

OBSOLETE

073T

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1. Introduction

073U In this chapter we put some lemmas that have become “obsolete” (see [Mil17]).

2. Homological algebra

0BFJ

076K **Remark 2.1.** The following remarks are obsolete as they are subsumed in Homology, Lemmas 21.11 and 22.6. Let \mathcal{A} be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Homology, Definition 9.1). Suppose that $K^{\bullet, \bullet}$ is a double complex to which Homology, Lemma 22.6 applies such that for some $r \geq 0$ all the objects $'E_r^{p,q}$ belong to \mathcal{C} . Then all the cohomology groups $H^n(sK^{\bullet})$ belong to \mathcal{C} . Namely, the assumptions imply that the kernels and images of $'d_r^{p,q}$ are in \mathcal{C} . Whereupon we see that each $'E_{r+1}^{p,q}$ is in \mathcal{C} . By induction we see that each $'E_{\infty}^{p,q}$ is in

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\mathcal{C} . Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in \mathcal{C} . Using that \mathcal{C} is closed under extensions we conclude that $H^n(sK^\bullet)$ is in \mathcal{C} as claimed. The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if (K^\bullet, F) is a filtered complex to which Homology, Lemma 21.11 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to \mathcal{C} , then each $H^n(K^\bullet)$ is an object of \mathcal{C} .

3. Obsolete algebra lemmas

088X

055Z **Lemma 3.1.** *Let M be an R -module of finite presentation. For any surjection $\alpha : R^{\oplus n} \rightarrow M$ the kernel of α is a finite R -module.*

Proof. This is a special case of Algebra, Lemma 5.3. □

00I5 **Lemma 3.2.** *Let $\varphi : R \rightarrow S$ be a ring map. If*

- (1) *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ , and*
- (2) *for any $x \in \text{Ker}(\varphi)$ there exists $n > 0$ such that $x^n = 0$,*

then φ induces a homeomorphism on spectra. Given a prime number p such that

- (a) *S is generated as an R -algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
- (b) *the kernel of φ is generated by nilpotent elements,*

then (1) and (2) hold, and for any ring map $R \rightarrow R'$ the ring map $R' \rightarrow R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. This is a combination of Algebra, Lemmas 45.3 and 45.6. □

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

00SX **Lemma 3.3.** *Let $R \rightarrow S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over \mathfrak{p} . Assume $S_{\mathfrak{q}}$ is essentially of finite type over $R_{\mathfrak{p}}$. Assume given*

- (1) *an integer $n \geq 0$,*
- (2) *a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \dots, x_n]$,*
- (3) *a surjective $\kappa(\mathfrak{p})$ -homomorphism*

$$\psi : (\kappa(\mathfrak{p})[x_1, \dots, x_n])_{\mathfrak{a}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}},$$

and

- (4) *elements $\bar{f}_1, \dots, \bar{f}_e$ in $\text{Ker}(\psi)$.*

Then there exist

- (1) *an integer $m \geq 0$,*
- (2) *and element $g \in S$, $g \notin \mathfrak{q}$,*
- (3) *a map*

$$\Psi : R[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}] \longrightarrow S_g,$$

and

- (4) *elements $f_1, \dots, f_e, f_{e+1}, \dots, f_{e+m}$ of $\text{Ker}(\Psi)$*

such that

(1) the following diagram commutes

$$\begin{array}{ccc} R[x_1, \dots, x_{n+m}] & \xrightarrow{x_{n+j} \mapsto 0} & (\kappa(\mathfrak{p})[x_1, \dots, x_n])_{\mathfrak{a}}, \\ \Psi \downarrow & & \downarrow \psi \\ S_g & \longrightarrow & S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \end{array}$$

(2) the element f_i , $i \leq n$ maps to a unit times \bar{f}_i in the local ring

$$(\kappa(\mathfrak{p})[x_1, \dots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})},$$

(3) the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and

(4) the induced map $R[x_1, \dots, x_{n+m}]_{\mathfrak{b}} \rightarrow S_{\mathfrak{q}}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals \mathfrak{p} and \mathfrak{q} . Namely, suppose we have constructed

$$\Psi' : R_{\mathfrak{p}}[x_1, \dots, x_{n+m}] \longrightarrow S_{\mathfrak{q}}$$

and $f'_1, \dots, f'_{e+m} \in R_{\mathfrak{p}}[x_1, \dots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \dots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R -algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak{q}}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals \mathfrak{p} and \mathfrak{q} . Pick $y_1, \dots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \dots, y_{n+m} \in S$ be elements which generate an R -subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R . Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \dots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \dots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \dots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \dots, x_n]$ of g_j and d . Set $y'_{n+j} = D(y_1, \dots, y_n)y_{n+j} - G_j(y_1, \dots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \dots, y_n, y'_n, \dots, y'_{n+m}$ generate an R -subalgebra of S whose localization is S . We define

$$\Psi : R[x_1, \dots, x_{n+m}] \rightarrow S$$

to be the map that sends x_i to y_i for $i = 1, \dots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \dots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal \mathfrak{b} maps onto the ideal $(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \dots, x_{n+m}]$.

Denote $J = \text{Ker}(\Psi)$. We have a short exact sequence

$$0 \rightarrow J_{\mathfrak{b}} \rightarrow R[x_1, \dots, x_{n+m}]_{\mathfrak{b}} \rightarrow S_{\mathfrak{q}} \rightarrow 0.$$

The surjectivity comes from our choice of $y_1, \dots, y_n, y'_n, \dots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \rightarrow \kappa(\mathfrak{p})[x_1, \dots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \dots, x_{n+m})} \rightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \rightarrow 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. \square

01DE **Remark 3.4** (Projective resolutions). Let R be a ring. For any set S we let $F(S)$ denote the free R -module on S . Then any left R -module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \rightarrow F(M) \rightarrow M \rightarrow 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

02CA **Lemma 3.5.** *Let S be a multiplicative set of A . Then the map*

$$f : \operatorname{Spec}(S^{-1}A) \longrightarrow \operatorname{Spec}(A)$$

induced by the canonical ring map $A \rightarrow S^{-1}A$ is a homeomorphism onto its image and $\operatorname{Im}(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. This is a duplicate of Algebra, Lemma 16.5. □

05IP **Lemma 3.6.** *Let $A \rightarrow B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A -algebra.*

Proof. Special case of More on Flatness, Proposition 13.10. □

053F **Lemma 3.7.** *Let R be a domain with fraction field K . Let $S = R[x_1, \dots, x_n]$ be a polynomial ring over R . Let M be a finite S -module. Assume that M is flat over R . If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S .*

Proof. This lemma is true because M is finitely presented even without the assumption that $M \otimes_R R'$ is finitely presented for every R' as in the statement of the lemma. This follows from More on Flatness, Proposition 13.10. Originally this lemma had an erroneous proof (thanks to Ofer Gabber for finding the gap) and was used in an alternative proof of the proposition cited. To reinstate this lemma, we need a correct argument in case R is a local normal domain using only results from the chapters on commutative algebra; please email stacks.project@gmail.com if you have an argument. □

02TQ **Lemma 3.8.** *Let $A \rightarrow B$ be a ring map. Let $f \in B$. Assume that*

- (1) $A \rightarrow B$ is flat,
- (2) f is a nonzerodivisor, and
- (3) $A \rightarrow B/fB$ is flat.

Then for every ideal $I \subset A$ the map $f : B/IB \rightarrow B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A . In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB . Hence the result follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \end{array}$$

with exact rows. □

051A **Lemma 3.9.** *If $R \rightarrow S$ is a faithfully flat ring map then for every R -module M the map $M \rightarrow S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.*

Proof. This lemma is a duplicate of Algebra, Lemma 81.11. □

07C2 **Remark 3.10.** This reference/tag used to refer to a Section in the chapter Smoothing Ring Maps, but the material has since been subsumed in Algebra, Section 126.

