all the fibres have degree \( r \).

\[ \square \]

Lemma 38.9. Let \( f : X \to S \) be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets
\[ \emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S \]
such that with \( S_r = Z_r \setminus Z_{r-1} \) the stratification \( S = \coprod S_r \) is characterized by the following universal property: Given a morphism \( g : T \to S \) the projection \( X \times_S T \to T \) is finite locally free of degree \( r \) if and only if \( g(T) \subset S_r \) (set theoretically). Moreover, the inclusion maps \( S_r \to S \) are quasi-compact.

Proof. The question is local on \( S \), hence we may assume that \( S \) is affine. By Morphisms, Lemma \( \ref{morphisms-flat-desc-finite-deg} \) the fibres of \( f \) are universally bounded in this case. Hence the existence of the stratification follows from Lemma \( \ref{morphisms-flat-desc-finite-deg} \).

We will show that \( U_r = S \setminus Z_r \to S \) is quasi-compact for each \( r \geq 0 \). This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that \( U_r \to U_{r-1} \) is quasi-compact. Choose an affine open \( W \subset U_{r-1} \). Write \( W = \text{Spec}(A) \). Then \( Z_r \cap W = V(I) \) for some ideal \( I \subset A \) and \( X \times_S \text{Spec}(A/I) \to \text{Spec}(A) \) is finite locally free of degree \( r \). Note that \( A/I = \text{colim} A/I_i \) where \( I_i \subset I \) runs through the finitely generated ideals. By Limits, Lemma \( \ref{limits-flat-desc-finite-deg} \) we see that \( X \times_S \text{Spec}(A/I_i) \to \text{Spec}(A/I_i) \) is finite locally free of degree \( r \) for some \( i \). (This uses that \( X \to S \) is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence \( \text{Spec}(A/I_i) \to \text{Spec}(A) = W \) factors (set theoretically) through \( Z_r \cap W \). It
follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of $A$. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. \hfill \qed

**Lemma 38.10.** Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes

$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$

such that a morphism $\Spec(k) \to S$ factors through $U_d$ if and only if $X \times_S \Spec(k)$ has degree $\geq d$ over $k$.

**Proof.** The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on $S$ and assume $S$ is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 38.8. Since $U_d = \bigcup_U U_d(W)$ the result follows. \hfill \qed

**Lemma 38.11.** Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram

$$
\begin{array}{ccc}
V & \longrightarrow & X \\
\pi \downarrow & & \downarrow f \\
U & \longrightarrow & S,
\end{array}
$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_* \mathcal{O}_V$ whose image contains $g|_V$.

**Proof.** We may assume $X$ and $S$ affine. We obtain a filtration $\emptyset = Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 38.8 and 38.9. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\Im(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that $T$ is the support of $g$ as a section of $\mathcal{O}_X$ (Modules, Definition 5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let $r$ be the smallest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of $T$ such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace $S$ by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \Spec(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 29.1). Since the support of $g$ is contained in $f^{-1}V(I)$ by our choice of $r$ we see that there exists an integer $N$ such that $a_j^N g = 0$ for $j = 1, \ldots, m$. Replacing $a_j$ by $a_j^r$ we may assume that $Ig = 0$. For any $A$-module $M$ write $M[I]$ for the $I$-torsion of $M$, i.e., $M[I] = \{ m \in M \mid Im = 0 \}$. Write $X = \Spec(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of $Z_r$, the $A/I$-module $B/IB$ is finite locally free of rank $r$. Hence after replacing $S$ by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/I A)^{\oplus r}$ as $A/I$-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo $I$ to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \longrightarrow B[I]$$

is what we want.

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the $A$-module $A[I]$ and the map $\mathcal{F} \oplus r \to \pi_* O_V$ the one corresponding to $A[I] \oplus r \subset A \oplus r \to B$. $\square$

**Lemma 38.12.** Let $f : X \to Y$ be a separated, locally quasi-finite morphism with $Y$ affine. Then every finite set of points of $X$ is contained in an open affine of $X$.

**Proof.** Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 38.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 29.5. $\square$

**Lemma 38.13.** Let $U \to X$ be a surjective étale morphism of schemes. Assume $X$ is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through $U$. In fact, we may assume $Y \to X$ is finite and of finite presentation.

**Proof.** Since $X$ is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \bigsqcup U_i \to X$ is surjective. After replacing $U$ by $U'$, we see that we may assume $U$ is affine. Then there exists an integer $d$ bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 53.10). We will prove the lemma by induction on $d$ for all quasi-compact and separated schemes $U$ mapping surjective and étale onto $X$. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 38.2 and we obtain a factorization

$$
\begin{array}{ccc}
U & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \pi \\
X
\end{array}
$$

with $\pi$ integral and $j$ a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in $Y$. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 21.11 and open because it is étale as a morphism between schemes étale over $Y$) and the second summand is the (open and closed) complement. The image $V \subset Y$ of $W$ is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of $Y$ for example by Lemma 38.8 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with $Z$ a scheme, which Zariski locally factors through $W$. Choose a factorization $Z \to Z' \to Y$ with $Z'$ integral and $Z \to Z'$ open immersion (Lemma 38.2). After replacing $Z'$ by the scheme theoretic closure of $Z$ in $Z'$ we may assume that $Z$ is scheme theoretically dense in $Z'$. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$
This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through $U$. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then $z$ maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of $z$ on which the morphism factors through $U$. If $z \in Z$, then we have a neighbourhood $V \subset Z$ which factors through $W \subset U \times_X Y$ and hence through $U$. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through $U$. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $U$. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 7.2. For large enough $i$ we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in $Y$ recovers $V_j$, see Limits, Lemma 4.11. For even larger $i$ the morphisms $V_j \to U$ over $X$ come from morphisms $V_{i,j} \to U$ over $X$, see Limits, Proposition 6.1. This finishes the proof. □

39. Application to morphisms with connected fibres

In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

Lemma 39.1. Consider a diagram of morphisms of schemes

\[
\begin{array}{ccc}
Z & \xrightarrow{\sigma} & X \\
& \searrow & \downarrow \\
& & Y
\end{array}
\]

an a point $y \in Y$. Assume

1. $X \to Y$ is of finite presentation and flat,
2. $Z \to Y$ is finite locally free,
3. $Z_y \neq \emptyset$,
4. all fibres of $X \to Y$ are geometrically reduced, and
5. $X_y$ is geometrically connected over $k(y)$.

Then there exists an open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 24.7, 20.4, 45.4, 24.5, 20.3, 45.3, 24.9, and 42.10.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section $s$ coming from $\sigma$. Let $X^0_Z$ denote the subset of $X_Z$ defined in Situation 27.1. By Lemma 27.6 it is an open subset of $X_Z$.

The pullback $X_{Z \times_Y Z}$ of $X$ to $Z \times_Y Z$ comes equipped with two sections $s_0, s_1$, namely the base changes of $s$ by $\text{pr}_0, \text{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 27.1 gives two subsets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$. By Lemma 27.2 these are the inverse images of $X^0_Z$ under the morphisms $1_X \times \text{pr}_0, 1_X \times \text{pr}_1 : X_{Z \times_Y Z} \to X_Z$. In particular these subsets are open.
Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As $X_y$ is geometrically connected, we see that the fibres of $(X_{Z \times_Y Z})_y$ and $(X_{Z \times_Y Z})_0$ over each $z_i$ agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z \times_Y Z})_y^0 \quad \text{and} \quad s_1(z_i) \in (X_{Z \times_Y Z})_y^0.$$  

Since the sets $(X_{Z \times_Y Z})_y^0$ and $(X_{Z \times_Y Z})_y^0$ are open in $X_{Z \times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z \times_Y Z})_y^0 \quad \text{and} \quad s_1(W) \subset (X_{Z \times_Y Z})_y^0.$$  

Then it follows directly from the construction in Situation 27.1 that

$$p^{-1}(W) \cap (X_{Z \times_Y Z})_y^0 = p^{-1}(W) \cap (X_{Z \times_Y Z})_y^0,$$

where $p : X_{Z \times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an open neighbourhood $V \subset Y$ of $y$ such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace $Y$ by $V$. After we do this we see that $X_y^0 \subset Y_Z$ is an open such that

$$(1_X \times \text{pr}_0)^{-1}(X_y^0) = (1_X \times \text{pr}_1)^{-1}(X_y^0).$$

This means that the image $X_y^0 \subset X$ of $X_y^0$ is an open such that $(X_y \to X)^{-1}(X_y^0) = X_y^0$, see Descent, Lemma 10.2. At this point it is clear that $X_y^0$ is the desired open subscheme. \hfill $\square$

055W Lemma 39.2. Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume

1. $h$ is flat and of finite presentation,
2. all fibres of $h$ are geometrically reduced, and
3. $T$ is geometrically connected over $\kappa(s)$.

Then we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that

1. all fibres of $V \to S'$ are geometrically connected,
2. $V_{s'} = T \times_s s'$.

Proof. The problem is clearly local on $S$, hence we may replace $S$ by an affine open neighbourhood of $s$. The topology on $Y_s$ is induced from the topology on $X$, see Schemes, Lemma 18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of $h$ to $V$ is quasi-compact (as $S$ affine and $V$ quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing $Y$ by $V$ we may assume, in addition to (1) and (2) that $Y_s = T$ and $S$ affine.

Pick a point $y \in Y_s$ such that $h$ is Cohen-Macaulay at $y$, see Lemma 20.7. By Lemma 21.3 there exists a diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & \end{array}
$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{z\}$. Apply Lemma 36.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$
lying over \( s \). Note that \( Z' \to S' \) is also locally of finite presentation and flat (as an open of the base change of \( Z \to S \)), hence \( Z' \to S' \) is finite locally free, see Morphisms, Lemma 45.2. Note that \( Y_{S'} \to S' \) is flat and of finite presentation with geometrically reduced fibres as a base change of \( h \). Also \( Y' = Y_s \) is geometrically connected. Apply Lemma 45.1 to \( Z' \to Y_{S'} \) over \( S' \) to get \( V \subset Y_{S'} \) satisfying (2) whose fibres over \( S' \) are either empty or geometrically connected. As \( V \to S' \) is open (Morphisms, Lemma 24.9), after shrinking \( S' \) we may assume \( V \to S' \) is surjective, whence (1) holds. □

**Lemma 39.3.** Let \( h : Y \to S \) be a morphism of schemes. Let \( s \in S \) be a point. Let \( T \subset Y_s \) be an open subscheme. Assume

1. \( h \) is of finite presentation,
2. \( h \) is normal, and
3. \( T \) is geometrically irreducible over \( \kappa(s) \).

Then we can find an elementary étale neighbourhood \( (S', s') \to (S, s) \) and an open \( V \subset Y_{S'} \) such that

- all fibres of \( V \to S' \) are geometrically integral,
- \( V_s = T \times_s s' \).

**Proof.** Apply Lemma 39.2 to find an elementary étale neighbourhood \( (S', s') \to (S, s) \) and an open \( V \subset Y_{S'} \) such that all fibres of \( V \to S' \) are geometrically integral and \( V_s = T \times_s s' \). Note that \( V \to S' \) is open, see Morphisms, Lemma 24.9. Hence after replacing \( S' \) by the image of \( V \to S' \) we see that all fibres of \( V \to S' \) are nonempty. As \( V \) is an open of the base change of \( h \) all fibres of \( V \to S' \) are geometrically normal, see Lemma 18.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if \( t \in S' \) then \( V_t \) is of finite type over \( \kappa(t) \) and hence \( V_t \times_{\kappa(t)} \kappa(t) \) is of finite type over \( \kappa(t) \) hence Noetherian. By choice of \( S' \to S \) the scheme \( V_t \times_{\kappa(t)} \kappa(t) \) is connected. Hence \( V_t \times_{\kappa(t)} \kappa(t) \) is irreducible by Properties, Lemma 7.6 and we win. □

### 40. Application to the structure of finite type morphisms

The result in this section can be found in [GR71]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

**Lemma 40.1.** Let \( f : X \to S \) be a morphism. Let \( x \in X \) and set \( s = f(x) \). Assume that \( f \) is locally of finite type and that \( n = \dim_x(X_s) \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{g} & X' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
Y & \xleftarrow{h} & Y' \\
\downarrow{y} & & \downarrow{y'} \\
S & \xrightarrow{=} & S' \\
\end{array}
\]

and a point \( x' \in X' \) with \( g(x') = x \) such that with \( y = \pi(x') \) we have
(1) \( h : Y \to S \) is smooth of relative dimension \( n \),
(2) \( g : (X', x') \to (X, x) \) is an elementary étale neighbourhood,
(3) \( \pi \) is finite, and \( \pi^{-1}(\{y\}) = \{x'\} \), and
(4) \( \kappa(y) \) is a purely transcendental extension of \( \kappa(s) \).

Moreover, if \( f \) is locally of finite presentation then \( \pi \) is of finite presentation.

**Proof.** The problem is local on \( X \) and \( S \), hence we may assume that \( X \) and \( S \) are affine. By Algebra, Lemma 124.3 after replacing \( X \) by a standard open neighbourhood of \( x \) in \( X \) we may assume there is a factorization

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathbb{A}^n_S \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi} & S
\end{array}
\]

such that \( \pi \) is quasi-finite and such that \( \kappa(\pi(x)) \) is purely transcendental over \( \kappa(s) \). By Lemma 36.1 there exists an elementary étale neighbourhood

\[
(Y, y) \to (\mathbb{A}^n_S, \pi(x))
\]

and an open \( X' \subset X \times \mathbb{A}^n_S \) \( Y \) which contains a unique point \( x' \) lying over \( y \) such that \( X' \to Y \) is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma 20.11

\[\square\]

057K **Lemma 40.2.** Let \( f : X \to S \) be a morphism. Let \( x \in X \) and set \( s = f(x) \). Assume that \( f \) is locally of finite type and that \( n = \dim_x(X_s) \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xleftarrow{h} & Y' \\
\downarrow & & \downarrow \\
S & \xleftarrow{e} & S'
\end{array}
\]

and a point \( x' \in X' \) with \( g(x') = x \) such that with \( y' = \pi(x') \), \( s' = h(y') \) we have

(1) \( h : Y' \to S' \) is smooth of relative dimension \( n \),
(2) all fibres of \( Y' \to S' \) are geometrically integral,
(3) \( g : (X', x') \to (X, x) \) is an elementary étale neighbourhood,
(4) \( \pi \) is finite, and \( \pi^{-1}(\{y'\}) = \{x'\} \),
(5) \( \kappa(y') \) is a purely transcendental extension of \( \kappa(s') \), and
(6) \( e : (S', s') \to (S, s) \) is an elementary étale neighbourhood.

Moreover, if \( f \) is locally of finite presentation, then \( \pi \) is of finite presentation.

**Proof.** The question is local on \( S \), hence we may replace \( S \) by an affine open neighbourhood of \( s \). Next, we apply Lemma 10.1 to get a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xleftarrow{h} & Y \\
\downarrow & & \downarrow \\
S & \xleftarrow{e} & S
\end{array}
\]
where \( h \) is smooth of relative dimension \( n \) and \( \kappa(y) \) is a purely transcendental extension of \( \kappa(s) \). Since the question is local on \( X \) also, we may replace \( Y \) by an affine neighbourhood of \( y \) (and \( X' \) by the inverse image of this under \( \pi \)). As \( S \) is affine this guarantees that \( Y \to S \) is quasi-compact, separated and smooth, in particular of finite presentation. Let \( T \) be the connected component of \( Y_s \) containing \( y \). As \( Y_s \) is Noetherian we see that \( T \) is open. We also see that \( T \) is geometrically connected over \( \kappa(s) \) by Varieties, Lemma 7.14. Since \( T \) is also smooth over \( \kappa(s) \) it is geometrically normal, see Varieties, Lemma 25.4. We conclude that \( \kappa \) is normal by Lemma 18.3. At this point we have verified all assumption of Lemma \( \text{Properties, Lemma 7.6} \). Finally, note that the smooth morphism \( \kappa \) is geometrically irreducible, see Varieties, Lemma 25.4. We conclude that \( \kappa \) is irreducible, such that \( \kappa(s) \) is also, we may replace \( S' \) by an open subschemes of geometrically integral fibres of \( Y' \to S' \), and hence geometrically integral. Thus we may replace \( S' \) by \( U'' \) and \( Y' \) by \( h^{-1}(U'') \cap V' \). □

The significance of the property \( \pi^{-1}(\{y'\}) = \{x'\} \) is partially explained by the following lemma.

**Lemma 40.3.** Assumption and notation as in Lemma 40.2 In addition to properties (1) – (6) we may also arrange it so that

(7) \( S', Y', X' \) are affine.

**Proof.** Note that if \( Y' \) is affine, then \( X' \) is affine as \( \pi \) is finite. Choose an affine open neighbourhood \( U' \subseteq S' \) of \( s' \). Choose an affine open neighbourhood \( V' \subseteq h^{-1}(U') \) of \( y' \). Let \( W' = h(V') \). This is an open neighbourhood of \( s' \) in \( S' \), see Morphisms, Lemma 32.10, contained in \( U' \). Choose an affine open neighbourhood \( U'' \subseteq W' \) of \( s' \). Then \( h^{-1}(U'') \cap V' \) is affine because it is equal to \( U'' \times_{U'} V' \). By construction \( h^{-1}(U'') \cap V' \to U'' \) is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of \( Y' \to S' \), and hence geometrically integral. Thus we may replace \( S' \) by \( U'' \) and \( Y' \) by \( h^{-1}(U'') \cap V' \). □

The significance of the property \( \pi^{-1}(\{y'\}) = \{x'\} \) is partially explained by the following lemma.

**Lemma 40.4.** Let \( \pi : X \to Y \) be a finite morphism. Let \( x \in X \) with \( y = \pi(x) \) such that \( \pi^{-1}(\{y\}) = \{x\} \). Then

(1) For every neighbourhood \( U \subseteq X \) of \( x \) in \( X \), there exists a neighbourhood \( V \subseteq Y \) of \( y \) such that \( \pi^{-1}(V) \subseteq U \).
(2) The ring map \( \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \) is finite.
(3) If \( \pi \) is of finite presentation, then \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) is of finite presentation.
(4) For any quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) we have \( \mathcal{F}_x = \pi_* \mathcal{F}_y \) as \( \mathcal{O}_{Y,y} \)-modules.

**Proof.** The first assertion is purely topological; use that \( \pi \) is a continuous and closed map such that \( \pi^{-1}([y]) = \{x\} \). To prove the second and third parts we may assume \( X = \text{Spec}(B) \) and \( Y = \text{Spec}(A) \). Then \( A \to B \) is a finite ring map and \( y \) corresponds to a prime \( \mathfrak{p} \) of \( A \) such that there exists a unique prime \( \mathfrak{q} \) of \( B \) lying over \( \mathfrak{p} \). Then \( B_\mathfrak{q} = B_\mathfrak{p} \), see Algebra, Lemma [10.11]. In other words, the map \( A_\mathfrak{p} \to B_\mathfrak{q} \) is equal to the map \( A_\mathfrak{p} \to B_\mathfrak{p} \) you get from localizing \( A \to B \) at \( \mathfrak{p} \). Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that \( \mathcal{F} = \tilde{M} \) for some \( B \)-module \( M \). Then \( \mathcal{F} = M_\mathfrak{q} \) and \( \pi_* \mathcal{F}_y = M_\mathfrak{p} \). By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that \( \mathcal{F}_x = \pi_* \mathcal{F}_y \) directly. \( \square \)

### 41. Application to the fppf topology

**Lemma 41.1.** Let \( S \) be a scheme. Let \( \{S_i \to S\}_{i \in I} \) be an fppf covering. Then there exist

1. an \( \acute{e}tale \) covering \( \{S'_i \to S\} \),
2. surjective finite locally free morphisms \( V_a \to S'_a \),

such that the fppf covering \( \{V_a \to S\} \) refines the given covering \( \{S_i \to S\} \).

**Proof.** We may assume that each \( S_i \to S \) is locally quasi-finite, see Lemma [21.6].

Fix a point \( s \in S \). Pick an \( i \in I \) and a point \( s_i \in S_i \) mapping to \( s \). Choose an elementary \( \acute{e}tale \) neighbourhood \( (S', s) \to (S, s) \) such that there exists an open

\[ S_i \times_S S' \supset V \]

which contains a unique point \( v \in V \) mapping to \( s \in S' \) and such that \( V \to S' \) is finite, see Lemma [36.1]. Then \( V \to S' \) is finite locally free, because it is finite and because \( S_i \times_S S' \to S' \) is flat and locally of finite presentation as a base change of the morphism \( S_i \to S \), see Morphisms, Lemmas [20.4] [24.7] and [15.2]. Hence \( V \to S' \) is open, and after shrinking \( S' \) we may assume that \( V \to S' \) is surjective finite locally free. Since we can do this for every point of \( S \) we conclude that \( \{S_i \to S\} \) can be refined by a covering of the form \( \{V_a \to S\}_{a \in A} \) where each \( V_a \to S \) factors as \( V_a \to S'_a \to S \) étale and \( V_a \to S'_a \) surjective finite locally free. \( \square \)

**Lemma 41.2.** Let \( S \) be a scheme. Let \( \{S_i \to S\}_{i \in I} \) be an fppf covering. Then there exist

1. a Zariski open covering \( S = \bigcup U_j \),
2. surjective finite locally free morphisms \( W_j \to U_j \),
3. Zariski open coverings \( W_j = \bigcup_{k} W_{j,k} \),
4. surjective finite locally free morphisms \( T_{j,k} \to W_{j,k} \)

such that the fppf covering \( \{T_{j,k} \to S\} \) refines the given covering \( \{S_i \to S\} \).
Let \( \{ V_a \to S \}_{a \in A} \) be the fpqc covering found in Lemma 41.1. In other words, this covering refines \( \{ S_i \to S \} \) and each \( V_a \to S \) factors as \( V_a \to S_a' \to S \) with \( S_a' \to S \) étale and \( V_a \to S_a' \) surjective finite locally free.

By Remark 35.3 there exists a Zariski open covering \( S = \bigcup U_j \), for each \( j \) a finite locally free, surjective morphism \( W_j \to U_j \), and for each \( j \) a Zariski open covering \( \{ W_{j,k} \to W_j \} \) such that the family \( \{ W_{j,k} \to S \} \) refines the étale covering \( \{ S_a' \to S \} \), i.e., for each pair \( j, k \) there exists an \( a(j, k) \) and a factorization \( W_{j,k} \to S_a' \to S \) of the morphism \( W_{j,k} \to S \). Set \( T_{j,k} = W_{j,k} \times_{S_a'} V_a \) and everything is clear. \( \square \)

Lemma 41.3. Let \( S \) be a scheme. If \( U \subset S \) is open and \( V \to U \) is a surjective integral morphism, then there exists a surjective integral morphism \( \overline{V} \to V \) with \( \overline{V} \times_S U \) isomorphic to \( V \) as schemes over \( U \).

Proof. Let \( V' \to S \) be the normalization of \( S \) in \( V \), see Morphisms, Section 50. By construction \( V' \to S \) is integral. By Morphisms, Lemmas 50.6 and 50.12 we see that the inverse image of \( U \) in \( V' \) is \( V \). Let \( Z \) be the reduced induced scheme structure on \( S \setminus U \). Then \( \overline{V} = V' \amalg Z \) works. \( \square \)

Lemma 41.4. Let \( S \) be a quasi-compact and quasi-separated scheme. If \( U \subset S \) is a quasi-compact open and \( V \to U \) is a surjective finite morphism, then there exists a surjective finite morphism \( \overline{V} \to V \) with \( \overline{V} \times_S U \) isomorphic to \( V \) as schemes over \( U \).

Proof. By Zariski’s Main Theorem (Lemma 38.3) we can assume \( V \) is a quasi-compact open in a scheme \( V' \) finite over \( S \). After replacing \( V' \) by the scheme theoretic image of \( V \) we may assume that \( V \) is dense in \( V' \). It follows that \( V' \times_S U = V \) because \( V \to V' \times_S U \) is closed as \( V \) is finite over \( U \). Let \( Z \) be the reduced induced scheme structure on \( S \setminus U \). Then \( \overline{V} = V' \amalg Z \) works. \( \square \)

Lemma 41.5. Let \( S \) be a scheme. Let \( \{ S_i \to S \}_{i \in I} \) be an fpqc covering. Then there exists a surjective integral morphism \( S' \to S \) and an open covering \( S' = \bigcup U'_a \) such that for each \( \alpha \) the morphism \( U'_a \to S \) factors through \( S_i \to S \) for some \( i \).

Proof. Choose \( S' = \bigcup U_j \), \( W_j \to U_j \), \( W_j = \bigcup W_{j,k} \), and \( T_{j,k} \to W_{j,k} \) as in Lemma 41.2. By Lemma 41.3 we can extend \( W_j \to U_j \) to a surjective integral morphism \( W_j \to S \). After this we can extend \( T_{j,k} \to W_{j,k} \) to a surjective integral morphism \( T_{j,k} \to W_{j,k} \). We set \( T_j \) equal to the product of all the schemes \( T_{j,k} \) over \( W_j \) (Limits, Lemma 3.1). Then we set \( S' \) equal to the product of all the schemes \( T_j \) over \( S \). If \( x \in S' \), then there is a \( j \) such that \( x \) lies in \( U_j \). Hence there is a \( k \) such that the image of \( x \) under the projection \( S' \to W_j \) lies in \( W_{j,k} \). Hence under the projection \( S' \to T_j \) the point \( x \) ends up in \( T_{j,k} \). And \( T_{j,k} \to S \) factors through \( S_i \) for some \( i \). Finally, the morphism \( S' \to S \) is integral and surjective by Limits, Lemmas 3.3 and 3.2. \( \square \)

Lemma 41.6. Let \( S \) be a quasi-compact and quasi-separated scheme. Let \( \{ S_i \to S \}_{i \in I} \) be an fpqc covering. Then there exists a surjective finite morphism \( S' \to S \) of finite presentation and an open covering \( S' = \bigcup U'_a \) such that for each \( \alpha \) the morphism \( U'_a \to S \) factors through \( S_i \to S \) for some \( i \).

