DERIVED CATEGORIES OF SCHEMES

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1. Introduction

In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [TT90], [BN93], [BV03], and [LN07]. Of course there are many other references.

2. Conventions

If $A$ is an abelian category and $M$ is an object of $A$ then we also denote $M$ the object of $K(A)$ and/or $D(A)$ corresponding to the complex which has $M$ in degree 0 and is zero in all other degrees.

If we have a ring $A$, then $K(A)$ denotes the homotopy category of complexes of $A$-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, O_X)$ the symbol $K(O_X)$ denotes the homotopy category of complexes of $O_X$-modules and $D(O_X)$ the associated derived category.

3. Derived category of quasi-coherent modules

In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme $X$. A reference is [TT90, Appendix B]. By the discussion in Schemes, Section 24 the embedding $QCoh(O_X) \subset Mod(O_X)$ exhibits $QCoh(O_X)$ as a weak Serre subcategory of the category of $O_X$-modules. Denote $D_{QCoh}(O_X) \subset D(O_X)$ the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 17. Thus we obtain a canonical functor

$$D(QCoh(O_X)) \longrightarrow D_{QCoh}(O_X)$$

see Derived Categories, Equation (17.1.1).

Lemma 3.1. Let $X$ be a scheme. Then $D_{QCoh}(O_X)$ has direct sums.

Proof. By Injectives, Lemma 13.4 the derived category $D(O_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24.

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of $D_{QCoh}$.

Lemma 3.2. Let $X$ be a scheme. Let $(K_n)$ be an inverse system of $D_{QCoh}(O_X)$ with derived limit $K = R\lim K_n$ in $D(O_X)$. Assume $H^q(K_{n+1}) \rightarrow H^q(K_n)$ is surjective for all $q \in \mathbb{Z}$ and $n \geq 1$. Then

1. $H^q(K) = \lim H^q(K_n)$,
2. $R\lim H^q(K_n) = \lim H^q(K_n)$, and
3. for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.

Proof. Let $B$ be the set of affine opens of $X$. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in B$ by Cohomology of Schemes, Lemma 2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \rightarrow H^0(U, H^q(K_n))$ are surjective for $U \in B$ by Schemes, Lemma 7.5. Part (1) follows from Cohomology, Lemma 34.11 whose
The following lemma will help us to “compute” a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

**Lemma 3.3.** Let $X$ be a scheme. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism.

**Proof.** Denote $H^i = H^i(E)$ the $i$th cohomology sheaf of $E$. Let $\mathcal{B}$ be the set of affine open subsets of $X$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbb{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 2.2. Thus the lemma follows from Cohomology, Lemma 34.9. □

**Lemma 3.4.** Let $X$ be a scheme. Let $F : \text{Mod}(\mathcal{O}_X) \to \text{Ab}$ be an additive functor and $N \geq 0$ an integer. Assume that

1. $F$ commutes with countable direct products,
2. $R^p F(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.

Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^p F(E) \to R^p F(\tau_{\geq p - N + 1} E)$ are isomorphisms.

**Proof.** By shifting the complex we see it suffices to prove the assertion for $p = 0$. Write $E_n = \tau_{\geq -n} E$. We have $E = R\lim E_n$, see Lemma 3.3. Thus $RF(E) = R\lim RF(E_n)$ in $D(\text{Ab})$ by Injectives, Lemma 13.6. Thus we have a short exact sequence

$$0 \to R^1 \lim R^{-1} F(E_n) \to R^0 F(E) \to \lim R^0 F(E_n) \to 0$$

see More on Algebra, Remark 77.9. To finish the proof we will show that the term on the left is zero and that the term on the right equals $R^0 F(E_{N-1})$.

We have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^p F(H^{-n}(E)[n]) = R^{p+n} F(H^{-n}(E)) = 0$$

for $p + n \geq N$ and

$$R^p F(H^{-n}(E)[n+1]) = R^{p+n+1} F(H^{-n}(E)) = 0$$

for $p + n + 1 \geq N$. We conclude that

$$R^p F(E_n) \to R^p F(E_{n-1})$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus the systems $R^p F(E_n)$ all satisfy the ML condition and $R^1 \lim$ gives zero (see discussion in More on Algebra, Section 77). Moreover, the system $R^p F(\tau_{\geq -n} E)$ is constant starting with $n = N - 1$ as desired. □

The following lemma is the key ingredient to many of the results in this chapter.

---

1In particular, $E$ has a K-injective representative as in Cohomology, Lemma 35.1
Lemma 3.5. Let $X = \text{Spec}(A)$ be an affine scheme. All the functors in the diagram
\[
\begin{array}{ccc}
D(\text{QCoh}(\mathcal{O}_X)) & \xrightarrow{(3.0.1)} & D_{\text{QCoh}}(\mathcal{O}_X) \\
\downarrow \sim & & \downarrow \sim \\
D(A) & \xrightarrow{R\Gamma(X,-)} & D_{\text{QCoh}}(\mathcal{O}_X)
\end{array}
\]
are equivalences of triangulated categories. Moreover, for $E$ in $D_{\text{QCoh}}(\mathcal{O}_X)$ we have $H^0(X,E) = H^0(X,\mathcal{O}_X)$.

**Proof.** The functor $R\Gamma(X,-)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor
\[(3.5.1) \quad R\Gamma(X,-) : D_{\text{QCoh}}(\mathcal{O}_X) \to D(A).\]
We will show this functor is quasi-inverse to (3.0.1) via the equivalence between quasi-coherent modules on $X$ and the category of $A$-modules.

Elucidation. Denote $(Y,\mathcal{O}_Y)$ the one point space with sheaf of rings given by $A$. Denote $\pi : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X,-)$ can be identified with $R\pi_*\pi^*$ and the functor (3.0.1) via the equivalence $\text{Mod}(\mathcal{O}_Y) = \text{Mod}_A = \text{QCoh}(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = \sim$ (see Modules, Lemma 10.5 and Schemes, Lemmas 7.1 and 7.5). Thus the functors
\[
\begin{array}{ccc}
D(A) & \xrightarrow{(3.0.1)} & D_{\text{QCoh}}(\mathcal{O}_X) \\
\downarrow \sim & & \downarrow \sim \\
D(\mathcal{O}_X) & \xrightarrow{R\Gamma(X,-)} & D_{\text{QCoh}}(\mathcal{O}_X)
\end{array}
\]
are adjoint (by Cohomology, Lemma 28.1). In particular we obtain canonical adjunction mappings
\[a : R\Gamma(X,E) \to E\]
for $E$ in $D(\mathcal{O}_X)$ and
\[b : M^* \to R\Gamma(X,\tilde{M})\]
for $M^*$ a complex of $A$-modules.

Let $E$ be an object of $D_{\text{QCoh}}(\mathcal{O}_X)$. We may apply Lemma 3.4 to the functor $F(-) = \Gamma(X,-)$ with $N = 1$ by Cohomology of Schemes, Lemma 2.2. Hence
\[R^0\Gamma(X,E) = R^0\Gamma(X,\tau_{\geq 0}E) = \Gamma(X,H^0(E))\]
(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings $a$ and $b$ induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus $a$ and $b$ are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that $\sim$ is an exact functor (Schemes, Lemma 5.4). Namely, this implies that
\[H^0\left(R\Gamma(X,E)\right) = R^0\Gamma(X,E) = \Gamma(X,\tilde{H^0(E)})\]
which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have
\[H^0(R\Gamma(X,\tilde{M})) = R^0\Gamma(X,\tilde{M}) = \Gamma(X,H^0(\tilde{M})) = \Gamma(X,\tilde{H^0(M)}) = H^0(M)\]
which proves that $H^0(b)$ is an isomorphism. \qed
Lemma 3.6. Let $X = \text{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a $K$-flat complex of $A$-modules, then $\tilde{K}^\bullet$ is a $K$-flat complex of $\mathcal{O}_X$-modules.

Proof. By More on Algebra, Lemma 57.5 we see that $K^\bullet \otimes_A A_p$ is a $K$-flat complex of $A_p$-modules for every $p \in \text{Spec}(A)$. Hence we conclude from Cohomology, Lemma 26.4 (and Schemes, Lemma 5.4) that $\tilde{K}^\bullet$ is $K$-flat. □

Lemma 3.7. If $f : X \to Y$ is a morphism of affine schemes given by the ring map $A \to B$, then the diagram

\[
\begin{array}{ccc}
D(B) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_X) \\
\downarrow & & \downarrow Rf_* \\
D(A) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_Y)
\end{array}
\]

commutes.

Proof. Follows from Lemma 3.5 using that $R\Gamma(Y, Rf_* K) = R\Gamma(X, K)$ by Cohomology, Lemma 32.5. □

Lemma 3.8. Let $f : Y \to X$ be a morphism of schemes.

(1) The functor $Lf^*$ sends $D_{\text{QCoh}}(\mathcal{O}_X)$ into $D_{\text{QCoh}}(\mathcal{O}_Y)$.

(2) If $X$ and $Y$ are affine and $f$ is given by the ring map $A \to B$, then the diagram

\[
\begin{array}{ccc}
D(B) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_Y) \\
\downarrow \varphi_{XY}^* B & & \downarrow Lf^* \\
D(A) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_X)
\end{array}
\]

commutes.

Proof. We first prove the diagram

\[
\begin{array}{ccc}
D(B) & \longrightarrow & D(\mathcal{O}_Y) \\
\downarrow \varphi_{XY}^* B & & \downarrow Lf^* \\
D(A) & \longrightarrow & D(\mathcal{O}_X)
\end{array}
\]

commutes. This is clear from Lemma 3.6 and the constructions of the functors in question. To see (1) let $E$ be an object of $D_{\text{QCoh}}(\mathcal{O}_X)$. To see that $Lf^* E$ has quasi-coherent cohomology sheaves we may work locally on $X$. Note that $Lf^*$ is compatible with restricting to open subschemes. Hence we can assume that $f$ is a morphism of affine schemes as in (2). Then we can apply Lemma 3.5 to see that $E$ comes from a complex of $A$-modules. By the commutativity of the first diagram of the proof the same holds for $Lf^* E$ and we conclude (1) is true. □

Lemma 3.9. Let $X$ be a scheme.

(1) For objects $K, L$ of $D_{\text{QCoh}}(\mathcal{O}_X)$ the derived tensor product $K \otimes^L_{\mathcal{O}_X} L$ is in $D_{\text{QCoh}}(\mathcal{O}_X)$.

(2) If $X = \text{Spec}(A)$ is affine then

\[
\tilde{M}^\bullet \otimes^L_{\mathcal{O}_X} \tilde{K}^\bullet = \tilde{M}^\bullet \otimes^L_A K^\bullet
\]

for any pair of complexes of $A$-modules $K^\bullet, M^\bullet$. 

The following lemma is the analogue of Cohomology of Schemes, Lemma 4.5.

Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is quasi-separated and quasi-compact.

(1) The functor $Rf_*$ sends $\text{QCoh}(\mathcal{O}_X)$ into $\text{QCoh}(\mathcal{O}_S)$.

(2) If $S$ is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object $E$ of $\text{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.

(3) In fact, if $S$ is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of $f$.

Proof. Let $E$ be an object of $\text{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that $Rf_*E$ has quasi-coherent cohomology sheaves. The question is local on $S$, hence we may assume $S$ is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 4.5. Thus $R^p f_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$ and the same remains true after base change.

First, assume $E$ is bounded below. We will show (1) and (2) and (3) hold for such $E$ with our choice of $N$. In this case we can for example use the spectral sequence

$$R^q f_* H^p(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 21.3), the quasi-coherence of $R^p f_* H^q(E)$, and the vanishing of $R^p f_* H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let $U \subset S$ be affine open. By Cohomology of Schemes, Lemma 4.6 and our choice of $N$ we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence we may apply Lemma 3.4 to the functor $\Gamma(f^{-1}(U), -)$ to see that

$$Rf(U, Rf_* E) = R\Gamma(f^{-1}(U), E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all $U \subset S$ affine open we conclude that $H^m(Rf_* E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1} E \to E \to \tau_{\geq -n} E \to (\tau_{\leq -n-1} E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 12.4. By (2) we see that $Rf_* \tau_{\leq -n-1} E$ has vanishing cohomology sheaves in degrees $\geq -n + N$. Thus, given an integer $q$ we see that $R^q f_* E$ is equal to $R^q f_* \tau_{\geq -n} E$ for some $n$ and the result above applies. □

4. Total direct image

Lemma 4.1. Let $f : X \to S$ be a morphism of schemes. Assume that $f$ is quasi-separated and quasi-compact.

Proof. Let $E$ be an object of $\text{QCoh}(\mathcal{O}_X)$. To see (1) let $K, L$ be objects of $\text{D}_{\text{QCoh}}(\mathcal{O}_X)$. To check that $K \otimes^L L$ is in $\text{D}_{\text{QCoh}}(\mathcal{O}_X)$ we may work locally on $X$, hence we may assume $X = \text{Spec}(A)$ is affine. By Lemma 3.5 we may represent $K$ and $L$ by complexes of $A$-modules. Then part (2) implies the result. □

Lemma 4.2. Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : \text{D}_{\text{QCoh}}(\mathcal{O}_X) \to \text{D}_{\text{QCoh}}(\mathcal{O}_S)$ commutes with direct sums.
Proof. Let $E_i$ be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_* E_i \longrightarrow Rf_* E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer $N$ as in Lemma 4.1 Then $R^0 f_* E = R^0 f_* \tau_{\geq -N} E$ and $R^0 f_* E_i = R^0 f_* \tau_{\geq -N} E_i$ by the lemma cited. Observe that $\tau_{\geq -N} E = \bigoplus \tau_{\geq -N} E_i$. Thus we may assume all of the $E_i$ have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \quad \text{and} \quad R^p f_* H^q(E_i) \Rightarrow R^{p+q} f_* E_i$$

(Derived Categories, Lemma 21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 6.1. \qed

5. Affine morphisms

0AVV In this section we collect some information about pushforward along an affine morphism of schemes.