07K3 **Lemma 3.11.** *Let (R, \mathfrak{m}) be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak{m}$ be a nonzerodivisor. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the minimal primes of R . Then*

$$\text{length}_R(R/(x)) = \sum_i \text{ord}_{R/\mathfrak{q}_i}(x)$$

Proof. Special (very easy) case of Chow Homology, Lemma 3.2. \square

0AXG **Lemma 3.12.** *Let A be a Noetherian local normal domain of dimension 2. For $f \in \mathfrak{m}$ nonzero denote $\text{div}(f) = \sum n_i(\mathfrak{p}_i)$ the divisor associated to f on the punctured spectrum of A . We set $|f| = \sum n_i$. There exist integers N and M such that $|f + g| \leq M$ for all $g \in \mathfrak{m}^N$.*

Proof. Pick $h \in \mathfrak{m}$ such that f, h is a regular sequence in A (this follows from Algebra, Lemmas 151.4 and 71.7). We will prove the lemma with $M = \text{length}_A(A/(f, h))$ and with N any integer such that $\mathfrak{m}^N \subset (f, h)$. Such an integer N exists because $\sqrt{(f, h)} = \mathfrak{m}$. Note that $M = \text{length}_A(A/(f + g, h))$ for all $g \in \mathfrak{m}^N$ because $(f, h) = (f + g, h)$. This moreover implies that $f + g, h$ is a regular sequence in A too, see Algebra, Lemma 103.2. Now suppose that $\text{div}(f + g) = \sum m_j(\mathfrak{q}_j)$. Then consider the map

$$c : A/(f + g) \longrightarrow \prod A/\mathfrak{q}_j^{(m_j)}$$

where $\mathfrak{q}_j^{(m_j)}$ is the symbolic power, see Algebra, Section 63. Since A is normal, we see that $A_{\mathfrak{q}_i}$ is a discrete valuation ring and hence

$$A_{\mathfrak{q}_i}/(f + g) = A_{\mathfrak{q}_i}/\mathfrak{q}_i^{m_i} A_{\mathfrak{q}_i} = (A/\mathfrak{q}_i^{(m_i)})_{\mathfrak{q}_i}$$

Since $V(f + g, h) = \{\mathfrak{m}\}$ this implies that c becomes an isomorphism on inverting h (small detail omitted). Since h is a nonzerodivisor on $A/(f + g)$ we see that the length of $A/(f + g, h)$ equals the Herbrand quotient $e_A(A/(f + g), 0, h)$ as defined in Chow Homology, Section 2). Similarly the length of $A/(h, \mathfrak{q}_j^{(m_j)})$ equals $e_A(A/\mathfrak{q}_j^{(m_j)}, 0, h)$. Then we have

$$\begin{aligned} M &= \text{length}_A(A/(f + g, h)) \\ &= e_A(A/(f + g), 0, h) \\ &= \sum_i e_A(A/\mathfrak{q}_i^{(m_i)}, 0, h) \\ &= \sum_i \sum_{m=0, \dots, m_j-1} e_A(\mathfrak{q}_j^{(m)}/\mathfrak{q}_j^{(m+1)}, 0, h) \end{aligned}$$

The equalities follow from Chow Homology, Lemmas 2.3 and 2.4 using in particular that the cokernel of c has finite length as discussed above. It is straightforward to prove that $e_A(\mathfrak{q}^{(m)}/\mathfrak{q}^{(m+1)}, 0, h)$ is at least 1 by Nakayama's lemma. This finishes the proof of the lemma. \square

0BJK **Lemma 3.13.** *Let $A \rightarrow B$ be a flat local homomorphism of Noetherian local rings. If A and $B/\mathfrak{m}_A B$ are Gorenstein, then B is Gorenstein.*

Proof. Follows immediately from Dualizing Complexes, Lemma 21.8. \square

0DXL **Lemma 3.14.** *Let (A, \mathfrak{m}) be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A -module. Let s be an integer. Assume*

- (1) A has a dualizing complex,
- (2) if $\mathfrak{p} \notin V(I)$ and $V(\mathfrak{p}) \cap V(I) \neq \{\mathfrak{m}\}$, then $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) > s$.

Then there exists an $n > 0$ and an ideal $J \subset A$ with $V(J) \cap V(I) = \{\mathfrak{m}\}$ such that JI^n annihilates $H_{\mathfrak{m}}^i(M)$ for $i \leq s$.

Proof. According to Local Cohomology, Lemma 8.4 we have to show this for the finite A -module $E^i = \text{Ext}_A^{-i}(M, \omega_A^\bullet)$ for $i \leq s$. The support Z of $E^0 \oplus \dots \oplus E^s$ is closed in $\text{Spec}(A)$ and does not contain any prime as in (2). Hence it is contained in $V(JI^n)$ for some J as in the statement of the lemma. \square

0EFS **Lemma 3.15.** *Let (A, \mathfrak{m}) be a Noetherian local ring. Let $I \subset A$ be an ideal. Let M be a finite A -module. Let s and d be integers. Assume*

- (a) *A has a dualizing complex,*
- (b) *$cd(A, I) \leq d$,*
- (c) *if $\mathfrak{p} \notin V(I)$ then $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) > s$ or $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) > d + s$.*

Then the assumptions of Algebraic and Formal Geometry, Lemma 10.4 hold for A, I, \mathfrak{m}, M and $H_{\mathfrak{m}}^i(M) \rightarrow \lim H_{\mathfrak{m}}^i(M/I^n M)$ is an isomorphism for $i \leq s$ and these modules are annihilated by a power of I .

Proof. The assumptions of Algebraic and Formal Geometry, Lemma 10.4 by the more general Algebraic and Formal Geometry, Lemma 10.5. Then the conclusion of Algebraic and Formal Geometry, Lemma 10.4 gives the second statement. \square

0EFZ **Lemma 3.16.** *In Algebraic and Formal Geometry, Situation 10.1 we have $H_{\mathfrak{a}}^s(M) = \lim H_{\mathfrak{a}}^s(M/I^n M)$.*

Proof. This is immediate from Algebraic and Formal Geometry, Theorem 10.8. The original version of this lemma, which had additional assumptions, was superseded by the this theorem. \square

4. Lemmas related to ZMT

073V The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 122.13.

00PR **Lemma 4.1.** *Let $\varphi : R \rightarrow S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_n t + \varphi(a_{n-1})$, and so on till $u_1 = u_2 t + \varphi(a_1)$. Then all of u_n, u_{n-1}, \dots, u_1 and $u_n t, u_{n-1} t, \dots, u_1 t$ are integral over R , and the ideals $(\varphi(a_0), \dots, \varphi(a_n))$ and (u_n, \dots, u_1) of S are equal.*

Proof. We prove this by induction on n . As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 122.1 that $u_n t$ is integral over R . Of course $u_n = \varphi(a_n)$ is integral over R . Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 35.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \rightarrow S$ where S' is the integral closure of R in S and the displayed equation we see that u_{n-1}, \dots, u_1 and $u_{n-1}t, \dots, u_1t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. \square

00PS **Lemma 4.2.** *Let $\varphi : R \rightarrow S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \dots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R .*

Proof. This is immediate from Lemma 4.1 since one of the elements u_i will not be in J . \square

The following two lemmas are a way of describing closed subschemes of \mathbf{P}_R^1 cut out by one (nondegenerate) equation.