Proof. Let \( Y \to X \) be the integral surjective morphism found in Lemma 41.5. Choose a finite affine open covering \( Y = \bigcup V_j \) such that \( V_j \to X \) factors through \( S_{i(j)} \). We can write \( Y = \lim Y_\lambda \) with \( Y_\lambda \to X \) finite and of finite presentation,
see Limits, Lemma 7.2. For large enough \( \lambda \) we can find affine opens \( V_{\lambda,j} \subset Y_{\lambda} \) whose inverse image in \( Y \) recovers \( V_j \), see Limits, Lemma 4.11. For even larger \( \lambda \) the morphisms \( V_j \to S_{i(j)}(j) \) over \( X \) come from morphisms \( V_{\lambda,j} \to S_{i(j)}(j) \) over \( X \), see Limits, Proposition 6.1. Setting \( S' = Y_{\lambda} \) for this \( \lambda \) finishes the proof.

\[ \square \]

Lemma 41.7. An fppf covering of schemes is a ph covering.

Proof. Let \( \{ T_i \to T \} \) be an fppf covering of schemes, see Topologies, Definition 7.1. Observe that \( T_i \to T \) is locally of finite type. Let \( U \subset T \) be an affine open. It suffices to show that \( \{ T_i \times_T U \to U \} \) can be refined by a standard ph covering, see Topologies, Definition 8.4. This follows immediately from Lemma 41.6 and the fact that a finite morphism is proper (Morphisms, Lemma 42.10).

\[ \square \]

Remark 41.8. As a consequence of Lemma 41.7 we obtain a comparison morphism

\[ \epsilon : (\text{Sch}/S)_{\text{ph}} \longrightarrow (\text{Sch}/S)_{\text{fppf}} \]

This is the morphism of sites given by the identity functor on underlying categories (with suitable choices of sites as in Topologies, Remark 10.1). The functor \( \epsilon_* \) is the identity on underlying presheaves and the functor \( \epsilon^{-1} \) associated to an fppf sheaf its ph sheafification. By composition we can in addition compare the ph topology with the syntomic, smooth, étale, and Zariski topologies.

42. Quasi-projective schemes

The term “quasi-projective scheme” has not yet been defined. A possible definition could be a scheme which has an ample invertible sheaf. However, if \( X \) is a scheme over a base scheme \( S \), then we say that \( X \) is quasi-projective over \( S \) if the morphism \( X \to S \) is quasi-projective (Morphisms, Definition 38.1). Since the identity morphism of any scheme is quasi-projective, we see that a scheme quasi-projective over \( S \) doesn’t necessarily have an ample invertible sheaf. For this reason it seems better to leave the term “quasi-projective scheme” undefined.

Lemma 42.1. Let \( S \) be a scheme which has an ample invertible sheaf. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent

1. \( X \to S \) is quasi-projective,
2. \( X \to S \) is H-quasi-projective,
3. there exists a quasi-compact open immersion \( X \to X' \) of schemes over \( S \) with \( X' \to S \) projective,
4. \( X \to S \) is of finite type and \( X \) has an ample invertible sheaf, and
5. \( X \to S \) is of finite type and there exists an \( f \)-very ample invertible sheaf.

Proof. The implication (2) \( \Rightarrow \) (1) is Morphisms, Lemma 38.3. The implication (1) \( \Rightarrow \) (2) is Morphisms, Lemma 41.16. The implication (2) \( \Rightarrow \) (3) is Morphisms, Lemma 41.11.

Assume \( X \subset X' \) as in (3). In particular \( X \to S \) is of finite type. By Morphisms, Lemma 41.11 the morphism \( X \to S \) is H-projective. Thus there exists a quasi-compact immersion \( i : X \to \mathbb{P}^n_S \). Hence \( L = i^*\mathcal{O}_{\mathbb{P}^n_S}(1) \) is \( f \)-very ample. As \( X \to S \) is quasi-compact we conclude from Morphisms, Lemma 36.2 that \( L \) is \( f \)-ample. Thus \( X \to S \) is quasi-projective by definition.

The implication (4) \( \Rightarrow \) (2) is Morphisms, Lemma 37.3.
Assume the equivalent conditions (1), (2), (3) hold. Choose an immersion \( i : X \to \mathbb{P}^n_S \) over \( S \). Let \( \mathcal{L} \) be an ample invertible sheaf on \( S \). To finish the proof we will show that \( \mathcal{N} = f^* \mathcal{L} \otimes O_X = i^* O_{\mathbb{P}^n_X}(1) \) is ample on \( X \). By Properties, Lemma 26.14 we reduce to the case \( X = \mathbb{P}^n_S \). Let \( \mathcal{L} \) be an ample invertible sheaf on \( S \). To finish the proof we will show that \( N = f^* \mathcal{L} \otimes O_X = \text{Spec}(\mathcal{O} \otimes \mathcal{L})(1) \) is ample on \( X \). By Properties, Lemma 26.14, we reduce to the case \( X = \mathbb{P}^n_S \). Let \( s \in \Gamma(S, \mathcal{L} \otimes \mathcal{O}(1)) \) be a section such that the corresponding open \( S_s \) is affine. Say \( S_s = \text{Spec}(A) \). Recall that \( \mathbb{P}^n_S \) is the projective bundle associated to \( \mathcal{O}_S T_0 \oplus \cdots \oplus \mathcal{O}_S T_n \), see Constructions, Lemma 21.4 and its proof. Let \( s_i \in \Gamma(X, \mathcal{O}(1)) \) be the global section corresponding to the section \( T_i \) of \( \mathcal{O}_S T_0 \oplus \cdots \oplus \mathcal{O}_S T_n \). Then we see that \( f^* s_i \otimes s_i \) is affine because it is equal to \( \text{Spec}(A[T_0/T_i, \ldots, T_n/T_i]) \). This proves that \( N \) is ample by definition.

The equivalence of (1) and (5) follows from Morphisms, Lemmas 36.2 and 37.5.

**Lemma 42.2.** Let \( \mathcal{L} \) be an ample invertible sheaf. Let \( \mathcal{N} \) be the full subcategory of the category of schemes over \( S \) satisfying the equivalent conditions of Lemma 42.1.

1. if \( S' \to S \) is a morphism of schemes and \( S' \) has an ample invertible sheaf, then base change determines a functor \( \mathcal{N} \to \mathcal{N}' \),
2. the category \( \mathcal{N} \) is closed under fibre products,
3. the category \( \mathcal{N} \) is closed under finite disjoint unions,
4. if \( X \to S \) is projective, then \( X \in \mathcal{N} \),
5. if \( X \to S \) is quasi-affine of finite type, then \( X \in \mathcal{N} \),
6. if \( X \to S \) is quasi-projective and proper, then \( X \in \mathcal{N} \),
7. if \( X \to S \) is quasi-projective and separated, then \( X \in \mathcal{N} \),
8. if \( X \to S \) is a quasi-compact immersion, then \( X \in \mathcal{N} \),
9. add more here.

**Proof.** Part (1) follows from Morphisms, Lemma 38.2.

Part (2) follows from the fourth characterization of Lemma 42.1. If \( X \to S \) and \( Y \to S \) are quasi-projective, then \( X \times S Y \to Y \) is quasi-projective by Morphisms, Lemma 38.2. Hence (3) follows from (2).

If \( X = Y \amalg Z \) is a disjoint union of schemes and \( \mathcal{L} \) is an invertible \( \mathcal{O}_X \)-module such that \( \mathcal{L}|_Y \) and \( \mathcal{L}|_Z \) are ample, then \( \mathcal{L} \) is ample (details omitted). Thus part (4) follows from the fourth characterization of Lemma 42.1.

Part (5) follows from Morphisms, Lemma 41.10.

Part (6) follows from Morphisms, Lemma 38.7.

Part (7) follows from part (6) and Lemma 38.2.

Part (8) follows from part (7) and Morphisms, Lemma 19.16.

### 43. Projective schemes

**Lemma 43.1.** Let \( \mathcal{L} \) be an ample invertible sheaf. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent

1. \( X \to S \) is projective,
2. \( X \to S \) is H-projective,
3. \( X \to S \) is quasi-projective and proper,
4. \( X \to S \) is H-quasi-projective and proper,
(5) $X \to S$ is proper and $X$ has an ample invertible sheaf,
(6) $X \to S$ is proper and there exists an $f$-ample invertible sheaf,
(7) $X \to S$ is proper and there exists an $f$-very ample invertible sheaf,
(8) there is a quasi-coherent graded $\mathcal{O}_S$-algebra $A$ generated by $A_1$ over $A_0$ with $A_1$ a finite type $\mathcal{O}_S$-module such that $X = \text{Proj}_S(A)$.

Proof. Observe first that in each case the morphism $f$ is proper, see Morphisms, Lemmas 41.3 and 41.5. Hence it suffices to prove the equivalence of the notions in case $f$ is a proper morphism. We will use this without further mention in the following.

The equivalences (1) $\iff$ (3) and (2) $\iff$ (4) are Morphisms, Lemma 41.13.

The implication (2) $\Rightarrow$ (1) is Morphisms, Lemma 41.3.

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are Morphisms, Lemma 41.16.

The implication (1) $\Rightarrow$ (7) is immediate from Morphisms, Definitions 41.1 and 36.1.

The conditions (3) and (6) are equivalent by Morphisms, Definition 38.1.

Thus (1) – (4), (6) are equivalent and imply (7). By Lemma 42.1 conditions (3), (5), and (7) are equivalent. Thus we see that (1) – (7) are equivalent.

By Divisors, Lemma 27.5 we see that (8) implies (1). Conversely, if (2) holds, then we can choose a closed immersion
$$i : X \to P^n_S = \text{Proj}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$
See Constructions, Lemma 21.4 for the equality. By Divisors, Lemma 28.1 we see that $X$ is the relative $\text{Proj}$ of a quasi-coherent graded quotient algebra $A$ of $\mathcal{O}_S[T_0, \ldots, T_n]$. Then $A$ satisfies the conditions of (8). □

Lemma 43.2. Let $S$ be a scheme which has an ample invertible sheaf. Let $P_S$ be the full subcategory of the category of schemes over $S$ satisfying the equivalent conditions of Lemma 43.1

(1) if $S' \to S$ is a morphism of schemes and $S'$ has an ample invertible sheaf, then base change determines a functor $P_S \to P_{S'}$,
(2) if $X \in P_S$ and $Y \in P_X$, then $Y \in P_S$,
(3) the category $P_S$ is closed under fibre products,
(4) the category $P_S$ is closed under finite disjoint unions,
(5) if $X \to S$ is finite, then $X$ is in $P_S$,
(6) add more here.

Proof. Part (1) follows from Morphisms, Lemma 41.9.

Part (2) follows from the fifth characterization of Lemma 43.1 and the fact that compositions of proper morphisms are proper (Morphisms, Lemma 39.4).

If $X \to S$ and $Y \to S$ are projective, then $X \times_SY \to Y$ is projective by Morphisms, Lemma 41.9. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fifth characterization of Lemma 43.1.

Part (5) follows from Morphisms, Lemma 42.15. □

Here is a slightly different type of result.
Lemma 43.3. Let \( f : X \to Y \) be a proper morphism of schemes. Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module. Let \( y \in Y \) be a point such that \( \mathcal{L}_y \) is ample on \( X_y \). Then there is an open neighbourhood \( V \subset Y \) of \( y \) such that \( \mathcal{L}_{f^{-1}(V)} \) is ample on \( f^{-1}(V)/V \).

Proof. We may assume \( Y \) is affine. Then we find a directed set \( I \) and an inverse system of morphisms \( X_i \to Y_i \) of schemes with \( Y_i \) of finite type over \( \mathbf{Z} \), with affine transition morphisms \( X_i \to X_i' \) and \( Y_i \to Y_i' \), with \( X_i \to Y_i \) proper, such that \( X \to Y = \lim(X_i \to Y_i) \). See Limits, Lemma \[13.3\] After shrinking \( I \) we can assume we have a compatible system of invertible \( \mathcal{O}_{X_i} \)-modules \( \mathcal{L}_i \) pulling back to \( \mathcal{L} \), see Limits, Lemma \[10.3\] Let \( y_i \in Y_i \) be the image of \( y \). Then \( \kappa(y) = \colim \kappa(y_i) \). Hence for some \( i \) we have \( \mathcal{L}_{i,y_i} \) is ample on \( X_{i,y_i} \), by Limits, Lemma \[11.15\] By Cohomology of Schemes, Lemma \[21.4\] we find an open neighbourhood \( V_i \subset Y_i \) of \( y_i \) such that \( \mathcal{L}_i \) restricted to \( f^{-1}(V_i) \) is ample relative to \( V_i \). Letting \( V \subset Y \) be the inverse image of \( V_i \) finishes the proof (hints: use Morphisms, Lemma \[35.9\] and the fact that \( X \to Y \times Y, X_i \) is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma \[35.7\].)

44. Closed points in fibres

Lemma 44.1. Let \( f : X \to S \) be a morphism of schemes. Let \( Z \subset X \) be a closed subscheme. Let \( s \in S \). Assume

1. \( Z \) is irreducible with generic point \( \eta \),
2. \( X \) is irreducible,
3. \( f \) is dominant,
4. \( f \) is locally of finite type,
5. \( \dim(X_s) \leq \dim(X_\eta) \),
6. \( Z \) is locally principal in \( X \), and
7. \( Z_\eta = \emptyset \).

Then the fibre \( Z_s \) is (set theoretically) a union of irreducible components of \( X_s \).

Proof. Let \( X_{\text{red}} \) denote the reduction of \( X \). Then \( Z \cap X_{\text{red}} \) is a locally principal closed subscheme of \( X_{\text{red}} \), see Divisors, Lemma \[13.11\] Hence we may assume that \( X \) is reduced. In other words \( X \) is integral, see Properties, Lemma \[3.4\] In this case the morphism \( X \to S \) factors through \( S_{\text{red}} \), see Schemes, Lemma \[12.6\] Thus we may replace \( S \) by \( S_{\text{red}} \) and assume that \( S \) is integral too.

The assertion that \( f \) is dominant signifies that the generic point of \( X \) is mapped to \( \eta \), see Morphisms, Lemma \[8.5\] Moreover, the scheme \( X_\eta \) is an integral scheme which is locally of finite type over the field \( \kappa(\eta) \). Hence \( d = \dim(X_\eta) \geq 0 \) is equal to \( \dim_X(x_\eta) \) for every point \( x \) of \( X_\eta \), see Algebra, Lemmas \[113.4\] and \[113.5\] In view of Morphisms, Lemma \[27.4\] and condition (5) we conclude that \( \dim_X(X_s) = d \) for every \( x \in X_s \).

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists \( x \in Z_s \) which is a generic point of an irreducible component of \( Z_s \) but not a generic point of any irreducible component of \( X_s \). Then we see that \( \dim_X(Z_s) \leq d - 1 \), because \( \dim_X(X_s) = d \) and in a neighbourhood of \( x \) in \( X_s \) the closed subscheme \( Z_s \) does not contain any of the irreducible components of \( X_s \). Hence after replacing \( X \) by an open neighbourhood of \( x \) we may assume
that \( \dim_z(Z_{f(z)}) \leq d - 1 \) for all \( z \in Z \), see Morphisms, Lemma 27.4. Let \( \xi' \in Z \) be a generic point of an irreducible component of \( Z \) and set \( s' = f(\xi) \). As \( Z \neq X \) is locally principal we see that \( \dim(\mathcal{O}_{X, \xi}) = 1 \), see Algebra, Lemma 59.10 (this is where we use \( X \) is Noetherian). Let \( \xi \in X \) be the generic point of \( X \) and let \( \xi_1 \) be a generic point of any irreducible component of \( X_{s'} \) which contains \( \xi' \). Then we see that we have the specializations

\[
\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.
\]

As \( \dim(\mathcal{O}_{X, \xi}) = 1 \) one of the two specializations has to be an equality. By assumption \( s' \neq \eta \), hence the first specialization is not an equality. Hence \( \xi' = \xi_1 \) is a generic point of an irreducible component of \( X_{s'} \). Applying Morphisms, Lemma 27.4 one more time this implies \( \dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_{\eta}) = d \) which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point \( x \in X \) lying over \( s \) such that \( x \) is a generic point of an irreducible component of \( Z \), but not a generic point of any of the irreducible components of \( X_{s'} \). Let \( U \subset S \) be an affine neighbourhood of \( s \) and let \( V \subset X \) be an affine neighbourhood of \( x \) with \( f(V) \subset U \). Write \( U = \text{Spec}(A) \) and \( V = \text{Spec}(B) \) so that \( f|_V \) is given by a ring map \( A \to B \). Let \( q \subset B \), resp. \( p \subset A \) be the prime corresponding to \( x \), resp. \( s \). After possibly shrinking \( V \) we may assume \( Z \cap V \) is cut out by some element \( q \in B \). Denote \( K \) the fraction field of \( A \). What we know at this point is the following:

1. \( A \subset B \) is a finitely generated extension of domains,
2. the element \( g \otimes 1 \) is invertible in \( B \otimes_A K \),
3. \( d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(p)) \),
4. \( g \otimes 1 \) is not a unit of \( B \otimes_A \kappa(p) \), and
5. \( g \otimes 1 \) is not in any of the minimal primes of \( B \otimes_A \kappa(p) \).

We are seeking a contradiction.

Pick elements \( x_1, \ldots, x_n \in B \) which generate \( B \) over \( A \). For a finitely generated \( \mathbb{Z} \)-algebra \( A_0 \subset A \) let \( B_0 \subset B \) be the \( A_0 \)-subalgebra generated by \( x_1, \ldots, x_n \), denote \( K_0 \) the fraction field of \( A_0 \), and set \( p_0 = A_0 \cap p \). We claim that when \( A_0 \) is large enough then (1) – (5) also hold for the system \((A_0 \subset B_0, g, p_0)\).

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write \((g \otimes 1)h = 1\) for some \( h \otimes 1/a \in B \otimes_A K \). Write \( g = \sum a_j x^j \), \( h = \sum a'_j x^j \) (multi-index notation) for some coefficients \( a_j, a'_j \in A \). As soon as \( A_0 \) contains \( a \) and the \( a_j, a'_j \) then (2) holds because \( B_0 \otimes_{A_0} K_0 \subset B \otimes_A K \) (as localizations of the injective map \( B_0 \to B \)). To achieve (3) consider the exact sequence

\[
0 \to I \to A[X_1, \ldots, X_n] \to B \to 0
\]

which defines \( I \) where the second map sends \( X_i \) to \( x_i \). Since \( \otimes \) is right exact we see that \( I \otimes_A K \), respectively \( I \otimes_A \kappa(p) \) is the kernel of the surjection \( K[X_1, \ldots, X_n] \to B \otimes_A K \), respectively \( \kappa(p)[X_1, \ldots, X_n] \to B \otimes_A \kappa(p) \). As a polynomial ring over a field is Noetherian there exist finitely many elements \( h_j \in I \), \( j = 1, \ldots, m \) which generate \( I \otimes_A K \) and \( I \otimes_A \kappa(p) \). Write \( h_j = \sum a_{j, l} X^l \). As soon as \( A_0 \) contains all \( a_{j, l} \) we get to the situation where

\[
B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(p_0) \otimes_{\kappa(p_0)} \kappa(p) = B \otimes_A \kappa(p).
\]
By either Morphisms, Lemma 27.3 or Algebra, Lemma 115.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(p_0) \subset B \otimes_A \kappa(p)$ each minimal prime of $B_0 \otimes_{A_0} \kappa(p_0)$ lies under a minimal prime of $B \otimes_A \kappa(p)$ by Algebra, Lemma 29.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. \hfill \square

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 14.3.

**Lemma 44.2.** Let $A \to B$ be a local homomorphism of local rings, and $g \in m_B$.

Assume

1. $A$ and $B$ are domains and $A \subset B$,
2. $B$ is essentially of finite type over $A$,
3. $g$ is not contained in any minimal prime over $m_AB$, and
4. $\dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B)) = \text{trdeg}_A(B)$.

Then $A \subset B/gB$, i.e., the generic point of $\text{Spec}(A)$ is in the image of the morphism $\text{Spec}(B/gB) \to \text{Spec}(A)$.

**Proof.** Note that the two assertions are equivalent by Algebra, Lemma 29.6. To start the proof let $C$ be an $A$-algebra of finite type and $q$ a prime of $C$ such that $B = C_q$. Of course we may assume that $C$ is a domain and that $g \in C$. After replacing $C$ by a localization we see that $\dim(C/m_AC) = \dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B))$, see Morphisms, Lemma 27.1. Setting $K$ equal to the fraction field of $A$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $\text{Spec}(C) \to \text{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_q \otimes_A K$. As $C_q$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \not\in q$. Thus after replacing $C$ by $C_h$ we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of $C$ to make sure that the minimal primes of $C/m_AC$ correspond one-to-one with the minimal primes of $B/m_AB$. At this point we apply Lemma 14.1 to $X = \text{Spec}(C) \to \text{Spec}(A) = S$ and the locally closed subscheme $Z = \text{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(m_A)$ has to contain an irreducible component of $X \otimes \kappa(m_A) = \text{Spec}(C/m_AC)$. But this contradicts the assumption that $g$ is not contained in any prime minimal over $m_AB$. The lemma follows. \hfill \square

**Lemma 44.3.** Let $A \to B$ be a local homomorphism of local rings. Assume

1. $A$ and $B$ are domains and $A \subset B$,
2. $B$ is essentially of finite type over $A$, and
3. $B$ is flat over $A$.

Then we have

$$\dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B)) = \text{trdeg}_A(B).$$

**Proof.** Let $C$ be an $A$-algebra of finite type and $q$ a prime of $C$ such that $B = C_q$. We may assume $C$ is a domain. We have $\dim_q(C/m_AC) = \dim(B/m_AB) + \text{trdeg}_{\kappa(m_A)}(\kappa(m_B))$, see Morphisms, Lemma 27.1. Setting $K$ equal to the fraction field of $A$ we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_q(C/m_AC) = \dim(C \otimes_A K)$. Choose a valuation
ring $A'$ in $K$ dominating $A$, see Algebra, Lemma [49.2]. Set $C' = C \otimes_A A'$. Choose a prime $q'$ of $C'$ lying over $q$; such a prime exists because

\[ C'/m_A C' = C/m_A C \otimes_{k(m_A)} k(A') \]

which proves that $C/m_A C \to C'/m_A C'$ is faithfully flat. This also proves that $\dim(q(C/m_A C)) = \dim(q'(C'/m_A C'))$, see Algebra, Lemma [115.6]. Note that $B' = C'_q$ is a localization of $B \otimes_A A'$. Hence $B'$ is flat over $A'$. The generic fibre $B' \otimes_A K$ is a localization of $B \otimes_A K$. Hence $B'$ is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{q'}(C'/m_A C') = \dim(C' \otimes_A K)$ which implies the desired equality $\dim_{q}(C/m_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where $A$ is a valuation ring.

Let $A \subset B$ be as in the lemma with $A$ a valuation ring. As before write $B = C_q$ for some domain $C$ of finite type over $A$. By Algebra, Lemma [124.9] we obtain $\dim(C/m_A C) = \dim(C \otimes_A K)$ and we win. \[ \square \]

**Lemma 44.4.** Let $f : X \to S$ be a morphism of schemes. Let $x \leadsto x'$ be a specialization of points in $X$. Set $s = f(x)$ and $s' = f(x')$. Assume

1. $x'$ is a closed point of $X_{s'}$, and
2. $f$ is locally of finite type.

Then the set

\[ \{x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \leadsto x_1 \leadsto x' \} \]

is dense in the closure of $x$ in $X_s$.

**Proof.** We apply Schemes, Lemma [20.4] to the specialization $x \leadsto x'$. This produces a morphism $\varphi : \text{Spec}(B) \to X$ where $B$ is a valuation ring such that $\varphi$ maps the generic point to $x$ and the closed point to $x'$. We may also assume that $\kappa(x)$ is the fraction field of $B$. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see Algebra, Lemma [49.6]) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$. Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(B) & \longrightarrow & X_A & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & S
\end{array}
\]

The generic (resp. closed) point of $B$ maps to a point $x_A$ (resp. $x_A'$) of $X_A$ lying over the generic (resp. closed) point of $\text{Spec}(A)$. Note that $x_A'$ is a closed point of the special fibre of $X_A$ by Morphisms, Lemma [19.4]. Note that the generic fibre of $X_A \to \text{Spec}(A)$ is isomorphic to $X_s$. Thus we have reduced the lemma to the case where $S$ is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_{x}(X_{\eta})$. If $\dim_{x}(X_{\eta}) = 0$, then there are no other points of $X_{\eta}$ specializing to $x$ and $x$ is closed in its fibre, see Morphisms, Lemma [19.6] and the result holds. Assume $\dim_{x}(X_{\eta}) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\{x\}$ of $X$, see Schemes, Definition [12.5]. To prove the lemma we may replace $X$ by $X'$ as this only decreases $\dim_{x}(X_{\eta})$. Hence we may also assume that $X$ is an integral scheme and that $x$ is its generic point. In addition, we may replace
X by an affine neighbourhood of \(x'\). Thus we have \(X = \text{Spec}(B)\) where \(A \subset B\) is a finite type extension of domains. Note that in this case \(\dim_x(X_\eta) = \dim(X_\eta) = \dim(X'_\eta)\), and that in fact \(X'_\eta\) is equidimensional, see Algebra, Lemma [124.9]

Let \(W \subset X_\eta\) be a proper closed subset (this is the subset we want to “avoid”). As \(X_\eta\) is of finite type over a field we see that \(W\) has finitely many irreducible components \(W = W_1 \cup \ldots \cup W_n\). Let \(q_j \subset B\), \(j = 1, \ldots, r\) be the corresponding prime ideals. Let \(q \subset B\) be the maximal ideal corresponding to the point \(x'\). Let \(p_1, \ldots, p_s \subset B\) be the minimal primes lying over \(m_A B\). There are finitely many as these correspond to the irreducible components of the Noetherian scheme \(X'\).