0818 **Lemma 5.1.** Let $f : X \rightarrow S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.

**Proof.** The statement means that a morphism $\alpha : E \rightarrow F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_* \alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on $S$. Hence we may assume $S$ and therefore $X$ is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 3.5. Some details omitted. \qed

0819 **Lemma 5.2.** Let $f : X \rightarrow S$ be an affine morphism of schemes. For $E$ in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_* Lf^* E = E \otimes^L_{\mathcal{O}_S} f_* \mathcal{O}_X$.

**Proof.** Since $f$ is affine the map $f_* \mathcal{O}_X \rightarrow Rf_* \mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 2.3). There is a canonical map $E \otimes^L f_* \mathcal{O}_X = E \otimes^L Rf_* \mathcal{O}_X \rightarrow Rf_* Lf^* E$ adjoint to the map

$$Lf^*(E \otimes^L f_* \mathcal{O}_X) = Lf^* E \otimes^L Lf^* Rf_* \mathcal{O}_X \longrightarrow Lf^* E \otimes^L \mathcal{O}_X = Lf^* E$$

coming from $1 : Lf^* E \rightarrow Lf^* E$ and the canonical map $Lf^* Rf_* \mathcal{O}_X \rightarrow \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on $S$. Hence we may assume $S$ and therefore $X$ is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor $Lf^*$ given in Lemmas 3.5 and 3.8. Some details omitted. \qed

Let $Y$ be a scheme. Let $A$ be a sheaf of $\mathcal{O}_Y$-algebras. We will denote $D_{QCoh}(A)$ the inverse image of $D_{QCoh}(\mathcal{O}_X)$ under the restriction functor $D(A) \rightarrow D(\mathcal{O}_X)$. In other words, $K \in D(A)$ is in $D_{QCoh}(A)$ if and only if its cohomology sheaves are quasi-coherent as $\mathcal{O}_X$-modules. If $A$ is quasi-coherent itself this is the same as asking the cohomology sheaves to be quasi-coherent as $A$-modules, see Morphisms, Lemma 11.6.

0AVV **Lemma 5.3.** Let $f : X \rightarrow Y$ be an affine morphism of schemes. Then $f_*$ induces an equivalence $\Phi : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(f_* \mathcal{O}_X)$.
whose composition with \( D_{QCoh}(f_*\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y) \) is \( Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y) \).

**Proof.** Recall that \( Rf_* \) is computed on an object \( K \in D_{QCoh}(\mathcal{O}_X) \) by choosing a \( \mathcal{K} \)-injective complex \( \mathcal{I}^\bullet \) of \( \mathcal{O}_X \)-modules representing \( K \) and taking \( f_*\mathcal{I}^\bullet \). Thus we let \( \Phi(K) \) be the complex \( f_*\mathcal{I}^\bullet \) viewed as a complex of \( f_*\mathcal{O}_X \)-modules. Denote \( g : (X,\mathcal{O}_X) \to (Y,f_*\mathcal{O}_X) \) the obvious morphism of ringed spaces. Then \( g \) is a flat morphism of ringed spaces (see below for a description of the stalks) and \( \Phi \) is the restriction of \( Rg_* \) to \( D_{QCoh}(\mathcal{O}_X) \). We claim that \( Lg^* \) is a quasi-inverse. First, observe that \( Lg^* \) sends \( D_{QCoh}(f_*\mathcal{O}_X) \) into \( D_{QCoh}(\mathcal{O}_X) \) because \( g^* \) transforms quasi-coherent modules into quasi-coherent modules (Modules, Lemma [10.4]). To finish the proof it suffices to show that the adjunction mappings

\[
Lg^*\Phi(K) = Lg^*Rg_*K \to K \quad \text{and} \quad M \to Rg_*Lg^*M = \Phi(Lg^*M)
\]

are isomorphisms for \( K \in D_{QCoh}(\mathcal{O}_X) \) and \( M \in D_{QCoh}(f_*\mathcal{O}_X) \). This is a local question, hence we may assume \( Y \) and therefore \( X \) are affine.

Assume \( Y = \text{Spec}(B) \) and \( X = \text{Spec}(A) \). Let \( p = x \in \text{Spec}(A) = X \) be a point mapping to \( q = y \in \text{Spec}(B) = Y \). Then \((f_*\mathcal{O}_X)_y = A_q \) and \( \mathcal{O}_{X,x} = A_p \) hence \( g \) is flat. Hence \( g^* \) is exact and \( H^i(Lg^*M) = g^*H^i(M) \) for any \( M \) in \( D(f_*\mathcal{O}_X) \). For \( K \in D_{QCoh}(\mathcal{O}_X) \) we see that

\[
H^i(\Phi(K)) = H^i(Rf_*K) = f_*H^i(K)
\]

by the vanishing of higher direct images (Cohomology of Schemes, Lemma [2.3] and Lemma [3.4]) Thus it suffices to show that

\[
g^*g_*\mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{G} \to g_*g^*\mathcal{F}
\]

are isomorphisms where \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-module and \( \mathcal{G} \) is a quasi-coherent \( f_*\mathcal{O}_X \)-module. This follows from Morphisms, Lemma [11.6].

\[\square\]

6. The coherator

Let \( X \) be a scheme. The coherator is a functor

\[
Q_X : \text{Mod}(\mathcal{O}_X) \to \text{Coh}(\mathcal{O}_X)
\]

which is right adjoint to the inclusion functor \( \text{Coh}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X) \). It exists for any scheme \( X \) and moreover the adjunction mapping \( Q_X(\mathcal{F}) \to \mathcal{F} \) is an isomorphism for every quasi-coherent module \( \mathcal{F} \), see Properties, Proposition [23.4]. Since \( Q_X \) is left exact (as a right adjoint) we can consider its right derived extension

\[
RQ_X : D(\mathcal{O}_X) \to D(\text{Coh}(\mathcal{O}_X)).
\]

Since \( Q_X \) is right adjoint to the inclusion functor \( \text{Coh}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X) \) we see that \( RQ_X \) is right adjoint to the canonical functor \( D(\text{Coh}(\mathcal{O}_X)) \to D(\mathcal{O}_X) \) by Derived Categories, Lemma [29.5].

In this section we will study the functor \( RQ_X \). In Section [20] we will study the (closely related) right adjoint to the inclusion functor \( D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X) \) (when it exists).
Lemma 6.1. Let $f : X \to Y$ be an affine morphism of schemes. Then $f_*$ defines a derived functor $f_* : D(QCoh(O_X)) \to D(QCoh(O_Y))$. This functor has the property that

$$
D(QCoh(O_X)) \xrightarrow{f_*} D(QCoh(O_Y)) \xrightarrow{RF_*} D(QCoh(O_Y))
$$

Proof. The functor $f_* : QCoh(O_X) \to QCoh(O_Y)$ is exact, see Cohomology of Schemes, Lemma 2.3. Hence $f_*$ defines a derived functor $f_* : D(QCoh(O_X)) \to D(QCoh(O_Y))$ by simply applying $f_*$ to any representative complex, see Derived Categories, Lemma 16.7. For any complex of $O_X$-modules $F^\bullet$ there is a canonical map $f_* F^\bullet \to Rf_* F^\bullet$. To finish the proof we show this is a quasi-isomorphism.

Lemma 6.2. Let $f : X \to Y$ be a morphism of schemes. Assume $f$ is quasi-compact, quasi-separated, and flat. Then, denoting

$$
\Phi : D(QCoh(O_X)) \to D(QCoh(O_Y))
$$

the right derived functor of $f_* : QCoh(O_X) \to QCoh(O_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(O_Y)) \to D(O_X)$.

Since $f$ is quasi-compact and quasi-separated, we see that $f_*$ preserves quasi-coherence, see Schemes, Lemma 24.1. Hence any $K$ in $D(QCoh(O_X))$ can be represented by a $K$-injective complex $\mathcal{J}^\bullet$ of $QCoh(O_X)$, see Injectives, Theorem 12.6.

Let $A$ be an object of $D(QCoh(O_Y))$ and $E$ an object of $D(O_X)$. Then

$$
\text{Hom}_{D(QCoh(O_Y))}(A, RQ_Y(Rf_* E)) = \text{Hom}_{D(QCoh(O_Y))}(A, Rf_* E)
= \text{Hom}_{D(O_X)}(f^* A, E)
= \text{Hom}_{D(QCoh(O_X))}(f^* A, RQ_X(E))
= \text{Hom}_{D(QCoh(O_Y))}(A, \Phi(RQ_X(E)))
$$

This implies what we want.
Lemma 6.3. Let $X = \text{Spec}(A)$ be an affine scheme. Then

1. $Q_X : \text{Mod}(\mathcal{O}_X) \to \text{QCoh}(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the $A$-module $\Gamma(X, \mathcal{F})$,
2. $RQ_X : D(\mathcal{O}_X) \to D(\text{QCoh}(\mathcal{O}_X))$ is the functor which sends $E$ to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,
3. restricted to $D_{\text{QCoh}}(\mathcal{O}_X)$ the functor $RQ_X$ defines a quasi-inverse to $\text{(3.0.1)}$.

Proof. The functor $Q_X$ is the functor $\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$ by Schemes, Lemma [7.1]. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 3.5. \qed

Definition 6.4. Let $X$ be a scheme. Let $E$ be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say $E$ is supported on $T$ if the cohomology sheaves $H^i(E)$ are supported on $T$.

With this definition in hand, we are ready to prove a criterion for when the functor $D(\text{QCoh}(\mathcal{O}_X)) \to D_{\text{QCoh}}(\mathcal{O}_X)$ is an equivalence.

Lemma 6.5. Let $X$ be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor

$$\Phi : D(\text{QCoh}(\mathcal{O}_U)) \to D(\text{QCoh}(\mathcal{O}_X))$$

of the left exact functor $j_* : \text{QCoh}(\mathcal{O}_U) \to \text{QCoh}(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{array}{ccc}
D(QCoh(\mathcal{O}_U)) & \xrightarrow{\Phi} & D_{QCoh}(\mathcal{O}_U) \\
\downarrow \Phi & & \downarrow Rj_* \\
D(QCoh(\mathcal{O}_X)) & \xrightarrow{i_X} & D_{QCoh}(\mathcal{O}_X)
\end{array}$$

Then the functor $\text{(3.0.1)}$

$$D(\text{QCoh}(\mathcal{O}_X)) \to D_{\text{QCoh}}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by $RQ_X$.

Proof. Let $E$ be an object of $D_{\text{QCoh}}(\mathcal{O}_X)$ and let $A$ be an object of $D(\text{QCoh}(\mathcal{O}_X))$.

We have to show that the adjunction maps $RQ_X(i_X(A)) \to A$ and $E \to i_X(RQ_X(E))$ are isomorphisms. Consider the hypothesis $H_n$: the adjunction maps above are isomorphisms whenever $E$ and $i_X(A)$ are supported on a closed subset of $X$ which is contained in the union of $n$ affine opens of $X$. We will prove $H_n$ by induction on $n$.

Base case: $n = 0$. In this case $E = 0$, hence the map $E \to i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $\text{QCoh}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of $A$ are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \to A$ is an isomorphism as well.
Induction step. Suppose that $E$ and $i_X(A)$ are supported on a closed subset $T$ of $X$ contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. Consider the distinguished triangles

$$A \to \Phi(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where $\Phi$ is as in the statement of the lemma. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ \Phi = Rj_* \circ i_U$ by assumption and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(\Phi(A|_U))$ is a quasi-isomorphism over $U$. Hence $i_X(A')$ and $E'$ are supported on the closed subset $T \setminus U$ of $X$ which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for $A'$ and $E'$. By Derived Categories, Lemma 11.11 it suffices to prove the maps

$$RQ_X(i_X(\Phi(A|_U))) \to \Phi(A|_U) \quad \text{and} \quad Rj_*(E|_U) \to i_X(RQ_X(Rj_*(E|_U)))$$

are isomorphisms. By assumption and by Lemma 6.2 (the inclusion morphism $j : U \to X$ is flat, quasi-compact, and quasi-separated) we have

$$RQ_X(i_X(\Phi(A|_U))) = RQ_X(Rj_*(i_U(\Phi(A|_U))) = \Phi(RQ_U(i_U(\Phi(A|_U))))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(\Phi(RQ_U(E|_U))) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \to A|_U \quad \text{and} \quad E|_U \to i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 6.3. The result follows.

**Proposition 6.6.** Let $X$ be a quasi-compact scheme with affine diagonal. Then the functor

$$D(QCoh(O_X)) \to D_{QCoh}(O_X)$$

is an equivalence with quasi-inverse given by $RQ_X$.

**Proof.** Let $U \subset X$ be an affine open. Then the morphism $U \to X$ is affine by Morphisms, Lemma 11.11 Thus the assumption of Lemma 6.5 holds by Lemma 6.1 and we win.

**Lemma 6.7.** Let $f : X \to Y$ be a morphism of schemes. Assume $X$ and $Y$ are quasi-compact and have affine diagonal. Then, denoting

$$\Phi : D(QCoh(O_X)) \to D(QCoh(O_Y))$$

the right derived functor of $f_* : QCoh(O_X) \to QCoh(O_Y)$ the diagram

$$\begin{array}{ccc}
D(QCoh(O_X)) & \xrightarrow{\Phi} & D_{QCoh}(O_X) \\
\downarrow & & \downarrow Rf_* \\
D(QCoh(O_Y)) & \to & D_{QCoh}(O_Y)
\end{array}$$

is commutative.

**Proof.** Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 6.6. Hence we can identify these categories (and similarly for other quasi-compact schemes with affine diagonal). The statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all $K$ in $D(QCoh(O_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle
in $D(QCoh(O_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $U \subset X$ be an affine open. Since the diagonal of $X$ is affine, the inclusion morphism $j : U \to X$ is affine (Morphisms, Lemma 11.11). Similarly, the composition $g = f \circ j : U \to Y$ is affine. Let $\mathcal{I}^*$ be a K-injective complex in $QCoh(O_U)$.