00Q4 **Lemma 4.3.** *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d . Assume that for every prime \mathfrak{p} of R at least one coefficient of F is not in \mathfrak{p} . Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R -module S_n is finite locally free of rank d .*

Proof. The R -module S_n has a presentation

$$R[X, Y]_{n-d} \rightarrow R[X, Y]_n \rightarrow S_n \rightarrow 0.$$

Thus by Algebra, Lemma 78.3 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \rightarrow \kappa(\mathfrak{p})[X, Y]$ for all primes \mathfrak{p} . This is clear from the assumption that F does not map to the zero polynomial mod \mathfrak{p} . The assertion on ranks is clear from this as well. \square

00Q5 **Lemma 4.4.** *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e . Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define “relatively prime”. If you have another definition, then you can show it is equivalent to this one. \square

00Q6 **Lemma 4.5.** *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d . Let $S = R[X, Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in \mathfrak{p} . There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e , and a $G \in R[X, Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \rightarrow (S_{n+e})_f$ for all $n \geq d$.*

Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R . In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 4.3. Pick any $G \in R[X, Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X, Y]$ is relatively prime to the image of $F(X, Y)$ (this is possible for some e). Apply Algebra, Lemma 78.3 to the map induced by multiplication by G from $S_d \rightarrow S_{d+e}$. By our choice of G and Lemma 4.4 we see $S_d \otimes \kappa(\mathfrak{p}) \rightarrow S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \rightarrow S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X, Y]$ is relatively prime to the image of F for all primes \mathfrak{p}' of R . And then by Algebra, Lemma 78.3 again we see that all the maps $G : S_d \rightarrow S_{d+e}$, $n \geq d$ are isomorphisms. \square

00Q7 **Remark 4.6.** Let R be a ring. Suppose that we have $F \in R[X, Y]_d$ and $G \in R[X, Y]_e$ such that, setting $S = R[X, Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms $S_n \rightarrow S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R -algebra A as follows:

- (1) as an R -module $A = S_{ed}$, and
- (2) multiplication $A \times A \rightarrow A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed} .

This makes sense because multiplication by G^d induces a bijective map $S_{de} \rightarrow S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A .

00Q3 **Lemma 4.7.** *Let R be a ring, let $f \in R$. Suppose we have S , S' and the solid arrows forming the following commutative diagram of rings*

$$\begin{array}{ccccc}
 & & S'' & & \\
 & \swarrow & \vdots & \searrow & \\
 R & \xrightarrow{\quad} & & \xrightarrow{\quad} & S \\
 \downarrow & & \downarrow & & \downarrow \\
 R_f & \longrightarrow & S' & \longrightarrow & S_f
 \end{array}$$

Assume that $R_f \rightarrow S'$ is finite. Then we can find a finite ring map $R \rightarrow S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f , $i = 1, \dots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j < d_i} (a_{ij}/f^n)t^j$ for some uniform n . Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i}g_i^{d_i} + \sum_{j < d_i} f^{n(d_i-j)}a_{ij}g_i^j$ of S is killed by a power of f . Hence upon increasing n to n' , which replaces g_i by $f^{n'-n}g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j < d_i} f^{n(d_i-j)}a_{ij}t^j$ with coefficients in R . Also, by construction $Q_i(f^n g_i) = 0$ in S . Thus we get a finite R -algebra $S'' = R[z_1, \dots, z_w]/(Q_1(z_1), \dots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \rightarrow S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \rightarrow S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N . Thus $f^N h$ is an element of the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S'' . OK, and it is easy to see that S''/J does the job. \square

5. Formally smooth ring maps

07GD

00TO **Lemma 5.1.** *Let R be a ring. Let S be a R -algebra. If S is of finite presentation and formally smooth over R then S is smooth over R .*

Proof. See Algebra, Proposition 136.13. \square

6. Sites and sheaves

0EGM

0931 **Remark 6.1** (No map from lower shriek to pushforward). Let U be an object of \mathcal{C} . For any abelian sheaf \mathcal{G} on \mathcal{C}/U one may wonder whether there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

To construct such a thing is the same as constructing a map $j_U^{-1}j_U!\mathcal{G} \rightarrow \mathcal{G}$. Note that restriction commutes with sheafification. Thus we can use the presheaf of Modules on Sites, Lemma 19.2. Hence it suffices to define for V/U a map

$$\bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{G}(V/U)$$

compatible with restrictions. It looks like we can take the which is zero on all summands except for the one where φ is the structure morphism $\varphi_0 : V \rightarrow U$ where we take 1. However, this isn't compatible with restriction mappings: namely, if $\alpha : V' \rightarrow V$ is a morphism of \mathcal{C} , then denote V'/U the object of \mathcal{C}/U with structure morphism $\varphi'_0 = \varphi_0 \circ \alpha$. We need to check that the diagram

$$\begin{array}{ccc} \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) & \longrightarrow & \mathcal{G}(V/U) \\ \downarrow & & \downarrow \\ \bigoplus_{\varphi' \in \text{Mor}_{\mathcal{C}}(V',U)} \mathcal{G}(V' \xrightarrow{\varphi'} U) & \longrightarrow & \mathcal{G}(V'/U) \end{array}$$

commutes. The problem here is that there may be a morphism $\varphi : V \rightarrow U$ different from φ_0 such that $\varphi \circ \alpha = \varphi'_0$. Thus the left vertical arrow will send the summand corresponding to φ into the summand on which the lower horizontal arrow is equal to 1 and almost surely the diagram doesn't commute.

7. Cohomology

0BM0 The following lemma computes the cohomology sheaves of the derived limit in a special case.

0A08 **Lemma 7.1.** *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

- (1) K_n is an object of $D^+(\mathcal{O})$ for all n ,
- (2) for $q \in \mathbf{Z}$ there exists $n(q)$ such that $H^q(K_{n+1}) \rightarrow H^q(K_n)$ is an isomorphism for $n \geq n(q)$,
- (3) every object of \mathcal{C} has a covering whose members are elements of \mathcal{B} ,
- (4) for every $U \in \mathcal{B}$ we have $H^p(U, H^q(K_n)) = 0$ for $p > d$ and all q .

Then we have $H^m(R \lim K_n) = \lim H^m(K_n)$ for all $m \in \mathbf{Z}$.

Proof. Set $K = R \lim K_n$. Let $U \in \mathcal{B}$. For each n there is a spectral sequence

$$H^p(U, H^q(K_n)) \Rightarrow H^{p+q}(U, K_n)$$

which converges as K_n is bounded below, see Derived Categories, Lemma 21.3. If we fix $m \in \mathbf{Z}$, then we see from our assumption (4) that only $H^p(U, H^q(K_n))$ contribute to $H^m(U, K_n)$ for $0 \leq p \leq d$ and $m - d \leq q \leq m$. By assumption (2) this implies that $H^m(U, K_{n+1}) \rightarrow H^m(U, K_n)$ is an isomorphism as soon as $n \geq \max n(m), \dots, n(m-d)$. The functor $R\Gamma(U, -)$ commutes with derived limits by Injectives, Lemma 13.6. Thus we have

$$H^m(U, K) = H^m(R \lim R\Gamma(U, K_n))$$

On the other hand we have just seen that the complexes $R\Gamma(U, K_n)$ have eventually constant cohomology groups. Thus by More on Algebra, Remark 77.9 we find that $H^m(U, K)$ is equal to $H^m(U, K_n)$ for all $n \gg 0$ for some bound independent of $U \in \mathcal{B}$. Pick such an n . Finally, recall that $H^m(K)$ is the sheafification of

the presheaf $U \mapsto H^m(U, K)$ and $H^m(K_n)$ is the sheafification of the presheaf $U \mapsto H^m(U, K_n)$. On the elements of \mathcal{B} these presheaves have the same values. Therefore assumption (3) guarantees that the sheafifications are the same too. The lemma follows. \square