Moreover, each of these irreducible components has dimension \(> 0\) (see above) hence we see that \(p_i \neq q\) for all \(i\). Now, pick an element \(g \in q\) such that \(g \notin q_j\) for all \(j\) and \(g \notin p_i\) for all \(i\), see Algebra, Lemma [14.2]. Denote \(Z \subset X\) the locally principal closed subscheme defined by \(g\). Let \(Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}\), \(n \geq 0\) be the decomposition of the generic fibre of \(Z\) into irreducible components (finitely many as the generic fibre is Noetherian). Denote \(Z_1 \subset X\) the closure of \(Z_{1,\eta}\). After replacing \(X\) by a smaller affine neighbourhood we may assume that \(x' \in Z_i\) for each \(i = 1, \ldots, n\). By construction \(Z \cap X'_\eta\) does not contain any irreducible component of \(X'_\eta\). Hence by Lemma [44.1] we conclude that \(Z_\eta \neq \emptyset\). In other words \(n \geq 1\).

Letting \(x_1 \in Z_1\) be the generic point we see that \(x_1 \rightsquigarrow x'\) and \(f(x_1) = \eta\). Also, by construction \(Z_{1,\eta} \cap W_j \subset W_j\) is a proper closed subset. Hence every irreducible component of \(Z_{1,\eta} \cap W_j\) has codimension \(\geq 2\) in \(X_\eta\) whereas \(\text{codim}(Z_{1,\eta}, X_\eta) = 1\) by Algebra, Lemma [59.10]. Thus \(W \cap Z_{1,\eta}\) is a proper closed subset. At this point we see that the induction hypothesis applies to \(Z_1 \to S\) and the specialization \(x_1 \rightsquigarrow x'\). This produces a closed point \(x_2\) of \(Z_{1,\eta}\) not contained in \(W\) which specializes to \(x'\). Thus we obtain \(x \rightsquigarrow x_2 \rightsquigarrow x'\), the point \(x_2\) is closed in \(X_\eta\), and \(x_2 \notin W\) as desired. \(\square\)

**Remark 44.5.** The proof of Lemma [44.4] actually shows that there exists a sequence of specializations

\[x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_n \rightsquigarrow x'\]

where all \(x_i\) are in the fibre \(X_\eta\), each specialization is immediate, and \(x_n\) is a closed point of \(X_\eta\). The integer \(d = \text{trdeg}_{m_A}(\kappa(x)) = \dim(\{x\})\) where the closure is taken in \(X_\eta\). Moreover, the points \(x_i\) can be chosen to avoid any closed subset of \(X_\eta\) which does not contain the point \(x\).

Examples, Section [32] shows that the following lemma is false if \(A\) is not assumed Noetherian.

**Lemma 44.6.** Let \(\varphi : A \to B\) be a local ring map of local rings. Let \(V \subset \text{Spec}(B)\) be an open subscheme which contains at least one prime not lying over \(m_A\). Assume \(A\) is Noetherian, \(\varphi\) essentially of finite type, and \(A/m_A \subset B/m_B\) is finite. Then there exists a \(q \in V\), \(m_A \neq q \cap A\) such that \(A \to B/q\) is the localization of a quasi-finite ring map.

**Proof.** Since \(A\) is Noetherian and \(A \to B\) is essentially of finite type, we know that \(B\) is Noetherian too. By Properties, Lemma [6.4] the topological space \(\text{Spec}(B) \setminus \{m_B\}\) is Jacobson. Hence we can choose a closed point \(q\) which is contained in the nonempty open

\[V \setminus \{q \subset B \mid m_A = q \cap A\}.\]
Suppose that \( \text{Spec}(B/q) \) has two points, namely \( m_B \) and \( q \) and \( q \) does not lie over \( m_A \). Write \( B/q = C_m \) for some finite type \( A \)-algebra \( C \) and prime ideal \( m \). Then \( A \rightarrow C \) is quasi-finite at \( m \) by Algebra, Lemma 121.2 (2). Hence by Algebra, Lemma 122.14 we see that after replacing \( C \) by a principal localization the ring map \( A \rightarrow C \) is quasi-finite.

**Lemma 44.7.** Let \( f : X \rightarrow S \) be a morphism of schemes. Let \( x \in X \) with image \( s \in S \). Let \( U \subset X \) be an open subscheme. Assume \( f \) locally of finite type, \( S \) locally Noetherian, \( x \) a closed point of \( X_s \), and assume there exists a point \( x' \in U \) with \( x' \rightarrow x \) and \( f(x') \neq s \). Then there exists a closed subscheme \( Z \subset X \) such that (a) \( x \in Z \), (b) \( f|_Z : Z \rightarrow S \) is quasi-finite at \( x \), and (c) there exists a \( z \in Z \), \( z \in U \), \( z \rightarrow x \) and \( f(z) \neq s \).

**Proof.** This is a reformulation of Lemma 44.6. Namely, set \( A = \mathcal{O}_{S,x} \) and \( B = \mathcal{O}_{X,x} \). Denote \( V \subset \text{Spec}(B) \) the inverse image of \( U \). The ring map \( f^\sharp : A \rightarrow B \) is essentially of finite type. By assumption there exists at least one point of \( V \) which does not map to the closed point of \( \text{Spec}(A) \). Hence all the assumptions of Lemma 44.6 hold and we obtain a prime \( q \subset B \) which does not lie over \( m_A \) and such that \( A \rightarrow B/q \) is the localization of a quasi-finite ring map. Let \( z \in X \) be the image of the point \( q \) under the canonical morphism \( \text{Spec}(B) \rightarrow X \). Set \( Z = \{ z \} \) with the induced reduced scheme structure. As \( z \rightarrow x \) we see that \( x \in Z \) and \( \mathcal{O}_{Z,x} = B/q \). By construction \( Z \rightarrow S \) is quasi-finite at \( x \).

**Remark 44.8.** We can use Lemma 44.6 or its variant Lemma 44.7 to give an alternative proof of Lemma 44.4 in case \( S \) is locally Noetherian. Here is a rough sketch. Namely, first replace \( S \) by the spectrum of the local ring at \( s' \). Then we may use induction on \( \dim(S) \). The case \( \dim(S) = 0 \) is trivial because then \( s' = s \).

Replace \( X \) by the reduced induced scheme structure on \( \{ x \} \). Apply Lemma 44.7 to \( X \rightarrow S \) and \( x' \rightarrow s' \) and any nonempty open \( U \subset X \) containing \( x \). This gives us a closed subscheme \( x' \in Z \subset X \) a point \( z \in Z \) such that \( Z \rightarrow S \) is quasi-finite at \( x' \) and such that \( f(z) \neq s' \). Then \( z \) is a closed point of \( X_{f(z)} \) and \( z \rightarrow x' \). As \( f(z) \neq s' \) we see \( \dim(\mathcal{O}_{S,f(z)}) < \dim(S) \). Since \( x \) is the generic point of \( X \) we see \( x \rightarrow z \), hence \( s = f(x) \rightarrow f(z) \). Apply the induction hypothesis to \( s \rightarrow f(z) \) and \( z \rightarrow f(z) \) to win.

**Lemma 44.9.** Suppose that \( f : X \rightarrow S \) is locally of finite type, \( S \) locally Noetherian, \( x \in X \) a closed point of its fibre \( X_s \), and \( U \subset X \) an open subscheme such that \( U \cap X_s = \emptyset \) and \( x \in \overline{U} \), then the conclusions of Lemma 44.7 hold.

**Proof.** Namely, we can reduce this to the cited lemma as follows: First we replace \( X \) and \( S \) by affine neighbourhoods of \( x \) and \( s \). Then \( X \) is Noetherian, in particular \( U \) is quasi-compact (see Morphisms, Lemma 14.6 and Topology, Lemmas 9.2 and 12.13). Hence there exists a specialization \( x' \rightarrow x \) with \( x' \in U \) (see Morphisms, Lemma 6.5). Note that \( f(x') \neq s \). Thus we see all hypotheses of the lemma are satisfied and we win.

45. Stein factorization

Stein factorization is the statement that a proper morphism \( f : X \rightarrow S \) with \( f_* \mathcal{O}_X = \mathcal{O}_S \) has connected fibres.
Lemma 45.1. Let $S$ be a scheme. Let $f : X \to S$ be a universally closed and quasi-separated morphism. There exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f} & S' \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow{\pi} & S
\end{array}
$$

with the following properties:

1. The morphism $f'$ is universally closed, quasi-compact, quasi-separated, and surjective,
2. The morphism $\pi : S' \to S$ is integral,
3. We have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
4. We have $S' = \text{Spec}_S(f_*\mathcal{O}_X)$, and
5. $S'$ is the normalization of $S$ in $X$, see Morphisms, Definition 50.3.

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms, Lemma 39.10 the morphism $f$ is quasi-compact. Hence the normalization $S'$ of $S$ in $X$ is defined (Morphisms, Definition 50.3) and we have the factorization $X \to S' \to S$. By Morphisms, Lemma 50.11 we have (2), (4), and (5). The morphism $f'$ is universally closed by Morphisms, Lemma 39.7. It is quasi-compact by Schemes, Lemma 21.13 and quasi-separated by Schemes, Lemma 21.14.

To show the remaining statements we may assume the base scheme $S$ is affine, say $S = \text{Spec}(R)$. Then $S' = \text{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral $R$-algebra. Thus it is clear that $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 24.1 and hence equal to $A$). This proves (3).

Let us show that $f'$ is surjective. As $f'$ is universally closed (see above) the image of $f'$ is a closed subset $V(I) \subset S' = \text{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^2(h)$ is a global section of the structure sheaf of $X$ which vanishes at every point. As $X$ is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|_X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words $I$ is contained in the radical ideal of $A$ and we conclude that $V(I) = S'$ as desired. □

Lemma 45.2. In Lemma 45.1 assume in addition that $f$ is locally of finite type. Then for $y \in Y$ the fibre $\pi^{-1}(\{y\}) = \{y_1, \ldots, y_n\}$ is finite and the field extensions $\kappa(y_i)/\kappa(y)$ are finite.

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{y\})$, see Algebra, Lemma 35.20. As $f'$ is surjective, we find that $|X_y| \to \pi^{-1}(\{y\})$ is surjective. Observe that $X_y$ is a quasi-separated scheme of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus $X_y$ is Noetherian (Morphisms, Lemma 14.6). A topological argument (omitted) now shows that $\pi^{-1}(\{y\})$ is finite. For each $i$ we can pick a finite type point $x_i \in X_y$ mapping to $y_i$ (Morphisms, Lemma 15.7). We conclude that $\kappa(y_i)/\kappa(y)$ is finite: $x_i$ can be represented by a morphism $\text{Spec}(k_i) \to X_y$ of finite type (by our definition of finite type points) and hence $\text{Spec}(k_i) \to y = \text{Spec}(\kappa(y))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(y)$ is finite (Morphisms, Lemma 15.1). □
Lemma 45.3. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then $X_s$ is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre $X_u$.

Proof. If $X_s$ is geometrically connected, then any base change of it is connected. On the other hand, suppose that $X_s$ is not geometrically connected. Then by Varieties, Lemma 7.11 we see that $X_s \times_{\text{Spec} \kappa(s)} \text{Spec}(k)$ is disconnected for some finite separable field extension $\kappa(s) \subset k$. By Lemma 31.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k$. In this case $X_u$ is disconnected.

□

Theorem 45.4 (Stein factorization; Noetherian case). Let $S$ be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & S' \\
\downarrow f & & \downarrow \pi \\
S & &
\end{array}
$$

with the following properties:

1. the morphism $f'$ is proper with geometrically connected fibres,
2. the morphism $\pi : S' \to S$ is finite,
3. we have $f'_* \mathcal{O}_X = \mathcal{O}_{S'}$,
4. we have $S' = \text{Spec}_S(f_* \mathcal{O}_X)$, and
5. $S'$ is the normalization of $S$ in $X$, see Morphisms, Definition 50.3.

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 45.1. Note that besides the conclusions of Lemma 45.1 we also have that $f'$ is separated (Schemes, Lemma 21.14) and finite type (Morphisms, Lemma 14.8). Hence $f'$ is proper. By Cohomology of Schemes, Proposition 19.1 we see that $f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that $\pi$ is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of $f'$ are geometrically connected. It is clear from the discussion above that we may replace $S$ by $S'$, and we may therefore assume that $S$ is Noetherian, affine, $f : X \to S$ is proper, and $f_* \mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of $S$. We have to show that $X_s$ is geometrically connected. By Lemma 45.3 we see that it suffices to show $X_u$ is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume $U$ is affine. Thus $U$ is Noetherian (Morphisms, Lemma 14.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 39.5), and that also $(f_U)_* \mathcal{O}_{X_U} = \mathcal{O}_U$ (Cohomology of Schemes, Lemma 5.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre $X_s$ is connected.

At this point we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 20.7. It tells us that

$$
\mathcal{O}_{S,s}^\wedge = \lim_n H^0(X_n, \mathcal{O}_{X_n})
$$

where $X_n$ is the $n$th infinitesimal neighbourhood of $X_s$. Since the underlying topological space of $X_n$ is equal to that of $X_s$ we see that if $X_s = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all $n$. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$.
It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on $X_n$. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $O_{S,s}$ is a local ring. Thus the assumption was wrong, i.e., $X_s$ is connected, and we win. \hfill \Box

\textbf{Theorem 45.5} (Stein factorization; general case). Let $S$ be a scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & S' \\
\downarrow f & & \downarrow \pi \\
S & & 
\end{array}
$$

with the following properties:

1. the morphism $f'$ is proper with geometrically connected fibres,
2. the morphism $\pi : S' \to S$ is integral,
3. we have $f_*O_X = O_{S'}$,
4. we have $S' = \text{Spec}_S(f_*O_X)$, and
5. $S'$ is the normalization of $S$ in $X$, see Morphisms, Definition 50.3.

\textbf{Proof.} We may apply Lemma 45.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 45.1 we also have that $f'$ is separated (Schemes, Lemma 21.14) and finite type (Morphisms, Lemma 14.8). Hence $f'$ is proper. At this point we have proved all of the statements except for the statement that $f'$ has geometrically connected fibres.

We may assume that $S = \text{Spec}(R)$ is affine. Set $R' = \Gamma(X,O_X)$. Then $S' = \text{Spec}(R')$. Thus we may replace $S$ by $S'$ and assume that $S = \text{Spec}(R)$ is affine $R = \Gamma(X,O_X)$. Next, let $s \in S$ be a point. Let $U \to S$ be an étale morphism of affine schemes and let $u \in U$ be a point mapping to $s$. Let $X_U \to U$ be the base change of $X$. By Lemma 45.3 it suffices to show that the fibre of $X_U \to U$ over $u$ is connected. By Cohomology of Schemes, Lemma 5.2 we see that $\Gamma(X_U,O_{X_U}) = \Gamma(U,O_U)$. Hence we have to show: Given $S = \text{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X,O_X) = R$ and $s \in S$ is a point, the fibre $X_s$ is connected.

By Limits, Lemma 13.3 we can write $(X \to S) = \lim(X_i \to S_i)$ with $X_i \to S_i$ proper and of finite presentation and $S_i$ Noetherian. For $i$ large enough $S_i$ is affine (Limits, Lemma 4.13). Say $S_i = \text{Spec}(R_i)$. Let $R'_i = \Gamma(X_i,O_{X_i})$. Observe that we have ring maps $R_i \to R'_i \to R$. Namely, we have the first because $X_i$ is a scheme over $R_i$ and the second because we have $X \to X_i$ and $R = \Gamma(X,O_X)$. Note that $R = \text{colim} R'_i$ by Limits, Lemma 4.7. Then

$$
\begin{array}{ccc}
X & \xrightarrow{X_i} & S_i \\
\downarrow & & \downarrow \\
S & \xrightarrow{S'_i} & S_i
\end{array}
$$

is commutative with $S'_i = \text{Spec}(R'_i)$. Let $s'_i \in S'_i$ be the image of $s$. We have $X_s = \lim X_{i,s'_i}$ because $X = \lim X_i$, $S = \lim S'_i$, and $\kappa(s) = \text{colim} \kappa(s'_i)$. Now let $X_s = U \amalg V$ with $U$ and $V$ open and closed. Then $U,V$ are the inverse images of opens $U_i,V_i$ in $X_{i,s'_i}$ (Limits, Lemma 4.11). By Theorem 45.4 the fibres of $X_i \to S'_i$ are connected, hence either $U$ or $V$ is empty. This finishes the proof. \hfill \Box

Here is an application.
Lemma 45.6. Let \( f : X \rightarrow Y \) be a morphism of schemes. Assume

1. \( f \) is proper,
2. \( Y \) is integral with generic point \( \xi \),
3. \( Y \) is normal,
4. \( X \) is reduced,
5. every generic point of an irreducible component of \( X \) maps to \( \xi \),
6. we have \( H^0(X_\xi, \mathcal{O}) = \kappa(\xi) \).

Then \( f_*\mathcal{O}_X = \mathcal{O}_Y \) and \( f \) has geometrically connected fibres.

Proof. Apply Theorem 45.5 to get a factorization \( X \rightarrow Y' \rightarrow Y \). It is enough to show that \( Y' = Y \). This will follow from Morphisms, Lemma 51.8. Namely, \( Y' \) is reduced because \( X \) is reduced (Morphisms, Lemma 50.8). The morphism \( Y' \rightarrow Y \) is integral by the theorem cited above. Every generic point of \( Y' \) lies over \( \xi \) by Morphisms, Lemma 50.9 and assumption (5). On the other hand, since \( Y' \) is the relative spectrum of \( f_*\mathcal{O}_X \) we see that the scheme theoretic fibre \( Y'_\xi \) is the spectrum of \( H^0(X_\xi, \mathcal{O}) \) which is equal to \( \kappa(\xi) \) by assumption. Hence \( Y' \) is an integral scheme with function field equal to the function field of \( Y \). This finishes the proof. □

Here is another application.

Lemma 45.7. Let \( X \rightarrow S \) be a flat proper morphism of finite presentation. Let \( n_{X/S} \) be the function on \( Y \) counting the numbers of geometric connected components of fibres of \( f \) introduced in Lemma 26.3. Then \( n_{X/S} \) is lower semi-continuous.

Proof. Let \( s \in S \). Set \( n = n_{X/S}(s) \). Note that \( n < \infty \) as the geometric fibre of \( X \rightarrow S \) at \( s \) is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood \( V \) of \( s \) such that \( n_{X/S}|_V \geq n \). Let \( X \rightarrow S' \rightarrow S \) be the Stein factorization as in Theorem 45.5. By Lemma 45.2 there are finitely many points \( s'_1, \ldots, s'_m \in S' \) lying over \( s \) and the extensions \( \kappa(s'_i)/\kappa(s) \) are finite. Then Lemma 37.1 tells us that after replacing \( S \) by an étale neighbourhood of \( s \) we may assume \( S' = V_1 \amalg \ldots \amalg V_m \) as a scheme with \( s'_i \in V_i \) and \( \kappa(s'_i)/\kappa(s) \) purely inseparable. Then the schemes \( X_\xi \) are geometrically connected over \( \kappa(s) \), hence \( m = n \). The schemes \( X_i = (f')^{-1}(V_i) \), \( i = 1, \ldots, n \) are flat and of finite presentation over \( S \). Hence the image of \( X_i \rightarrow S \) is open (Morphisms, Lemma 24.9). Thus in a neighbourhood of \( s \) we see that \( n_{X/S} \) is at least \( n \). □

Lemma 45.8. Let \( f : X \rightarrow S \) be a morphism of schemes. Assume

1. \( f \) is proper, flat, and of finite presentation, and
2. the geometric fibres of \( f \) are reduced.

Then the function \( n_{X/S} : S \rightarrow \mathbb{Z} \) counting the numbers of geometric connected components of fibres of \( f \) is locally constant.

Proof. By Lemma 45.7 the function \( n_{X/S} \) is lower semicontinuous. For \( s \in S \) consider the \( \kappa(s) \)-algebra

\[
A = H^0(X_s, \mathcal{O}_{X_s})
\]

By Varieties, Lemma 9.3 and the fact that \( X_s \) is geometrically reduced \( A \) is a finite product of finite separable extensions of \( \kappa(s) \). Hence \( A \otimes_{\kappa(s)} \kappa(\bar{s}) \) is a product of \( \beta_0(s) = \dim_{\kappa(s)} H^0(E \otimes \kappa(s)) \) copies of \( \kappa(\bar{s}) \). Thus \( X_{\bar{s}} \) has \( \beta_0(s) = \dim_{\kappa(s)} A \) connected components. In other words, we have \( n_{X/S} = \beta_0 \) as functions on \( S \).
Thus $n^{-1}_{X/S}$ is upper semi-continuous by Derived Categories of Schemes, Lemma \[28.1\] This finishes the proof. □

A final application.

Lemma 45.9. Let $(A, I)$ be a henselian pair. Let $X \rightarrow \text{Spec}(A)$ be separated and of finite type. Set $X_0 = X \times_{\text{Spec}(A)} \text{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subscheme such that $Y \rightarrow \text{Spec}(A/I)$ is proper. Then there exists an open and closed subscheme $W \subset X$ which is proper over $A$ with $W \times_{\text{Spec}(A)} \text{Spec}(A/I) = Y$.

Proof. We will denote $T \mapsto T_0$ the base change by $\text{Spec}(A/I) \rightarrow \text{Spec}(A)$. By Chow's lemma (in the form of Limits, Lemma [12.1]) there exists a surjective proper morphism $\varphi : X' \rightarrow X$ such that $X'$ admits an immersion into $\mathbf{P}_A^n$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of $X'_0$. Suppose the lemma holds for $(X', Y')$. Let $W' \subset X'$ be the open and closed subscheme proper over $A$ such that $Y' = W'_0$. By Morphisms, Lemma [39.7] $W = \varphi(W') \subset X$ and $Q = \varphi(X' \setminus W') \subset X$ are closed subsets and by Morphisms, Lemma [39.8] $W$ is proper over $A$. The image of $W \cap Q$ in $\text{Spec}(A)$ is closed. Since $(A, I)$ is henselian, if $W \cap Q$ is nonempty, then we find that $W \cap Q$ has a point lying over $\text{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that $W$ is an open and closed subscheme of $X$ proper over $A$ with $W_0 = Y$. Thus we reduce to the case described in the next paragraph.

Assume there exists an immersion $j : X \rightarrow \mathbf{P}_A^n$ over $A$. Let $\overline{X}$ be the scheme theoretic image of $j$. Since $j$ is a quasi-compact morphism (Schemes, Lemma [21.15]) we see that $j : X \rightarrow \overline{X}$ is an open immersion (Morphisms, Lemma [7.7]). Hence the base change $j_0 : X_0 \rightarrow \overline{X}_0$ is an open immersion as well. Thus $j_0(Y) \subset \overline{X}_0$ is open. It is also closed by Morphisms, Lemma [39.7]. Suppose the lemma holds for $(\overline{X}, j_0(Y))$. Let $\overline{W} \subset \overline{X}$ be the corresponding open and closed subscheme proper over $A$ such that $j_0(Y) = \overline{W}_0$. Then $T = \overline{W} \setminus j(X)$ is closed in $\overline{W}$, hence has closed image in $\text{Spec}(A)$ by properness of $\overline{W}$ over $A$. Since $(A, I)$ is henselian, we find that if $T$ is nonempty, then there is a point of $T$ mapping into $\text{Spec}(A/I)$. This is impossible because $j_0(Y) = \overline{W}_0$ is contained in $j(X)$. Hence $\overline{W}$ is contained in $j(X)$ and we can set $W \subset X$ equal to the unique open and closed subscheme mapping isomorphically to $\overline{W}$ via $j$. Thus we reduce to the case described in the next paragraph.

Assume $X \subset \mathbf{P}_A^n$ is a closed subscheme. Then $X \rightarrow \text{Spec}(A)$ is a proper morphism. Let $Z = X_0 \setminus Y$. This is an open and closed subscheme of $X_0$ and $X_0 = Y \amalg Z$. Let $X \rightarrow X' \rightarrow \text{Spec}(A)$ be the Stein factorization as in Theorem [45.5]. Let $Y' \subset X'_0$ and $Z' \subset X'_0$ be the images of $Y$ and $Z$. Since the fibres of $X \rightarrow Z$ are geometrically connected, we see that $Y' \cap Z' = \emptyset$. Hence $X'_0 = Y' \amalg Z'$ as $X \rightarrow X'$ is surjective. Since $X' \rightarrow \text{Spec}(A)$ is integral, we see that $X'$ is the spectrum of an $A$-algebra integral over $A$. Recall that open and closed subsets of spectra correspond 1-to-1 with idempotents in the corresponding ring, see Algebra, Lemma [20.3]. Hence by More on Algebra, Lemma [10.8] we see that we may write $X' = W'' \amalg V''$ with $W''$ and $V''$ open and closed and with $Y' = W'_0$ and $Z' = V'_0$. Let $W$ be the inverse image in $X$ to finish the proof. □

46. Descending separated locally quasi-finite morphisms

02W7
In this section we show that “separated locally quasi-finite morphisms satisfy descent for fppf-coverings”. See Descent, Definition 33.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [GR71, Lemma 5.7.1]. It can also be found in [Mur95], and [ABD+66 Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

**Lemma 46.1.** Let $S$ be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.

**Proof.** Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 21.13 and Morphisms, Lemma 19.13. Hence Descent, Lemma 33.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let $(V, \varphi)$ be a descent datum relative to $X$ over $S$ and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \text{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

\[ W^1 \times_S X \xrightarrow{\varphi} \varphi(W^1 \times_S X) \]

\[ V \times_S X \]

\[ \varphi \]

\[ X \times_S V \]

\[ X \times_S X \]

\[ \xrightarrow{1} \]

\[ X \times_S X \]

\[ \xrightarrow{\text{pr}_0} \]

\[ X \]

\[ \varphi \]

\[ \xrightarrow{\text{pr}_1} \]

\[ X \times_S X \]

\[ \xrightarrow{1} \]

\[ X \times_S X \]

\[ W \]

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 24.9). Hence we conclude that $W$ is open. Moreover, it is also clearly the case that $W$ is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for $\varphi$. Hence we obtain a new descent datum $(W, \varphi')$ by restricting $\varphi$ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 38.2. Thus by Descent, Lemma 35.1 we see that the descent datum $(W, \varphi')$ is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens $W_i$ which are stable for the descent morphism $\varphi$. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data $(W, \varphi')$ is effective. This means the original descent datum is effective by glueing the schemes obtained from descending the opens $W_i$, see Descent, Lemma 32.13. \qed
47. Relative finite presentation

Let $R \to A$ be a finite type ring map. Let $M$ be an $A$-module. In More on Algebra, Section 71 we defined what it means for $M$ to be finitely presented relative to $R$. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

\textbf{Definition 47.1.} Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is \textit{finitely presented relative to $S$} or \textit{of finite presentation relative to $S$} if there exists an affine open covering $S = \bigcup V_i$ and for every $i$ an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to $S$, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to $S$.