Since $j_* : QCoh(O_U) \to QCoh(O_X)$ has an exact left adjoint $j^* : QCoh(O_X) \to QCoh(O_U)$ we see that $(j_*)\mathcal{I}^*$ is a K-injective complex in $QCoh(O_X)$, see Derived Categories, Lemma 30.9. It follows that

$$\Phi(j_*\mathcal{I}^*) = f_*j_*\mathcal{I}^* = g_*\mathcal{I}^*$$

By Lemma 3.4 we see that $j_*\mathcal{I}^*$ represents $Rj_*\mathcal{I}^*$ and $g_*\mathcal{I}^*$ represents $Rg_*\mathcal{I}^*$. On the other hand, we have $Rf_* \circ Rj_* = Rg_*$. Hence $f_*j_*\mathcal{I}^*$ represents $Rf_*(j_*\mathcal{I}^*)$. We conclude that the lemma is true for any complex of the form $j_*\mathcal{G}^*$ with $\mathcal{G}^*$ a complex of quasi-coherent modules on $U$. (Note that if $\mathcal{G}^* \to \mathcal{I}^*$ is a quasi-isomorphism, then $j_*\mathcal{G}^* \to j_*\mathcal{I}^*$ is a quasi-isomorphism as well since $j_*$ is an exact functor on quasi-coherent modules.)

Let $\mathcal{F}^*$ be a complex of quasi-coherent $O_X$-modules. Let $T \subset X$ be a closed subset such that the support of $\mathcal{F}^p$ is contained in $T$ for all $p$. We will use induction on the minimal number $n$ of affine opens $U_1, \ldots, U_n$ such that $T \subset U_1 \cup \ldots \cup U_n$. The base case $n = 0$ is trivial. If $n \geq 1$, then set $U = U_1$ and denote $j : U \to X$ the open immersion as above. We consider the map of complexes $c : \mathcal{F}^* \to j_*j^*\mathcal{F}^*$. We obtain two short exact sequences of complexes:

$$0 \to \text{Ker}(c) \to \mathcal{F}^* \to \text{Im}(c) \to 0$$

and

$$0 \to \text{Im}(c) \to j_*j^*\mathcal{F}^* \to \text{Coker}(c) \to 0$$

The complexes $\text{Ker}(c)$ and $\text{Coker}(c)$ are supported on $T \setminus U \subset U_2 \cup \ldots \cup U_n$ and the result holds for them by induction. The result holds for $j_*j^*\mathcal{F}^*$ by the discussion in the preceding paragraph. We conclude by looking at the distinguished triangles associated to the short exact sequences and using the initial remark of the proof. □

0CRY Remark 6.8 (Warning). Let $X$ be a quasi-compact scheme with affine diagonal. Even though we know that $D(QCoh(O_X)) = D_{QCoh}(O_X)$ by Proposition 6.9 strange things can happen and it is easy to make mistakes with this material. One pitfall is to carelessly assume that this equality means derived functors are the same. For example, suppose we have a quasi-compact open $U \subset X$. Then we can consider the higher right derived functors

$$R^i(QCoh)\Gamma(U, -) : QCoh(O_X) \to Ab$$

of the left exact functor $\Gamma(U, -)$. Since this is a universal $\delta$-functor, and since the functors $H^i(U, -)$ (defined for all abelian sheaves on $X$) restricted to $QCoh(O_X)$ form a $\delta$-functor, we obtain canonical tranformations

$$t^i : R^i(QCoh)\Gamma(U, -) \to H^i(U, -).$$

These transformations aren’t in general isomorphisms even if $X = \text{Spec}(A)$ is affine! Namely, we have $R^1(QCoh)\Gamma(U, \mathcal{I}) = 0$ if $\mathcal{I}$ an injective $A$-module by construction of right derived functors and the equivalence of $QCoh(O_X)$ and $\text{Mod}_A$. But Examples, Lemma 40.2 shows there exists $A$, $\mathcal{I}$, and $U$ such that $H^1(U, \mathcal{I}) \neq 0$. 

7. The coherator for Noetherian schemes

**Lemma 7.1.** Let $X$ be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $\text{QCoh}(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.

**Proof.** Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $I \subset X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 12.5). By Cohomology of Schemes, Lemma 10.4 the section $s$ corresponds to a map $\sigma : I^n \to \mathcal{J}$ for some $n$. As $\mathcal{J}$ is an injective object of $\text{QCoh}(\mathcal{O}_X)$ we can extend $\sigma$ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to $s$. \hfill $\square$

**Lemma 7.2.** Let $f : X \to Y$ be a morphism of Noetherian schemes. Then $f_* \text{ on quasi-coherent sheaves}$ has a right derived extension $\Phi : D(\text{QCoh}(\mathcal{O}_X)) \to D(\text{QCoh}(\mathcal{O}_Y))$ such that the diagram

$$
\begin{array}{ccc}
D(\text{QCoh}(\mathcal{O}_X)) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_X) \\
\Phi \downarrow & & \downarrow \text{Rf}_* \\
D(\text{QCoh}(\mathcal{O}_Y)) & \longrightarrow & D_{\text{QCoh}}(\mathcal{O}_Y)
\end{array}
$$

commutes.

**Proof.** Since $X$ and $Y$ are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 5.4 and Schemes, Remark 21.18). Thus $f_*$ preserve quasi-coherence, see Schemes, Lemma 24.1. Next, let $K$ be an object of $D(\text{QCoh}(\mathcal{O}_X))$. Since $\text{QCoh}(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 23.4), we can represent $K$ by a K-injective complex $I^\bullet$ such that each $I^n$ is an injective object of $\text{QCoh}(\mathcal{O}_X)$, see Injectives, Theorem 12.6. Thus we see that the functor $\Phi$ is defined by setting

$$
\Phi(K) = f_*I^\bullet
$$

where the right hand side is viewed as an object of $D(\text{QCoh}(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$
f_*I^\bullet \longrightarrow \text{Rf}_*I^\bullet
$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbb{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 4.1. Consider the short exact sequence

$$
0 \to \sigma_{\geq m-N-1}I^\bullet \to I^\bullet \to \sigma_{\leq m-N-2}I^\bullet \to 0
$$

of complexes of quasi-coherent sheaves on $X$. By Lemma 4.1 we see that the cohomology sheaves of $\text{Rf}_*\sigma_{\geq m-N-3}I^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $\text{Rf}_*I^\bullet$ is isomorphic to $\text{R}^m\text{f}_*I^\bullet$. In other words, we may assume that $I^\bullet$ is a bounded below complex of injective objects of $\text{QCoh}(\mathcal{O}_X)$. This follows from Leray’s acyclicity lemma (Derived Categories, Lemma 16.7) via Cohomology, Lemma 12.5 and Lemma 7.1. \hfill $\square$

**Proposition 7.3.** Let $X$ be a Noetherian scheme. Then the functor $D(\text{QCoh}(\mathcal{O}_X)) \to D_{\text{QCoh}}(\mathcal{O}_X)$ is an equivalence with quasi-inverse given by $\text{RQ}_X$. 

Proof. This follows from Lemma 6.5 and Lemma 7.2

8. Koszul complexes

Let $A$ be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of $A$. We have defined the Koszul complex $K_*(f_1, \ldots, f_r)$ in More on Algebra, Definition 28.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^*(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^*(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^*(f_1, \ldots, f_r) \to A \to K^*(f_1, \ldots, f_r) \to I^*(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^*(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^\oplus r \to A$ is given by multiplication by $f_i$ on the $i$th factor. Hence $I^*(f_1, \ldots, f_r) \to A$ factors as

$$I^*(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^*(f_1, \ldots, f_r)) \to H^0(I^*(f_1, \ldots, f_r)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^*(f_1, \ldots, f_r)) = H^{i-1}(K^*(f_1, \ldots, f_r))$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of $A$ there are canonical maps

$$I^*(f_1 g_1, \ldots, f_r g_r) \to I^*(f_1, \ldots, f_r) \quad \text{and} \quad K^*(f_1 g_1, \ldots, f_r g_r) \to K^*(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by $g_i$ on the $i$th summand of $I^0(f_1 g_1, \ldots, f_r g_r) = A^\oplus r$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

$$I^*(f_1, \ldots, f_r) \leftarrow I^*(f_1^2, \ldots, f_r^2) \leftarrow I^*(f_1^3, \ldots, f_r^3) \leftarrow \ldots$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

Situation 8.1. Here $A$ is a ring and $f_1, \ldots, f_r$ is a sequence of elements of $A$. We set $X = \text{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1, \ldots, r} D(f_i)$ the given open covering of $U$.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on $U$. Suppose given a complex $I^*$ of $A$-modules and an $A$-module $M$. Then we define $\text{Hom}_A(I^*, M)$ to be the complex with $n$th term

$$\text{Hom}_A(I^-, M)$$

differentialials given as the contragredients of the differentials on $I^*$. The Koszul complex

$$K^*(f_1, \ldots, f_r)$$

is defined in the natural way. We then define

$$\text{Hom}_A(I^*(f_1, \ldots, f_r), M)$$

which play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

Lemma 8.2. In Situation 8.1, Let $M$ be an $A$-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes

$$\colim_n \text{Hom}_A(I^*(f_1, \ldots, f_r), M) \to \check{C}_\mathcal{U}(\mathcal{F})$$

functorial in $M$. 


Proof. Recall that the alternating Ĉech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 23. As usual we view a $p$-cochain in $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ as an alternating function $s$ on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0 \ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0} \ldots f_{i_p}} = \mathcal{F}(U_{i_0 \ldots i_p})$. On the other hand, a $p$-cochain $t$ in $\text{Hom}_A(I^* (f_1^e, \ldots, f_r^e), M)$ is given by a map $t : \Lambda^{p+1}(A^\oplus r) \to M$. Write $[i] \in A^\oplus r$ for the $i$th basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \Lambda^{p+1}(A^\oplus r)$$

Then we send $t$ as above to $s$ with

$$s_{i_0 \ldots i_p} = \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \cdots f_{i_p}^e}$$

It is clear that $s$ so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^* (f_1^e, \ldots, f_r^e) \leftarrow I^* (f_1^{e+1}, \ldots, f_r^{e+1}),$$

of the $\{8.0.1\}$ sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p} [i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \ldots f_{i_p}}$ in Algebra, Lemma 9.9 that these maps define an isomorphism of cochain modules in degree $p$ in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$d(s)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

and $f_{i_0} \ldots f_{i_{p+1}}$,

$$d(t)_{i_0 \ldots i_{p+1}} = \frac{t([i_0, \ldots, i_{p+1}])}{f_{i_0}^e \cdots f_{i_{p+1}}^e}$$

The two formulas agree by inspection. □

Suppose given a finite complex $I^*$ of $A$-modules and a complex of $A$-modules $M^*$. We obtain a double complex $H^{\bullet, \bullet} = \text{Hom}_A(I^*, M^*)$ where $H^{p, q} = \text{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\text{Hom}_A(I^*, M^*)$ and the second from the differential on $M^*$. Associated to this double complex is the total complex with degree $n$ term given by

$$\bigoplus_{p+q=n} \text{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 23.3. As our complex $I^*$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 25.
functorial in $M^\bullet$. 

**Proof.** Immediate from Lemma 8.2 and our conventions for taking associated total complexes. □

**Lemma 8.4.** In Situation 8.1 Let $F^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism 

$$\text{Tot}(\mathcal{C}_\text{att}(U,F^\bullet)) \to R\Gamma(U,F^\bullet)$$

in $D(A)$ functorial in $F^\bullet$.

**Proof.** Let $E$ be the set of affine opens of $U$. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 2.2) this is a special case of Cohomology, Lemma 36.2. □

In Situation 8.1 denote $I_e$ the object of $D(\mathcal{O}_X)$ corresponding to the complex of $A$-modules $I^\bullet(f^e_1,\ldots,f^e_r)$ via the equivalence of Lemma 3.5. The maps (8.0.1) give a system 

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to $U$. Thus we see that for every object $E$ of $D(\mathcal{O}_X)$ there is a canonical map

$$\text{colim}_e \text{Hom}_{D(\mathcal{O}_X)}(I_e,E) \to H^0(U,E)$$

constructed by sending a map $I_e \to E$ to its restriction to $U$ and using that $\text{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U,E|_U) = H^0(U,E)$.

**Proposition 8.5.** In Situation 8.1 For every object $E$ of $D_{QCoh}(\mathcal{O}_X)$ the map (8.4.1) is an isomorphism.

**Proof.** By Lemma 3.5 we may assume that $E$ is given by a complex of quasi-coherent sheaves $F^\bullet$. Let $M^\bullet = \Gamma(X,F^\bullet)$ be the corresponding complex of $A$-modules. By Lemmas 8.3 and 8.4 we have quasi-isomorphisms 

$$\text{colim}_e \text{Tot}(\text{Hom}_A(I^\bullet(f^e_1,\ldots,f^e_r),M^\bullet)) \to \text{Tot}(\mathcal{C}_\text{att}(U,F^\bullet)) \to R\Gamma(U,F^\bullet)$$

Taking $H^0$ on both sides we obtain 

$$\text{colim}_e \text{Hom}_{D(A)}(I^\bullet(f^e_1,\ldots,f^e_r),M^\bullet) = H^0(U,E)$$

Since $\text{Hom}_{D(A)}(I^\bullet(f^e_1,\ldots,f^e_r),M^\bullet) = \text{Hom}_{D(\mathcal{O}_X)}(I_e,E)$ by Lemma 3.5 the lemma follows. □

In Situation 8.1 denote $K_e$ the object of $D(\mathcal{O}_X)$ corresponding to the complex of $A$-modules $K^\bullet(f^e_1,\ldots,f^e_r)$ via the equivalence of Lemma 3.5. Thus we have distinguished triangles 

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system 

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system $(I_e)$. Moreover, there is a compatible system of maps $K_e \to H^0(K_e) = \mathcal{O}_X/(f^e_1,\ldots,f^e_r)$

**Lemma 8.6.** In Situation 8.1 Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1,\ldots,0$. Then given $s \in H^0(X,E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that $s$ is in the image of $H^0(X,K_e) \to H^0(X,E)$.  