0D7P **Lemma 7.2.** *In Simplicial Spaces, Situation 3.3 let a_0 be an augmentation towards a site \mathcal{D} as in Simplicial Spaces, Remark 4.1. Suppose given strictly full weak Serre subcategories*

$$\mathcal{A} \subset \text{Ab}(\mathcal{D}), \quad \mathcal{A}_n \subset \text{Ab}(\mathcal{C}_n)$$

Then

- (1) *the collection of abelian sheaves \mathcal{F} on \mathcal{C}_{total} whose restriction to \mathcal{C}_n is in \mathcal{A}_n for all n is a strictly full weak Serre subcategory $\mathcal{A}_{total} \subset \text{Ab}(\mathcal{C}_{total})$.*

If a_n^{-1} sends \mathcal{A} into \mathcal{A}_n for all n , then

- (2) *a^{-1} sends \mathcal{A} into \mathcal{A}_{total} and*
 (3) *a^{-1} sends $D_{\mathcal{A}}(\mathcal{D})$ into $D_{\mathcal{A}_{total}}(\mathcal{C}_{total})$.*

If $R^q a_{n,*}$ sends \mathcal{A}_n into \mathcal{A} for all n, q , then

- (4) *$R^q a_*$ sends \mathcal{A}_{total} into \mathcal{A} for all q , and*
 (5) *Ra_* sends $D_{\mathcal{A}_{total}}^+(\mathcal{C}_{total})$ into $D_{\mathcal{A}}^+(\mathcal{D})$.*

Proof. The only interesting assertions are (4) and (5). Part (4) follows from the spectral sequence in Simplicial Spaces, Lemma 9.3 and Homology, Lemma 21.11. Then part (5) follows by considering the spectral sequence associated to the canonical filtration on an object K of $D_{\mathcal{A}_{total}}^+(\mathcal{C}_{total})$ given by truncations. We omit the details. \square

8. Simplicial methods

08Q0

01AA **Lemma 8.1.** *Assumptions and notation as in Simplicial, Lemma 32.1. There exists a section $g : U \rightarrow V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V . In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Simplicial, Lemmas 32.1 and 30.8. \square

018W **Lemma 8.2.** *Let \mathcal{C} be a category with finite coproducts and finite limits. Let X be an object of \mathcal{C} . Let $k \geq 0$. The canonical map*

$$\text{Hom}(\Delta[k], X) \longrightarrow \text{cosk}_1 \text{sk}_1 \text{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} \text{Mor}(V, \text{cosk}_1 \text{sk}_1 \text{Hom}(\Delta[k], X)) &= \text{Mor}(\text{sk}_1 V, \text{sk}_1 \text{Hom}(\Delta[k], X)) \\ &= \text{Mor}(i_{1!} \text{sk}_1 V, \text{Hom}(\Delta[k], X)) \\ &= \text{Mor}(i_{1!} \text{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk , the second equality by the adjointness of $i_{1!}$ and sk_1 , and the first equality by Simplicial, Definition 17.1 where the last X denotes the constant simplicial object with value X . By Simplicial, Lemma 20.2 an element in this set depends only on the terms of degree 0 and

1 of $i_{1!}sk_1V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see *Simplicial*, Lemma 21.3. Thus the set above is equal to $Mor(V \times \Delta[k], X) = Mor(V, Hom(\Delta[k], X))$. \square

018X **Lemma 8.3.** *Let \mathcal{C} be a category. Let X be an object of \mathcal{C} such that the self products $X \times \dots \times X$ exist. Let $k \geq 0$ and let $C[k]$ be as in *Simplicial*, Example 5.6. With notation as in *Simplicial*, Lemma 15.2 the canonical map*

$$Hom(C[k], X)_1 \longrightarrow (cosk_0sk_0 Hom(C[k], X))_1$$

is identified with the map

$$\prod_{\alpha: [k] \rightarrow [1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. This is shown in the proof of Hypercoverings, Lemma 7.3. \square

9. Obsolete lemmas on schemes

07VA Lemmas that seem superfluous.

03H1 **Lemma 9.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}_R^n$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. This is a special case of More on Morphisms, Theorem 46.5. \square

01YJ **Lemma 9.2.** *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ . Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *If \mathcal{P} holds for a direct sum of coherent sheaves then it holds for both.*
- (3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have \mathcal{P} for $(Z \rightarrow X)_*\mathcal{I}$.*
- (4) *There exists some coherent sheaf \mathcal{G} on X such that*
 - (a) *$Supp(\mathcal{G}) = Z_0$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and*
 - (c) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf \mathcal{F} on X whose support is contained in Z_0 .

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property \mathcal{P} .

Consider a coherent sheaf \mathcal{G} as in (3). By Cohomology of Schemes, Lemma 12.2 there exists a sheaf of ideals \mathcal{I} on Z_0 and a short exact sequence

$$0 \rightarrow ((Z_0 \rightarrow X)_*\mathcal{I})^{\oplus r} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

where the support of \mathcal{Q} is strictly contained in Z_0 . In particular $r > 0$ and \mathcal{I} is nonzero because the support of \mathcal{G} is equal to Z . Since \mathcal{Q} has property \mathcal{P} we conclude that also $((Z_0 \rightarrow X)_*\mathcal{I})^{\oplus r}$ has property \mathcal{P} . By (2) we deduce property \mathcal{P} for $(Z_0 \rightarrow X)_*\mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 12.5 at the appropriate point gives the lemma. Some details omitted. \square

01YK **Lemma 9.3.** *Let X be a Noetherian scheme. Let \mathcal{P} be a property of coherent sheaves on X such that*

- (1) *For any short exact sequence of coherent sheaves if two out of three of them have property \mathcal{P} then so does the third.*
- (2) *If \mathcal{P} holds for a direct sum of coherent sheaves then it holds for both.*
- (3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf \mathcal{G} such that*
 - (a) *$\text{Supp}(\mathcal{G}) = Z$,*
 - (b) *\mathcal{G}_ξ is annihilated by \mathfrak{m}_ξ , and*
 - (c) *property \mathcal{P} holds for \mathcal{G} .*

Then property \mathcal{P} holds for every coherent sheaf on X .

Proof. This follows from Lemma 9.2 in exactly the same way that Cohomology of Schemes, Lemma 12.6 follows from Cohomology of Schemes, Lemma 12.5. \square

01XP **Lemma 9.4.** *Let X be a scheme. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent \mathcal{O}_X -modules. Assume that*

- (1) *X is quasi-compact,*
- (2) *\mathcal{F} is of finite type, and*
- (3) *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on \mathcal{F} factors through \mathcal{F}' .

Proof. In other words we claim that $s^n \mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. In other words, we claim that the quotient map $\mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}'$ becomes zero after multiplying by a power of s . This follows from Properties, Lemma 17.3. \square

10. Functor of quotients

08J4

082R **Lemma 10.1.** *Let $S = \text{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S . Let $q_i : \mathcal{F} \rightarrow \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent \mathcal{O}_X -modules. Assume \mathcal{Q}_1 flat over S . Let $T \rightarrow S$ be a quasi-compact morphism of schemes such that there exists a factorization*

$$\begin{array}{ccc} & \mathcal{F}_T & \\ q_{1,T} \swarrow & & \searrow q_{2,T} \\ \mathcal{Q}_{1,T} & \leftarrow \cdots \cdots \cdots & \mathcal{Q}_{2,T} \end{array}$$

Then exists a closed subscheme $Z \subset S$ such that (a) $T \rightarrow S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\text{Ker}(q_2)$ is a finite type \mathcal{O}_X -module and X quasi-compact, then we can take $Z \rightarrow S$ of finite presentation.

Proof. Apply Flatness on Spaces, Lemma 8.2 to the map $\text{Ker}(q_2) \rightarrow \mathcal{Q}_1$. \square

11. Spaces and fpqc coverings

0ARG The material here was made obsolete by Gabber's argument showing that algebraic spaces satisfy the sheaf condition with respect to fpqc coverings. Please visit Properties of Spaces, Section 17.