\textbf{Lemma 47.2.} Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

1. $\mathcal{F}$ is of finite presentation relative to $S$,  
2. for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely presented relative to $\mathcal{O}_S(V)$.

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to $V$.

\textbf{Proof.} The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 47.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 71.8 it suffices to find a standard open covering $U = \bigcup U_k$ of $U$ such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick $i$ such that $f(u) \in V_i$ and then pick $j$ such that $u \in U_{ij}$. By Schemes, Lemma 17.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in $V'$ and $V_i$. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of $U$, resp. $U_{ij}$. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus $U'$ is also a standard affine open of $U$ and $U_{ij}$. By More on Algebra, Lemma 71.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V_i)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 71.3 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 71.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. \qed

\textbf{Lemma 47.3.} Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

1. If $f$ is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to $S$ if and only if $\mathcal{F}$ is of finite presentation.
2. The morphism $f$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to $S$. 

\textbf{Proof.}
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Proof. Follows immediately from the definitions, see discussion following More on Algebra, Definition 71.2.

Lemma 47.4. Let \( \pi : X \to Y \) be a finite morphism of schemes locally of finite type over a base scheme \( S \). Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Then \( \mathcal{F} \) is of finite presentation relative to \( S \) if and only if \( \pi_* \mathcal{F} \) is of finite presentation relative to \( S \).

Proof. Translation of the result of More on Algebra, Lemma 71.3 into the language of schemes.

Lemma 47.5. Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. Let \( S' \to S \) be a morphism of schemes, set \( X' = X \times_S S' \) and denote \( \mathcal{F}' \) the pullback of \( \mathcal{F} \) to \( X' \). If \( \mathcal{F} \) is of finite presentation relative to \( S \), then \( \mathcal{F}' \) is of finite presentation relative to \( S' \).

Proof. Translation of the result of More on Algebra, Lemma 71.5 into the language of schemes.

Lemma 47.6. Let \( X \to Y \to S \) be morphisms of schemes which are locally of finite type. Let \( \mathcal{G} \) be a quasi-coherent \( \mathcal{O}_Y \)-module. If \( f : X \to Y \) is locally of finite presentation and \( \mathcal{G} \) of finite presentation relative to \( S \), then \( f^* \mathcal{G} \) is of finite presentation relative to \( S \).

Proof. Translation of the result of More on Algebra, Lemma 71.6 into the language of schemes.

Lemma 47.7. Let \( X \to Y \to S \) be morphisms of schemes which are locally of finite type. Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. If \( Y \to S \) is locally of finite presentation and \( \mathcal{F} \) is of finite presentation relative to \( Y \), then \( \mathcal{F} \) is of finite presentation relative to \( S \).

Proof. Translation of the result of More on Algebra, Lemma 71.7 into the language of schemes.

Lemma 47.8. Let \( X \to S \) be a morphism of schemes which is locally of finite type. Let \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) be a short exact sequence of quasi-coherent \( \mathcal{O}_X \)-modules.

1. If \( \mathcal{F}', \mathcal{F}'' \) are finitely presented relative to \( S \), then so is \( \mathcal{F} \).
2. If \( \mathcal{F}' \) is a finite type \( \mathcal{O}_X \)-module and \( \mathcal{F} \) is finitely presented relative to \( S \), then \( \mathcal{F}'' \) is finitely presented relative to \( S \).

Proof. Translation of the result of More on Algebra, Lemma 71.9 into the language of schemes.

Lemma 47.9. Let \( X \to S \) be a morphism of schemes which is locally of finite type. Let \( \mathcal{F}, \mathcal{F}' \) be quasi-coherent \( \mathcal{O}_X \)-modules. If \( \mathcal{F} \oplus \mathcal{F}' \) is finitely presented relative to \( S \), then so are \( \mathcal{F} \) and \( \mathcal{F}' \).

Proof. Translation of the result of More on Algebra, Lemma 71.10 into the language of schemes.
48. Relative pseudo-coherence

This section is the analogue of More on Algebra, Section 72 for schemes. We strongly urge the reader to take a look at that section first. Although we have developed the material in this section and the material on pseudo-coherent complexes in Cohomology, Sections 40, 41, 42, and 43 for arbitrary complexes of \( \mathcal{O}_X \)-modules, if \( X \) is a scheme then working exclusively with objects in \( D_{\text{QCoh}}(\mathcal{O}_X) \) greatly simplifies many of the lemmas and arguments, often reducing the problem at hand immediately to the algebraic counterpart. Moreover, one of the first thing we do is to show that being relatively pseudo-coherent implies the cohomology sheaves are quasi-coherent, see Lemma 48.3. Hence, on a first reading we suggest the reader work exclusively with objects in \( D_{\text{QCoh}}(\mathcal{O}_X) \).

**Lemma 48.1.** Let \( X \to S \) be a finite type morphism of affine schemes. Let \( E \) be an object of \( D(\mathcal{O}_X) \). Let \( m \in \mathbb{Z} \). The following are equivalent

1. for some closed immersion \( i : X \to \mathbb{A}_S^n \) the object \( R^i_*E \) of \( D(\mathcal{O}_{\mathbb{A}_S^n}) \) is \( m \)-pseudo-coherent, and
2. for all closed immersions \( i : X \to \mathbb{A}_S^n \) the object \( R^i_*E \) of \( D(\mathcal{O}_{\mathbb{A}_S^n}) \) is \( m \)-pseudo-coherent.

**Proof.** Say \( S = \text{Spec}(R) \) and \( X = \text{Spec}(A) \). Let \( i \) correspond to the surjection \( \alpha : R[x_1, \ldots, x_n] \to A \) and let \( X \to \mathbb{A}_S^n \) correspond to \( \beta : R[y_1, \ldots, y_m] \to A \). Choose \( f_j \in R[x_1, \ldots, x_n] \) with \( \alpha(f_j) = \beta(y_j) \) and \( g_i \in R[y_1, \ldots, y_m] \) with \( \beta(g_i) = \alpha(x_i) \). Then we get a commutative diagram

\[
\begin{array}{ccc}
R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow{y_j \mapsto f_j} & R[x_1, \ldots, x_n] \\
\downarrow \alpha & & \downarrow \\
R[y_1, \ldots, y_m] & \xrightarrow{x_i \mapsto g_i} & A
\end{array}
\]

corresponding to the commutative diagram of closed immersions

\[
\begin{array}{ccc}
\mathbb{A}_S^{n+m} & \xleftarrow{\mathbb{A}_S^n} & \mathbb{A}_S^n \\
\downarrow & & \downarrow \\
\mathbb{A}_S^m \xleftarrow{X} & & \mathbb{A}_S^n
\end{array}
\]

Thus it suffices to show that under a closed immersion

\[
f : \mathbb{A}_S^m \to \mathbb{A}_S^{n+m}
\]

an object \( E \) of \( D(\mathcal{O}_{\mathbb{A}_S^m}) \) is \( m \)-pseudo-coherent if and only if \( Rf_*E \) is \( m \)-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 11.5 and the fact that \( f_*\mathcal{O}_{\mathbb{A}_S^m} \) is a pseudo-coherent \( \mathcal{O}_{\mathbb{A}_S^{n+m}} \)-module. The pseudo-coherence of \( f_*\mathcal{O}_{\mathbb{A}_S^m} \) is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 9.2 and More on Algebra, Lemma 72.3.\[\square\]

Recall that if \( f : X \to S \) is a morphism of scheme which is locally of finite type, then for every pair of affine opens \( U \subset X \) and \( V \subset S \) such that \( f(U) \subset V \), the ring map \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is of finite type (Morphisms, Lemma 14.2). Hence there always exist closed immersions \( U \to \mathbb{A}_V^n \) and the following definition makes sense.
Definition 48.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E$ be an object of $D(O_X)$. Let $\mathcal{F}$ be an $O_X$-module. Fix $m \in \mathbb{Z}$.

1. We say $E$ is $m$-pseudo-coherent relative to $S$ if there exists an affine open covering $S = \bigcup V_i$ and for each $i$ an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 48.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$.

2. We say $E$ is pseudo-coherent relative to $S$ if $E$ is $m$-pseudo-coherent relative to $S$ for all $m \in \mathbb{Z}$.

3. We say $\mathcal{F}$ is $m$-pseudo-coherent relative to $S$ if $\mathcal{F}$ viewed as an object of $D(O_X)$ is $m$-pseudo-coherent relative to $S$.

4. We say $\mathcal{F}$ is pseudo-coherent relative to $S$ if $\mathcal{F}$ viewed as an object of $D(O_X)$ is pseudo-coherent relative to $S$.

If $X$ is quasi-compact and $E$ is $m$-pseudo-coherent relative to $S$ for some $m$, then $E$ is bounded above. If $E$ is pseudo-coherent relative to $S$, then $E$ has quasi-coherent cohomology sheaves.

Lemma 48.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If $E$ in $D(O_X)$ is $m$-pseudo-coherent relative to $S$, then $H^i(E)$ is a quasi-coherent $O_S$-module for $i > m$. If $E$ is pseudo-coherent relative to $S$, then $E$ is an object of $D_{QCoh}(O_X)$.

Proof. Choose an affine open covering $S = \bigcup V_i$ and for each $i$ an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 48.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$. Since being quasi-coherent is local on $X$, we may assume that there exists an closed immersion $i : X \to \mathbb{A}^n_S$ such that $R_i E$ is $m$-pseudo-coherent on $\mathbb{A}^n_S$. By Derived Categories of Schemes, Lemma 9.1 this means that $H^q(R_i E)$ is quasi-coherent for $q > m$. Since $R_i$ is an exact functor, we have $i_* H^q(E) = H^q(R_i E)$ is quasi-coherent on $\mathbb{A}^n_S$. By Morphisms, Lemma 4.1 this implies that $H^q(E)$ is quasi-coherent as desired (strictly speaking it implies there exists some quasi-coherent $O_X$-module $\mathcal{F}$ such that $i_* \mathcal{F} = i_* H^q(E)$ and then Modules, Lemma 13.3 tells us that $\mathcal{F} \cong H^q(E)$ hence the result).

Next, we prove the condition of relative pseudo-coherence localizes well.

Lemma 48.4. Let $S$ be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let $E$ be an object of $D(O_X)$. If the equivalent conditions of Lemma 48.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 48.1 are satisfied for the pair $(U \to S, E|_U)$.

Proof. Write $S = \text{Spec}(R)$, $V = D(f)$, $X = \text{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 48.1 are satisfied for the pair $(X \to V, E)$.

Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(f x_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to \mathbb{A}_V^n \to \mathbb{A}_S^{n+1}$$

and we can apply Derived Categories of Schemes, Lemma 11.5 to conclude that the pushforward $E'$ of $E$ to $\mathbb{A}_S^{n+1}$ is $m$-pseudo-coherent.
Choose an element $g' \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $\mathbb{R}[x_0, \ldots, x_{n+1}] \to \mathbb{R}[x_0, \ldots, x_n, 1/g']$. We obtain

\[
\begin{array}{ccc}
X & \leftarrow & U \\
\downarrow & & \downarrow \\
\mathbb{A}^{n+1}_S & \leftarrow & D(g') \rightarrow \mathbb{A}^{n+2}_S
\end{array}
\]

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'|D(g')$ to $\mathbb{A}^{n+2}_S$ is $m$-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbb{A}^{n+2}_S$ we conclude the lemma is true.

**Lemma 48.5.** Let $X \to S$ be a finite type morphism of affine schemes. Let $E$ be an object of $D(O_X)$. Let $m \in \mathbb{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent

1. the equivalent conditions of Lemma 48.1 hold for the pairs $(U_i \to S, E|_{U_i})$,
2. the equivalent conditions of Lemma 48.1 hold for the pair $(X \to S, E)$.

**Proof.** The implication (2) $\Rightarrow$ (1) is Lemma 48.4. Assume (1). Say $S = \text{Spec}(R)$ and $X = \text{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in $A$. Consider the surjections

\[R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.\]

which sends $y_i$ to $f_i$ and $z_i$ to $g_i$. Note that $R[x_i, y_i, z_i]/(\sum y_i z_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of $E$ to $T = \text{Spec}(R[x_i, y_i, z_i]/(\sum y_i z_i - 1))$ is $m$-pseudo-coherent, see Derived Categories of Schemes, Lemma 11.5. For each $i_0$ it suffices to prove the restriction of this pushforward to $W_{i_0} = \text{Spec}(R[x_i, y_i, z_i, 1/y_i]/(\sum y_i z_i - 1))$ is $m$-pseudo-coherent. Note that there is a commutative diagram

\[
\begin{array}{ccc}
X & \leftarrow & U_{i_0} \\
\downarrow & & \downarrow \\
T & \leftarrow & W_{i_0}
\end{array}
\]

which implies that the pushforward of $E$ to $T$ restricted to $W_{i_0}$ is the pushforward of $E|_{U_{i_0}}$ to $W_{i_0}$. Since $R[x_i, y_i, z_i, 1/y_i]/(\sum y_i z_i - 1)$ is isomorphic to a polynomial ring over $R$ this proves what we want.

**Lemma 48.6.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E$ be an object of $D(O_X)$. Fix $m \in \mathbb{Z}$. The following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $S$,
2. for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 48.1 are satisfied for the pair $(U \to V, E|_U)$.

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is $m$-pseudo-coherent relative to $V$.

**Proof.** The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 48.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 48.5 it suffices to find a standard open covering $U = \bigcup U_k$ of
Let such that Lemma 48.8. Throughout the following, we will use the fact that the assumption that the equivalent conditions of Lemma 48.1 are satisfied for the pair \((U \rightarrow \overline{V}, E|_{\overline{U}})\) implies that the equivalent conditions of Lemma 48.1 are satisfied for the pair \((U', \overline{V}, E|_{U'})\). □

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative \(m\)-pseudo-coherence to the notion defined in More on Algebra, Definition 72.4. We will use the fact that for an affine scheme \(U = \text{Spec}(A)\) the functor \(R\Gamma(U, -)\) induces an equivalence between \(D_{QCoh}(\mathcal{O}_U)\) and \(D(A)\), see Derived Categories of Schemes, Lemma 3.5. This functor is compatible with pullbacks: if \(E\) is an object of \(D_{QCoh}(\mathcal{O}_U)\) and \(A \rightarrow B\) is a ring map corresponding to a morphism of affine schemes \(g : V = \text{Spec}(B) \rightarrow \text{Spec}(A) = U\), then \(R\Gamma(V, g^*E) = R\Gamma(U, E) \otimes_A^L B\). See Derived Categories of Schemes, Lemma 3.8.

Let \(f : X \rightarrow S\) be a morphism of schemes which is locally of finite type. Let \(E\) be an object of \(D_{QCoh}(\mathcal{O}_X)\). Fix \(m \in \mathbb{Z}\). The following are equivalent

1. \(E\) is \(m\)-pseudo-coherent relative to \(S\),
2. there exists an affine open covering \(S = \bigcup V_i\) and for each \(i\) an affine open covering \(f^{-1}(V_i) = \bigcup U_{ij}\) such that the complex of \(\mathcal{O}_X(U_{ij})\)-modules \(R\Gamma(U_{ij}, E)\) is \(m\)-pseudo-coherent relative to \(\mathcal{O}_S(V_i)\), and
3. for every affine opens \(U \subset X\) and \(V \subset S\) with \(f(U) \subset V\) the complex of \(\mathcal{O}_X(U)\)-modules \(R\Gamma(U, E)\) is \(m\)-pseudo-coherent relative to \(\mathcal{O}_S(V)\).

**Proof.** Let \(U\) and \(V\) be as in (2) and choose a closed immersion \(i : U \rightarrow A^n_v\). A formal argument, using Lemma 48.6 shows it suffices to prove that \(R_i(i^*E|_U)\) is quasi-isomorphic to the complex of quasi-coherent sheaves on \(U\) associated to the object \(R\Gamma(U, E)\) of \(D(A)\). Note that \(R\Gamma(U, E) = R\Gamma(\mathcal{A}_v^n, R_i(i^*E|_U))\) as \(i\) is a closed immersion (and hence \(i^*\) is exact). Thus \(R_i(E)\) is associated to \(R\Gamma(U, E)\) viewed as an object of \(D(R[x_1, \ldots, x_n])\). We conclude as \(m\)-pseudo-coherence of \(R_i(i^*E|_U)\) is equivalent to \(m\)-pseudo-coherence of \(R\Gamma(U, E)\) in \(D(R[x_1, \ldots, x_n])\) by Derived Categories of Schemes, Lemma 9.2 which is equivalent to \(m\)-pseudo-coherence relative to \(\mathcal{O}_S(V)\) by definition. □

Let \(i : X \rightarrow Y\) morphism of schemes locally of finite type over a base scheme \(S\). Assume that \(i\) induces a homeomorphism of \(X\) with a closed subset of \(Y\). Let \(E\) be an object of \(D(\mathcal{O}_X)\). Then \(E\) is \(m\)-pseudo-coherent relative to \(S\) if and only if \(R_i(E)\) is \(m\)-pseudo-coherent relative to \(S\).

**Proof.** By Morphisms, Lemma 43.4 the morphism \(i\) is affine. Thus we may assume \(S, Y\), and \(X\) are affine. Say \(S = \text{Spec}(R), Y = \text{Spec}(A),\) and \(X = \text{Spec}(B)\). The condition means that \(A/\text{rad}(A) \rightarrow B/\text{rad}(B)\) is surjective. As \(B\) is of finite type
over $A$, we can find $b_1, \ldots, b_m \in \text{rad}(B)$ which generate $B$ as an $A$-algebra. Say $b_j^N = 0$ for all $j$. Consider the diagram of rings

$$
\begin{array}{cccccc}
B & \leftarrow & R[x_i, y_j]/(y_j^N) & \leftarrow & R[x_i, y_j] \\
\uparrow & & \uparrow & & \\
A & \leftarrow & R[x_i] \\
\end{array}
$$

which translates into a diagram

$$
\begin{array}{cccccc}
X & \rightarrow & T & \rightarrow & \mathbb{A}^{n+m}_S \\
\downarrow & & \downarrow & & \\
Y & \rightarrow & \mathbb{A}^n_S \\
\end{array}
$$

of affine schemes. By Lemma 48.6 we see that $E$ is $m$-pseudo-coherent relative to $S$ if and only if its pushforward to $\mathbb{A}^{n+m}_S$ is $m$-pseudo-coherent. By Derived Categories of Schemes, Lemma 11.5 we see that this is true if and only if its pushforward to $T$ is $m$-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to $\mathbb{A}^n_S$ is $m$-pseudo-coherent. Again by Lemma 48.6 this holds if and only if $R_iE$ is $m$-pseudo-coherent relative to $S$. □

Lemma 48.9. Let $\pi : X \rightarrow Y$ be a finite morphism of schemes locally of finite type over a base scheme $S$. Let $E$ be an object of $D_{\text{QCoh}}(\mathcal{O}_X)$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if $R\pi_*E$ is $m$-pseudo-coherent relative to $S$.

Proof. Translation of the result of More on Algebra, Lemma 72.5 into the language of schemes. Observe that $R\pi_*$ indeed maps $D_{\text{QCoh}}(\mathcal{O}_X)$ into $D_{\text{QCoh}}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Lemma 4.1. To do the translation use Lemma 48.6. □

Lemma 48.10. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $(E, E', E'')$ be a distinguished triangle of $D(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$.

1. If $E$ is $(m+1)$-pseudo-coherent relative to $S$ and $E'$ is $m$-pseudo-coherent relative to $S$ then $E''$ is $m$-pseudo-coherent relative to $S$.
2. If $E$, $E''$ are $m$-pseudo-coherent relative to $S$, then $E'$ is $m$-pseudo-coherent relative to $S$.
3. If $E'$ is $(m+1)$-pseudo-coherent relative to $S$ and $E''$ is $m$-pseudo-coherent relative to $S$, then $E$ is $(m+1)$-pseudo-coherent relative to $S$.

Moreover, if two out of three of $E, E', E''$ are pseudo-coherent relative to $S$, the so is the third.

Proof. Immediate from Lemma 48.6 and Cohomology, Lemma 41.4. □

Lemma 48.11. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then

1. $\mathcal{F}$ is $m$-pseudo-coherent relative to $S$ for all $m > 0$.
2. $\mathcal{F}$ is 0-pseudo-coherent relative to $S$ if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $\mathcal{F}$ is $(-1)$-pseudo-coherent relative to $S$ if and only if $\mathcal{F}$ is quasi-coherent and finitely presented relative to $S$.

**Proof.** Part (1) is immediate from the definition. To see part (3) we may work locally on $X$ (both properties are local). Thus we may assume $X$ and $S$ are affine. Choose a closed immersion $i : X \to \mathbb{A}^n_S$. Then we see that $\mathcal{F}$ is $(-1)$-pseudo-coherent relative to $S$ if and only if $i_*\mathcal{F}$ is $(-1)$-pseudo-coherent, which is true if and only if $i_*\mathcal{F}$ is an $\mathcal{O}_{\mathbb{A}^n_S}$-module of finite presentation, see Cohomology, Lemma 41.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 11.2. By Morphisms, Lemma 48.12 we see that $\mathcal{F}$ is quasi-coherent if and only if $i_*\mathcal{F}$ is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved. □

**Lemma 48.12.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $E, K$ be objects of $D(\mathcal{O}_X)$. If $E \otimes K$ is $m$-pseudo-coherent relative to $S$ so are $E$ and $K$.

**Proof.** Follows from Cohomology, Lemma 41.6 and the definitions. □

**Lemma 48.13.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent relative to $S$ for all $i$. Then $\mathcal{F}^\bullet$ is $m$-pseudo-coherent relative to $S$.

**Proof.** Follows from Cohomology, Lemma 41.7 and the definitions. □

**Lemma 48.14.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(\mathcal{O}_X)$. If $E$ is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to $S$ for all $i$, then $E$ is $m$-pseudo-coherent relative to $S$.

**Proof.** Follows from Cohomology, Lemma 41.8 and the definitions. □

**Lemma 48.15.** Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(\mathcal{O}_X)$ which is $m$-pseudo-coherent relative to $S$. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote $E'$ the derived pullback of $E$ to $X'$. If $S'$ and $X$ are Tor independent over $S$, then $E'$ is $m$-pseudo-coherent relative to $S'$.

**Proof.** The problem is local on $X$ and $X'$ hence we may assume $X, S, S'$, and $X'$ are affine. Choose a closed immersion $i : X \to \mathbb{A}^n_S$ and denote $i' : X' \to \mathbb{A}^n_{S'}$ the base change to $S'$. Denote $g : X' \to X$ and $g' : \mathbb{A}^n_{S'} \to \mathbb{A}^n_S$ the projections, so $E' = Lg^*E$. Since $X$ and $S'$ are tor-independent over $S$, the base change map (Cohomology, Remark 29.3) induces an isomorphism

$$Ri'_*(Lg^*E) = (Lg')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at $x'$ is the map

$$E_x \otimes_{\mathcal{O}_{\mathbb{A}^n_S,x}} \mathcal{O}_{\mathbb{A}^n_{S'},x'} \to E_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

coming from the closed immersions $i$ and $i'$. Note that the source is quasi-isomorphic to a localization of $E_x \otimes_{\mathcal{O}_{\mathbb{A}^n_S,x}} \mathcal{O}_{S',x'}$ which is isomorphic to the target as $\mathcal{O}_{X',x'}$ is isomorphic to (the same) localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,x}} \mathcal{O}_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma 41.3 □
Lemma 48.16. Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base $S$. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(O_Y)$. Assume

1. $O_X$ is pseudo-coherent relative to $Y$, and
2. $E$ is $m$-pseudo-coherent relative to $S$.

Then $L_! f^* E$ is $m$-pseudo-coherent relative to $S$.

Proof. The problem is local on $X$. Thus we may assume $X$, $Y$, and $S$ are affine. Arguing as in the proof of More on Algebra, Lemma 72.13 we can find a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & A_n^d \\
\downarrow{f} & & \downarrow{p} \\
Y & \xrightarrow{j} & A_n^S
\end{array}
$$

Observe that

$$R_i_* L_! f^* E = R_i_* L j^* L p^* E = L p^* E \otimes_{O_{A_n^S}} R_i_* O_X$$

by Cohomology, Lemma 45.4. By assumption and the fact that $Y$ is affine, we can represent $R i_* O_X = i_* O_X$ by a complexes of finite free $O_{A_n^S}$-modules $F^\bullet$, with $F^q = 0$ for $q > 0$ (details omitted; use Derived Categories of Schemes, Lemma 9.2 and More on Algebra, Lemma 72.7). By assumption $E$ is bounded above, say $H^q(E) = 0$ for $q > a$. Represent $E$ by a complex $E^\bullet$ of $O_Y$-modules with $E^q = 0$ for $q > a$. Then the derived tensor product above is represented by Tot($p^* E^\bullet \otimes_{O_{A_n^S}} F^\bullet$).