Proof. Since $U$ is covered by $r$ affine opens we have $H^i(U, F) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 4.2). By Lemma 3.4 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1} E)$. There is a spectral sequence

\[ H^i(U, H^j(\tau_{\geq -r+1} E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N} E) \]

see Derived Categories, Lemma 21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of $E$. We conclude that $s|_U = 0$. Think of $s$ as a morphism $O_X \to E$ in $D(O_X)$. By Proposition 8.5 the composition $I_e \to O_X \to E$ is zero for some $e$. By the distinguished triangle $I_e \to O_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that $s$ is the composition $O_X \to K_e \to E$. □

9. Pseudo-coherent and perfect complexes

08E4 In this section we make the connection between the general notions defined in Cohomology, Sections 12, 13, 14, 15 and the corresponding notions for complexes of modules in More on Algebra, Sections 62, 63, and 69.

08E5 Lemma 9.1. Let $X$ be a scheme. If $E$ is an $m$-pseudo-coherent object of $D(O_X)$, then $H^i(E)$ is a quasi-coherent $O_X$-module for $i > m$ and $H^m(E)$ is a quotient of a quasi-coherent $O_X$-module. If $E$ is pseudo-coherent, then $E$ is an object of $D_{QCoh}(O_X)$.

Proof. Locally on $X$ there exists a strictly perfect complex $\mathcal{E}^\bullet$ such that $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ for $i > m$ and $H^m(E)$ is a quotient of $H^m(\mathcal{E}^\bullet)$. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

08E7 Lemma 9.2. Let $X = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(O_X)$. Then $E$ is an $m$-pseudo-coherent (resp. pseudo-coherent) as an object of $D(O_X)$ if and only if $M^\bullet$ is $m$-pseudo-coherent (resp. pseudo-coherent) as a complex of $A$-modules.

Proof. It is immediate from the definitions that if $M^\bullet$ is $m$-pseudo-coherent, so is $E$. To prove the converse, assume $E$ is $m$-pseudo-coherent. As $X = \text{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}^i_\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}^i_\bullet \to E|_{U_i}$ inducing isomorphisms on $H^j$ for $j > m$ and surjections on $H^m$. By Cohomology, Lemma 12.8 after refining the open covering we may assume $\alpha_i$ is given by a map of complexes $\mathcal{E}^i_\bullet \to M^\bullet|_{U_i}$ for each $i$. By Modules, Lemma 14.6 the terms $\mathcal{E}^i_\bullet$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}^i_\bullet$ is a finite free $O_{U_i}$-module. From the definition it follows that $M^\bullet_{U_i}$ is an $m$-pseudo-coherent complex of $A_{U_i}$-modules. We conclude by applying More on Algebra, Lemma 62.15. □

The case “pseudo-coherent” follows from the fact that $E$ is pseudo-coherent if and only if $E$ is $m$-pseudo-coherent for all $m$ (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 62.5.

08E8 Lemma 9.3. Let $X$ be a Noetherian scheme. Let $E$ be an object of $D_{QCoh}(O_X)$. For $m \in \mathbb{Z}$ the following are equivalent

1. $H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and
2. $E$ is $m$-pseudo-coherent.
In particular, $E$ is pseudo-coherent if and only if $E$ is an object of $D_{\text{QCoh}}(\mathcal{O}_X)$.

**Proof.** As $X$ is quasi-compact we see that in both (1) and (2) the object $E$ is bounded above. Thus the question is local on $X$ and we may assume $X$ is affine. Say $X = \text{Spec}(A)$ for some Noetherian ring $A$. In this case $E$ corresponds to a complex of $A$-modules $M^\bullet$ by Lemma 3.5. By Lemma 9.2 we see that $E$ is $m$-pseudo-coherent if and only if $M^\bullet$ is $m$-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite $A$-module (Properties, Lemma 16.1). Thus the result follows from More on Algebra, Lemma 62.18. □

**Lemma 9.4.** Let $X = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then

1. $E$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.
2. $E$ has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.

**Proof.** Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{\text{QCoh}}(X)$ of Lemma 3.5 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$, then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat $A$-modules with $K^i = 0$ for $i \notin [a, b]$, see More on Algebra, Lemma 63.3. Then $E$ is isomorphic to $K^\bullet$. Since each $K^i$ is a flat $\mathcal{O}_X$-module, we see that $E$ has tor amplitude in $[a, b]$ by Cohomology, Lemma 44.3.

Assume that $E$ has tor amplitude in $[a, b]$. Then $E$ is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of $A$-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free $A$-modules. Then for any $A$-module $N$ we have

$$E \otimes_{\mathcal{O}_X} \tilde{N} \cong M^\bullet \otimes_{\mathcal{O}_X} \tilde{N} \cong M^\bullet \otimes_A N$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. □

**Lemma 9.5.** Let $f : X \to S$ be a morphism of affine schemes corresponding to the ring map $R \to A$. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then

1. $E$ as an object of $D(f^{-1}\mathcal{O}_S)$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$ as an object of $D(R)$.
2. $E$ locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ if and only if $M^\bullet$ has finite tor dimension as an object of $D(R)$.

**Proof.** Consider a prime $q \subset A$ lying over $p \subset R$. Let $x \in X$ and $s = f(x) \in S$ be the corresponding points. Then $(f^{-1}\mathcal{O}_S)_x = \mathcal{O}_{S, s} = R_p$ and $E_x = M^\bullet_q$. Keeping this in mind we can see the equivalence as follows.

If $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of $R$-modules, then the same is true for the localization of $M^\bullet$ at any prime of $A$. Then we conclude by Cohomology, Lemma 44.5 that $E$ has tor amplitude in $[a, b]$ as a complex of sheaves of $f^{-1}\mathcal{O}_S$-modules. Conversely, assume that $E$ has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_S)$. We conclude (using the last cited lemma) that $M^\bullet_q$ has tor amplitude in $[a, b]$ as a complex of $R_p$-modules for every prime $q \subset A$ lying over $p \subset R$. By More on Algebra, Lemma 63.15 we find that $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of $R$-modules. This finishes the proof of (1).
Since $X$ is quasi-compact, if $E$ locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_S$-modules, then actually $E$ has tor amplitude in $[a,b]$ for some $a,b$ as a complex of $f^{-1}\mathcal{O}_S$-modules. Thus (2) follows from (1). □

**Lemma 9.6.** Let $X$ be a quasi-separated scheme. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent

1. $E$ has tor amplitude in $[a,b]$, and
2. for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{L}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \not\in [a,b]$.

**Proof.** It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As $X$ is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence $j_* \otimes \otimes$ transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{L}_{\mathcal{O}_U} \mathcal{G})$ for $i \not\in [a,b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \text{Spec}(A)$ and let $M^\bullet$ be the complex of $A$-modules corresponding to $E|_U$ by Lemma 3.5. We have just shown that $M^\bullet \otimes^{L}_A X$ has vanishing cohomology groups outside the range $[a,b]$, in other words $M^\bullet$ has tor amplitude in $[a,b]$. By Lemma 9.4 we conclude that $E|_U$ has tor amplitude in $[a,b]$. This proves the lemma. □

**Lemma 9.7.** Let $X = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of $A$-modules and let $E$ be the corresponding object of $D(\mathcal{O}_X)$. Then $E$ is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.

**Proof.** This is a logical consequence of Lemmas 9.2 and 9.4 Cohomology, Lemma 45.5 and More on Algebra, Lemma 69.2. □

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

**Lemma 9.8.** Let $X$ be a scheme.

1. If $L$ in $D^+_{QCoh}(\mathcal{O}_X)$ and $K$ in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K,L)$ is in $D_{QCoh}(\mathcal{O}_X)$ and locally bounded below.
2. If $L$ in $D_{QCoh}(\mathcal{O}_X)$ and $K$ in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K,L)$ is in $D_{QCoh}(\mathcal{O}_X)$.
3. If $X = \text{Spec}(A)$ is affine and $K,L \in D(A)$ then

$$R\mathcal{H}om(\widetilde{K},\widetilde{L}) = \widehat{R\mathcal{H}om}_A(K,L)$$

in the following two cases

(a) $K$ is pseudo-coherent and $L$ is bounded below,
(b) $K$ is perfect and $L$ arbitrary.
4. If $X = \text{Spec}(A)$ and $K,L$ are in $D(A)$, then the $n$th cohomology sheaf of $R\mathcal{H}om(K,L)$ is the sheaf associated to the presheaf

$$X \ni D(f) \mapsto \text{Ext}_A^n(K \otimes_A L, f)$$

for $f \in A$.

**Proof.** The construction of the internal hom in the derived category of $\mathcal{O}_X$ commutes with localization (see Cohomology, Section 38). Hence to prove (1) and (2) we may replace $X$ by an affine open. By Lemmas 3.5, 9.2, and 9.7 in order to prove (1) and (2) it suffices to prove (3).
Part (3) follows from the computation of the internal hom of Cohomology, Lemma 42.10 by representing $K$ by a bounded above (resp. finite) complex of finite projective $A$-modules and $L$ by a bounded below (resp. arbitrary) complex of $A$-modules.

To prove (4) recall that on any ringed space the $n$th cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf

$$U \mapsto \mathcal{H}om_{D(U)}(A|_U, B|_U[n]) = \text{Ext}^n_D(O_U)(A|_U, B|_U)$$

See Cohomology, Section 38. On the other hand, the restriction of $\tilde{K}$ to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for $L$. Hence (4) follows from the equivalence of categories of Lemma 3.5. □

Lemma 9.9. Let $X$ be a scheme. Let $K, L, M$ be objects of $D_{QCoh}(\mathcal{O}_X)$. The map

$$K \otimes^L_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^L_{\mathcal{O}_X} L)$$

of Cohomology, Lemma 38.7 is an isomorphism in the following cases

1. $M$ perfect, or
2. $K$ is perfect, or
3. $M$ is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and $K$ has finite tor dimension.

Proof. Lemma 9.8 reduces cases (1) and (3) to the affine case which is treated in More on Algebra, Lemma 88.3 (You also have to use Lemmas 9.2, 9.7, and 9.4 to do the translation into algebra.) If $K$ is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because we can work locally and reduce to the case that $K$ is a finite complex of finite free modules in which case it is clear. □

10. Derived category of coherent modules

Let $X$ be a locally Noetherian scheme. In this case the category $\text{Coh}(\mathcal{O}_X) \subset \text{Mod}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 10 and Cohomology of Schemes, Lemma 9.2. Denote

$$D_{\text{Coh}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 17. Thus we obtain a canonical functor

$$D(\text{Coh}(\mathcal{O}_X)) \longrightarrow D_{\text{Coh}}(\mathcal{O}_X)$$

see Derived Categories, Equation (17.1.1).

Lemma 10.1. Let $X$ be a Noetherian scheme. Then the functor

$$D^-(\text{Coh}(\mathcal{O}_X)) \longrightarrow D^-(\text{Coh}(\mathcal{O}_X))(\text{QCoh}(\mathcal{O}_X))$$

is an equivalence.

Proof. Observe that $\text{Coh}(\mathcal{O}_X) \subset \text{QCoh}(\mathcal{O}_X)$ is a Serre subcategory, see Homology, Definition 10.1 and Lemma 10.2 and Cohomology of Schemes, Lemmas 9.2 and 9.3. On the other hand, if $\mathcal{G} \rightarrow \mathcal{F}$ is a surjection from a quasi-coherent $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X$-module, then there exists a coherent submodule $\mathcal{G}' \subset \mathcal{G}$ which surjects onto $\mathcal{F}$. Namely, we can write $\mathcal{G}$ as the filtered union of its coherent submodules by Properties, Lemma 22.3 and then one of these will do the job. Thus the lemma follows from Derived Categories, Lemma 17.4. □
Proposition 10.2. Let $X$ be a Noetherian scheme. Then the functors
\[ D^-(\text{Coh}(\mathcal{O}_X)) \rightarrow D^-_{\text{Coh}}(\mathcal{O}_X) \quad \text{and} \quad D^b(\text{Coh}(\mathcal{O}_X)) \rightarrow D^b_{\text{Coh}}(\mathcal{O}_X) \]
are equivalences.

Proof. Consider the commutative diagram
\[
\begin{array}{ccc}
D^-(\text{Coh}(\mathcal{O}_X)) & \longrightarrow & D^-_{\text{Coh}}(\mathcal{O}_X) \\
\downarrow & & \downarrow \\
D^-(\text{QCoh}(\mathcal{O}_X)) & \longrightarrow & D^-_{\text{QCoh}}(\mathcal{O}_X)
\end{array}
\]
By Lemma 10.1 the left vertical arrow is fully faithful. By Proposition 7.3 the bottom arrow is an equivalence. By construction the right vertical arrow is fully faithful. We conclude that the top horizontal arrow is fully faithful. If $K$ is an object of $D^{-}(\text{Coh}(\mathcal{O}_X))$ then the object $K'$ of $D^-(\text{QCoh}(\mathcal{O}_X))$ which corresponds to it by Proposition 7.3 will have coherent cohomology sheaves. Hence $K'$ is in the essential image of the left vertical arrow by Lemma 10.1 and we find that the top horizontal arrow is essentially surjective. This finishes the proof for the bounded above case. The bounded case follows immediately from the bounded above case. □

Lemma 10.3. Let $S$ be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $E$ be an object of $D^b_{\text{Coh}}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over $S$ for all $i$. Then $Rf_*E$ is an object of $D^b_{\text{Coh}}(\mathcal{O}_S)$.

Proof. Consider the spectral sequence
\[ R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \]
see Derived Categories, Lemma 21.3. By assumption and Cohomology of Schemes, Lemma 26.10 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D^b_{\text{Coh}}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 4.5 or from Lemma 4.1. □

Lemma 10.5. Let $X$ be a locally Noetherian scheme. If $L$ is in $D^+_{\text{Coh}}(\mathcal{O}_X)$ and $K$ in $D^-_{\text{Coh}}(\mathcal{O}_X)$, then $R\text{Hom}(K, L)$ is in $D^+_{\text{Coh}}(\mathcal{O}_X)$.