03W9 **Lemma 11.1.** *Let S be a scheme. Let X be an algebraic space over S . Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a fpqc covering of schemes over S . Then the map*

$$\text{Mor}_S(T, X) \longrightarrow \prod_{i \in I} \text{Mor}_S(T_i, X)$$

is injective.

Proof. Immediate consequence of Properties of Spaces, Proposition 17.1. \square

03WA **Lemma 11.2.** *Let S be a scheme. Let X be an algebraic space over S . Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. This is true because all algebraic spaces satisfy the sheaf property for the fpqc topology, see Properties of Spaces, Proposition 17.1. \square

03WB **Lemma 11.3.** *Let S be a scheme. Let X be an algebraic space over S . If X is Zariski locally quasi-separated over S , then X satisfies the sheaf condition for the fpqc topology.*

Proof. Immediate consequence of the general Properties of Spaces, Proposition 17.1. \square

03WC **Remark 11.4.** This remark used to discuss to what extend the original proof of Lemma 11.3 (of December 18, 2009) generalizes.

12. Very reasonable algebraic spaces

07T6 Material that is somewhat obsolete.

03IN **Lemma 12.1.** *Let S be a scheme. Let X be a reasonable algebraic space over S . Then $|X|$ is Kolmogorov (see Topology, Definition 8.4).*

Proof. Follows from the definitions and Decent Spaces, Lemma 12.3. \square

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \rightarrow X$, then X is very reasonable, see Decent Spaces, Lemma 4.7.

03I9 **Lemma 12.2.** *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. \square

03IA **Lemma 12.3.** *Let S be a scheme. Let X be an algebraic space over S . If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.2. \square

03IB **Lemma 12.4.** *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Decent Spaces, Lemma 5.1. \square

03JF **Lemma 12.5.** *Let S be a scheme. Let X, Y be algebraic spaces over S . Let $Y \rightarrow X$ be a representable morphism. If X is very reasonable, so is Y .*

Proof. This is case (ϵ) of Decent Spaces, Lemma 5.3. \square

03IC **Remark 12.6.** Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U . Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence X is very reasonable. On the other hand, the diagonal $U \times_X U \rightarrow U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 21.4. Let $0_1, 0_2$ be the two zeros of U . Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2 . Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

03K7 **Lemma 12.7.** *Let S be a scheme. Let X be a very reasonable algebraic space over S . There exists a set of schemes U_i and morphisms $U_i \rightarrow X$ such that*

- (1) *each U_i is a quasi-compact scheme,*
- (2) *each $U_i \rightarrow X$ is étale,*
- (3) *both projections $U_i \times_X U_i \rightarrow U_i$ are quasi-compact, and*
- (4) *the morphism $\coprod U_i \rightarrow X$ is surjective (and étale).*

Proof. Decent Spaces, Definition 6.1 says that there exist $U_i \rightarrow X$ such that (2), (3) and (4) hold. Fix i , and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \rightarrow U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i -invariant open (see Groupoids, Lemma 19.2). Hence W' is a quasi-compact scheme, $W' \rightarrow X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \rightarrow W'$ are quasi-compact. This means the family of $W' \rightarrow X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. \square

13. Obsolete lemma on algebraic spaces

0D45 Lemmas that seem superfluous.

07V2 **Lemma 13.1.** *In Cohomology of Spaces, Situation 16.1 the morphism $p : X \rightarrow \text{Spec}(A)$ is surjective.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 16.7 but now is a consequence of it. \square

07V3 **Lemma 13.2.** *In Cohomology of Spaces, Situation 16.1 the morphism $p : X \rightarrow \text{Spec}(A)$ is universally closed.*

Proof. This lemma was originally used in the proof of Cohomology of Spaces, Proposition 16.7 but now is a consequence of it. \square

14. Variants of cotangent complexes for schemes

08T5 This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 24 for applications.

Let $f : X \rightarrow Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

$$08T6 \quad (14.0.1) \quad \begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & A \\ \downarrow & & \downarrow & \searrow^{i} & \\ Y & \longleftarrow & V & & \end{array}$$

of schemes where

- (1) U is an open subscheme of X ,
- (2) V is an open subscheme of Y , and
- (3) there exists an isomorphism $A = V \times \text{Spec}(P)$ over V where P is a polynomial algebra over \mathbf{Z} (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V . Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (14.0.1), is often denoted $U \rightarrow A$ where it is understood that (a) U is an open subscheme of X , (b) $U \rightarrow A$ is a morphism over Y , (c) the image of the structure morphism $A \rightarrow Y$ is an open $V \subset Y$, and (d) $A \rightarrow V$ is an affine space. We'll write $U \rightarrow A/V$ to indicate $V \subset Y$ is the image of $A \rightarrow Y$. Recall that X_{Zar} denotes the small Zariski site X . There are forgetful functors

$$\mathcal{C}_{X/Y} \rightarrow X_{Zar}, (U \rightarrow A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar}, (U \rightarrow A/V) \mapsto V.$$

08T7 **Lemma 14.1.** *Let $X \rightarrow Y$ be a morphism of schemes.*

- (1) *The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar} .*
- (2) *The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar} .*
- (3) *The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.*

Proof. Ad (1). Given an object $U \rightarrow A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \rightarrow U$ of X_{Zar} consider the object $i' : U' \rightarrow A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \rightarrow U$. The morphism $(U' \rightarrow A) \rightarrow (U \rightarrow A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar} .

Ad (2). Given an object $U \rightarrow A/V$ and $V' \rightarrow V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \rightarrow A') \rightarrow (U \rightarrow A)$ over $V' \rightarrow V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \rightarrow A/V$ and a morphism $(U', V') \rightarrow (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \rightarrow A'/V') \rightarrow (U \rightarrow A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. \square

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \rightarrow A_i) \rightarrow (U \rightarrow A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

- (1) $U = \bigcup U_i$, and
- (2) $A_i = A$ for all i .

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \rightarrow Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

- (1) The sheaf \mathcal{O} given by the rule $(U \rightarrow A) \mapsto \mathcal{O}(A)$.
- (2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \rightarrow A) \mapsto \mathcal{O}(U)$.
- (3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \rightarrow A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

$$\begin{array}{ccc}
 & (Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) & \xrightarrow{i} (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\
 08T8 \quad (14.1.1) & \downarrow \pi & \\
 & (Sh(X_{Zar}), \mathcal{O}_X) &
 \end{array}$$

The morphism i is the identity on underlying topoi and $i^\# : \mathcal{O} \rightarrow \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 31.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \rightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \rightarrow D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \rightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \rightarrow D(\underline{\mathcal{O}}_X)$.

08TA **Remark 14.2.** We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar} . There is a third topology $\tau_{X \rightarrow Y}$ where a family of morphisms $\{(U_i \rightarrow A_i) \rightarrow (U \rightarrow A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$ and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 14.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \rightarrow (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)$$

(recall that τ_X is our “default” topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$\begin{array}{ccccc}
 Sh(X_{Zar}) & \xleftarrow{\pi} & Sh(\mathcal{C}_{X/Y}) & \xleftarrow{\quad} & Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y}) \\
 \downarrow f & & & & \downarrow \\
 Sh(Y_{Zar}) & \xleftarrow{\quad} & & \xleftarrow{\quad} & Sh(\mathcal{C}_{X/Y}, \tau_Y)
 \end{array}$$

is **not** commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y})$, but we can certainly find examples where such a sheaf pulls back to zero on X . Note that any presheaf \mathcal{F} on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \rightarrow A/V)$ the set $\mathcal{F}(V)$. Even if \mathcal{F} happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of \mathcal{F} to $Sh(\mathcal{C}_{X/Y}, \tau_{X \rightarrow Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}}_Y$. But what is true is that there is a map $\underline{\mathcal{F}} \rightarrow \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

15. Deformations and obstructions of flat modules

08VZ In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ . This material is obsolete due to the improved discussion in Quot, Section 6.