Since $j$ is a closed immersion, the functor $j_*$ is exact and $R j_*$ is computed by applying $j_*$ to any representing complex of sheaves. Thus we have to show that $j_* \text{Tot}(p^* E^\bullet \otimes_{O_{A_n^S}} F^\bullet)$ is $m$-pseudo-coherent as a complex of $O_{A_n^S}$-modules. Note that $\text{Tot}(p^* E^\bullet \otimes_{O_{A_n^S}} F^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $p^* E^\bullet \otimes_{O_{A_n^S}} F^q[-q]$. Note that for $q < 0$ the complexes $p^* E^\bullet \otimes_{O_{A_n^S}} F^q[-q]$ have zero cohomology in degrees $\leq m$ and hence are $m$-pseudo-coherent.

Hence, applying Lemma 48.15 and induction, it suffices to show that $p^* E^\bullet \otimes_{O_{A_n^S}} F^q[-q]$ is pseudo-coherent relative to $S$ for all $q$. Note that $F^q = 0$ for $q > 0$. Since also $F^q$ is finite free this reduces to proving that $p^* E^\bullet$ is $m$-pseudo-coherent relative to $S$ which follows from Lemma 48.15 for instance. \hfill $\square$

Lemma 48.17. Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base $S$. Let $m \in \mathbb{Z}$. Let $E$ be an object of $D(O_X)$. Assume $O_Y$ is pseudo-coherent relative to $S$. Then the following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $Y$, and
2. $E$ is $m$-pseudo-coherent relative to $S$.

---

7This means $f$ is pseudo-coherent, see Definition 49.2
8This means $Y \to S$ is pseudo-coherent, see Definition 49.2
The question is local on $X$, hence we may assume $X, Y,$ and $S$ are affine. Arguing as in the proof of More on Algebra, Lemma [72.13] we can find a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{A}_S^m \\
\downarrow f & & \downarrow j \\
Y & \xrightarrow{p} & \mathbb{A}_S^n
\end{array}
\]

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to $S$ implies that $\mathcal{O}_{\mathbb{A}_S^n}$ is pseudo-coherent relative to $\mathbb{A}_S^m$ (by flat base change; this can be seen by using for example Lemma [48.15]). This in turn implies that $j_*\mathcal{O}_{\mathbb{A}_S^n}$ is pseudo-coherent as an $\mathcal{O}_{\mathbb{A}_S^m}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma [11.5].

**Proof.** The question is local on $X$, hence we may assume $X, Y,$ and $S$ are affine. Arguing as in the proof of More on Algebra, Lemma [72.13] we can find a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{A}_S^m \\
\downarrow f & & \downarrow j \\
Y & \xrightarrow{p} & \mathbb{A}_S^n \\
\downarrow & & \downarrow \\
S & & \mathbb{A}_S^n
\end{array}
\]

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to $S$ implies that $\mathcal{O}_{\mathbb{A}_S^n}$ is pseudo-coherent relative to $\mathbb{A}_S^m$ (by flat base change; this can be seen by using for example Lemma [48.15]). This in turn implies that $j_*\mathcal{O}_{\mathbb{A}_S^n}$ is pseudo-coherent as an $\mathcal{O}_{\mathbb{A}_S^m}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma [11.5].

**Lemma 48.18.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow & & \downarrow \\
S & & \mathbb{A}_S^n
\end{array}
\]

be a commutative diagram of schemes. Assume $i$ is a closed immersion and $P \rightarrow S$ flat and locally of finite presentation. Let $E$ be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

1. $E$ is $m$-pseudo-coherent relative to $S$,
2. $Ri_*E$ is $m$-pseudo-coherent relative to $S$, and
3. $Ri_*E$ is $m$-pseudo-coherent on $P$.

**Proof.** The equivalence of (1) and (2) is Lemma [48.9]. The equivalence of (2) and (3) follows from Lemma [48.17] applied to $\text{id} : P \rightarrow P$ provided we can show that $\mathcal{O}_P$ is pseudo-coherent relative to $S$. This follows from More on Algebra, Lemma [73.3] and the definitions.

**49. Pseudo-coherent morphisms**

Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections [60] 72 and [73]. For now the only thing you need to know is that a ring map $A \rightarrow B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and $B$ as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

**Lemma 49.1.** Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent

1. there exist an affine open covering $S = \bigcup V_j$ and for each $j$ an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_{ji})$ is a pseudo-coherent ring map,
2. for every pair of affine opens $U \subset X, V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is pseudo-coherent, and
3. $f$ is locally of finite type and $\mathcal{O}_X$ is pseudo-coherent relative to $S$.

**Proof.** To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition [13.1] for the property of being a pseudo-coherent
ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas [72.12][72.11] and [72.16].

If (1) holds, then $f$ is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma [48.7] and the definitions that $\mathcal{O}_X$ is pseudo-coherent relative to $S$. Conversely, if (3) holds, then we see that for every $U$ and $V$ as in (2) the ring $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_S(V)$ and $\mathcal{O}_X(U)$ is as a module pseudo-coherent relative to $\mathcal{O}_S(V)$, see Lemmas [48.6] and [48.7]. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold.

\textbf{Definition 49.2.} A morphism of schemes $f : X \to S$ is called \textit{pseudo-coherent} if the equivalent conditions of Lemma [49.1] are satisfied. In this case we also say that $X$ is pseudo-coherent over $S$.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

\textbf{Lemma 49.3.} A flat base change of a pseudo-coherent morphism is pseudo-coherent.

\textbf{Proof.} This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma [72.12].

\textbf{Lemma 49.4.} A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.

\textbf{Proof.} This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma [72.13] or More on Algebra, Lemma [72.15].

\textbf{Lemma 49.5.} A pseudo-coherent morphism is locally of finite presentation.

\textbf{Proof.} Immediate from the definitions.

\textbf{Lemma 49.6.} A flat morphism which is locally of finite presentation is pseudo-coherent.

\textbf{Proof.} This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma [73.4].

\textbf{Lemma 49.7.} Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme $S$. Then $f$ is pseudo-coherent.

\textbf{Proof.} This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma [72.15].

\textbf{Lemma 49.8.} Let $f : X \to S$ be a finite morphism of schemes. Then $f$ is pseudo-coherent if and only if $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_S$-module.

\textbf{Proof.} Translated into algebra this lemma says the following: If $R \to A$ is a finite ring map, then $R \to A$ is pseudo-coherent as a ring map (which means by definition that $A$ as an $A$-module is pseudo-coherent relative to $R$) if and only if $A$ is pseudo-coherent as an $R$-module. This follows from the more general More on Algebra, Lemma [72.9].
Lemma 49.9. Let $f : X \to S$ be a morphism of schemes. If $S$ is locally Noetherian, then $f$ is pseudo-coherent if and only if $f$ is locally of finite type.

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with $R$ Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 60.18 that $A$ is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. □

Lemma 49.10. The property $P(f) = "f is pseudo-coherent"$ is fpqc local on the base.

Proof. We will use the criterion of Descent, Lemma 19.4 to prove this. By Definition 49.2 being pseudo-coherent is Zariski local on the base. By Lemma 49.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 19.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an $A$-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then $C$ is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 60.16. □

Lemma 49.11. Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if $I$ is pseudo-coherent as a $B$-module.

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that $B$ is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 49.6. Note that $A \to B/I$ is pseudo-coherent if and only if $B/I$ is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 60.12 we see this is equivalent to the condition that $B/I$ is pseudo-coherent as an $B$-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that $I$ is pseudo-coherent if and only if $B/I$ is (see More on Algebra, Lemma 60.7). □

The following lemma will be obsoleted by the stronger Lemma 49.13.

Lemma 49.12. The property $P(f) = "f is pseudo-coherent"$ is syntomic local on the source.

Proof. We will use the criterion of Descent, Lemma 23.4 to prove this. It follows from Lemmas 49.6 and 49.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 29.7 and 29.6). It is clear from Definition 49.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 23.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 11.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 134.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic
morphisms $W_i \to A^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$
\begin{array}{ccc}
X'_i & \longrightarrow & W_i \\
\downarrow & \ & \downarrow \\
X & \longrightarrow & A^n_Y
\end{array}
$$

After replacing $X'$ by $\bigsqcup X'_i$ and setting $W = \bigsqcup W_i$ we obtain a fibre product diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & \ & \downarrow h \\
X & \longrightarrow & A^n_Y
\end{array}
$$

with $W \to A^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to A^n_Y$ is open (see Morphisms, Lemma 24.9) and $X' \to X$ is surjective we can find $f \in \Gamma(A^n_Y, O)$ such that $X \subset D(f) \subset \text{Im}(h)$. Write $Y = \text{Spec}(R)$, $X = \text{Spec}(A)$, $X' = \text{Spec}(A')$ and $W = \text{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$
\begin{array}{ccc}
A' = B/IB & \longrightarrow & B \\
\downarrow & \ & \downarrow \\
A = R[x_1, \ldots, x_n]/I & \leftarrow & R[x_1, \ldots, x_n]
\end{array}
$$

By Lemma 49.11 we see that $IB$ is pseudo-coherent as a $B$-module. The ring map $R[x_1, \ldots, x_n]/f \to B/IB$ is faithfully flat by our choice of $f$ above. This implies that $I_f \subset R[x_1, \ldots, x_n]/f$ is pseudo-coherent, see More on Algebra, Lemma 60.16. Applying Lemma 49.11 one more time we see that $R \to A$ is pseudo-coherent.

0699 Lemma 49.13. The property $P(f) = "f \text{ is pseudo-coherent}"$ is fpqc local on the source.

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fpqc covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 41.2 there exist

1. a Zariski open covering $X = \bigcup U_j$,
2. surjective finite locally free morphisms $W_j \to U_j$,
3. Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
4. surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fpqc covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both $X_i$ and $T_{j,k}$ are pseudo-coherent over $X$ by Lemma 49.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 49.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 49.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.
Assume that \( X' \to X \to S \) is a sequence of morphisms of schemes with \( X' \to X \) surjective finite locally free and \( X' \to Y \) pseudo-coherent. Our goal is to show that \( X \to S \) is pseudo-coherent. Note that by Descent, Lemma 11.3 the morphism \( X \to S \) is locally of finite presentation. It is clear that the problem reduces to the case that \( X', X \) and \( S \) are affine and \( X' \to X \) is free of some rank \( r > 0 \). The corresponding algebra problem is the following: Suppose \( R \to A \to A' \) are ring maps such that \( R \to A' \) is pseudo-coherent, \( R \to A \) is of finite presentation, and \( A' \cong A'^{\oplus r} \) as an \( A \)-module. Goal: Show \( R \to A \) is pseudo-coherent. The assumption that \( R \to A' \) is pseudo-coherent means that \( A' \) as an \( A' \)-module is pseudo-coherent relative to \( R \). By More on Algebra, Lemma 72.5 this implies that \( A' \) as an \( A \)-module is pseudo-coherent relative to \( R \). Since \( A' \cong A'^{\oplus r} \) as an \( A \)-module we see that \( A \) as an \( A \)-module is pseudo-coherent relative to \( R \), see More on Algebra, Lemma 72.8. This by definition means that \( R \to A \) is pseudo-coherent and we win. \( \square \)

50. Perfect morphisms

In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map \( A \to B \) is perfect if and only if it is pseudo-coherent and \( B \) has finite tor dimension as an \( A \)-module.

**Lemma 50.1.** Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. The following are equivalent

1. there exist an affine open covering \( S = \bigcup V_j \) and for each \( j \) an affine open covering \( f^{-1}(V_j) = \bigcup U_{ij} \) such that \( \mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij}) \) is a perfect ring map, and
2. for every pair of affine opens \( U \subset X, V \subset S \) such that \( f(U) \subset V \) the ring map \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is perfect.

**Proof.** Assume (1) and let \( U, V \) be as in (2). It follows from Lemma 49.1 that \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 13.1. These properties follow from More on Algebra, Lemmas 61.11, 61.14, and 61.16. Some details omitted. \( \square \)

**Definition 50.2.** A morphism of schemes \( f : X \to S \) is called **perfect** if the equivalent conditions of Lemma 50.1 are satisfied. In this case we also say that \( X \) is perfect over \( S \).

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

**Lemma 50.3.** A flat base change of a perfect morphism is perfect.

**Proof.** This translates into the following algebra result: Let \( A \to B \) be a perfect ring map. Let \( A \to A' \) be flat. Then \( A' \to B \otimes_A A' \) is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 49.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 61.14. \( \square \)

**Lemma 50.4.** A composition of perfect morphisms of schemes is perfect.
**Proof.** This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 49.4 and its proof. By assumption there exist integers $n$, $m$ such that $B$ has tor dimension $\leq n$ over $A$ and $C$ has tor dimension $\leq m$ over $B$. Then for any $A$-module $M$ we have

$$M \otimes^L_A C = (M \otimes^L_A B) \otimes^L_B C$$

and the spectral sequence of More on Algebra, Example 58.4 shows that $\text{Tor}_p^A(M, C) = 0$ for $p > n + m$ as desired.

**Lemma 50.5.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. $f$ is flat and perfect, and
2. $f$ is flat and locally of finite presentation.

**Proof.** The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 73.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 49.5.

**Lemma 50.6.** Let $f : X \to S$ be a morphism of schemes. Assume $S$ is regular and $f$ is locally of finite type. Then $f$ is perfect.

**Proof.** See More on Algebra, Lemma 73.5.

**Lemma 50.7.** A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.

**Proof.** Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of $A$. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of $A/I$ by a polynomial algebra over $A$. Hence it suffices to see that $A/I$ is pseudo-coherent as an $A$-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex $K(A, f_1, \ldots, f_r)$ is a finite free resolution of $A/I$. Hence $A/I$ is a perfect complex of $A$-modules and we win.

**Lemma 50.8.** Let

$$\begin{tikzcd}
X \ar{r}{f} \ar{dr} & Y \\
& S \ar{ur}
\end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

**Proof.** We can factor $f$ as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 32.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 50.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 32.6. Hence $i$ is a section of a smooth morphism, therefore
i is a regular immersion, see Divisors, Lemma 22.7. This implies that i is perfect, see Lemma 50.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 50.4

069A Remark 50.9. It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is S = Spec(k), X = Spec(k), Y = Spec(k[x]/(x^2)) and X → Y the unique S-morphism.

069B Lemma 50.10. The property P(f) = “f is perfect” is fpqc local on the base.

Proof. We will use the criterion of Descent, Lemma 19.4 to prove this. By Definition 50.2 being perfect is Zariski local on the base. By Lemma 50.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 19.4 translates into the following algebra statement: Let A → B be a faithfully flat ring map. Let C = A[x_1, ..., x_n]/I be an A-algebra. If C ⊗_A B is perfect as an B[x_1, ..., x_n]-module, then C is perfect as a A[x_1, ..., x_n]-module. This is More on Algebra, Lemma 67.12.

069C Lemma 50.11. Let f : X → S be a pseudo-coherent morphism of schemes. The following are equivalent

1. f is perfect,
2. O_X locally has finite tor dimension as a sheaf of f^{-1}O_S-modules, and
3. for all x ∈ X the ring O_{X,x} has finite tor dimension as an O_{S,f(x)}-module.

Proof. The problem is local on X and S. Hence we may assume that X = Spec(B), S = Spec(A) and f corresponds to a pseudo-coherent ring map A → B.

If (1) holds, then B has finite tor dimension d as A-module. Then B_q has tor dimension d as an A_p-module for all primes q ⊂ B with p = A ∩ q, see More on Algebra, Lemma 61.15. Then O_X has tor dimension d as a sheaf of f^{-1}O_S-modules by Cohomology, Lemma 42.5. Thus (1) implies (2).

By Cohomology, Lemma 42.5 (2) implies (3).

Assume (3). We cannot use More on Algebra, Lemma 61.15 to conclude as we are not given that the tor dimension of B_q over A_p is bounded independent of q. Choose a presentation A[x_1, ..., x_n] → B. Then B is pseudo-coherent as a A[x_1, ..., x_n]-module. Let q ⊂ A[x_1, ..., x_n] be a prime ideal lying over p ⊂ A. Then either B_q is zero or by assumption it has finite tor dimension as an A_p-module. Since the fibres of A → A[x_1, ..., x_n] have finite global dimension, we can apply More on Algebra, Lemma 69.8 to A_p → A[x_1, ..., x_n]_q to see that B_q is a perfect A[x_1, ..., x_n]_q-module. Hence B is a perfect A[x_1, ..., x_n]-module by More on Algebra, Lemma 69.7. Thus A → B is a perfect ring map by definition.

066G Lemma 50.12. Let S be a Noetherian scheme. Let f : X → S be a perfect proper morphism of schemes. Let E ∈ D(O_X) be perfect. Then Rf_*E is a perfect object of D(O_S).

Proof. We claim that Derived Categories of Schemes, Lemma 23.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of O_X-modules. Thus it suffices to show that O_X has finite tor dimension as a sheaf of f^{-1}O_S-modules. This is equivalent to being perfect by Lemma 50.11.
Lemma 50.13. The property $P(f) = \"f is perfect\"$ is fppf local on the source.

Proof. Let $\{g_i : X_i \to X\}_{i \in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 49.13 we conclude that $f$ is pseudo-coherent. Hence by Lemma 50.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to $x$. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 61.17.

Lemma 50.14. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume

(1) $X$ is locally Noetherian,

(2) $j \circ i$ is a regular immersion, and

(3) $i$ is perfect.

Then $i$ and $j$ are regular immersions.

Proof. Since $X$ (and hence $Y$) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 20.8. Thus the result follows from Divided Power Algebra, Lemma 7.5.

51. Local complete intersection morphisms

In Divisors, Section 21 we have defined 4 different types of regular immersions: regular, Koszul-regular, $H_1$-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on $X$ factors as

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{A}^S \\
\downarrow & & \downarrow S \\
S
\end{array}
$$

where $i$ is a $*$-regular immersion for $* \in \{\emptyset, \text{Koszul}, H_1, \text{quasi}\}$. However, we don’t know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require $i$ to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = \text{Koszul}$ or $* = \emptyset$. Hence we will define a local complete intersection morphism or Koszul morphism to be a morphism of schemes $f : X \to S$ that locally on $X$ has a factorization as above with $i$ a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

Lemma 51.1. Let $S$ be a scheme. Let $U$, $P$, $P'$ be schemes over $S$. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over $S$. Assume $P$ and $P'$ smooth over $S$. Then the following are equivalent

(1) $i$ is a Koszul-regular immersion in a neighbourhood of $x$, and

(2) $i'$ is a Koszul-regular immersion in a neighbourhood of $x$.

Proof. Assume $i$ is a Koszul-regular immersion in a neighbourhood of $x$. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 22.8 that $j$ is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 22.11 that $i'$ is a Koszul-regular immersion.
Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbb{A}^n_S$ followed by the projection $\mathbb{A}^n_S \to S$ which is smooth. Picture

$$
\begin{array}{ccc}
X & \xleftarrow{U} & \mathbb{A}^n_S = P \\
\downarrow & \downarrow & \downarrow \\
S & \xrightarrow{\pi} & 
\end{array}
$$

In fact you can do this with any affine open neighbourhood $U$ of $x$ in $X$, see Morphisms, Lemma 37.2.

**Definition 51.2.** Let $f : X \to S$ be a morphism of schemes.

1. Let $x \in X$. We say that $f$ is Koszul at $x$ if $f$ is of finite type at $x$ and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.

2. We say $f$ is a Koszul morphism, or that $f$ is a local complete intersection morphism if $f$ is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion $i$ followed by a smooth morphism $\pi$, whether or not $i$ is Koszul regular in a neighbourhood of $x$ is an intrinsic property of $f$ at $x$. Let us record this here explicitly as a lemma so that we can refer to it.

**Lemma 51.3.** Let $f : X \to S$ be a local complete intersection morphism. Let $P$ be a scheme smooth over $S$. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over $S$. Then $i$ is a Koszul-regular immersion.

**Proof.** This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

**Lemma 51.4.** Let $f : X \to S$ be a local complete intersection morphism. Then

1. $f$ is locally of finite presentation,
2. $f$ is pseudo-coherent, and
3. $f$ is perfect.

**Proof.** Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that $f$ factors as $\pi \circ i$ where $\pi$ is smooth and $i$ is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 50.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 50.5. Finally a composition of perfect morphisms is perfect, see Lemma 50.4. □

**Lemma 51.5.** Let $f : X = \text{Spec}(B) \to S = \text{Spec}(A)$ be a morphism of affine schemes. Then $f$ is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 30.3.

**Proof.** Follows immediately from the definitions. □
Beware that a base change of a Koszul morphism is not Koszul in general.

**Lemma 51.6.** A flat base change of a local complete intersection morphism is a local complete intersection morphism.

**Proof.** Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.3.

**Lemma 51.7.** A composition of local complete intersection morphisms is a local complete intersection morphism.

**Proof.** Let \( g : Y \to S \) and \( f : X \to Y \) be local complete intersection morphisms. Let \( x \in X \) and set \( y = f(x) \). Choose an open neighbourhood \( V \subset Y \) of \( y \) and a factorization \( g|_V = \pi \circ i \) for some Koszul-regular immersion \( i : V \to P \) and smooth morphism \( \pi : P \to S \). Next choose an open neighbourhood \( U \) of \( x \in X \) and a factorization \( f|_U = i' \circ \pi' \) for some Koszul-regular immersion \( i' : U \to P' \) and smooth morphism \( \pi' : P' \to Y \). In fact, we may assume that \( P' = A^n_{V'} \), see discussion preceding and following Definition 51.2. Picture:

\[
\begin{array}{cccccc}
X & \leftarrow & U & \rightarrow & P' = A^n_{V'} \\
\downarrow & & \downarrow & & \downarrow \\
Y & \leftarrow & V & \rightarrow & P \\
\downarrow & & \downarrow & & \downarrow \\
S & \leftarrow & & & S \\
\end{array}
\]

Set \( P'' = A^n_{P'} \). Then \( U \to P' \to P'' \) is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely \( i' \) and the flat base change of \( i \) via \( P'' \to P \), see Divisors, Lemma 21.3 and Divisors, Lemma 21.7. Also \( P'' \to P \to S \) is smooth as a composition of smooth morphisms, see Morphisms, Lemma 32.4. Hence we conclude that \( X \to S \) is Koszul at \( x \) as desired.

**Lemma 51.8.** Let \( f : X \to S \) be a morphism of schemes. The following are equivalent:

1. \( f \) is flat and a local complete intersection morphism, and
2. \( f \) is syntomic.

**Proof.** Working affine locally this is More on Algebra, Lemma 30.5. We also give a more geometric proof.

Assume (2). By Morphisms, Lemma 29.10 for every point \( x \) of \( X \) there exist affine open neighbourhoods \( U \) of \( x \) and \( V \) of \( f(x) \) such that \( f|_U : U \to V \) is standard syntomic. This means that \( U = \text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \text{Spec}(R) \) where \( R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \) is a relative global complete intersection over \( R \). By Algebra, Lemma 34.13 the sequence \( f_1, \ldots, f_c \) is a regular sequence in each local ring \( R[x_1, \ldots, x_n]_q \) for every prime \( q \supset (f_1, \ldots, f_c) \). Consider the Koszul complex \( K_* = K_*(R[x_1, \ldots, x_n], f_1, \ldots, f_c) \) with homology groups \( H_i = H_i(K_*) \). By More on Algebra, Lemma 27.2 we see that \( (H_i)_q = 0, i > 0 \) for every \( q \) as above. On the other hand, by More on Algebra, Lemma 26.6 we see that \( H_i \) is annihilated by \( (f_1, \ldots, f_c) \). Hence we see that \( H_i = 0, i > 0 \) and \( f_1, \ldots, f_c \) is a Koszul-regular
This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then $f$ is a flat and locally of finite presentation (Lemma 51.4). Hence, according to Morphisms, Lemma 29.10 it suffices to show that the local rings $\mathcal{O}_{X,x}$ are local complete intersection rings. Choose, locally on $X$, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over $S$, see Divisors, Definition 22.2. Hence according to Divisors, Lemma 22.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of $X$ are local complete intersections. □

069L **Lemma 51.9.** A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.

**Proof.** Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 21.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

069M **Lemma 51.10.** Let

![Diagram](image)

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

**Proof.** Immediate from the definitions. □

The following lemma is of a different nature.

09RL **Lemma 51.11.** Let

![Diagram](image)

be a commutative diagram of morphisms of schemes. Assume

1. $S$ is locally Noetherian,
2. $Y \to S$ is locally of finite type,
3. $f : X \to Y$ is perfect,
4. $X \to S$ is a local complete intersection morphism.

Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

**Proof.** In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 27.7. Moreover, whether an ideal (resp. ideal sheaf) is regular may
be checked on local rings (resp. stalks), see Algebra, Lemma 67.6 (resp. Divisors, Lemma 20.8)

The question is local. Hence we may assume $S, X, Y$ are affine. In this situation we may choose a commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}^{n+m}_S & \xrightarrow{j} & X \\
\downarrow & & \downarrow \\
\mathbb{A}^n_S & \xleftarrow{j} & Y \\
S & \xleftarrow{f} & S \\
\end{array}
$$

whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings

$$
\begin{array}{ccc}
J & \xrightarrow{\Delta} & \mathcal{O}_{\mathbb{A}^{n+m}_S,x} \\
\downarrow & & \downarrow \\
I & \xrightarrow{\Delta} & \mathcal{O}_{\mathbb{A}^n_S,f(x)} \\
\end{array}
$$

where $J$ and $I$ are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal $J$ is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X,x}$ has finite tor dimension over $\mathcal{O}_{Y,f(x)}$. Hence we may apply Divided Power Algebra, Lemma 7.6 to conclude that $I$ and $J/I$ are generated by regular sequences. By our initial remarks, this finishes the proof. □

**Lemma 51.12.** The property $\mathcal{P}(f) = "f"$ is a local complete intersection morphism" is fpqc local on the base.