Proof. The proof is the same as the proof of Lemma 10.3. You can also deduce it from Lemma 10.3 by considering what the exact functor $Rf_*$ does to the distinguished triangles $\tau_{\leq a} E \rightarrow E \rightarrow \tau_{\geq a+1} E \rightarrow \tau_{\leq a} E[1]$. □
in $D^+_{\text{Coh}}(A)$, then $R\text{Hom}_A(K, L)$ is in $D^+_{\text{Coh}}(A)$. Since $L$ is bounded below and $K$ is bounded below there is a convergent spectral sequence

$$\text{Ext}^n_A(K, H^q(L)) \Rightarrow \text{Ext}^{n+q}_A(K, L)$$

and there are convergent spectral sequences

$$\text{Ext}^n_A(H^{-j}(K), H^q(L)) \Rightarrow \text{Ext}^{n+j}_A(K, H^q(L))$$

This finishes the proof as the modules $\text{Ext}^n_A(M, N)$ are finite for finite $A$-modules $M, N$ by Algebra, Lemma 70.9.

\[\text{Lemma 10.6.} \quad \text{Let } X \text{ be a Noetherian ring. Let } X \text{ be a proper scheme over } A. \text{ For } L \text{ in } D^+_{\text{Coh}}(\mathcal{O}_X) \text{ and } K \text{ in } D^+_{\text{Coh}}(\mathcal{O}_X), \text{ the } A\text{-modules } \text{Ext}^n_{\mathcal{O}_X}(K, L) \text{ are finite.}\]

\[\text{Proof.} \quad \text{Recall that} \]

$$\text{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\text{Hom}_{\mathcal{O}_X}(K, L)) = H^n(\text{Spec}(A), Rf_* R\text{Hom}_{\mathcal{O}_X}(K, L))$$

see Cohomology, Lemma 38.1 and Cohomology, Section 13. Thus the result follows from Lemmas 10.5 and 10.4.

\[\text{Lemma 10.7.} \quad \text{Let } X \text{ be a Noetherian regular scheme of finite dimension. Then every object of } D^b_{\text{Coh}}(\mathcal{O}_X) \text{ is perfect and conversely every perfect object of } D(\mathcal{O}_X) \text{ is in } D^b_{\text{Coh}}(\mathcal{O}_X).\]

\[\text{Proof.} \quad \text{Combine More on Algebra, Lemma 69.13 with Lemma 9.7.}\]

11. Descent finiteness properties of complexes

This section is the analogue of Descent, Section 7 for objects of the derived category of a scheme. The easiest such result is probably the following.

\[\text{Lemma 11.1.} \quad \text{Let } f : X \to Y \text{ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let } E \in D(\mathcal{O}_Y). \text{ Let } a, b \in \mathbb{Z}. \text{ Then } E \text{ has tor-amplitude in } [a, b] \text{ if and only if } Lf^* E \text{ has tor-amplitude in } [a, b].\]

\[\text{Proof.} \quad \text{Pullback always preserves tor-amplitude, see Cohomology, Lemma 44.4. We may check tor-amplitude in } [a, b] \text{ on stalks, see Cohomology, Lemma 44.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 38.17. Thus the result follows from More on Algebra, Lemma 63.17.}\]

\[\text{Lemma 11.2.} \quad \text{Let } \{f_i : X_i \to X\} \text{ be an fpqc covering of schemes. Let } E \in D_{\text{QCoh}}(\mathcal{O}_X). \text{ Let } m \in \mathbb{Z}. \text{ Then } E \text{ is } m\text{-pseudo-coherent if and only if each } Lf_i^* E \text{ is } m\text{-pseudo-coherent.}\]

\[\text{Proof.} \quad \text{Pullback always preserves } m\text{-pseudo-coherence, see Cohomology, Lemma 43.3. Conversely, assume that } Lf_i^* E \text{ is } m\text{-pseudo-coherent for all } i. \text{ Let } U \subset X \text{ be an affine open. It suffices to prove that } E|_U \text{ is } m\text{-pseudo-coherent. Since } \{f_i : X_i \to X\} \text{ is an fpqc covering, we can find finitely many affine open } V_j \subset X_{a(j)} \text{ such that } f_{a(j)}(V_j) \subset U \text{ and } U = \bigcup f_{a(j)}(V_j). \text{ Set } V = \prod V_i. \text{ Thus we may replace } X \text{ by } U \text{ and } \{f_i : X_i \to X\} \text{ by } \{V \to U\} \text{ and assume that } X \text{ is affine and our covering is given by a single surjective flat morphism } \{f : Y \to X\} \text{ of affine schemes. In this case the result follows from More on Algebra, Lemma 62.16 via Lemmas 3.5 and 9.2.}\]
Lemma 11.3. Let \( f_i : X_i \to X \) be an fpff covering of schemes. Let \( E \in D(\mathcal{O}_X) \). Let \( m \in \mathbb{Z} \). Then \( E \) is \( m \)-pseudo-coherent if and only if each \( Lf_i^*E \) is \( m \)-pseudo-coherent.

Proof. Pullback always preserves \( m \)-pseudo-coherence, see Cohomology, Lemma 43.3. Conversely, assume that \( Lf_i^*E \) is \( m \)-pseudo-coherent for all \( i \). Let \( U \subset X \) be an affine open. It suffices to prove that \( E|_U \) is \( m \)-pseudo-coherent. Since \( \{f_i : X_i \to X\} \) is an fpff covering, we can find finitely many affine open \( V_j \subset X_{a(j)} \) such that \( f_{a(j)}(V_j) \subset U \) and \( U = \bigcup f_{a(j)}(V_j) \). Set \( V = \bigsqcup V_i \). Thus we may replace \( X \) by \( U \) and \( \{f_i : X_i \to X\} \) by \( \{V \to U\} \) and assume that \( X \) is affine and our covering is given by a single surjective flat morphism \( f : Y \to X \) of finite presentation.

Since \( f \) is flat the derived functor \( Lf^* \) is just given by \( f^* \) and \( f^* \) is exact. Hence \( H^i(Lf^*E) = f^*H^i(E) \). Since \( Lf^*E \) is \( m \)-pseudo-coherent, we see that \( Lf^*E \in D^{-}(\mathcal{O}_Y) \). Since \( f \) is surjective and flat, we see that \( E \in D^{-}(\mathcal{O}_X) \). Let \( i \in \mathbb{Z} \) be the largest integer such that \( H^i(E) \) is nonzero. If \( i < m \), then we are done. Otherwise, \( f^*H^i(E) \) is a finite type \( \mathcal{O}_Y \)-module by Cohomology, Lemma 43.9. Then by Descent, Lemma 7.2 the \( \mathcal{O}_X \)-module \( H^i(E) \) is of finite type. Thus, after replacing \( X \) by the members of a finite affine open covering, we may assume there exists a map

\[ \alpha : \mathcal{O}_X^{-m}[i] \to E \]

such that \( H^i(\alpha) \) is a surjection. Let \( C \) be the cone of \( \alpha \) in \( D(\mathcal{O}_X) \). Pulling back to \( Y \) and using Cohomology, Lemma 43.4 we find that \( Lf^*C \) is \( m \)-pseudo-coherent. Moreover \( H^j(C) = 0 \) for \( j \geq i \). Thus by induction on \( i \) we see that \( C \) is \( m \)-pseudo-coherent. Using Cohomology, Lemma 43.4 again we conclude.

Lemma 11.4. Let \( \{f_i : X_i \to X\} \) be an fpqc covering of schemes. Let \( E \in D(\mathcal{O}_X) \). Then \( E \) is perfect if and only if each \( Lf_i^*E \) is perfect.

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 45.6. Conversely, assume that \( Lf_i^*E \) is perfect for all \( i \). Then the cohomology sheaves of each \( Lf_i^*E \) are quasi-coherent, see Lemma 9.1 and Cohomology, Lemma 45.5. Since the morphisms \( f_i \) are flat we see that \( H^p(Lf_i^*E) = f_i^*H^p(E) \). Thus the cohomology sheaves of \( E \) are quasi-coherent by Descent, Proposition 5.2. Having said this the lemma follows formally from Cohomology, Lemma 45.8 and Lemmas 11.1 and 11.2.

Lemma 11.5. Let \( i : Z \to X \) be a morphism of ringed spaces such that \( i \) is a closed immersion of underlying topological spaces and such that \( i_*\mathcal{O}_Z \) is \( m \)-pseudo-coherent as an \( \mathcal{O}_X \)-module. Let \( E \in D(\mathcal{O}_Z) \). Then \( E \) is \( m \)-pseudo-coherent if and only if \( Ri_*E \) is \( m \)-pseudo-coherent.

Proof. Throughout this proof we will use that \( i_* \) is an exact functor, and hence that \( Ri_* = i_* \), see Modules, Lemma 6.1.

Assume \( E \) is \( m \)-pseudo-coherent. Let \( x \in X \). We will find a neighbourhood of \( x \) such that \( i_*E \) is \( m \)-pseudo-coherent on it. If \( x \notin Z \) then this is clear. Thus we may assume \( x \in Z \). We will use that \( U \cap Z \) for \( x \in U \subset X \) open form a fundamental system of neighbourhoods of \( x \) in \( Z \). After shrinking \( X \) we may assume \( E \) is bounded above. We will argue by induction on the largest integer \( p \) such that \( H^p(E) \) is nonzero. If \( p < m \), then there is nothing to prove. If \( p \geq m \), then \( H^p(E) \) is an \( \mathcal{O}_Z \)-module of finite type, see Cohomology, Lemma 43.9. Thus
we may choose, after shrinking $X$, a map $\mathcal{O}_Z^\oplus[-p] \to E$ which induces a surjection $\mathcal{O}_Z^\oplus \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^\oplus[-p] \to E \to C \to \mathcal{O}_Z^\oplus[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that $C$ is $m$-pseudo-coherent by Cohomology, Lemma 43.4. By induction we see that $i_*C$ is $m$-pseudo-coherent on $X$. Since $i_*\mathcal{O}_Z$ is $m$-pseudo-coherent on $X$ as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^\oplus[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^\oplus[-p+1]$$

and Cohomology, Lemma 43.4 that $i_*E$ is $m$-pseudo-coherent.

Assume that $i_*E$ is $m$-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of $z$ such that $E$ is $m$-pseudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of $z$ in $Z$. After shrinking $X$ we may assume $i_*E$ and hence $E$ is bounded above. We will argue by induction on the largest integer $p$ such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 43.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing $E$.

We may choose, after shrinking $X$, a map $\alpha : \mathcal{O}_X^\oplus[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^\oplus \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^\oplus \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^\oplus \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^\oplus[-p] \to E \to C \to \mathcal{O}_Z^\oplus[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^\oplus[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^\oplus[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 43.4 we conclude that $i_*C$ is $m$-pseudo-coherent. By induction we conclude that $C$ is $m$-pseudo-coherent. By Cohomology, Lemma 43.4 again we conclude that $E$ is $m$-pseudo-coherent. \hfill \Box

**Lemma 11.6.** Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module.\footnote{This means that $f$ is pseudo-coherent, see More on Morphisms, Lemma 52.8} Let $E \in D_{\text{QCoh}}(\mathcal{O}_X)$. Then $E$ is $m$-pseudo-coherent if and only if $Rf_*E$ is $m$-pseudo-coherent.

**Proof.** This is a translation of More on Algebra, Lemma 62.12 into the language of schemes. To do the translation, use Lemmas 3.5 and 9.2. \hfill \Box

### 12. Lifting complexes

Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as $Rj_*$ is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

**Lemma 12.1.** Let $X$ be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors

$$D_{\text{QCoh}}(\mathcal{O}_X) \to D_{\text{QCoh}}(\mathcal{O}_U) \text{ and } D^+_{\text{QCoh}}(\mathcal{O}_X) \to D^+_{\text{QCoh}}(\mathcal{O}_U)$$

are essentially surjective. If $X$ is quasi-compact, then the functors

$$D^-_{\text{QCoh}}(\mathcal{O}_X) \to D^-_{\text{QCoh}}(\mathcal{O}_U) \text{ and } D^b_{\text{QCoh}}(\mathcal{O}_X) \to D^b_{\text{QCoh}}(\mathcal{O}_U)$$

are essentially surjective.
are essentially surjective.

Proof. The argument preceding the lemma applies for the first case because $Rj_*\alpha$ maps $D_{QCoh}(O_U)$ into $D_{QCoh}(O_X)$ by Lemma 9.1. It is clear that $Rj_*$ maps $D^+_{QCoh}(O_U)$ into $D^+_{QCoh}(O_X)$ which implies the statement on bounded below complexes. Finally, Lemma 4.1 guarantees that $Rj_*$ maps $D^-_{QCoh}(O_U)$ into $D^-_{QCoh}(O_X)$ if $X$ is quasi-compact. Combining these two we obtain the last statement. □

**Lemma 12.2.** Let $X$ be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object $E$ of $D(O_U)$ there exists a bounded above complex of finite free $O_X$-modules whose restriction to $U$ is isomorphic to $E$.

Proof. By Lemma 9.1 we see that $E$ is an object of $D_{QCoh}(O_U)$. By Lemma 12.1 we may assume $E = E'[U]$ for some object $E'$ of $D_{QCoh}(O_X)$. Write $X = \text{Spec}(A)$. By Lemma 3.5 we can find a complex $M^\bullet$ of $A$-modules whose associated complex of $O_X$-modules is a representative of $E'$.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 9.2 the complexes $M^\bullet_j$ are pseudo-coherent complexes of $A_{f_j}$-modules. Let $n$ be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each $F^i$ is a finite free $R$-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

\[
\begin{array}{cccccccc}
F^n & \longrightarrow & F^{n+1} & \longrightarrow & \cdots \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \cdots
\end{array}
\]

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on $n$ we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all $i$. The lemma will follow by taking $\tilde{F}^\bullet$.