Our setup will be the following. We assume given

- (1) a ring Λ ,
- (2) an algebraic space X over Λ ,
- (3) a Λ -algebra A , set $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$, and
- (4) a finitely presented \mathcal{O}_{X_A} -module \mathcal{F} flat over A .

In this situation we will consider all possible surjections

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$$

where A' is a Λ -algebra whose kernel I is an ideal of square zero in A' . Given A' we obtain a first order thickening $X_A \rightarrow X_{A'}$ of algebraic spaces over $\mathrm{Spec}(\Lambda)$. For each of these we consider the problem of lifting \mathcal{F} to a finitely presented module \mathcal{F}' on $X_{A'}$ flat over A' . We would like to replicate the results of Deformation Theory, Lemma 12.1 in this setting.

To be more precise let $Lift(\mathcal{F}, A')$ denote the category of pairs (\mathcal{F}', α) where \mathcal{F}' is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \rightarrow \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}'_1, \alpha_1) \rightarrow (\mathcal{F}'_2, \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \rightarrow \mathcal{F}'_2$ which are compatible with α_1 and α_2 . The set of isomorphism classes of $Lift(\mathcal{F}, A')$ is denoted $Lift(\mathcal{F}, A')$.

Let \mathcal{G} be a sheaf of $\mathcal{O}_X \otimes_{\Lambda} A$ -modules on $X_{\acute{e}tale}$ flat over A . We introduce the category $Lift(\mathcal{G}, A')$ of pairs (\mathcal{G}', β) where \mathcal{G}' is a sheaf of $\mathcal{O}_X \otimes_{\Lambda} A'$ -modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \rightarrow \mathcal{G}$.

08W0 **Lemma 15.1.** *Notation and assumptions as above. Let $p : X_A \rightarrow X$ denote the projection. Given A' denote $p' : X_{A'} \rightarrow X$ the projection. The functor p'_* induces an equivalence of categories between*

- (1) the category $Lift(\mathcal{F}, A')$, and
- (2) the category $Lift(p_*\mathcal{F}, A')$.

Proof. FIXME. □

Let \mathcal{H} be a sheaf of $\mathcal{O} \otimes_{\Lambda} A$ -modules on $\mathcal{C}_{X/\Lambda}$ flat over A . We introduce the category $Lift_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs (\mathcal{H}', γ) where \mathcal{H}' is a sheaf of $\mathcal{O} \otimes_{\Lambda} A'$ -modules flat over A' and $\gamma : \mathcal{H}' \otimes_{A'} A \rightarrow \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_{\Lambda} A$ -modules.

Let \mathcal{G} be a sheaf of $\mathcal{O}_X \otimes_{\Lambda} A$ -modules on $X_{\acute{e}tale}$ flat over A . Consider the morphisms i and π of Cotangent, Equation (26.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given by the rule $(U \rightarrow \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_{\Lambda} A$ -modules. Denote $i_*\underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_{\Lambda} A$ -modules.

08W1 **Lemma 15.2.** *Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between*

- (1) the category $Lift_{\mathcal{O}}(i_*\underline{\mathcal{G}}, A')$, and
- (2) the category $Lift(\underline{\mathcal{G}}, A')$.

Proof. FIXME. □

08W2 **Lemma 15.3.** *Notation and assumptions as in Lemma 15.2. Consider the object*

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_{\Lambda} A)$. Given a surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have

- (1) The category $\text{Lift}(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}_{\mathcal{O}_X \otimes A}^2(L, \mathcal{G} \otimes_A I)$ is zero.
- (2) If $\text{Lift}(\mathcal{G}, A')$ is nonempty, then $\text{Lift}(\mathcal{G}, A')$ is principal homogeneous under $\text{Ext}_{\mathcal{O}_X \otimes A}^1(L, \mathcal{G} \otimes_A I)$.
- (3) Given a lift \mathcal{G}' , the set of automorphisms of \mathcal{G}' which pull back to $\text{id}_{\mathcal{G}}$ is canonically isomorphic to $\text{Ext}_{\mathcal{O}_X \otimes A}^0(L, \mathcal{G} \otimes_A I)$.

Proof. FIXME. □

Finally, we put everything together as follows.

08W3 **Proposition 15.4.** *With Λ , X , A , \mathcal{F} as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have*

- (1) The category $\text{Lift}(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in \text{Ext}_{X_A}^2(L, \mathcal{F} \otimes_A I)$ is zero.
- (2) If $\text{Lift}(\mathcal{F}, A')$ is nonempty, then $\text{Lift}(\mathcal{F}, A')$ is principal homogeneous under $\text{Ext}_{X_A}^1(L, \mathcal{F} \otimes_A I)$.
- (3) Given a lift \mathcal{F}' , the set of automorphisms of \mathcal{F}' which pull back to $\text{id}_{\mathcal{F}}$ is canonically isomorphic to $\text{Ext}_{X_A}^0(L, \mathcal{F} \otimes_A I)$.

Proof. FIXME. □

08W4 **Lemma 15.5.** *In the situation of Proposition 15.4, if $X \rightarrow \text{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.*

Proof. FIXME. □

16. The stack of coherent sheaves in the non-flat case

0CXY In Quot, Theorem 5.12 the assumption that $f : X \rightarrow B$ is flat is not necessary. In this section we modify the method of proof based on ideas from derived algebraic geometry to get around the flatness hypothesis. An entirely different method is used in Quot, Section 6 to get exactly the same result; this is why the method from this section is obsolete.

The only step in the proof of Quot, Theorem 5.12 which uses flatness is in the application of Quot, Lemma 5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 23. The proof of the lemma relies on Deformation Theory, Lemmas 12.1 and 12.5 from Deformation Theory, Section 12. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 12.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 11.7 and 11.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S . Denote $X \otimes^{\mathbf{L}} A$ a “derived base change” of X to A and denote $i : X_A \rightarrow X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X \otimes^{\mathbf{L}} A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X \otimes^{\mathbf{L}} A}) = \text{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \rightarrow \text{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A -flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\text{Ext}_{X \otimes^{\mathbf{L}} A}^i(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where \mathcal{F}' is a flat deformation of \mathcal{F} . By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\text{Ext}_{X_A}^i(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Quot, Lemma 5.11 with the only change being that one replaces the first occurrence of \mathcal{F} by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a “direct” construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of \mathcal{F} . In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the ext groups $\text{Ext}_{X_A}^i(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \text{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

09DN **Remark 16.1** (Direct construction). Let S be a scheme. Let $f : X \rightarrow B$ be a morphism of algebraic spaces over S . Let U be another algebraic space over B . Denote $q : X \times_B U \rightarrow U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \rightarrow L_{X \times_B U/B} \rightarrow E \rightarrow Lq^*L_{U/B}[1]$$

of Cotangent, Section 27. For any sheaf \mathcal{F} of $\mathcal{O}_{X \times_B U}$ -modules we have the Atiyah class

$$\mathcal{F} \rightarrow L_{X \times_B U/B} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} \mathcal{F}[1]$$

see Cotangent, Section 18. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} E[1] \rightarrow E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \rightarrow Lq^*L_{U/B} \otimes_{\mathcal{O}_{X \times_B U}}^{\mathbf{L}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\ \uparrow e_{\mathcal{F}} & & \uparrow \text{Atiyah} & & \uparrow = \\ E(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \end{array}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