**Proof.** Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of $S$. Assume that each base change $f_i : X_i \to S_i$ of $f$ is a local complete intersection morphism. Note that this implies in particular that $f$ is locally of finite type, see Lemma 51.4 and Descent, Lemma 20.10. Let $x \in X$. Choose an open neighbourhood $U$ of $x$ and an immersion $j : U \to \mathbb{A}^n_S$ over $S$ (see discussion preceding Definition 51.2). We have to show that $j$ is a Koszul-regular immersion. Since $f_i$ is a local complete intersection morphism, we see that the base change $j_i : U \times_S S_i \to \mathbb{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 51.3. Because $\{\mathbb{A}^n_{S_i} \to \mathbb{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 20.32 that $j$ is a Koszul-regular immersion as desired. □

**Lemma 51.13.** The property $\mathcal{P}(f) = "f"$ is a local complete intersection morphism" is syntomic local on the source.

**Proof.** We will use the criterion of Descent, Lemma 23.4 to prove this. It follows from Lemmas 51.8 and 51.7 that being a local complete intersection morphism is preserved under precomposing with syntonic morphisms. It is clear from Definition 51.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 23.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism.
Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma [11.1]. Choose a closed immersion $X \to \mathbb{A}^n_Y$. By Algebra, Lemma [34.18] we can find an affine open covering $X' = \bigcup_{i=1}^n X'_i$ and syntomic morphisms $W_i \to \mathbb{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$
\begin{array}{ccc}
X'_i & \longrightarrow & W_i \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathbb{A}^n_Y
\end{array}
$$

After replacing $X'$ by $\bigsqcup X'_i$ and setting $W = \bigsqcup W_i$ we obtain a fibre product diagram of affine schemes

$$
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & & \downarrow h \\
X & \longrightarrow & \mathbb{A}^n_Y
\end{array}
$$

with $h : W \to \mathbb{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbb{A}^n_Y$ is open (see Morphisms, Lemma [24.9]) and $X' \to X$ is surjective we see that $X$ is contained in the image of $W \to \mathbb{A}^n_Y$. Choose a closed immersion $W \to \mathbb{A}^{n+m}_Y$ over $\mathbb{A}^n_Y$. Now the diagram looks like

$$
\begin{array}{ccc}
X' & \longrightarrow & W \\
\downarrow & & \downarrow h \\
X & \longrightarrow & \mathbb{A}^{n+m}_Y
\end{array}
$$

Because $h$ is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbb{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbb{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma [21.8] that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma [20.32]) we conclude that $X \to \mathbb{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. □

**Lemma 51.14.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. Assume both $X$ and $Y$ are flat and locally of finite presentation over $S$. Then the set

$$\{ x \in X \mid f \text{ Koszul at } x \}$$

is open in $X$ and its formation commutes with arbitrary base change $S' \to S$.

**Proof.** The set is open by definition (see Definition [51.2]). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of $f$. Let $x' \in X'$ be a point such that $f'$ is Koszul at $x'$. Denote $s' \in S'$, $x \in X$, $y \in Y$, $s \in S$ the image of $x'$. Note that $f$ is locally of finite presentation, see Morphisms, Lemma [20.11]. Hence we may choose an affine neighbourhood $U \subset X$ of $x$ and an immersion $i : U \to \mathbb{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbb{A}^n_{Y'}$ the base change of $i$. The assumption that $f'$ is Koszul at $x'$ implies that $i'$ is a Koszul-regular immersion in a neighbourhood of $x'$, see Lemma
51.3. The scheme $X'$ is flat and locally of finite presentation over $S'$ as a base change of $X$ (see Morphisms, Lemmas \ref{morphisms-flat} and \ref{morphisms-local}). Hence $i'$ is a relative $H_1$-regular immersion over $S'$ in a neighbourhood of $x$ (see Divisors, Definition \ref{divisors-definition}). Thus the base change $i'_*: U'_* \to A^n_{Y'}$ is a $H_1$-regular immersion in an open neighbourhood of $x'$, see Divisors, Lemma \ref{divisors-base} and the discussion following Divisors, Definition \ref{divisors-definition}. Since $s' = \text{Spec}(k(s')) \to \text{Spec}(k(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma \ref{morphisms-flat}) we conclude that the base change $i_*: U_* \to A^n_Y$ is an $H_1$-regular immersion in a neighbourhood of $x$, see Descent, Lemma \ref{descent-lemma}. Finally, note that $A^n_Y$ is flat and locally of finite presentation over $S$, hence Divisors, Lemma \ref{divisors-lemma} implies that $i$ is a (Koszul-)regular immersion in a neighbourhood of $x$, see \ref{divisors-definition}. Finally, note that $A^n_Y$ is flat and locally of finite presentation over $S$, hence Divisors, Lemma \ref{divisors-lemma} implies that $i$ is a (Koszul-)regular immersion in a neighbourhood of $x$, as desired. □

\textbf{Lemma 51.15.} Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then $f$ is unramified if and only if $f$ is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on $X$.

\textbf{Proof.} The first assertion follows immediately from Lemma \ref{morphisms-unramified} and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of $f$ we choose, locally on $X$, a factorization of $f$ as $f = p \circ i$ where $i : X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma \ref{divisors-lemma} we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as $i$ is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma \ref{divisors-lemma}. □

\textbf{Lemma 51.16.} Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence

$$0 \to i^* \mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma \ref{exact-sequence} is short exact.

\textbf{Proof.} The question is local on $Z$ hence we may assume there exists a factorization $Z \to A^m_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{i'} & A^m_Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
& & X
\end{array}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to A^m_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma \ref{divisors-lemma} the sequence

$$0 \to (i')^* \mathcal{C}_{A^m_Y/\mathcal{A}_{X}} \to \mathcal{C}_{Z/A_{X}} \to \mathcal{C}_{Z/A_{Y}} \to 0$$

is exact and locally split. Note that $i^* \mathcal{C}_{Y/X} = (i')^* \mathcal{C}_{A^m_Y/\mathcal{A}_{X}}$ by Lemma \ref{exact-sequence} and note that the diagram

$$\begin{array}{ccc}
(i')^* \mathcal{C}_{A^m_Y/\mathcal{A}_{X}} & \xrightarrow{\cong} & \mathcal{C}_{Z/A_{X}} \\
\downarrow & & \downarrow \\
{i^* \mathcal{C}_{Y/X}} & \to & \mathcal{C}_{Z/X}
\end{array}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. □
52. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

In the sequences below each of the maps are as constructed in either Morphisms, Lemma 31.8 or Lemma 7.5. Let \( g : Z \to Y \) and \( f : Y \to X \) be morphisms of schemes.

1. There is a canonical exact sequence

\[
g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,
\]

see Morphisms, Lemma 31.9. If \( g : Z \to Y \) is formally smooth, then this sequence is a short exact sequence, see Lemma 11.11.

2. If \( g \) is formally unramified, then there is a canonical exact sequence

\[
C_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,
\]

see Lemma 7.10. If \( f \circ g : Z \to X \) is formally smooth, then this sequence is a short exact sequence, see Lemma 11.12.

3. If \( g \) and \( f \circ g \) are formally unramified, then there is a canonical exact sequence

\[
C_{Z/X} \to C_{Z/Y} \to g^*\Omega_{Y/X} \to 0,
\]

see Lemma 7.11. If \( f : Y \to X \) is formally smooth, then this sequence is a short exact sequence, see Lemma 11.13.

4. If \( g \) and \( f \) are formally unramified, then there is a canonical exact sequence

\[
g^*C_{Y/X} \to C_{Z/X} \to C_{Z/Y} \to 0.
\]

see Lemma 7.12. If \( g : Z \to Y \) is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 51.16.

53. Weakly étale morphisms

A ring homomorphism \( A \to B \) is weakly étale if both \( A \to B \) and \( B \otimes_A B \to B \) are flat, see More on Algebra, Definition 87.1. The analogous notion for morphisms of schemes is the following.

Definition 53.1. A morphism of schemes \( X \to Y \) is weakly étale or absolutely flat if both \( X \to Y \) and the diagonal morphism \( X \to X \times_Y X \) are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if \( X \to Y \) is weakly étale, then \( L_{X/Y} = 0 \), as follows from Cotangent, Lemma 8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 9). In this section we stick with the basics.

Lemma 53.2. Let \( f : X \to Y \) be a morphism of schemes. The following are equivalent

1. \( X \to Y \) is weakly étale, and
2. for every \( x \in X \) the ring map \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is weakly étale.
Proof. Observe that under both assumptions (1) and (2) the morphism \( f \) is flat. Thus we may assume \( f \) is flat. Let \( x \in X \) with image \( y = f(x) \) in \( Y \). There are canonical maps of rings

\[
\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \to \mathcal{O}_{X,x}
\]

where the first map is a localization (hence flat) and the second map is a surjection (hence an epimorphism of rings). Condition (1) means that for all \( x \) the second arrow is flat. Condition (2) is that for all \( x \) the composition is flat. These conditions are equivalent by Algebra, Lemma 17.3 and More on Algebra, Lemma 87.2.

Lemma 53.3. Let \( X \to Y \) be a morphism of schemes such that \( X \to X \times_Y X \) is flat. Let \( F \) be an \( \mathcal{O}_X \)-module. If \( F \) is flat over \( Y \), then \( F \) is flat over \( X \).

Proof. Let \( x \in X \) with image \( y = f(x) \) in \( Y \). Since \( X \to X \times_Y X \) is flat, we see that \( \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \) is flat. Hence the result follows from More on Algebra, Lemma 87.2 and the definitions.

Lemma 53.4. Let \( f : X \to S \) be a morphism of schemes. The following are equivalent

1. The morphism \( f \) is weakly étale.
2. For every affine opens \( U \subset X \), \( V \subset S \) with \( f(U) \subset V \) the ring map \( \mathcal{O}_S(V) \to \mathcal{O}_X(U) \) is weakly étale.
3. There exists an open covering \( S = \bigcup_{j \in J} V_j \) and open coverings \( f^{-1}(V_j) = \bigcup_{i \in I_j} U_i \) such that each of the morphisms \( U_i \to V_j \), \( j \in J, i \in I_j \) is weakly étale.
4. There exists an affine open covering \( S = \bigcup_{j \in J} V_j \) and affine open coverings \( f^{-1}(V_j) = \bigcup_{i \in I_j} U_i \) such that the ring map \( \mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i) \) is of weakly étale, for all \( j \in J, i \in I_j \).

Moreover, if \( f \) is weakly étale then for any open subschemes \( U \subset X \), \( V \subset S \) with \( f(U) \subset V \) the restriction \( f|_U : U \to V \) is weakly-étale.

Proof. Suppose given open subschemes \( U \subset X \), \( V \subset S \) with \( f(U) \subset V \). Then \( U \times_Y V \subset X \times_Y X \) is open (Schemes, Lemma 17.3) and the diagonal \( \Delta_{U/V} \) of \( f|_U : U \to V \) is the restriction \( \Delta_{X/Y}|_U : U \to U \times_Y U \). Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 24.3) the final statement of the lemma is follows as well as the equivalence of (1) and (3). If \( X \) and \( Y \) are affine, then \( X \to Y \) is weakly étale if and only if \( \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \) is weakly étale (use again Morphisms, Lemma 24.3). Thus (1) and (3) are also equivalent to (2) and (4).

Lemma 53.5. Let \( X \to Y \to Z \) be morphisms of schemes.

1. If \( X \to X \times_Y X \) and \( Y \to Y \times_Z Y \) are flat, then \( X \to X \times_Z X \) is flat.
2. If \( X \to Y \) and \( Y \to Z \) are weakly étale, then \( X \to Z \) is weakly étale.

Proof. Part (1) follows from the factorization

\[
X \to X \times_Y X \to X \times_Z X
\]

of the diagonal of \( X \) over \( Z \), the fact that

\[
X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y;
\]

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 24.7 and 24.5). Part (2) follows
from part (1) and the fact (just used) that the composition of flat morphisms is flat. □

**Lemma 53.6.** Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of $X$.

1. If $X \to X \times_Y X$ is flat, then $X' \to X' \times_{Y'} X'$ is flat.
2. If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.

**Proof.** Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_{Y'} X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 24.7. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. □

**Lemma 53.7.** Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X \times_Z X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times_Z Y
\end{array}
\]

The top horizontal arrow is flat and the vertical arrows are flat. Hence $X$ is flat over $Y \times_Z Y$. By Morphisms, Lemma 24.12 we see that $Y$ is flat over $Y \times_Z Y$. □

**Lemma 53.8.** Let $f : X \to Y$ be a weakly étale morphism of schemes. Then $f$ is formally unramified, i.e., $\Omega_{X/Y} = 0$.

**Proof.** Recall that $f$ is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 6.7. Via Lemma 53.4 and Morphisms, Lemma 31.5 this follows from the case of rings which is More on Algebra, Lemma 87.12. □

**Lemma 53.9.** Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases

1. $X \to Y$ is a flat monomorphism,
2. $X \to Y$ is an open immersion,
3. $X \to Y$ is flat and unramified,
4. $X \to Y$ is étale.

**Proof.** If (1) holds, then $\Delta_{X/Y}$ is an isomorphism, hence certainly $f$ is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 33.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 34.5), hence (4) follows from (3). □

**Lemma 53.10.** Let $f : X \to Y$ be a morphism of schemes. If $Y$ is reduced and $f$ weakly étale, then $X$ is reduced.

**Proof.** Via Lemma 53.4 this follows from the case of rings which is More on Algebra, Lemma 87.8. □
Lemma 53.11. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

1. $f$ is weakly étale,
2. for $x \in X$ the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism on strict henselizations.

Proof. Let $x \in X$ be a point with image $y = f(x)$ in $Y$. Choose a separable algebraic closure $\kappa^{sep}$ of $\kappa(x)$. Let $\mathcal{O}^{sh}_{X,x}$ be the strict henselization corresponding to $\kappa^{sep}$ and $\mathcal{O}^{sh}_{Y,y}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in $\kappa^{sep}$. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}^{sh}_{X,x} \\
\uparrow & & \uparrow \\
\mathcal{O}_{Y,y} & \longrightarrow & \mathcal{O}^{sh}_{Y,y}
\end{array}
$$

local homomorphisms of local rings, see Algebra, Lemma 150.12. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 87.14 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 42.1.

Assume $f$ weakly étale. By Lemma 53.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 87.9 and 87.11 the right vertical arrow is weakly étale. By More on Algebra, Theorem 87.25 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}^{sh}_{X,x}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 87.10. Thus (2) implies (1) by Lemma 53.2. □

Lemma 53.12. Let $f : X \to Y$ be a morphism of schemes. If $Y$ is a normal scheme and $f$ weakly étale, then $X$ is a normal scheme.

Proof. By More on Algebra, Lemma 42.6 a scheme $S$ is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 53.11. □

Lemma 53.13. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of schemes over $S$. If $X$, $Y$ are weakly étale over $S$, then $f$ is weakly étale.

Proof. We will use Morphisms, Lemmas 24.7 and 24.5 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an immersion. Thus Lemma 53.3 implies, that since $X$ is flat over $X \times_S X$ it follows that $X$ is flat over $X \times_Y X$. □

54. Reduced fibre theorem

In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in
a straightforward manner from Epp’s result on eliminating ramification, see More on Algebra, Theorem 95.23.

Let \( A \) be a Dedekind domain with fraction field \( K \). Let \( X \) be a scheme flat and of finite type over \( A \). Let \( L \) be a finite extension of \( K \). Let \( B \) be the integral closure of \( A \) in \( L \). Then \( B \) is a Dedekind domain (Algebra, Lemma 119.18). Let \( Y_B = X \times_{\text{Spec}(A)} \text{Spec}(B) \) be the base change. Then \( Y_B \to \text{Spec}(B) \) is of finite type (Morphisms, Lemma 14.4). Hence \( Y_B \) is Noetherian (Morphisms, Lemma 14.6). Thus the normalization \( \nu : Y \to X_B \) exists (see Morphisms, Definition 51.1 and the discussion following). Picture

\[
\begin{array}{c}
Y \xrightarrow{\nu} X_B \\
\downarrow \quad \downarrow \\
\text{Spec}(B) \quad \text{Spec}(A)
\end{array}
\]

We sometimes call \( Y \) the normalized base change of \( X \). In general the morphism \( \nu \) may not be finite. But if \( A \) is a Nagata ring (a condition that is virtually always satisfied in practice) then \( \nu \) is of finite and \( Y \) is of finite type over \( B \), see Morphisms, Lemmas 51.10 and 17.1.

Taking the normalized base change commutes with composition. More precisely, if \( K \subset L \subset M \) are finite extensions of fields with integral closures \( A \subset B \subset C \) then the normalized base change \( Z \) of \( Y \to \text{Spec}(B) \) relative to \( L \subset M \) is equal to the normalized base change of \( X \to \text{Spec}(A) \) relative to \( K \subset M \).

**Theorem 54.1.** Let \( A \) be a Dedekind ring with fraction field \( K \). Let \( X \) be a scheme flat and of finite type over \( A \). Assume \( A \) is a Nagata ring. There exists a finite extension \( K \subset L \) such that the normalized base change \( Y \) is smooth over \( \text{Spec}(B) \) at all generic points of all fibres.

**Proof.** During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like \( X \to \text{Spec}(A) \) is smooth commutes with base change, see Morphisms, Lemma 32.13.

We first choose a finite extension \( K \subset L \) such that \((X_L)_{\text{red}} \) is geometrically reduced over \( L \), see Varieties, Lemma 6.11. Since \( Y \to (X_B)_{\text{red}} \) is birational we see applying Varieties, Lemma 6.8 that \( Y_L \) is geometrically reduced over \( L \) as well. Hence \( Y_L \to \text{Spec}(L) \) is smooth on a dense open \( V \subset Y_L \) by Varieties, Lemma 25.7. Thus the smooth locus \( U \subset Y \) of the morphism \( Y \to \text{Spec}(B) \) is open (by Morphisms, Definition 32.1) and is dense in the generic fibre. Replacing \( A \) by \( B \) and \( X \) by \( Y \) we reduce to the case treated in the next paragraph.

Assume \( X \) is normal and the smooth locus \( U \subset X \) of \( X \to \text{Spec}(A) \) is dense in the generic fibre. This implies that \( U \) is dense in all but finitely many fibres, see Lemma 22.3. Let \( x_1, \ldots, x_r \in X \setminus U \) be the finitely many generic points of irreducible components of \( X \setminus U \) which are moreover generic points of irreducible components of fibres of \( X \to \text{Spec}(A) \). Set \( \mathcal{O}_i = \mathcal{O}_{X,x_i} \). Let \( A_i \) be the localization of \( A \) at the maximal ideal corresponding to the image of \( x_i \) in \( \text{Spec}(A) \). By More on Algebra, Proposition 95.25 there exist finite extensions \( K \subset K_i \) which are solutions for the extension of discrete valuation rings \( A_i \to \mathcal{O}_i \). Let \( K \subset L \) be a finite extension dominating all of the extensions \( K \subset K_i \). Then \( K \subset L \) is still a solution for \( A_i \to \mathcal{O}_i \) by More on Algebra, Lemma 95.4.
Consider the diagram \([54.0.1]\) with the extension \(L/K\) we just produced. Note that \(U_B \subset X_B\) is smooth over \(B\), hence normal (for example use Algebra, Lemma 157.9). Thus \(Y \rightarrow X_B\) is an isomorphism over \(U_B\). Let \(y \in Y\) be a generic point of an irreducible component of a fibre of \(Y \rightarrow \text{Spec}(B)\) lying over the maximal ideal \(m \subset B\). Assume that \(y \notin U_B\). Then \(y\) maps to one of the points \(x_i\). It follows that \(\mathcal{O}_{Y,y}\) is a local ring of the integral closure of \(\mathcal{O}_i\) in \(R(X) \otimes_K L\) (details omitted). Hence because \(K \subset L\) is a solution for \(A_i \rightarrow \mathcal{O}_i\) we see that \(B_m \rightarrow \mathcal{O}_{Y,y}\) is formally smooth (this is the definition of being a "solution"). In other words, \(m\mathcal{O}_{Y,y} = m_y\) and the residue field extension is separable. Hence the local ring of the fibre at \(y\) is \(\kappa(y)\). This implies the fibre is smooth over \(\kappa(m)\) at \(y\) for example by Algebra, Lemma 138.5 This finishes the proof.

**Lemma 54.2** (Variant over curves). Let \(f : X \rightarrow S\) be a flat, finite type morphism of schemes. Assume \(S\) is Nagata, integral with function field \(K\), and regular of dimension 1. Then there exists a finite extension \(L/K\) such that in the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{f} \\
T & \xrightarrow{g} & S
\end{array}
\]

the morphism \(g\) is smooth at all generic points of fibres. Here \(T\) is the normalization of \(S\) in \(\text{Spec}(L)\) and \(\nu : Y \rightarrow X \times_S T\) is the normalization.

**Proof.** Choose a finite affine open covering \(S = \bigcup \text{Spec}(A_i)\). Then \(K\) is equal to the fraction field of \(A_i\) for all \(i\). Let \(X_i = X \times_S \text{Spec}(A_i)\). Choose \(L_i/K\) as in Theorem 54.1 for the morphism \(X_i \rightarrow \text{Spec}(A_i)\). Let \(B_i \subset L_i\) be the integral closure of \(A_i\) and let \(Y_i\) be the normalized base change of \(X\) to \(B_i\). Let \(L/K\) be a finite extension dominating each \(L_i\). Let \(T_i \subset T\) be the inverse image of \(\text{Spec}(A_i)\). For each \(i\) we get a commutative diagram

\[
\begin{array}{ccc}
g^{-1}(T_i) & \xrightarrow{g} & Y_i \\
\downarrow & & \downarrow \\
T_i & \xrightarrow{\nu} & \text{Spec}(B_i) \\
\downarrow & & \downarrow \\
& & \text{Spec}(A_i)
\end{array}
\]

and in fact the left hand square is a normalized base change as discussed at the beginning of the section. In the proof of Theorem 54.1 we have seen that the smooth locus of \(Y \rightarrow T\) contains the inverse image in \(g^{-1}(T_i)\) of the set of points where \(Y_i\) is smooth over \(B_i\). This proves the lemma.

**Lemma 54.3** (Variant with separable extension). Let \(A\) be a Dedekind ring with fraction field \(K\). Let \(X\) be a scheme flat and of finite type over \(A\). Assume \(A\) is a Nagata ring and that for every generic point \(\eta\) of an irreducible component of \(X\) the field extension \(K \subset \kappa(\eta)\) is separable. Then there exists a finite separable extension \(K \subset L\) such that the normalized base change \(Y\) is smooth over \(\text{Spec}(B)\) at all generic points of all fibres.

**Proof.** This is proved in exactly the same manner as Theorem 54.1 with a few minor modifications. The most important change is to use More on Algebra, Lemma 95.26 instead of More on Algebra, Proposition 95.25. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented)
morphism like $X \to \text{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 32.15.

Since $X$ is flat over $A$ every generic point $\eta$ of $X$ maps to the generic point of $\text{Spec}(A)$. After replacing $X$ by its reduction we may assume $X$ is reduced. In this case $X_K$ is geometrically reduced over $K$ by Varieties, Lemma 6.8. Hence $X_K \to \text{Spec}(K)$ is smooth on a dense open by Varieties, Lemma 25.7. Thus the smooth locus $U \subset X$ of the morphism $X \to \text{Spec}(A)$ is open (by Morphisms, Definition 32.1) and is dense in the generic fibre. This reduces us to the situation of the following paragraph.

Assume $X$ is normal and the smooth locus $U \subset X$ of $X \to \text{Spec}(A)$ is dense in the generic fibre. This implies that $U$ is dense in all but finitely many fibres, see Lemma 22.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \text{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Observe that the fraction field of $\mathcal{O}_i$ is the residue field of a generic point of $X$. Let $A_i$ be the localization of $A$ at the maximal ideal corresponding to the image of $x_i$ in $\text{Spec}(A)$. We may apply More on Algebra, Lemma 95.26 and we find finite separable extensions $K \subset K_i$ which are solutions for $A_i \to \mathcal{O}_i$. Let $K \subset L$ be a finite separable extension dominating all of the extensions $K \subset K_i$. Then $K \subset L$ is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 95.4.

Consider the diagram (54.0.1) with the extension $L/K$ we just produced. Note that $U_B \subset X_B$ is smooth over $B$, hence normal (for example use Algebra, Lemma 157.9). Thus $Y \to X_B$ is an isomorphism over $U_B$. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \text{Spec}(B)$ lying over the maximal ideal $m \subset B$. Assume that $y \notin U_B$. Then $y$ maps to one of the points $x_i$. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because $K \subset L$ is a solution for $A_i \to \mathcal{O}_i$ we see that $B_m \to \mathcal{O}_{Y,y}$ is formally smooth (this is the definition of being a "solution"). In other words, $m\mathcal{O}_{Y,y} = m_y$ and the residue field extension is separable. Hence the local ring of the fibre at $y$ is $\kappa(y)$. This implies the fibre is smooth over $\kappa(m)$ at $y$ for example by Algebra, Lemma 138.5. This finishes the proof. □

**Lemma 54.4** (Variant with separable extensions over curves). Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume $S$ is Nagata, integral with function field $K$, and regular of dimension 1. Assume the field extensions $K \subset \kappa(\eta)$ are separable for every generic point $\eta$ of an irreducible component of $X$. Then there exists a finite separable extension $L/K$ such that in the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\nu} & X \times_S T \\
\downarrow{g} & & \downarrow{f} \\
T & \rightarrow & S
\end{array}
\]

the morphism $g$ is smooth at all generic points of fibres. Here $T$ is the normalization of $S$ in $\text{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

**Proof.** This follows from Lemma 54.3 in exactly the same manner that Lemma 54.2 follows from Theorem 54.1 □
55. Ind-quasi-affine morphisms

Definition 55.1. A scheme \( X \) is ind-quasi-affine if every quasi-compact open of \( X \) is quasi-affine. Similarly, a morphism of schemes \( X \to Y \) is ind-quasi-affine if \( f^{-1}(V) \) is ind-quasi-affine for each affine open \( V \) in \( Y \).