The induction step will be to extend the diagram above by adding $F^{n-1}$. Let $C^\bullet$ be the cone on $\alpha$ (Derived Categories, Definition 9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 62.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 62.3 we see that $H^{-1}(C^\bullet_{f_j})$ is a finite $A_{f_j}$-module. Choose a finite free $A$-module $F^{n-1}$ and an $A$-module $\beta : F^{n-1} \to C^{-1}$ such that the composition $F^{n-1} \to C^{-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

\[
\begin{array}{cccccccc}
F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \cdots \\
\downarrow{\alpha^{n-1}} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
\cdots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \cdots
\end{array}
\]
Moreover, these maps define a morphism of distinguished triangles

\[
(F^n \to \ldots) \xrightarrow{\beta} (F^{n-1} \to \ldots) \xrightarrow{\beta} (F^n \to \ldots)[1]
\]

Hence our choice of \( \beta \) implies that the map of complexes \((F^{-1} \to \ldots) \to M^*\) induces an isomorphism on cohomology localized at \( f_j \) in degrees \( \geq n \) and a surjection in degree \(-1\). This finishes the proof of the lemma.

**Lemma 12.3.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( E \in D^b_{QCoh}(\mathcal{O}_X) \). There exists an integer \( n_0 > 0 \) such that \( \text{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0 \) for every finite locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) and every \( n \geq n_0 \).

**Proof.** Recall that \( \text{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) \). We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 33.3. Thus if \( X = U \cup V \) and the result of the lemma holds for \( E|_U, E|_V \), and \( E|_{U \cap V} \) for some bound \( n_0 \), then the result holds for \( E \) with bound \( n_0 + 1 \). Thus it suffices to prove the lemma when \( X \) is affine, see Cohomology of Schemes, Lemma 4.1.

Assume \( X = \text{Spec}(A) \) is affine. Choose a complex of \( A \)-modules \( M^* \) whose associated complex of quasi-coherent modules represents \( E \), see Lemma 3.5. Write \( \mathcal{E} = \mathcal{P} \) for some \( A \)-module \( P \). Since \( \mathcal{E} \) is finite locally free, we see that \( \mathcal{P} \) is a finite projective \( A \)-module. We have

\[
\text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) = \text{Hom}_{D(A)}(P, M^*[n])
\]

The first equality by Lemma 3.5, the second equality by Derived Categories, Lemma 19.8 and the final equality because \( \text{Hom}_A(P, -) \) is an exact functor. As \( E \) and hence \( M^* \) is bounded we get zero for all sufficiently large \( n \).

**Lemma 12.4.** Let \( X \) be an affine scheme. Let \( U \subset X \) be a quasi-compact open. For every perfect object \( E \) of \( D(\mathcal{O}_U) \) there exists an integer \( r \) and a finite locally free sheaf \( \mathcal{F} \) on \( U \) such that \( \mathcal{F}[-r] \oplus E \) is the restriction of a perfect object of \( D(\mathcal{O}_X) \).

**Proof.** Say \( X = \text{Spec}(A) \). Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 45.5. By Lemma 12.2 we can find a bounded above complex \( \mathcal{F}^* \) of finite free \( A \)-modules such that \( E \) is isomorphic to \( \mathcal{F}^*|_U \) in \( D(\mathcal{O}_U) \). By Cohomology, Lemma 45.5 and since \( U \) is quasi-compact, we see that \( E \) has finite tor dimension, say \( E \) has tor amplitude in \([a, b]\). Pick \( r < a \) and set

\[
\mathcal{F} = \text{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \text{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).
\]

Since \( E \) has tor amplitude in \([a, b]\) we see that \( \mathcal{F}|_U \) is flat (Cohomology, Lemma 44.2). Hence \( \mathcal{F}|_U \) is flat and of finite presentation, thus finite locally free (Properties, Lemma 20.2). It follows that

\[
\mathcal{F}|_U[-r - 1] \to E \to (\mathcal{F} \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]
\]

is a strictly perfect complex on \( U \) representing \( E \). We obtain a distinguished triangle

\[
\mathcal{F}|_U[-r - 1] \to E \to (\mathcal{F} \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]
\]
Note that \((\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)\) is a perfect complex on \(X\). To finish the proof it suffices to pick \(r\) such that the map \(\mathcal{F}|_U[-r-1] \to E|_U\) is zero in \(D(O_U)\), see Derived Categories, Lemma 14.10. By Lemma 12.3 this holds if \(r \ll 0\). \(\square\)

**Lemma 12.5.** Let \(X\) be an affine scheme. Let \(U \subset X\) be a quasi-compact open. Let \(E, E'\) be objects of \(D_{QCoh}(O_X)\) with \(E\) perfect. For every map \(\alpha : E|_U \to E'|_U\) there exist maps

\[
E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'
\]

of perfect complexes on \(X\) such that \(\beta : E_1 \to E\) restricts to an isomorphism on \(U\) and such that \(\alpha = \gamma|_U \circ \beta|_U^{-1}\). Moreover we can assume \(E_1 = E \otimes_{O_X} I\) for some perfect complex \(I\) on \(X\).

**Proof.** Write \(X = \text{Spec}(A)\). Write \(U = D(f_1) \cup \ldots \cup D(f_r)\). Choose finite complex of finite projective \(A\)-modules \(M^*\) representing \(E\) (Lemma 9.7). Choose a complex of \(A\)-modules \((M')^*\) representing \(E'\) (Lemma 3.5). In this case the complex \(H^* = \text{Hom}_A(M^*, (M')^*)\) is a complex of \(A\)-modules whose associated complex of quasi-coherent \(O_X\)-modules represents \(R\text{Hom}(E, E')\), see Cohomology, Lemma 42.9. Then \(\alpha\) determines an element \(s\) of \(H^0(U, R\text{Hom}(E, E'))\), see Cohomology, Lemma 38.1. There exists an \(e\) and a map

\[
\xi : I^*(f_1^e, \ldots, f_r^e) \to \text{Hom}_A(M^*, (M')^*)
\]

corresponding to \(s\), see Proposition 8.5. Letting \(E_1\) be the object corresponding to complex of quasi-coherent \(O_X\)-modules associated to

\[
\text{Tot}(I^*(f_1^e, \ldots, f_r^e) \otimes_A M^*)
\]

we obtain \(E_1 \to E\) using the canonical map \(I^*(f_1^e, \ldots, f_r^e) \to A\) and \(E_1 \to E'\) using \(\xi\) and Cohomology, Lemma 38.1. \(\square\)

**Lemma 12.6.** Let \(X\) be an affine scheme. Let \(U \subset X\) be a quasi-compact open. For every perfect object \(F\) of \(D(O_U)\) the object \(F \oplus F[1]\) is the restriction of a perfect object of \(D(O_X)\).

**Proof.** By Lemma 12.4 we can find a perfect object \(E\) of \(D(O_X)\) such that \(E|_U = \mathcal{F}[r] \oplus F\) for some finite locally free \(O_U\)-module \(\mathcal{F}\). By Lemma 12.5 we can find a morphism of perfect complexes \(\alpha : E_1 \to E\) such that \((E_1)|_U \cong E|_U\) and such that \(\alpha|_U\) is the map

\[
\begin{pmatrix}
\text{id}_{\mathcal{F}[r]} & 0 \\
0 & 0
\end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F
\]

Then the cone on \(\alpha\) is a solution. \(\square\)

**Lemma 12.7.** Let \(X\) be a quasi-compact and quasi-separated scheme. Let \(f \in \Gamma(X, O_X)\). For any morphism \(\alpha : E \to E'\) in \(D_{QCoh}(O_X)\) such that

1. \(E\) is perfect, and
2. \(E'\) is supported on \(T = V(f)\)

there exists an \(n \geq 0\) such that \(f^n\alpha = 0\).

**Proof.** We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 33.3. Thus if \(X = U \cup V\) and the result of the lemma holds for \(f|_{U}\), \(f|_{V}\), and \(f|_{U \cap V}\), then the result holds for \(f\). Thus it suffices to prove the lemma when \(X\) is affine, see Cohomology of Schemes, Lemma 4.1.
Let $X = \text{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{Q\text{Coh}}(X)$ of Lemma 3.5 without further mention. Represent $E$ by a finite complex of finite projective $A$-modules $P^\bullet$. This is possible by Lemma 9.3. Let $t$ be the largest integer such that $P^t$ is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1}P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective $A$-module $P$ in degree $0$. Represent $E'$ by a complex $M^\bullet$ of $A$-modules. Then $\alpha$ corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of $f$. Since $P$ is a finite $A$-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some $n$ as desired. 

08EK **Lemma 12.8.** Let $X$ be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object $F$ of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object $E$ of $D(\mathcal{O}_X)$ supported in $T$.

**Proof.** Say $T = V(g_1, \ldots, g_s)$. After replacing $g_j$ by a power we may assume multiplication by $g_j$ is zero on $F$, see Lemma 12.7. Choose $E$ as in Lemma 12.6. 

Note that $g_j : E \to E$ restricts to zero on $U$. Choose a distinguished triangle

$$E \overset{g_j}{\rightarrow} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 4.10 the object $C_1$ restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on $U$. Moreover, $g_j : C_1 \to C_1$ has square zero by Derived Categories, Lemma 4.5. 

Namely, the diagram

$$
\begin{array}{ccc}
E & \longrightarrow & C_1 \\
\downarrow & & \downarrow \\
0 & \overset{g_j}{\rightarrow} & 0 \\
\downarrow & & \downarrow \\
E & \longrightarrow & C_1 \\
& & \downarrow \\
& & E[1]
\end{array}
$$

is commutative since the compositions $E \overset{g_j}{\rightarrow} E \to C_1$ and $C_1 \to E[1] \overset{g_j}{\rightarrow} E[1]$ are zero. Continuing, setting $C_{i+1}$ equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex $C_s$ on $X$ supported on $T$ whose restriction to $U$ gives

$$F \oplus F[1] \oplus F[2] \oplus \ldots \oplus F[s]$$

Choose morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 12.5 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1] \oplus F[2] \oplus \ldots \oplus F[s] \to F \oplus F[1] \oplus F[2] \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for $F$ where it is zero. By Lemma 12.5 we also have $C' = C_s \otimes^L I$ for some perfect complex $I$ on $X$. Hence the nullity of $g^j_1|_{\text{id}_{C_s}}$ implies the same thing for $C'$. Thus $C'$ is supported on $T$ as well. Then $\text{Cone}(\gamma)$ is a solution.

A special case of the following lemma can be found in [Nee96].

09IM **Lemma 12.9.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in $X$. Let $E$ be an object of $D_{Q\text{Coh}}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where
$P$ is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where $R$ is a perfect object of $D(\mathcal{O}_X)$ supported on $T$ such that $P$ is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible $\alpha$ and $\beta|_U$.

**Proof.** Since $X$ is quasi-compact there exists an integer $m$ such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens $V_j$ of $X$. Arguing by induction on $m$ we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with $V$ affine. By Lemma [12.8] we can choose a perfect object $Q$ in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma [12.5] we can replace $Q$ by $Q \oplus I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \to P|_{U \cap V} \to E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma [41.1] we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_V$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus $R$ is perfect and supported on $T$ as desired. \qed

**Remark 12.10.** The proof of Lemma [12.9] shows that

$$R|_U = P \oplus P^{|_{\leq n_1}|}[1] \oplus \ldots \oplus P^{|_{\leq n_m}|}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of $P$. By repeating the construction for the map $P^{|_{\leq n_1}|}[1] \oplus \ldots \oplus P^{|_{\leq n_m}|}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and $P$ above any given degree.

### 13. Approximation by perfect complexes

**Definition 13.1.** Let $X$ be a scheme. Consider triples $(T, E, m)$ where

1. $T \subset X$ is a closed subset,
2. $E$ is an object of $D_{QCoh}(\mathcal{O}_X)$, and
3. $m \in \mathbb{Z}$.

We say *approximation holds for the triple $(T, E, m)$* if there exists a perfect object $P$ of $D(\mathcal{O}_X)$ supported on $T$ and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple $(T, E, m)$, then

1. $E$ is $m$-pseudo-coherent, see Cohomology, Definition [43.1] and
2. the cohomology sheaves $H^i(E)$ are supported on $T$ for $i \geq m$.

Moreover, the “support” of a perfect complex is a closed subscheme whose complement is retrocompact in $X$ (details omitted). Hence we cannot expect approximation to hold without this assumption on $T$. This partly explains the conditions in the following definition.

**Definition 13.2.** Let $X$ be a scheme. We say *approximation by perfect complexes holds on $X$* if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in $X$ there exists an integer $r$ such that for every triple $(T, E, m)$ as in Definition [13.1] with

1. $E$ is $(m - r)$-pseudo-coherent, and
2. $H^i(E)$ is supported on $T$ for $i \geq m - r$
Let $\text{triple } (T, E, m)$ be a triple as in Definition [13.1] If

1. $T \subset U$,
2. approximation holds for $(T, E|_U, m)$, and
3. the sheaves $H^i(E)$ for $i \geq m$ are supported on $T$,

then approximation holds for $(T, E, m)$.

**Proof.** Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over $U$, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of $(T, E, m)$ over $X$. See Cohomology, Lemmas [43.6] and [45.10] □

**Lemma 13.4.** Let $X$ be a scheme. Let $U \subset X$ be an open subscheme. Let $(T, E, m)$ be a triple as in Definition [13.1] If

1. $T \subset U$,
2. approximation holds for $(T, E|_U, m)$, and
3. the sheaves $H^i(E)$ for $i \geq m$ are supported on $T$,

then approximation holds for $(T, E, m)$.