09DP **Remark 16.2** (Construction of obstruction class). With notation as in Remark 16.1 let $i : U \rightarrow U'$ be a first order thickening of U over B . Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B' . The fundamental triangle

$$Li^*L_{U'/B} \rightarrow L_{U/B} \rightarrow L_{U/U'} \rightarrow Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \rightarrow \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \rightarrow \mathcal{I}[1]$. Combined with the map $e_{\mathcal{F}}$ of the previous remark we obtain

$$(\text{id}_{\mathcal{F}} \otimes Lq^*e_{U'}) \cup e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \text{Ext}_{\mathcal{O}_{X \times_B U}}^2(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

09DQ **Lemma 16.3.** *In the situation of Remark 16.2 assume that \mathcal{F} is flat over U . Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$ -module \mathcal{F}' flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.*

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 11.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{U'} \rightarrow \mathcal{O}_U \rightarrow 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\text{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \rightarrow q^*\mathcal{I} \rightarrow \mathcal{J} \rightarrow 0$$

where the $\text{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\text{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with \mathcal{F} we obtain the exact sequence

$$\mathcal{F} \otimes_{\mathcal{O}} \text{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \rightarrow \mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} \rightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J} \rightarrow 0$$

(Note that the roles of the letters \mathcal{I} and \mathcal{J} are reversed relative to the notation in Deformation Theory, Lemma 11.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\bar{z} = (\bar{x}, \bar{u}) : \text{Spec}(k) \rightarrow X \times_B U$. Set $R = \mathcal{O}_{B, \bar{b}}$, $A = \mathcal{O}_{X, \bar{x}}$, $B = \mathcal{O}_{U, \bar{u}}$, and $C = \mathcal{O}_{\bar{z}}$. By Cotangent, Lemma 27.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\bar{z}} = \mathcal{F}_{\bar{z}} \otimes \text{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\bar{z}} \otimes \text{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\bar{z}} \otimes_B \mathcal{I}_{\bar{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 11.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \rightarrow \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanishing of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J}$. A consideration of the spectral sequence

$$\text{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \text{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \rightarrow \text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \rightarrow \text{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \rightarrow \text{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 11.8 a verification we omit. \square

09DR **Lemma 16.4.** *In Quot, Situation 5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = \text{Coh}_{X/B}$. Then we have openness of versality for \mathcal{X} (see Artin's Axioms, Definition 13.1).*

Proof (sketch). Let $U \rightarrow S$ be of finite type morphism of schemes, x an object of \mathcal{X} over U and $u_0 \in U$ a finite type point such that x is versal at u_0 . After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 15.1) and $U = \text{Spec}(A)$ with $U \rightarrow S$ mapping into an affine open $\text{Spec}(\Lambda)$ of S . We will use Artin's Axioms, Lemma 23.4 to prove the lemma. Let \mathcal{F} be the coherent module on $X_A = \text{Spec}(A) \times_S X$ flat over A corresponding to the given object x .

Choose $E(\mathcal{F})$ and $e_{\mathcal{F}}$ as in Remark 16.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\text{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A -module M . Using this and using Deformation Theory, Lemma 11.7 we have an isomorphism of functors

$$T_x(M) = \text{Ext}_{X_A}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 16.3 given any surjection $A' \rightarrow A$ of Λ -algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \text{Ext}_{X_A}^2(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 23.3 to the computation of the Ext groups $\text{Ext}_{X_A}^i(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D_{Coh}^- ; hint: use Cotangent, Lemma 5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes_A^{\mathbf{L}} M) \longrightarrow \text{Ext}_{X_A}^i(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K , together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 23.2. Finally, condition (iv) of Artin's Axioms, Lemma 23.3 holds by a variant of Deformation Theory, Lemma 12.5 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 23.4 applies and the lemma is proved. \square

0CXZ **Theorem 16.5.** *Let S be a scheme. Let $f : X \rightarrow B$ be morphism of algebraic spaces over S . Assume that f is of finite presentation and separated. Then $\text{Coh}_{X/B}$ is an algebraic stack over S .*

Proof. This theorem is a copy of Quot, Theorem 6.1. The reason we have this copy here is that with the material in this section we get a second proof (as discussed at the beginning of this section). Namely, we argue exactly as in the proof of Quot, Theorem 5.12 except that we substitute Lemma 16.4 for Quot, Lemma 5.11. \square

17. Modifications

0AS3 Here is an obsolete result on the category of Restricted Power Series, Equation (11.0.1). Please visit Restricted Power Series, Section 11 for the current material.

0AE4 **Lemma 17.1.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The category of Restricted Power Series, Equation (11.0.1) for A is equivalent to the category Restricted Power Series, Equation (11.0.1) for the henselization A^h of A .*

Proof. This is a special case of Restricted Power Series, Lemma 11.3. \square

The following lemma on rational singularities is no longer needed in the chapter on resolving surface singularities.

0B50 **Lemma 17.2.** *In Resolution of Surfaces, Situation 9.1. Let M be a finite reflexive A -module. Let $M \otimes_A \mathcal{O}_X$ denote the pullback of the associated \mathcal{O}_S -module. Then $M \otimes_A \mathcal{O}_X$ maps onto its double dual.*

Proof. Let $\mathcal{F} = (M \otimes_A \mathcal{O}_X)^{**}$ be the double dual and let $\mathcal{F}' \subset \mathcal{F}$ be the image of the evaluation map $M \otimes_A \mathcal{O}_X \rightarrow \mathcal{F}$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

Since X is normal, the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings for points of codimension 1 (see Properties, Lemma 12.5). Hence $\mathcal{Q}_x = 0$ for such points by More on Algebra, Lemma 23.3. Thus \mathcal{Q} is supported in finitely many closed points and is globally generated by Cohomology of Schemes, Lemma 9.10. We obtain the exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}') \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{Q}) \rightarrow 0$$

because \mathcal{F}' is generated by global sections (Resolution of Surfaces, Lemma 9.2). Since $X \rightarrow \text{Spec}(A)$ is an isomorphism over the complement of the closed point, and since M is reflexive, we see that the maps

$$M \rightarrow H^0(X, \mathcal{F}') \rightarrow H^0(X, \mathcal{F})$$

induce isomorphisms after localization at any nonmaximal prime of A . Hence these maps are isomorphisms by More on Algebra, Lemma 23.13 and the fact that reflexive modules over normal rings have property (S_2) (More on Algebra, Lemma 23.18). Thus we conclude that $\mathcal{Q} = 0$ as desired. \square

18. Intersection theory

0AYK

02TL **Lemma 18.1.** *Let (S, δ) be as in Chow Homology, Situation 7.1. Let X be locally of finite type over S . Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \rightarrow X$ be the closed immersion of the zero scheme of a . Let $f \in R(X)^*$. In this case $i^* \text{div}_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Special case of Chow Homology, Lemma 28.1. \square

19. Dualizing modules on regular proper models

0C6D In Semistable Reduction, Situation 9.3 we let $\omega_{X/R}^\bullet = f^! \mathcal{O}_{\mathrm{Spec}(R)}$ be the relative dualizing complex of $f : X \rightarrow \mathrm{Spec}(R)$ as introduced in Duality for Schemes, Remark 12.5. Since f is Gorenstein of relative dimension 1 by Semistable Reduction, Lemma 9.2 we can use Duality for Schemes, Lemmas 26.10, 22.7, and 26.4 to see that

$$\omega_{X/R}^\bullet = \omega_X[1]$$

for some invertible \mathcal{O}_X -module ω_X . This invertible module is often called the *relative dualizing module of X over R* . Since R is regular (hence Gorenstein) of dimension 1 we see that $\omega_R^\bullet = R[1]$ is a normalized dualizing complex for R . Hence $\omega_X = H^{-2}(f^! \omega_R^\bullet)$ and we see that ω_X is not just a relative dualizing module but also a dualizing module, see Duality for Schemes, Example 23.1. Thus ω_X represents the functor

$$\mathrm{Coh}(\mathcal{O}_X) \rightarrow \mathrm{Sets}, \quad \mathcal{F} \mapsto \mathrm{Hom}_R(H^1(X, \mathcal{F}), R)$$

by Duality for Schemes, Lemma 23.5. This gives an alternative definition of the relative dualizing module in Semistable Reduction, Situation 9.3. The formation of ω_X commutes with arbitrary base change (for any proper Gorenstein morphism of given relative dimension); this follows from the corresponding fact for the relative dualizing complex discussed in Duality for Schemes, Remark 12.5 which goes back to Duality for Schemes, Lemma 12.4. Thus ω_X pulls back to the dualizing module ω_C of C over K discussed in Algebraic Curves, Lemma 4.2. Note that ω_C is isomorphic to $\Omega_{C/K}$ by Algebraic Curves, Lemma 4.1. Similarly $\omega_X|_{X_k}$ is the dualizing module ω_{X_k} of X_k over k .