An example of an ind-quasi-affine scheme is an open of an affine scheme. If \( X = \bigcup_{i \in I} U_i \) is a union of quasi-affine opens such that any two \( U_i \) are contained in a third, then \( X \) is ind-quasi-affine. An ind-quasi-affine scheme \( X \) is separated because any two affine opens \( U, V \) are contained in a separated open subscheme of \( X \), namely \( U \cup V \). Similarly an ind-quasi-affine morphism is separated.

Lemma 55.2. The property of being ind-quasi-affine is stable under base change.

Proof. Let \( f : X \to Y \) be an ind-quasi-affine morphism. Let \( Z \) be an affine scheme and let \( Z \to Y \) be a morphism. To show: \( Z \times_Y X \) is ind-quasi-affine. Let \( W \subset Z \times_Y X \) be a quasi-compact open. We can find finitely many affine opens \( V_1, \ldots, V_n \) of \( Y \) and finitely many quasi-compact opens \( U_i \subset f^{-1}(V_i) \) such that \( Z \) maps into \( \bigcup V_i \) and \( W \) maps into \( \bigcup U_i \). Then we may replace \( Y \) by \( \bigcup V_i \) and \( X \) by \( \bigcup U_i \). In this case \( f^{-1}(V_i) \) is quasi-compact open (details omitted; use that \( f \) is separated) and hence quasi-affine. Thus now \( X \to Y \) is a quasi-affine morphism (Morphisms, Lemma \ref{affine_lemma}) and the result follows from the fact that the base change of a quasi-affine morphism is quasi-affine (Morphisms, Lemma \ref{base_change_quasi-affine}). □

Lemma 55.3. The property of being ind-quasi-affine is fpqc local on the base.

Proof. Let \( f : X \to Y \) be a morphism of schemes. Let \( \{g_i : Y_i \to Y\} \) be an fpqc covering such that the base change \( f_i : X_i \to Y_i \) is ind-quasi-affine for all \( i \). We will show \( f \) is ind-quasi-affine. Namely, let \( U \subset X \) be a quasi-compact open mapping into an affine open \( V \subset Y \). We have to show that \( U \) is quasi-affine. Let \( V_j \subset Y_j, j = 1, \ldots, m \) be affine opens such that \( V = \bigcup g_{i_j}(V_j) \) (exist by definition of fpqc coverings). Then \( V_i \times_Y X \to V_i \) is ind-quasi-affine as well. Hence we may replace \( Y \) by \( V \) and \( \{g_i : Y_i \to Y\} \) by the finite covering \( \{V_j \to V\} \). We may replace \( X \) by \( U \), because \( V_j \times_Y U \subset V_j \times_Y X \) is open and hence \( V_j \times_Y U \to V_j \) is ind-quasi-affine as well (ind-quasi-affineness is inherited by opens). Hence we may assume \( X \) is quasi-compact and \( Y \) affine. In this case we have to show that \( X \) is quasi-affine and we know that \( X_i \) is quasi-affine. Thus the result follows from Descent, Lemma \ref{descent}. □

Lemma 55.4. A separated locally quasi-finite morphism of schemes is ind-quasi-affine.

Proof. Let \( f : X \to Y \) be a separated locally quasi-finite morphism of schemes. Let \( V \subset Y \) be affine and \( U \subset f^{-1}(V) \) quasi-compact open. We have to show \( U \) is quasi-affine. Since \( U \to V \) is a separated quasi-finite morphism of schemes, this follows from Zariski’s Main Theorem. See Lemma \ref{zariski-main-theorem}. □

56. Relative morphisms

In this section we prove a representability result which we will use in Fundamental Groups, Section \ref{fundamental-groups} to prove a result on the category of finite étale coverings of a...
scheme. The material in this section is discussed in the correct generality in Criteria for Representability, Section 10.

Let $S$ be a scheme. Let $Z$ and $X$ be schemes over $S$. Given a scheme $T$ over $S$ we can consider morphisms $b : T \times_S Z \to T \times_S X$ over $S$. Picture

\[
\begin{array}{ccc}
T \times_S Z & \xrightarrow{b} & T \times_S X \\
\downarrow & & \downarrow \\
Z & \hookrightarrow & X \\
\end{array}
\]

Of course, we can also think of $b$ as a morphism $b : T \times_S Z \to X$ such that

\[
\begin{array}{ccc}
T \times_S Z & \xrightarrow{b} & X \\
\downarrow & & \downarrow \\
Z & \hookrightarrow & X \\
\end{array}
\]

commutes. In this situation we can define a functor

\[
\text{Mor}_S(Z, X) : (\text{Sch}/S)^{opp} \to \text{Sets}, \ T \mapsto \{b \text{ as above}\}
\]

Here is a basic representability result.

**Lemma 56.1.** Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $\text{Mor}_S(Z, X)$ is representable by an affine scheme over $S$.

**Proof.** Write $S = \text{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over $R$. Choose a presentation

\[
\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).
\]

We will denote $x_i$ the image of $x_i$ in this quotient. Write

\[
P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].
\]

Consider the $R$-algebra map

\[
\Psi : R[\{x_i\}_{i \in I}] \to P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \mapsto \sum_j a_{ij} \otimes e_j.
\]

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements $c_{kj}$, $k \in K$, $1 \leq j \leq m$. We claim that $W = \text{Spec}(P/J)$ represents the functor $\text{Mor}_S(Z, X)$.

First, note that by construction $P/J$ is an $R$-algebra, hence a morphism $W \to S$. Second, by construction the map $\Psi$ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an $P/J$-algebra homomorphism

\[
P/J \otimes_R \Gamma(X, \mathcal{O}_X) \to P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)
\]

which determines a morphism $b_{\text{univ}} : W \times_S Z \to W \times_S X$. By the Yoneda lemma $b_{\text{univ}}$ determines a transformation of functors $W \to \text{Mor}_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to \text{Mor}_S(Z, X)(T)$ over any affine scheme $T$.

Suppose $T = \text{Spec}(R')$ is an affine scheme over $S$ and $b \in \text{Mor}_S(Z, X)(T)$. The structure morphism $T \to S$ defines an $R$-algebra structure on $R'$ and $b$ defines an $R'$-algebra map

\[
b^t : R' \otimes_R \Gamma(X, \mathcal{O}_X) \to R' \otimes_R \Gamma(Z, \mathcal{O}_Z).
\]
In particular we can write $b^t(1 \otimes \pi_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an $R$-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient $P/J$ by the construction of the ideal $J$ to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that $b$ is the pullback of $b_{\text{univ}}$. Some details omitted.

**Lemma 56.2.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $\text{Mor}_S(Z, X)$ is representable by a scheme affine over $S$.

**Proof.** Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z \times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset \text{Mor}_S(Z, X)$ be the subfunctor which assigns to $T/S$ the empty set if $T \to S$ does not factor through $U_i$ and $\text{Mor}_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma. Condition (2)(a) follows from Lemma 56.1 and the other two follow from straightforward arguments.

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: $X$ is quasi-affine, $f$ is quasi-affine, $f$ is quasi-projective, $f$ is locally projective, there exists an ample invertible sheaf on $X$, there exists an $f$-ample invertible sheaf on $X$, or there exists an $f$-very ample invertible sheaf on $X$.

**Lemma 56.3.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume

1. $Z \to S$ is finite locally free, and
2. for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.

Then $\text{Mor}_S(Z, X)$ is representable by a scheme.

**Proof.** Consider the set $I$ of pairs $(U, V)$ where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through $V$. For $i \in I$ denote $(U_i, V_i)$ the corresponding pair. Set $F_i = \text{Mor}_{V_i}(Z_{V_i}, U_i)$. It is immediate that $F_i$ is a subfunctor of $\text{Mor}_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 15.4 which proves the lemma.

Condition (2)(a) follows from Lemma 56.2

To check condition (2)(b) consider $T/S$ and $b \in \text{Mor}_S(Z, X)$. Thinking of $b$ as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i, b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i, b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider $T/S$ and $b \in \text{Mor}_S(Z, X)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i, b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some $i$ where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since $p$ is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image $s$ of $t$ in $S$. Thus our condition on $X \to S$ exactly shows a suitable pair exists. □
Lemma 56.4. Let \( Z \to S \) and \( X \to S \) be morphisms of schemes. Assume \( Z \to S \) is finite locally free and \( X \to S \) is separated and locally quasi-finite. Then \( \text{Mor}_S(Z,X) \) is representable by a scheme.

**Proof.** This follows from Lemmas 56.3 and 38.12. \( \square \)

57. Characterizing pseudo-coherent complexes, III

In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. This is a continuation of Derived Categories of Schemes, Section 30. A basic tool will be to reduce to the case of projective space using a derived version of Chow’s lemma, see Lemma 57.2.

Lemma 57.1. Consider a commutative diagram of schemes

\[
\begin{array}{ccc}
Z' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X' & \longrightarrow & S'
\end{array}
\]

Let \( S \to S' \) be a morphism. Denote by \( X \) and \( Y \) the base changes of \( X' \) and \( Y' \) to \( S \). Assume \( Y' \to S' \) and \( Z' \to X' \) are flat. Then \( X \times_S Y \) and \( Z' \) are Tor independent over \( X' \times_{S'} Y' \).

**Proof.** The question is local, hence we may assume all schemes are affine (some details omitted). Observe that \( X \times_S Y \to X' \times_{S'} Y' \) is cartesian with flat vertical arrows. Write \( X = \text{Spec}(A) \), \( X' = \text{Spec}(A') \), \( Y' = \text{Spec}(B') \). Then \( X \times_S Y = \text{Spec}(A \otimes_{A'} B') \). Write \( Z' = \text{Spec}(C') \). We have to show

\[
\text{Tor}_p^{B'}(A \otimes_{A'} B', C') = 0, \quad \text{for } p > 0
\]

Since \( A' \to B' \) is flat we have \( A \otimes_{A'} B' = A \otimes_{A'}^L B' \). Hence

\[
(A \otimes_{A'} B') \otimes_{B'}^{L} C' = (A \otimes_{A'}^{L} B') \otimes_{B'}^{L} C' = A \otimes_{A'}^{L} C' = A \otimes_{A'} C'
\]

The second equality by More on Algebra, Lemma 56.4. The last equality because \( A' \to C' \) is flat. This proves the lemma. \( \square \)

Lemma 57.2 (Derived Chow’s lemma). Let \( A \) be a ring. Let \( X \) be a separated scheme of finite presentation over \( A \). Let \( x \in X \). Then there exist an open neighbourhood \( U \subset X \) of \( x \), an \( n \geq 0 \), an open \( V \subset \mathbb{P}_A^n \), a closed subscheme \( Z \subset X \times_A \mathbb{P}_A^n \), a point \( z \in Z \), and an object \( E \) in \( D(\mathcal{O}_{X \times_A \mathbb{P}_A^n}) \) such that

1. \( Z \to X \times_A \mathbb{P}_A^n \) is of finite presentation,
2. \( b : Z \to X \) is an isomorphism over \( U \) and \( b(z) = x \),
3. \( c : Z \to \mathbb{P}_A^n \) is a closed immersion over \( V \),
4. \( b^{-1}(U) = c^{-1}(V) \), in particular \( c(z) \in V \),
5. \( E|_{X \times_A V} \cong (b,c)_*\mathcal{O}_Z|_{X \times_A V} \),
6. \( E \) is pseudo-coherent and supported on \( Z \).
Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and a scheme $X'$ separated and of finite presentation over $A'$ whose base change to $A$ is $X$. See Limits, Lemmas \ref{limits-locally-separated} and \ref{limits-separated}. Let $x' \in X'$ be the image of $x$. If we can prove the lemma for $x' \in X'/A'$, then the lemma follows for $x \in X/A$. Namely, if $U', n', V', Z', z'$, $E'$ provide the solution for $x' \in X'/A'$, then we can let $U \subset X$ be the inverse image of $U'$, let $n = n'$, let $V \subset P_A^n$ be the inverse image of $V'$, let $Z \subset X \times P^n$ be the scheme theoretic inverse image of $Z'$, let $z \in Z$ be the unique point mapping to $x$, and let $E$ be the derived pullback of $E'$. Observe that $E$ is pseudo-coherent by Cohomology, Lemma \ref{cohomology-pseudo-coherent}. It only remains to check (5). To see this set $W = b^{-1}(U) = c^{-1}(V)$ and $W' = (b')^{-1}(U) = (c')^{-1}(V')$ and consider the cartesian square

$$
\begin{array}{ccc}
W & \to & W' \\
(b,c) \downarrow & & \downarrow (b',c') \\
X \times_A V & \to & X' \times_{A'} V'
\end{array}
$$

By Lemma \ref{limits-separated}, the schemes $X \times_A V$ and $W'$ are Tor independent over $X' \times_{A'} V'$. Hence the derived pullback of $(b',c'), \mathcal{O}_{W'}$ to $X \times_A V$ is $(b,c), \mathcal{O}_W$ by Derived Categories of Schemes, Lemma \ref{derived-cat-separated}. This also uses that $R(b',c')_* \mathcal{O}_{Z'} = (b',c'), \mathcal{O}_{Z'}$ because $(b',c')$ is a closed immersion and similarly for $(b,c), \mathcal{O}_Z$. Since $E'_|_{U \times_A V'} = (b',c'), \mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b,c), \mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume $A$ is of finite type over $\mathbf{Z}$. Choose an affine open neighbourhood $U \subset X$ of $x$. Then $U$ is of finite type over $A$. Choose a closed immersion $U \to A^n_A$ and denote $j : U \to P^n_A$ the immersion we get by composing with the open immersion $A^n_A \to P^n_A$. Let $Z$ be the scheme theoretic closure of $(\text{id}_U, j) : U \to X \times_A P^n_A$

Since the projection $X \times P^n \to X$ is separated, we conclude from Morphisms, Lemma \ref{morphisms-separated} that $b : Z \to X$ is an isomorphism over $U$. Let $z \in Z$ be the unique point lying over $x$.

Let $Y \subset P^n_A$ be the scheme theoretic closure of $j$. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\text{id}_U, j) : U \to X \times_A Y$. As $X$ is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by the same lemma we used above. Choose $V \subset P^n_A$ open with $V \cap Y = j(U)$. Then we see that (3) and (4) hold.

Because $A$ is Noetherian we see that $X$ and $X \times_A P^n_A$ are Noetherian schemes. Hence we can take $E = (b,c), \mathcal{O}_Z$ in this case, see Derived Categories of Schemes, Lemma \ref{derived-cat-separated}. This finishes the proof. \hfill $\square$

**Lemma 57.3.** Let $A$, $x \in X$, and $U, n, V, Z, z, E$ be as in Lemma 57.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have

$$
Rq_*(Lp^* K \otimes^L E)|_V = R(U \to V)_* K|_U
$$

where $p : X \times_A P^n_A \to X$ and $q : X \times_A P^n_A \to P^n_A$ are the projections and where the morphism $U \to V$ is the finitely presented closed immersion $c \circ (b|_U)^{-1}$.
Proof. Since $b^{-1}(U) = c^{-1}(V)$ and since $c$ is a closed immersion over $V$, we see that $c \circ (b|_{U})^{-1}$ is a closed immersion. It is of finite presentation because $U$ and $V$ are of finite presentation over $A$, see Morphisms, Lemma 20.11. First we have

$$Rq_*(Lp^*K \otimes^L E)|_V = Rq'_*( (Lp^*K \otimes^L E)|_{X \times_A V})$$

where $q' : X \times_AV \to V$ is the projection because formation of total direct image commutes with localization. Set $W = b^{-1}(U) = c^{-1}(V)$ and denote $i : W \to X \times_A V$ the closed immersion $i = (b,c)|_W$. Then

$$Rq'_* ( (Lp^*K \otimes^L E)|_{X \times_A V}) = Rq'_* (Lp^*K|_{X \times_A V} \otimes^L i_*O_W)$$

by property (5). Since $i$ is a closed immersion we have $i_*O_W = Ri_*O_W$. Using Derived Categories of Schemes, Lemma 21.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(U \to V)_*K|_U$$

which is what we want. \qed

Lemma 57.4. Let $A$ be a ring. Let $X$ be a scheme separated and of finite presentation over $A$. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^L K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E$ in $D(\mathcal{O}_X)$, then $K$ is pseudo-coherent relative to $A$.

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^L K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E$ in $D(\mathcal{O}_X)$. Let $x \in X$. We will show that $K$ is pseudo-coherent relative to $A$ in a neighbourhood of $x$ and this will prove the lemma.

Choose $U,n,V,z,E$ as in Lemma 57.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^L E) \otimes^L O_{\mathbf{P}^n_A}(i))$$

$$= R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^L E \otimes^L Lq^*O_{\mathbf{P}^n_A}(i))$$

$$= R\Gamma(X, K \otimes^L Rq_*(E \otimes^L Lq^*O_{\mathbf{P}^n_A}(i)))$$

by Derived Categories of Schemes, Lemma 21.1. By Derived Categories of Schemes, Lemma 26.5 the complex $Rq_*(E \otimes^L Lq^*O_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on $X$. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 30.2 we conclude that $Rq_*(Lp^*K \otimes^L E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 57.3 we have

$$Rq_*(Lp^*K \otimes^L E)|_{X \times_A V} = R(U \to V)_*K|_U$$

Since $U \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_U$ is pseudo-coherent relative to $A$ by Lemma 48.18. \qed

58. Descent finiteness properties of complexes

This section is the continuation of Derived Categories of Schemes, Section 11.

Lemma 58.1. Let $X \to S$ be locally of finite type. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if each $Lf_i^*E$ is $m$-pseudo-coherent relative to $S$.\]
Proof. Assume $E$ is $m$-pseudo-coherent relative to $S$. The morphisms $f_i$ are pseudo-coherent by Lemma \[49.6\]. Hence $Lf_i^*E$ is $m$-pseudo-coherent relative to $S$ by Lemma \[48.16\].

Conversely, assume that $Lf_i^*E$ is $m$-pseudo-coherent relative to $S$ for each $i$. Pick $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, $T_{j,k} \to W_{j,k}$, and morphisms $\alpha_{j,k} : T_{j,k} \to X_{i(j,k)}$ over $S$ as in Lemma \[58.1\]. Since the morphism $T_{j,k} \to S$ is flat and of finite presentation, we see that $\alpha_{j,k}$ is pseudo-coherent by Lemma \[49.7\]. Hence

$$L\alpha_{j,k}^*Lf_{i(j,k)}^*E = L(T_{i,k} \to S)^*E$$

is $m$-pseudo-coherent relative to $S$ by Lemma \[48.16\]. Now we want to descend this property through the coverings $\{T_{j,k} \to W_{j,k}\}$, $W_j = \bigcup W_{j,k}$, $\{W_j \to U_j\}$, and $S = \bigcup U_j$. Since for Zariski coverings the result is true (by the definition of $m$-pseudo-coherence relative to $S$), this means we may assume we have a single surjective finite locally free morphism $\pi : Y \to X$ such that $L\pi^*E$ is pseudo-coherent relative to $S$. In this case $R\pi_*L\pi^*E$ is pseudo-coherent relative to $S$ by Lemma \[48.9\] (this is the first time we use that $E$ has quasi-coherent cohomology sheaves). We have $R\pi_*L\pi^*E = E \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_Y$ for example by Derived Categories of Schemes, Lemma \[21.1\] and locally on $X$ the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is the inclusion of a direct summand. Hence we conclude by Lemma \[48.12\].

0CSP Lemma \[58.2\]. Let $X \to T \to S$ be morphisms of schemes. Assume $T \to S$ is flat and locally of finite presentation and $X \to T$ locally of finite type. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if $E$ is $m$-pseudo-coherent relative to $T$.

Proof. Locally on $X$ we can choose a closed immersion $i : X \to \mathbb{A}^n_S$. Then $\mathbb{A}^n_S \to S$ is flat and locally of finite presentation. Thus we may apply Lemma \[48.17\] to see the equivalence holds.

0CSQ Lemma \[58.3\]. Let $f : X \to S$ be locally of finite type. Let $\{S_i \to S\}$ be an fpqc covering of schemes. Denote $f_i : X_i \to S_i$ the base change of $f$ and $g_i : X_i \to X$ the projection. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbb{Z}$. Then $E$ is $m$-pseudo-coherent relative to $S$ if and only if each $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$.

Proof. This follows formally from Lemmas \[58.1\] and \[58.2\]. Namely, if $E$ is $m$-pseudo-coherent relative to $S$, then $Lg_i^*E$ is $m$-pseudo-coherent relative to $S$ (by the first lemma), hence $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$ (by the second). Conversely, if $Lg_i^*E$ is $m$-pseudo-coherent relative to $S_i$, then $Lg_i^*E$ is $m$-pseudo-coherent relative to $S$ (by the second lemma), hence $E$ is $m$-pseudo-coherent relative to $S$ (by the first lemma).

59. Relatively perfect objects

0DJW This section is a continuation of the discussion in Derived Categories of Schemes, Section \[31\].

0DJX Lemma \[59.1\]. Let $i : X \to X'$ be a finite order thickening of schemes. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then $K'$ is pseudo-coherent.
Proof. We first prove $K'$ has quasi-coherent cohomology sheaves. To do this, we may reduce to the case of a first order thickening, see Section 2. Let $I \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out $X$. Tensoring the short exact sequence

$$0 \to I \to \mathcal{O}_{X'} \to i_* \mathcal{O}_X \to 0$$

with $K'$ we obtain a distinguished triangle

$$K' \otimes_{\mathcal{O}_{X'}}^L I \to K' \to K' \otimes_{\mathcal{O}_{X'}}^L i_* \mathcal{O}_X \to (K' \otimes_{\mathcal{O}_{X'}}^L I)[1]$$

Since $i_* = R i_*$ and since we may view $I$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_* (K \otimes_{\mathcal{O}_X}^L I) \to K' \to i_* K \to i_* (K \otimes_{\mathcal{O}_X}^L I)[1]$$

Please use Cohomology, Lemma 45.4 to identify the terms. Since $K$ is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that $K'$ is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Schemes, Lemmas 9.1, 3.9, and 4.1.

Assume $K'$ is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is local on $X'$ hence we may assume $X'$ is affine. Say $X' = \text{Spec}(A')$ and $X = \text{Spec}(A)$ with $A = A'/I$ and $I$ nilpotent. Then $K'$ comes from an object $M' \in D(A')$, see Derived Categories of Schemes, Lemma 3.5. Thus $M = M' \otimes_{A'}^L A$ is a pseudo-coherent object of $D(A)$ by Derived Categories of Schemes, Lemma 9.2, and our assumption on $K$. Hence we can represent $M$ by a bounded above complex of finite free $A$-modules $E'$, see More on Algebra, Lemma 60.5. By More on Algebra, Lemma 68.3 we conclude that $M'$ is pseudo-coherent as desired.

\[ \text{Lemma 59.2.} \quad \text{Consider a cartesian diagram} \]

\[
\begin{array}{ccc}
X & \xrightarrow{i} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{j} & Y'
\end{array}
\]

of schemes. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is $Y$-perfect, then $E'$ is $Y'$-perfect.

Proof. Recall that being $Y$-perfect for $E$ means $E$ is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Derived Categories of Schemes, Definition 31.1). By Lemma 59.1, we find that $E'$ is pseudo-coherent. In particular, $E'$ is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Lemma 9.1. To prove that $E'$ locally has finite tor dimension we may work locally on $X'$. Hence we may assume $X$, $S'$, $X$, $S$ are affine, say given by rings $A'$, $R'$, $A$, $R$. Then we reduce to the commutative algebra version by Derived Categories of Schemes, Lemma 31.3. The commutative algebra version in More on Algebra, Lemma 74.8.

\[ \text{Lemma 59.3.} \quad \text{Let } (R, I) \text{ be a pair consisting of a ring and an ideal } I \text{ contained in the Jacobson radical. Set } S = \text{Spec}(R) \text{ and } S_0 = \text{Spec}(R/I). \text{ Let } f : X \to S \text{ be proper, flat, and of finite presentation. Denote } X_0 = S_0 \times_S X. \text{ Let } E \in D(\mathcal{O}_X) \text{ be pseudo-coherent. If the derived restriction } E_0 \text{ of } E \text{ to } X_0 \text{ is } S_0\text{-perfect, then } E \text{ is } S\text{-perfect.} \]
In this section we study proper morphisms $f : X \to Y$ whose fibres have dimension $\leq 1$ having $R^1f_*\mathcal{O}_X = 0$. To understand the title of this section, please take a look at Algebraic Curves, Sections 22, 23, and 24.

**Lemma 60.1.** Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$. If

- (1) $R^1f_*\mathcal{O}_X = 0$, or more generally
- (2) there is a morphism $g : Y' \to Y$ such that $y$ is in the image of $g$ and such that $R^1f'_*\mathcal{O}_{X'} = 0$ where $f' : X' \to Y'$ is the base change of $f$ by $g$.

Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$.

**Proof.** Suppose we have $g : Y' \to Y$ as in (2). Let $y' \in Y'$ map to $y$. If we can show that $H^1(X'_{y'}, \mathcal{O}_{X'_{y'}}) = 0$, then it follows that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ by flat base change (Cohomology of Schemes, Lemma 5.2) as $X'_{y'} = X_y \otimes_{\kappa(y)} \kappa(y')$. Thus it suffices to show that $R^1f_*\mathcal{O}_X = 0$ implies $H^1(X_y, \mathcal{O}_{X_y}) = 0$.