**Proof.** Say $X = \text{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma [9.2] and the definitions.) Let $(T, E, m)$ be a triple as in the lemma. Let $t$ be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on $t$. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma [43.9] the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma [16.1]). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma [3.3]) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that $s$ is in the image of $H^t(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma [8.6]. Taking a finite direct sum of these maps we obtain a map $P \to E$ where $P$ is a perfect complex supported on $T$, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then $E'$ is $m$-pseudo-coherent (Cohomology, Lemma [43.4]). $H^i(E') = 0$ for $i \geq t$, and $H^t(E')$ is supported on $T$ for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of $(T, E', m)$. Fit the composition $P' \to E' \to P[1]$ into a distinguished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$ and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$
\begin{array}{c}
P \to P'' \to P' \to P[1] \\
\downarrow \downarrow \downarrow \downarrow \\
P \to E \to E' \to P[1]
\end{array}
$$

using TR3. Then $P''$ is a perfect complex (Cohomology, Lemma [45.7]) supported on $T$. An easy diagram chase shows that $P'' \to E$ is the desired approximation. □
Lemma 13.5. Let $X$ be a scheme. Let $X = U \cup V$ be an open covering with $U$ quasi-compact, $V$ affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on $U$, then approximation holds on $X$.

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in $X$. Let $r_U$ be the integer of Definition 13.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on $T$. Let $r'$ be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair $(X, T)$.

To see this choose a triple $(T, E, m)$ such that $E$ is $(m-r)$-pseudo-coherent and $H^i(E)$ is supported on $T$ for $i \geq m - r$. Let $t$ be the largest integer such that $H^i(E)|_T$ is nonzero. (Such an integer exists as $U$ is quasi-compact and $E|_V$ is $(m - r)$-pseudo-coherent.) We will prove that $E$ can be approximated by induction on $t$.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on $T'$ for $i \geq m - r'$. Hence Lemma 13.4 guarantees the existence of an approximation $P \to E|_T$ of $(T', E|_V, m)$ on $V$. Applying Lemma 13.3 we see that $(T', E, m)$ can be approximated. Such an approximation is also an approximation of $(T, E, m)$.

Induction step. Choose an approximation $P \to E|_T$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^i(P) \to H^i(E|_V)$. By Lemma 12.8 we can choose a perfect object $Q$ in $D(C_X)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 12.5 we can replace $Q$ by $Q \otimes^L I$ and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \to P|_{U \cap V} \to E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 41.1 we find a morphism $a : R \to E$ of $D(C_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus $R$ is perfect and supported on $T$ and the morph $H^i(R) \to H^i(E)$ is surjective on restriction to $U$. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then $E'$ is $(m-r)$-pseudo-coherent (Cohomology, Lemma 43.4), $H^i(E')|_T = 0$ for $i \geq t$, and $H^i(E')$ is supported on $T$ for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of $(T, E', m)$. Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$R \to R'' \to R' \to R[1]$$

using TR3. Then $R''$ is a perfect complex (Cohomology, Lemma 45.7) supported on $T$. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

Theorem 13.6. Let $X$ be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on $X$.

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 41 and Lemmas 13.5 and 13.4. □
14. Generating derived categories

09IP In this section we prove that the derived category \( D\text{-}QCoh(O_X) \) of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [BV03].

09IQ Lemma 14.1. Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( U \) be a quasi-compact open subscheme. Let \( P \) be a perfect object of \( D(O_U) \). Then \( P \) is a direct summand of a perfect object of \( D(O_X) \).

Proof. Special case of Lemma 12.9 □

09IR Lemma 14.2. In Situation 8.1 denote \( j : U \to X \) the open immersion and let \( K \) be the perfect object of \( D(O_X) \) corresponding to the Koszul complex on \( f_1, \ldots, f_r \) over \( A \). For \( E \in D_{QCoh}(O_X) \) the following are equivalent

1. \( E = Rj_*(E|_U) \), and
2. \( \text{Hom}_{D(O_X)}(K[n], E) = 0 \) for all \( n \in \mathbb{Z} \).

Proof. Choose a distinguished triangle \( E \to Rj_*(E|_U) \to N \to E[1] \). Observe that

\[ \text{Hom}_{D(O_X)}(K[n], Rj_*(E|_U)) = \text{Hom}_{D(O_U)}(K|_U[n], E) = 0 \]

for all \( n \) as \( K|_U = 0 \). Thus it suffices to prove the result for \( N \). In other words, we may assume that \( E \) restricts to zero on \( U \). Observe that there are distinguished triangles

\[ K^*(f_1^i, \ldots, f_r^i) \to K^*(f_1^{i_1}, \ldots, f_r^{i_r}) \to K^*(f_1^{i_1}, \ldots, f_r^{i_r}) \to \ldots \]

of Koszul complexes, see More on Algebra, Lemma 28.1. Hence if \( \text{Hom}_{D(O_X)}(K[n], E) = 0 \) for all \( n \in \mathbb{Z} \) then the same thing is true for the \( K \) replaced by \( K_e \) as in Lemma 8.6. Thus our lemma follows immediately from that one and the fact that \( E \) is determined by the complex of \( A \)-modules \( R^\ell(X, E) \), see Lemma 3.5 □

09IS Theorem 14.3. Let \( X \) be a quasi-compact and quasi-separated scheme. The category \( D_{QCoh}(O_X) \) can be generated by a single perfect object. More precisely, there exists a perfect object \( P \) of \( D(O_X) \) such that for \( E \in D_{QCoh}(O_X) \) the following are equivalent

1. \( E = 0 \), and
2. \( \text{Hom}_{D(O_X)}(P[n], E) = 0 \) for all \( n \in \mathbb{Z} \).

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1

If \( X \) is affine, then \( O_X \) is a perfect generator. This follows from Lemma 3.5

Assume that \( X = U \cup V \) is an open covering with \( U \) quasi-compact such that the theorem holds for \( U \) and \( V \) is an affine open. Let \( P \) be a perfect object of \( D(O_U) \) which is a generator for \( D_{QCoh}(O_U) \). Using Lemma 14.1 we may choose a perfect object \( Q \) of \( D(O_X) \) whose restriction to \( U \) is a direct sum one of whose summands is \( P \). Say \( V = \text{Spec}(A) \). Let \( Z = X \setminus U \). This is a closed subset of \( V \) with \( V \setminus Z \) quasi-compact. Choose \( f_1, \ldots, f_r \in A \) such that \( Z = V(f_1, \ldots, f_r) \). Let \( K \in D(O_V) \) be the perfect object corresponding to the Koszul complex on \( f_1, \ldots, f_r \) over \( A \). Note that since \( K \) is supported on \( Z \subset V \) closed, the pushforward \( K' = R(V \to X)_*K \)
is a perfect object of $D(O_X)$ whose restriction to $V$ is $K$ (see Cohomology, Lemma 45.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(O_X)$. Let $E$ be an object of $D_{QCoh}(O_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into $E$. By Cohomology, Lemma 33.6 we have $K' = R(V \to X)_*K$ and hence

$$\text{Hom}_{D(O_X)}(K'[n], E) = \text{Hom}_{D(O_V)}(K[n], E|_V)$$

Thus by Lemma 14.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_*E|_{U \cap V}$. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

$$\text{Hom}_{D(O_X)}(Q[n], E) = \text{Hom}_{D(O_U)}(Q|_U[n], E|_U)$$

which contains $\text{Hom}_{D(O_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of $P$ the vanishing of these groups implies that $E|_U$ is zero. Hence $E$ is zero. □

The following result is an strengthening of Theorem 14.3 proved using exactly the same methods. Let $T \subset X$ be a closed subset of a scheme $X$. Let’s denote $D_T(O_X)$ the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on $T$.

**Lemma 14.4.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(O_X)$ is generated by a single perfect object.

**Proof.** We will prove this using the induction principle of Cohomology of Schemes, Lemma 4.1.

Assume $X = \text{Spec}(A)$ is affine. In this case there exist $f_1, \ldots, f_r \in A$ such that $T = V(f_1, \ldots, f_r)$. Let $K$ be the Koszul complex on $f_1, \ldots, f_r$ as in Lemma 14.2. Then $K$ is a perfect object with cohomology supported on $T$ and hence a perfect object of $D_{QCoh,T}(O_X)$. On the other hand, if $E \in D_{QCoh,T}(O_X)$ and $\text{Hom}(K, E[n]) = 0$ for all $n$, then Lemma 14.2 tells us that $E = Rj_*(E|_{X \setminus T}) = 0$. Hence $K$ generates $D_{QCoh,T}(O_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 34.6).

Assume that $X = U \cup V$ is an open covering with $V$ affine and $U$ quasi-compact such that the lemma holds for $U$. Let $P$ be a perfect object of $D(O_V)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T \cap U}(O_U)$. Using Lemma 12.9 we may choose a perfect object $Q$ of $D(O_X)$ supported on $T$ whose restriction to $U$ is a direct sum of whose summands is $P$. Write $V = \text{Spec}(B)$. Let $Z = X \setminus U$. Then $Z$ is a closed subset of $V$ such that $V \setminus Z$ is quasi-compact. As $X$ is quasi-separated, it follows that $Z \cap T$ is a closed subset of $V$ such that $V = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $Z \cap T = V(g_1, \ldots, g_s)$. Let $K \in D(O_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over $B$. Note that since $K$ is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_*K$ is a perfect object of $D(O_X)$ whose restriction to $V$ is $K$ (see Cohomology, Lemma 45.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(O_X)$.

Let $E$ be an object of $D_{QCoh,T}(O_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into $E$. By Cohomology, Lemma 33.6 we have $K' = R(V \to X)_*K$ and hence

$$\text{Hom}_{D(O_X)}(K'[n], E) = \text{Hom}_{D(O_V)}(K[n], E|_V)$$
Thus by Lemma 14.2 we have $E|_V = Rj_*E|_W$ where $j : W \to V$ is the inclusion.

Picture

\[
\begin{array}{ccc}
W & \xrightarrow{j} & V \\
j' & \quad & j'' \\
U \cap V & \quad & Z \cap T \\
\end{array}
\]

Since $E$ is supported on $T$ we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in $W$. We conclude that

\[
E|_V = Rj_* (E|_W) = Rj_* (Rj'_*(E|_{U \cap V})) = Rj''_*(E|_{U \cap V})
\]

where the second equality is part (1) of Cohomology, Lemma 33.6. This implies that $E = R(U \to X)_*E|_U$ (small detail omitted). If this is the case then

\[
\text{Hom}_{D(O_X)}(Q[n], E) = \text{Hom}_{D(O_U)}(Q|_U[n], E|_U)
\]

which contains $\text{Hom}_{D(O_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of $P$ the vanishing of these groups implies that $E|_U$ is zero. Whence $E$ is zero. □

15. An example generator

0BQQ In this section we prove that the derived category of projective space over a ring is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf.

The following lemma says that $\bigoplus_{n \geq 0} \mathcal{L}^\otimes -n$ is a generator if $\mathcal{L}$ is ample.

0BQR **Lemma 15.1.** Let $X$ be a scheme and $\mathcal{L}$ an ample invertible $O_X$-module. If $K$ is a nonzero object of $D_{QCoh}(O_X)$, then for some $n \geq 0$ and $p \in \mathbb{Z}$ the cohomology group $H^p(X, K \otimes_{O_X} \mathcal{L}^\otimes n)$ is nonzero.

**Proof.** Recall that as $X$ has an ample invertible sheaf, it is quasi-compact and separated (Properties, Definition 26.1 and Lemma 26.7). Thus we may apply Proposition 6.6 and represent $K$ by a complex $F^\bullet$ of quasi-coherent modules. Pick any $p$ such that $H^p = \ker(F^p \to F^{p+1})/\text{im}(F^{p-1} \to F^p)$ is nonzero. Choose a point $x \in X$ such that the stalk $\mathcal{H}^x_p$ is nonzero. Choose an $n \geq 0$ and $s \in \Gamma(X, \mathcal{L}^\otimes n)$ such that $X_s$ is an affine open neighbourhood of $x$. Choose $\tau \in \mathcal{H}^p(X_s)$ which maps to a nonzero element of the stalk $\mathcal{H}^x_p$; this is possible as $\mathcal{H}^p$ is quasi-coherent and $X_s$ is affine. Since taking sections over $X_s$ is an exact functor on quasi-coherent modules, we can find a section $\tau' \in \mathcal{F}^p(X_s)$ mapping to zero in $\mathcal{F}^{p+1}(X_s)$ and mapping to $\tau$ in $\mathcal{H}^p(X_s)$. By Properties, Lemma 17.2 there exists an $m$ such that $\tau' \otimes s^\otimes m$ is the image of a section $\tau'' \in \Gamma(X, F^p \otimes \mathcal{L}^\otimes m)$.

Applying the same lemma once more, we find $l \geq 0$ such that $\tau'' \otimes s^\otimes l$ maps to zero in $\mathcal{F}^{p+1} \otimes \mathcal{L}^\otimes (m+l)$. Then $\tau''$ gives a nonzero class in $H^p(X, K \otimes_{O_X} \mathcal{L}^\otimes (n+l))$ as desired. □

0BQS **Lemma 15.2.** Let $A$ be a ring. Let $X = \mathbb{P}^n_A$. For every $a \in \mathbb{Z}$ there exists an exact complex

\[
0 \to O_X(a) \to \ldots \to O_X(a + i)^\oplus (i+1) \to \ldots \to O_X(a + n + 1) \to 0
\]

of vector bundles on $X$.

**Proof.** Recall that $\mathbb{P}^n_A$ is $\text{Proj}(A[X_0, \ldots, X_n])$, see Constructions, Definition 13.2.

Consider the Koszul complex

\[
K_\bullet = K_\bullet(A[X_0, \ldots, X_n], X_0, \ldots, X_n)
\]
over $S = A[X_0, \ldots, X_n]$ on $X_0, \ldots, X_n$. Since $X_0, \ldots, X_n$ is clearly a regular sequence in the polynomial ring $S$, we see that (More on Algebra, Lemma 29.2) that the Koszul complex $K_\bullet$ is exact, except in degree 0 where the cohomology is $S/(X_0, \ldots, X_n)$. Note that $K_\bullet$ becomes a complex of graded modules if we put the generators of $K_i$ in degree $+i$. In other words an exact complex

$$0 \to S(-n - 1) \to \cdots \to S(-n - 1 + i) \oplus (\tau_i) \to \cdots \to S \to S/(X_0, \ldots, X_n) \to 0$$

Applying the exact functor $\tau$ functor of Constructions, Lemma 8.4 and using that the last term is in the kernel of this functor, we obtain the exact complex

$$0 \to \mathcal{O}_X(-n - 1) \to \cdots \to \mathcal{O}_X(-n - 1 + i) \oplus (\tau_i) \to \cdots \to \mathcal{O}_X \to 0$$

Twisting by the invertible sheaves $\mathcal{O}_X(n + a)$ we get the exact complexes of the lemma.

\[\square\]

**Lemma 15.3.** Let $A$ be a ring. Let $X = \mathbb{P}_A^n$. Then

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n)$$

is a generator (Derived Categories, Definition 34.2) of $D_{QCoh}(X)$.