0C6E **Lemma 19.1.** *In Semistable Reduction, Situation 9.3 the dualizing module of C_i over k is*

$$\omega_{C_i} = \omega_X(C_i)|_{C_i}$$

where ω_X is as above.

Proof. Let $t : C_i \rightarrow X$ be the closed immersion. Since t is the inclusion of an effective Cartier divisor we conclude from Duality for Schemes, Lemmas 9.7 and 14.2 that we have $t^!(\mathcal{L}) = \mathcal{L}(C_i)|_{C_i}$ for every invertible \mathcal{O}_X -module \mathcal{L} . Consider the commutative diagram

$$\begin{array}{ccc} C_i & \xrightarrow{t} & X \\ g \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{s} & \mathrm{Spec}(R) \end{array}$$

Observe that C_i is a Gorenstein curve (Semistable Reduction, Lemma 9.2) with invertible dualizing module ω_{C_i} characterized by the property $\omega_{C_i}[0] = g^! \mathcal{O}_{\mathrm{Spec}(k)}$. See Algebraic Curves, Lemma 4.1, its proof, and Algebraic Curves, Lemmas 4.2 and 5.2. On the other hand, $s^!(R[1]) = k$ and hence

$$\omega_{C_i}[0] = g^! s^!(R[1]) = t^! f^!(R[1]) = t^! \omega_X$$

Combining the above we obtain the statement of the lemma. \square

20. Duplicate and split out references

09AQ This section is a place where we collect duplicates and references which used to say several things at the same time but are now split into their constituent parts.

03IF **Lemma 20.1.** *Let S be a scheme. Let X be an algebraic space over S . The map $\{\mathrm{Spec}(k) \rightarrow X \text{ monomorphism}\} \rightarrow |X|$ is injective.*

Proof. This is a duplicate of Properties of Spaces, Lemma 4.11. □

03QZ **Theorem 20.2.** *Let $S = \mathrm{Spec}(K)$ with K a field. Let \bar{s} be a geometric point of S . Let $G = \mathrm{Gal}_{\kappa(\bar{s})}$ denote the absolute Galois group. Then there is an equivalence of categories $\mathrm{Sh}(S_{\acute{e}tale}) \rightarrow G\text{-Sets}$, $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$.*

Proof. This is a duplicate of Étale Cohomology, Theorem 55.3. □

06IF **Remark 20.3.** You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 12 for the actual content.

08E6 **Lemma 20.4.** *Let X be a locally ringed space. A direct summand of a finite free \mathcal{O}_X -module is finite locally free.*

Proof. This is a duplicate of Modules, Lemma 14.6. □

08XU **Lemma 20.5.** *Let R be a ring. Let E be an R -module. The following are equivalent*

- (1) *E is an injective R -module, and*
- (2) *given an ideal $I \subset R$ and a module map $\varphi : I \rightarrow E$ there exists an extension of φ to an R -module map $R \rightarrow E$.*

Proof. This is Baer's criterion, see Injectives, Lemma 2.6. □

02PI **Lemma 20.6.** *Let R be a local ring.*

- (1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \mathrm{length}_R(M) - \mathrm{length}_R(N)$.*
- (2) *If (M, φ, ψ) is a $(2, 1)$ -periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.*
- (3) *Suppose that we have a short exact sequence of 2-periodic complexes*

$$0 \rightarrow (M_1, N_1, \varphi_1, \psi_1) \rightarrow (M_2, N_2, \varphi_2, \psi_2) \rightarrow (M_3, N_3, \varphi_3, \psi_3) \rightarrow 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. This follows from Chow Homology, Lemmas 2.3 and 2.4. □

0D3H **Remark 20.7.** This tag used to point to a section describing several examples of deformation problems. These now each have their own section. See Deformation Problems, Sections 4, 5, 6, and 7.

0D3M **Lemma 20.8.** *Deformation Problems, Examples 4.1, 5.1, 6.1, and 7.1 satisfy the Rim-Schlessinger condition (RS).*

Proof. This follows from Deformation Problems, Lemmas 4.2, 5.2, 6.2, and 7.2. □

0D3N **Lemma 20.9.** *We have the following canonical k -vector space identifications:*

- (1) In *Deformation Problems*, Example 4.1 if $x_0 = (k, V)$, then $T_{x_0}\mathcal{F} = (0)$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{End}_k(V)$ are finite dimensional.
- (2) In *Deformation Problems*, Example 5.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = \text{Ext}_{k[\Gamma]}^1(V, V) = H^1(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H^0(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is finitely generated.
- (3) In *Deformation Problems*, Example 6.1 if $x_0 = (k, V, \rho_0)$, then $T_{x_0}\mathcal{F} = H_{\text{cont}}^1(\Gamma, \text{End}_k(V))$ and $\text{Inf}_{x_0}(\mathcal{F}) = H_{\text{cont}}^0(\Gamma, \text{End}_k(V))$ are finite dimensional if Γ is topologically finitely generated.
- (4) In *Deformation Problems*, Example 7.1 if $x_0 = (k, P)$, then $T_{x_0}\mathcal{F}$ and $\text{Inf}_{x_0}(\mathcal{F}) = \text{Der}_k(P, P)$ are finite dimensional if P is finitely generated over k .

Proof. This follows from *Deformation Problems*, Lemmas 4.3, 5.3, 6.3, and 7.3. \square

21. Other chapters

Preliminaries

- (1) Introduction
- (2) Conventions
- (3) Set Theory
- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces
- (7) Sites and Sheaves
- (8) Stacks
- (9) Fields
- (10) Commutative Algebra
- (11) Brauer Groups
- (12) Homological Algebra
- (13) Derived Categories
- (14) Simplicial Methods
- (15) More on Algebra
- (16) Smoothing Ring Maps
- (17) Sheaves of Modules
- (18) Modules on Sites
- (19) Injectives
- (20) Cohomology of Sheaves
- (21) Cohomology on Sites
- (22) Differential Graded Algebra
- (23) Divided Power Algebra
- (24) Hypercoverings

Schemes

- (25) Schemes
- (26) Constructions of Schemes
- (27) Properties of Schemes
- (28) Morphisms of Schemes
- (29) Cohomology of Schemes
- (30) Divisors
- (31) Limits of Schemes

(32) Varieties

- (33) Topologies on Schemes
- (34) Descent
- (35) Derived Categories of Schemes
- (36) More on Morphisms
- (37) More on Flatness
- (38) Groupoid Schemes
- (39) More on Groupoid Schemes
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Topics in Scheme Theory

- (41) Chow Homology
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- (44) Adequate Modules
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- (47) Discriminants and Differents
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- (51) Resolution of Surfaces
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Algebraic Spaces

- (57) Algebraic Spaces
- (58) Properties of Algebraic Spaces
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- (62) Limits of Algebraic Spaces
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References

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