Assume $R^1f_*\mathcal{O}_X = 0$. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that $R^1f'_*\mathcal{O}_X = 0$ where $f' : X' \to \text{Spec}(\mathcal{O}_{Y,y})$ is the base change to the local ring. Thus we may assume $Y$ is the spectrum of a local ring and $y$ is the closed point. Then there is a short exact sequence

$$0 \to \mathfrak{m}_y \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X_y} \to 0$$
Let over the Noetherian scheme $\mathcal{X}$ for some $i$ immersions, see Limits, Lemma 13.2. Taking fibres at $y$ over $\mathcal{Y}$, write $\mathcal{X}$ as a cofiltered limit with $\mathcal{X}_i$ proper and of finite presentation over $\mathcal{Y}$ and where the morphisms $\mathcal{X} \to \mathcal{X}_i$ and the transition morphisms are closed immersions, see Limits, Lemma [13.2]. Taking fibres at $y$ we see that $\mathcal{X}_y = \lim_{i,y} \mathcal{X}_i$. For some $i$ there is a morphism $\mathcal{X}_{i,y} \to \mathcal{X}_y$ left inverse to the projection $\mathcal{X}_y \to \mathcal{X}_{i,y}$ by Limits, Proposition [6.1] (use that $\mathcal{X}_y$ is proper and hence of finite presentation over the Noetherian scheme $y$). Since $\mathcal{X}_y \to \mathcal{X}_{i,y}$ is a closed immersion, we find that it is an isomorphism. Thus, by the result of the previous paragraph, it suffices to prove the result for $\mathcal{X}_i \to \mathcal{Y}$ thereby reducing us to the case discussed in the next paragraph.

Assume $\mathcal{X} \to \mathcal{Y}$ is proper of finite presentation, $\dim(\mathcal{X}_y) \leq 1$ and $H^1(\mathcal{X}_y, \mathcal{O}_{\mathcal{X}_y}) = 0$. The question (to find the open $\mathcal{V}$) is local on $\mathcal{Y}$ hence we may assume $\mathcal{Y}$ is affine. Write $\mathcal{Y} = \lim_{i \in I} \mathcal{Y}_i$ as a cofiltered limit of affine schemes with $\mathcal{Y}_i$ the spectrum of a Noetherian ring (for example a finite type $\mathcal{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $\mathcal{X}_0 \to \mathcal{Y}_0$ such that $\mathcal{X} \cong \mathcal{Y} \times_{\mathcal{Y}_0} \mathcal{X}_0$, see Limits, Lemma [10.1]. After increasing $0$ we may assume $\mathcal{X}_0 \to \mathcal{Y}_0$ is proper (Limits, Lemma [13.1]). Let $y_0 \in \mathcal{Y}_0$ be the image of $y$. Then $H^1(\mathcal{X}_{0,y_0}, \mathcal{O}_{\mathcal{X}_{0,y_0}}) = 0$ by flat base change (see above). Clearly it suffices to prove the result for $\mathcal{X}_0 \to \mathcal{Y}_0$ and $y_0$ (i.e., find a suitable open $\mathcal{V}_0 \subset \mathcal{Y}_0$). This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{X} \to \mathcal{Y}$ is proper of finite presentation, $\mathcal{Y}$ affine Noetherian, $\dim(\mathcal{X}_y) \leq 1$ and $H^1(\mathcal{X}_y, \mathcal{O}_{\mathcal{X}_y}) = 0$. After shrinking $\mathcal{Y}$ we may assume every fibre of $\mathcal{X} \to \mathcal{Y}$ has dimension $\leq 1$ (use Morphisms, Lemma [27.4] and properness of $\mathcal{X} \to \mathcal{Y}$). Then
$R^p f_* \mathcal{F} = 0$ for $p > 1$ and all quasi-coherent modules on $X$ and this remains true after any base change, see Limits, Lemma [16.4]. We claim that

$$(R^1 f_* \mathcal{O}_X)_y = 0$$

This claim will finish the proof: since $R^1 f_* \mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition [19.1] the claim means that there is an open $V \subset Y$ such that $R^1 f_* \mathcal{O}_X|_V = 0$. Then we find $H^1(X_v, \mathcal{O}_{X_v}) = 0$ for all $v \in V$ by Lemma [60.1]. This property of the fibres is inherited by the fibres of any base change $X' \to Y'$ by a morphism $g : Y' \to V$ (since fibres of $X' \to Y'$ are flat base changes of the fibres of $f^{-1} V \to V$). Hence, if $Y'$ is Noetherian, then we get the vanishing of $R^1 f'_* \mathcal{O}_{X'}$ from the claim. Finally, if $Y'$ is general, then it suffices to prove the vanishing of $R^1 f'_* \mathcal{O}_{X'}$ locally on $Y'$. Thus we may assume $Y'$ is affine. Then $R^1 f'_* \mathcal{O}_{X'}$ is the quasi-coherent $\mathcal{O}_{Y'}$-module associated to $H^1(X', \mathcal{O}_{X'})$. Write $Y' = \lim V'_i$ with $V'_i$ affine and of finite type over $V$. Setting $X'_i = Y'_i \times_Y X$ we find $H^1(X'_i, \mathcal{O}_{X'_i}) = \colim H^1(X'_i, \mathcal{O}_{X'_i})$, see Cohomology, Lemma [20.2] and see the discussion in Limits, Section 4 on topology. Thus the desired vanishing follows from the vanishing in the Noetherian case. Some details omitted.

Proof of the claim. We will use the theorem on formal functions to prove it. We first do a flat base change by $\Spec(\mathcal{O}_{Y,y}) \to Y$ to reduce to the case where $Y$ is the spectrum of a Noetherian local ring and $y \in Y$ is the closed point (compare with proof of Lemma [60.1]). We have already observed that $R^1 f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module and hence its stalk is a finite $\mathcal{O}_y$-module. By Nakayama’s lemma (Algebra, Lemma [19.1]) if the completion of a finite module over a local ring is zero, then the module is zero. Thus Cohomology of Schemes, Lemma [20.7] tells us it suffices to prove

$$H^1(X, \mathcal{O}_X/m^n_y \mathcal{O}_X) = 0$$

for all $n \geq 1$. We are going to use induction on $n$. In the base case $n = 1$ we have $\mathcal{O}_X/m^n_y \mathcal{O}_X = \mathcal{O}_{X_y}$ and the vanishing holds by assumption. For general $n$ we consider the short exact sequence

$$0 \to m^n_y \mathcal{O}_X/m^{n+1}_y \mathcal{O}_X \to \mathcal{O}_X/m^{n+1}_y \mathcal{O}_X \to \mathcal{O}_X/m^n_y \mathcal{O}_X \to 0$$

Observe that we have a surjection

$$\mathcal{O}_{X_y}^{\oplus r_n} \cong m^n_y/m^{n+1}_y \otimes_{\kappa(y)} \mathcal{O}_{X_y} \longrightarrow m^n_y \mathcal{O}_X/m^{n+1}_y \mathcal{O}_X$$

for some integers $r_n \geq 0$. Since $\dim(X_y) \leq 1$ this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees $\geq 2$ coming from Cohomology, Proposition [21.7]). Hence the $H^1$ of the middle term in the short exact sequence is zero by induction and we win. □

0E7H Lemma 60.3. Let $f : X \to Y$ be a proper morphism of schemes such that $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Let $\mathcal{F}$ be quasi-coherent on $X$. Then

1. $R^p f_* \mathcal{F} = 0$ for $p > 1$, and
2. $R^1 f_* \mathcal{F} = 0$ if there is a surjection $f^* \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ quasi-coherent on $Y$.

If $Y$ is affine, then we also have

3. $H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and
4. $H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.
Proof. The vanishing in (1) is Limits, Lemma 16.4. To prove (2) we may work locally on $Y$ and assume $Y$ is affine. Then $R^1f_*\mathcal{F}$ is the quasi-coherent module on $Y$ associated to the module $H^1(X, \mathcal{F})$. Here we use that $Y$ is affine, quasi-coherence of higher direct images (Cohomology of Schemes, Lemma 4.5), and Cohomology of Schemes, Lemma 4.6. Since $Y$ is affine, the quasi-coherent module $\mathcal{G}$ is globally generated, and hence so is $f^*\mathcal{G}$ and $\mathcal{F}$. In this way we see that (4) implies (2). Part (3) follows from (1) as well as the remarks on quasi-coherence of direct images just made. Thus all that remains is the prove (4). If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on $H^1$. Since $H^1(X, \mathcal{O}_X) = 0$ because $R^1f_*\mathcal{O}_X = 0$ by Lemma 60.2, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.1) we conclude. 

Lemma 60.4. Let $f : X \to Y$ be a proper morphism of schemes. Assume

1. for $y \in Y$ we have $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$, and
2. $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Then $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is surjective for any base change $f' : X' \to Y'$ of $f$.

Proof. We may assume $Y$ and $Y'$ affine. Then we can choose a closed immersion $Y' \to Y''$ with $Y'' \to Y$ a flat morphism of affines. By flat base change (Cohomology of Schemes, Lemma 5.2) we see that the result holds for $X'' \to Y''$. Thus we may assume $Y'$ is a closed subscheme of $Y$. Let $I \subset \mathcal{O}_Y$ be the ideal cutting out $Y'$. Then there is a short exact sequence

$$0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

where we view $\mathcal{O}_{X'}$ as a quasi-coherent module on $X$. By Lemma 60.3 we have $H^1(X, I\mathcal{O}_X) = 0$. It follows that

$$H^0(Y, \mathcal{O}_Y) \to H^0(Y, f_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{X'})$$

is surjective as desired. The first arrow is surjective as $Y$ is affine and since we assumed $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and the second by the long exact sequence of cohomology associated to the short exact sequence above and the vanishing just proved.

Lemma 60.5. Consider a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{s} & Y 
\end{array}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

1. $X \to S$ is locally of finite presentation and flat at points of $X_s$,
2. $f$ is proper,
3. the fibres of $f_s : X_s \to Y_s$ have dimension $\leq 1$ and $R^1f_{s,*}\mathcal{O}_{X_s} = 0$,
4. $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is surjective.

Then there is an open $Y_s \subset V \subset Y$ such that (a) $f^{-1}(V)$ is flat over $S$, (b) $\dim(X_y) \leq 1$ for $y \in V$, (c) $R^1f_*\mathcal{O}_X|_V = 0$, (d) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is surjective, and (b), (c), and (d) remain true after base change by any $Y' \to V$. 

□
Proof. Let \( y \in Y \). It suffices to find an open neighbourhood of \( y \) with the desired properties. As a first step, we replace \( Y \) by the open \( V \) found in Lemma 10.1 so that \( R^1f_*\mathcal{O}_X \) is zero universally (the hypothesis of the lemma holds by Lemma 60.1). We also shrink \( Y \) so that all fibres of \( f \) have dimension \( \leq 1 \) (use Morphisms, Lemma 27.4 and properness of \( f \)). We may also assume \( Y \) and \( S \) are affine.

Please skip this paragraph on a first reading; it is here because we did not assume \( Y \to S \) is locally of finite presentation (the only case of practical interest). Since \( X \to Y \) is proper we may choose a closed immersion \( X \to X' \) where \( X' \to Y \) is proper of finite presentation (Limits, Lemma 13.2). Write \( Y = \lim_{i \in I} Y_i \) as a cofiltered limit with \( Y_i \) affine and \( Y_i \to S \) of finite presentation. Choose \( i \in I \) and a morphism \( X'_i \to Y_i \) of finite presentation with \( X' = Y \times_{Y_i} X'_i \), see Limits, Lemma 10.1. After increasing \( i \) we may assume \( X'_i \to Y_i \) is proper, see Limits, Lemma 13.1. Since \( X' = \lim_{i \in I} X'_i \) and \( X \to X' \) is a closed immersion we find an \( i \) such that \( X \to X'_i \) is a closed immersion, see Limits, Lemma 4.16. For this \( i \) we see that \( f_i : X \to Y_i \) is proper. Since \( Y_0 = \lim Y_{0,i} \) we may apply Limits, Lemma 16.1 to the system of morphisms \((f_{i,s} : X_s \to Y_{0,s})\) and we find that for \( i \) large enough the fibres of \( f_{i,s} \) have dimension \( \leq 1 \). Finally, consider the maps

\[
H^0(Y_{1,s}, \mathcal{O}_{Y_{1,s}}) \to H^0(Y_s, \mathcal{O}_Y) \to H^0(X_s, \mathcal{O}_X)
\]

Since \( Y_{1,s} \) is a Noetherian affine scheme and since \( X_s \) is a proper scheme over it, we know that \( H^0(X_s, \mathcal{O}_X) \) is a finite module over \( H^0(Y_{1,s}, \mathcal{O}_{Y_{1,s}}) \) (Cohomology of Schemes, Proposition 19.1). The second arrow is surjective by assumption. Hence we can find finitely many elements in \( H^0(Y_s, \mathcal{O}_Y) \) whose images in \( H^0(X_s, \mathcal{O}_X) \) generate as a module over \( H^0(Y_{1,s}, \mathcal{O}_{Y_{1,s}}) \). Then there exists an \( i' \geq i \) such that these elements are in the image of \( H^0(Y_{1,s}, \mathcal{O}_{Y_{1,s}}) \to H^0(Y_s, \mathcal{O}_Y) \) as \( Y_s = \lim Y_{s,i} \). In other words, we have checked the assumptions of the lemma hold for \( X \to Y_i \to S \).

An straightforward argument (omitted) shows that the truth of the lemma for \( X \to Y_i \to S \) implies the truth of the lemma for \( X \to Y \to S \). In this way we reduce to the case discussed in the next paragraph.

Assume \( S \) and \( Y \) affine and \( Y \to S \) of finite presentation. Write \( S = \lim S_i \) as a cofiltered of affine Noetherian schemes \( S_i \). By Limits, Lemma 10.1 there exists an element \( 0 \in I \) and a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow & & \downarrow \\
S_0
\end{array}
\]

of finite type morphisms of schemes whose base change to \( S \) is the diagram of the lemma. After increasing \( 0 \) we may assume \( Y_0 \) is affine and \( X_0 \to Y_0 \) proper, see Limits, Lemmas 13.1 and 4.13. Let \( s_0 \in S_0 \) be the image of \( s \). As \( Y_s \) is affine, we see that \( R^1f_{s_*}\mathcal{O}_{X_s} = 0 \) is equivalent to \( H^1(X_s, \mathcal{O}_{X_s}) = 0 \). Since \( X_s \) is the base change of \( X_{0,s_0} \) by the faithfully flat map \( \kappa(s_0) \to \kappa(s) \) we see that \( H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0 \) and hence \( R^1f_{s_*}\mathcal{O}_{X_{0,s_0}} = 0 \). Similarly, the surjectivity of \( \mathcal{O}_{Y_s} \to f_{s_*}\mathcal{O}_{X_s} \) implies the surjectivity of \( \mathcal{O}_{Y_{0,s_0}} \to f_{s_*}\mathcal{O}_{X_{0,s_0}} \). Since the dimension of the fibres of \( X_s \) are at most 1, the same is true for the morphism \( X_{0,s_0} \to Y_{0,s_0} \). Finally, there is a quasi-compact open \( U \subset X \) containing \( X_s \) such that \( U \to S \) is flat, see Theorem 15.1. After increasing \( 0 \) we may assume there is an open \( U_0 \subset X_0 \) whose inverse image in \( X \) is \( U \), see Limits, Lemma 4.11. Observe that \( X_{0,s_0} \subset U_0 \) as \( X_s \to X_{0,s_0} \).
is surjective. Increasing 0 once more, we may assume $U_0$ is flat over $S_0$, see Limits, Lemma [8.7]. Thus it suffices to prove the lemma for $X_0 \to Y_0 \to S_0$ and the point $s_0$.

Combining the reduction arguments above we reduce to the case where $S$ and $Y$ affine, $Y \to S$ of finite presentation, $S$ Noetherian, fibre of $f$ have dimension $\leq 1$, and $R^1 f_* O_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$O_{Y,y} \to (f_* O_X)_y$$

is surjective. The claim implies the lemma. Namely, since $f_* O_X$ is coherent (Cohomology of Schemes, Proposition [19.1]) the claim means there is an open $V \subset Y$ containing $y$ such that $O_V \to f_* O_X|_V$ is surjective. Then we can apply Lemma [60.4] to see that this remains true after any base change.

Proof of the claim. We first do a flat base change by Spec$(O_{S,s}) \to S$ to reduce to the case where $S$ is the spectrum of a Noetherian local ring and $s \in S$ is the closed point. Next, if $U \subset X$ denotes the open subset of points where $X \to S$ is flat (Theorem [15.1]), then since $X_y \subset X_s \subset U$ and since $f$ is proper, we see that after replacing $Y$ by a smaller affine open we may assume $U = X$. Then we consider the short exact sequence

$$0 \to m_s O_X \to O_X \to O_{X_s} \to 0$$

Choose generators $a_1, \ldots, a_r \in m_s$ and consider the short exact sequence

$$0 \to \mathcal{K} \to O_X^{\oplus r} \to m_s O_X \to 0$$

By flatness of $X$ over $S$ this determines a short exact sequence

$$0 \to g^* \mathcal{K} \to O_X^{\oplus r} \to m_s O_X \to 0$$

where $g : X \to S$ is the given morphism. By Lemma [60.3] we have $H^1(X, m_s O_X) = 0$ and $H^1(X, g^* \mathcal{K}) = 0$ as globally generated quasi-coherent modules. Combining this with the long exact sequences of cohomology for the short exact sequences above we find an exact sequence

$$H^0(X, O_{X,s}^{\oplus r}) \to H^0(X, O_X) \to H^0(X_s, O_{X_s}) \to 0$$

in other words, we see that $H^0(X_s, O_{X_s}) = H^0(X, O_X)/m_s H^0(X, O_X)$. Since $H^0(Y, O_Y) \to H^0(Y_s, O_{Y_s}) \to H^0(X_s, O_{X_s})$ is surjective, we conclude from Nakayama's lemma (Algebra, Lemma [19.1]) that $H^0(Y, O_Y) \to H^0(X, O_X)$ is surjective at least after localizing at the prime ideal corresponding to $y$. This finishes the proof. \qed

**0E7K Lemma 60.6.** Consider a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{g} & Y
\end{array}$$

of morphisms of schemes. Assume $X \to S$ is flat, $f$ is proper, $\dim(X_y) \leq 1$ for $y \in Y$, and $R^1 f_* O_X = 0$. Then $f_* O_X$ is $S$-flat and formation of $f_* O_X$ commutes with arbitrary base change $S' \to S$.

**Proof.** We may assume $Y$ and $S$ are affine, say $S = \text{Spec}(A)$. To show the quasi-coherent $O_Y$-module $f_* O_X$ is flat relative to $S$ it suffices to show that $H^0(X, O_X)$ is flat over $A$ (some details omitted). By Lemma [60.3] we have $H^1(X, O_X \otimes_A M) = 0$ for every $A$-module $M$. Since also $O_X$ is flat over $A$ we deduce the functor $M \mapsto$
Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & & \\
\end{array}
\]

of morphisms of schemes. Let \( s \in S \) be a point. Assume

1. \( X \to S \) is locally of finite presentation and flat at points of \( X_s \).
2. \( Y \to S \) is locally of finite presentation,
3. \( f \) is proper,
4. the fibres of \( f_s : X_s \to Y_s \) have dimension \( \leq 1 \) and \( R^1f_{s,*}\mathcal{O}_{X_s} = 0 \),
5. \( \mathcal{O}_Y \to f_{s,*}\mathcal{O}_{X_s} \) is an isomorphism.

Then there is an open \( Y_s \subset V \subset Y \) such that (a) \( V \) is flat over \( S \), (b) \( f^{-1}(V) \) is flat over \( S \), (c) \( \dim(X_y) \leq 1 \) for \( y \in V \), (d) \( R^1f_{s,*}\mathcal{O}_X|_V = 0 \), (e) \( \mathcal{O}_V \to f_{s,*}\mathcal{O}_X|_V \) is an isomorphism, and (a) – (e) remain true after base change of \( f^{-1}(V) \to V \) by any \( S' \to S \).

**Proof.** Let \( y \in Y_s \). We may always replace \( Y \) by an open neighbourhood of \( y \). Thus we may assume \( Y \) and \( S \) affine. We may also assume that \( X \) is flat over \( S \), \( \dim(X_y) \leq 1 \) for \( y \in Y \), \( R^1f_{s,*}\mathcal{O}_X = 0 \) universally, and that \( \mathcal{O}_Y \to f_{s,*}\mathcal{O}_X \) is surjective, see Lemma \( \text{[60.3]} \) (We won’t use all of this.)

Assume \( S \) and \( Y \) affine. Write \( S = \text{lim}_i S_i \) as a cofiltered of affine Noetherian schemes \( S_i \). By Limits, Lemma \( \text{[10.1]} \) there exists an element \( 0 \in I \) and a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow & & \downarrow \\
S_0 & & \\
\end{array}
\]

of finite type morphisms of schemes whose base change to \( S \) is the diagram of the lemma. After increasing 0 we may assume \( Y_0 \) is affine and \( X_0 \to S_0 \) proper, see Limits, Lemmas \( \text{[13.1]} \) and \( \text{[13.13]} \). Let \( s_0 \in S_0 \) be the image of \( s \). As \( Y_s \) is affine, we see that \( R^1f_{s,*}\mathcal{O}_{X_s} |_{X_{0,s_0}} = 0 \) is equivalent to \( H^1(X_{s_0}, \mathcal{O}_{X_{s_0}}) = 0 \). Since \( X_s \) is the base change of \( X_{0,s_0} \) by the faithfully flat map \( \kappa(s_0) \to \kappa(s) \) we see that \( H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0 \) and hence \( R^1f_{0,*}\mathcal{O}_{X_{0,s_0}} = 0 \). Similarly, as \( \mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s} \) is an isomorphism, so is \( \mathcal{O}_{Y_{0,s_0}} \to f_{0,*}\mathcal{O}_{X_{0,s_0}} \). Since the dimensions of the fibres of \( X_s \to Y_s \) are at most 1, the same is true for the morphism \( X_{0,s_0} \to Y_{0,s_0} \). Finally, since \( X \to S \) is flat, after increasing 0 we may assume \( X_0 \) is flat over \( S_0 \), see Limits, Lemma \( \text{[8.7]} \). Thus it suffices to prove the lemma for \( X_0 \to Y_0 \to S_0 \) and the point \( s_0 \).
Combining the reduction arguments above we reduce to the case where $S$ and $Y$ affine, $S$ Noetherian, the fibres of $f$ have dimension $\leq 1$, and $R^1f_*\mathcal{O}_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$$

is an isomorphism. The claim implies the lemma. Namely, since $f_*\mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition 19.1) the claim means we can replace $Y$ by an open neighbourhood of $y$ and obtain an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we conclude that $Y$ is flat over $S$ by Lemma 60.6. Finally, the isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ remains an isomorphism after any base change $S' \to S$ by the final statement of Lemma 60.6.

Proof of the claim. We already know that $\mathcal{O}_{Y,y} \twoheadrightarrow (f_*\mathcal{O}_X)_y$ is surjective (Lemma 60.5) and that $(f_*\mathcal{O}_X)_y$ is $\mathcal{O}_{S,s}$-flat (Lemma 60.6) and that the induced map

$$\mathcal{O}_{Y,y} = \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y/\mathfrak{m}_s(f_*\mathcal{O}_X)_y = (f_*\mathcal{O}_X)_y$$

is injective by the assumption in the lemma. Then it follows from Algebra, Lemma 98.1 that $\mathcal{O}_{Y,y} \twoheadrightarrow (f_*\mathcal{O}_X)_y$ is injective as desired. □

0E24

Lemma 60.8. Let $f : X \to Y$ be a proper morphism of Noetherian schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, such that the fibres of $f$ have dimension $\leq 1$, and such that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Then $f^* : \text{Pic}(Y) \to \text{Pic}(X)$ is a bijection onto the subgroup of $\mathcal{L} \in \text{Pic}(X)$ with $\mathcal{L}|_{X_y} = \mathcal{O}_{X_y}$ for all $y \in Y$.

Proof. By the projection formula (Cohomology, Lemma 45.2) we see that $f_*f^*\mathcal{N} \cong \mathcal{N}$ for $\mathcal{N} \in \text{Pic}(Y)$. We claim that for $\mathcal{L} \in \text{Pic}(X)$ with $\mathcal{L}|_{X_y} = \mathcal{O}_{X_y}$ for all $y \in Y$ we have $\mathcal{N} = f_*\mathcal{L}$ is invertible and $\mathcal{L} \cong f^*\mathcal{N}$. This will finish the proof.

The $\mathcal{O}_Y$-module $\mathcal{N} = f_*\mathcal{L}$ is coherent by Cohomology of Schemes, Proposition 19.1. Thus to see that it is an invertible $\mathcal{O}_Y$-module, it suffices to check on stalks (Algebra, Lemma 77.2). Since the map from a Noetherian local ring to its completion is faithfully flat, it suffices to check the completion $(f_*\mathcal{L})_y^\wedge$ is free (see Algebra, Section 96 and Lemma 77.5). For this we will use the theorem of formal functions as formulated in Cohomology of Schemes, Lemma 20.7. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$ and hence $(f_*\mathcal{O}_X)_y^\wedge \cong \mathcal{O}_{Y,y}^\wedge$, it suffices to show that $\mathcal{L}|_{X_y} = \mathcal{O}_{X_y}$ for each $n$ (compatibly for varying $n$). By Lemma 4.1 we have an exact sequence

$$H^1(X_y, m^n_y\mathcal{O}_X/m^{n+1}_y\mathcal{O}_X) \to \text{Pic}(X_{n+1}) \to \text{Pic}(X_n)$$

with notation as in the theorem on formal functions. Observe that we have a surjection

$$\mathcal{O}_{X_y}^{\oplus r_n} \cong m^n_y/m^{n+1}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{O}_{X_y} \longrightarrow m^n_y\mathcal{O}_X/m^{n+1}_y\mathcal{O}_X$$

for some integers $r_n \geq 0$. Since $\dim(X_y) \leq 1$ this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees $\geq 2$ coming from Cohomology, Proposition 21.7). Hence the $H^1$ in the sequence is zero and the transition maps $\text{Pic}(X_{n+1}) \to \text{Pic}(X_n)$ are injective as desired. We still have to show that $f^*\mathcal{N} \cong \mathcal{L}$. This is proved by the same method and we omit the details. □

61. Other chapters

Preliminaries (1) Introduction
References