**Proof.** Let $K \in D_{QCoh}(\mathcal{O}_X)$. Assume $\text{Hom}(E, K[p]) = 0$ for all $p \in \mathbb{Z}$. We have to show that $K = 0$. By Derived Categories, Lemma 34.3 we see that $\text{Hom}(E', K[p])$ is zero for all $E' \in \langle E \rangle$ and $p \in \mathbb{Z}$. By Lemma 15.2 applied with $a = -n - 1$ we see that $\mathcal{O}_X(-n - 1) \in \langle E \rangle$ because it is quasi-isomorphic to a finite complex whose terms are finite direct sums of summands of $E$. Repeating the argument with $a = -n - 2$ we see that $\mathcal{O}_X(-n - 2) \in \langle E \rangle$. Arguing by induction we find that $\mathcal{O}_X(-m) \in \langle E \rangle$ for all $m \geq 0$. Since

$$\text{Hom}(\mathcal{O}_X(-m), K[p]) = H^p(X, K \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)) = H^p(X, K \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m})$$

we conclude that $K = 0$ by Lemma 15.1 (This also uses that $\mathcal{O}_X(1)$ is an ample invertible sheaf on $X$ which follows from Properties, Lemma 26.12) \[\square\]

**Remark 15.4.** Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 14.3). Then the following are equivalent

1. for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
2. $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
3. $Lf^*E$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that $Lf^*$ is the left adjoint to $Rf_*$. These conditions hold if $f$ is affine (Lemma 5.1) or if $f$ is an open immersion, or if $f$ is a composition of such. We conclude that

1. if $X$ is a quasi-affine scheme then $\mathcal{O}_X$ is a generator for $D_{QCoh}(\mathcal{O}_X)$,
2. if $X \subset \mathbb{P}_A^n$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n)$ is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 15.3.
16. Compact and perfect objects

09M0 Let $X$ be a Noetherian scheme of finite dimension. By Cohomology, Proposition 20.7 and Cohomology on Sites, Lemma 47.3 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect objects of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

09M1 **Proposition 16.1.** Let $X$ be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.

**Proof.** If $K$ is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology, Lemma 45.11) we have

$$\text{Hom}_{D(\mathcal{O}_X)}(K,M) = H^0(X,K^\vee \otimes^L \mathcal{O}_X M)$$

 functorially in $M$. Since $K^\vee \otimes^L \mathcal{O}_X \to M$ commutes with direct sums and since $H^0(X,-)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 45.2 we conclude that $K$ is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let $K$ be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that $K$ is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 45.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 45.2. Thus the adjointness of restriction to $U$ and $Rj_*$ implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that $X$ is affine.

Assume $X = \text{Spec}(A)$ is affine. By Lemma 35.6 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 72.3.

The following result is a strengthening of Proposition 16.1. Let $T \subset X$ be a closed subset of a scheme $X$. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory consisting of complexes whose cohomology sheaves are supported on $T$. Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0A9B **Lemma 16.2.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.

**Proof.** We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 16.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 14.4. Hence by the above, $E$ is compact. Then it follows from Derived Categories, Proposition 35.6 that $E$ is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of $E$ by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows.
The following lemma is an application of the ideas that go into the proof of the preceding lemma.

**Lemma 16.3.** Let $X$ be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either

1. $P$ is perfect and $E$ supported on $T$, or
2. $P$ pseudo-coherent, $E$ supported on $T$, and $E$ bounded below.

Then there exists a perfect complex of $\mathcal{O}_X$-modules $I$ and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^L P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

**Proof.** Set $D = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R \text{Hom}(P, E)$ is an object of $D$. See Lemma $9.8$ for quasi-coherence. It is clear that $K$ is supported on $T$ as formation of $R \text{Hom}$ commutes with restriction to opens. The map $\alpha$ defines an element of $H^0(K) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in D$ be a perfect generator, see Lemma $14.4$. Write $K = \text{hocolim} K_n$ as in Derived Categories, Lemma $35.3$ using the generator $E$. Since the functor $D \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an $n$ and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to $\alpha$, see Derived Categories, Lemma $32.9$. By Derived Categories, Lemma $35.4$ applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that $\alpha_n$ factors as $\mathcal{O}_X[0] \to Q \to K_n$ where $Q$ is an object of $\langle E \rangle$. We conclude that $Q$ is a perfect complex supported on $T$.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction $I$ is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over $U$, and the composition $I \to K$ is zero as $\alpha$ factors through $Q$. This proves the lemma. \hfill \Box

### 17. Derived categories as module categories

In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

**Lemma 17.1.** Let $X$ be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $\left( E, d \right) = \text{Hom}_{\mathcal{O}_X}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor

$$- \otimes^L_{E} K^\bullet : D(E, d) \to D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma $26.3$ has image contained in $D_{QCoh}(\mathcal{O}_X)$.

**Proof.** Let $P$ be a differential graded $E$-module with property (P) and let $F_i^\bullet$ be a filtration on $P$ as in Differential Graded Algebra, Section $13$. Then we have

$$P \otimes^L_{E} K^\bullet = \text{hocolim} F_i P \otimes^L_{E} K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. \hfill \Box
Let $0 < M < 0 < \text{compact object of } D(O_X)$. Then

1. There exist integers $a \leq b$ such that $\text{Hom}_{D(O_X)}(K, L) = 0$ for $L \in D_{QCoh}(O_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and
2. If $L$ is bounded, then $\text{Ext}_{D(O_X)}^n(K, L)$ is zero for all but finitely many $n$.

**Proof.** Part (2) follows from (1) as $\text{Ext}_{D(O_X)}^n(K, L) = \text{Hom}_{D(O_X)}(K, L[n])$. We prove (1). Since $K$ is perfect we have

$$\text{Hom}_{D(O_X)}(K, L) = H^0(X, K \otimes \Omega^L_{O_X} L)$$

where $K^\vee$ is the “dual” perfect complex to $K$, see Cohomology, Lemma 45.11. Note that $K^\vee \otimes \Omega^L_{O_X} L$ is in $D_{QCoh}(X)$ by Lemmas 3.9 and 9.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes \Omega^L_{O_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes \Omega^L_{O_X} L)$$

shows that $H^j(K^\vee \otimes \Omega^L_{O_X} L)$ is zero if $H^q(L) = 0$ for $q \in [j-b, j-a]$. Let $N$ be the integer $d$ of Cohomology of Schemes, Lemma 4.4. Then $H^0(X, K^\vee \otimes \Omega^L_{O_X} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes \Omega^L_{O_X} L), \ldots, H^{0}(K^\vee \otimes \Omega^L_{O_X} L)$$

are zero. Namely, by the lemma cited and Lemma 3.4 we have

$$H^0(X, K^\vee \otimes \Omega^L_{O_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes \Omega^L_{O_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq -1}(K^\vee \otimes \Omega^L_{O_X} L))$ which is zero by Derived Categories, Lemma 16.1. It follows that $\text{Hom}_{D(O_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b-N, a]$. □

The following result is taken from [BV03].

**Theorem 17.3.** Let $X$ be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra $(E, d)$ with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(O_X)$ is equivalent to $D(E, d)$.

**Proof.** Let $K^\bullet$ be a K-injective complex of $O$-modules which is perfect and generates $D_{QCoh}(O_X)$. Such a thing exists by Theorem 14.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, d) = \text{Hom}_{\text{Comp}^{dg}(O_X)}(K^\bullet, K^\bullet)$$

where $\text{Comp}^{dg}(O_X)$ is the differential graded category of complexes of $O$-modules. Please see Differential Graded Algebra, Section 20. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \text{Ext}_{D(O_X)}^n(K^\bullet, K^\bullet)$$

for all $n \in \mathbb{Z}$. Only a finite number of these Exts are nonzero by Lemma 17.2. Consider the functor

$$- \otimes^L K^\bullet : D(E, d) \to D(O_X)$$

of Differential Graded Algebra, Lemma 26.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(O_X)$, see Proposition 16.1. Combined with (17.3.1) the functor
above is fully faithful as follows from Differential Graded Algebra, Lemmas \(26.6\).
It has a right adjoint
\[
R\text{Hom}(K^\bullet, -) : D(O_X) \rightarrow D(E, d)
\]
by Differential Graded Algebra, Lemmas \(26.5\) which is a left quasi-inverse functor
by generalities on adjoint functors. On the other hand, it follows from Lemma \(17.1\)
that we obtain
\[
- \otimes^L_E K^\bullet : D(E, d) \rightarrow D_{Qcoh}(O_X)
\]
and by our choice of \(K^\bullet\) as a generator of \(D_{Qcoh}(O_X)\) the kernel of the adjoint
restricted to \(D_{Qcoh}(O_X)\) is zero. A formal argument shows that we obtain the
desired equivalence, see Derived Categories, Lemma \(7.2\). □

0DJL Remark 17.4 (Variant with support). Let \(X\) be a quasi-compact and quasi-separated scheme. Let \(T \subset X\) be a closed subset such that \(X \setminus T\) is quasi-compact.
The analogue of Theorem \(17.3\) holds for \(D_{Qcoh,T}(O_X)\). This follows from the exact
same argument as in the proof of the theorem, using Lemmas \(14.4\) and \(16.2\) and a
variant of Lemma \(17.1\) with supports. If we ever need this, we will precisely state
the result here and give a detailed proof.

09SU Remark 17.5 (Uniqueness of dga). Let \(X\) be a quasi-compact and quasi-separated scheme over a ring \(R\). By the construction of the proof of Theorem \(17.3\) there
exists a differential graded algebra \((A, d)\) over \(R\) such that \(D_{Qcoh}(X)\) is \(R\)-linearly equivalent to \(D(A, d)\) as a triangulated category. One may ask: how unique is
\((A, d)\)? The answer is (only) slightly better than just saying that \((A, d)\) is well
defined up to derived equivalence. Namely, suppose that \((B, d)\) is a second such pair. Then we have
\[
(A, d) = \text{Hom}_{\text{Comp}^{dg}(O_X)}(K^\bullet, K^\bullet)\]
and
\[
(B, d) = \text{Hom}_{\text{Comp}^{dg}(O_X)}(L^\bullet, L^\bullet)
\]
for some K-injective complexes \(K^\bullet\) and \(L^\bullet\) of \(O_X\)-modules corresponding to perfect
generators of \(D_{Qcoh}(O_X)\). Set
\[
\Omega = \text{Hom}_{\text{Comp}^{dg}(O_X)}(K^\bullet, L^\bullet) \quad \Omega' = \text{Hom}_{\text{Comp}^{dg}(O_X)}(L^\bullet, K^\bullet)
\]
Then \(\Omega\) is a differential graded \(B^{opp} \otimes_R A\)-module and \(\Omega'\) is a differential graded
\(A^{opp} \otimes_R B\)-module. Moreover, the equivalence
\[
D(A, d) \rightarrow D_{Qcoh}(O_X) \rightarrow D(B, d)
\]
is given by the functor \(- \otimes^L_B \Omega'\) and similarly for the quasi-inverse. Thus we are
in the situation of Differential Graded Algebra, Remark \(28.10\) If we ever need this
remark we will provide a precise statement with a detailed proof here.

18. Characterizing pseudo-coherent complexes, I

0DJM We can use the methods above to characterize pseudo-coherent objects as derived
homotopy limits of approximations by perfect objects.

0DJN Lemma 18.1. Let \(X\) be a quasi-compact and quasi-separated scheme. Let \(K \in D(O_X)\). The following are equivalent
1. \(K\) is pseudo-coherent, and
2. \(K = \text{hocolim} K_n\) where \(K_n\) is perfect and \(\tau_{\geq -n} K_n \rightarrow \tau_{\geq -n} K\) is an isomorphism for all \(n\).
Let us prove that the derived category of projective space is perfect complex.

In Section 15 we proved that the derived category of projective space 

\[(n + 1)\text{-pseudo-coherent, hence } K \text{ is } (n + 1)\text{-pseudo-coherent. This is true for all } n, \text{ hence } K \text{ is pseudo-coherent, see Cohomology, Definition 15.1.}

Assume (1). We start by choosing an approximation \( K_1 \to K \) of \((X, K, -2)\) by a perfect complex \( K_1 \), see Definitions 13.1 and 13.2 and Theorem 13.6. Suppose by induction we have

\[ K_1 \to K_2 \to \ldots \to K_n \to K \]

with \( K_i \) perfect such that such that \( \tau_{\geq -i} K_i \to \tau_{\geq -i} K \) is an isomorphism for all \( 1 \leq i \leq n \). Then we pick \( a \leq b \) as in Lemma 17.2 for the perfect object \( K_n \). Choose an approximation \( K_{n+1} \to K \) of \((X, K, \min(a - 1, -n - 1))\). Choose a distinguished triangle

\[ K_{n+1} \to K \to C \to K_{n+1}[1] \]

Then we see that \( C \in D_{QCoh}(\mathcal{O}_X) \) has \( H^i(C) = 0 \) for \( i \geq 0 \). Thus by our choice of \( a, b \) we see that \( \text{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0 \). Hence the composition \( K_n \to K \to C \) is zero. Hence by Derived Categories, Lemma 4.2 we can factor \( K_n \to K \) through \( K_{n+1} \) proving the induction step.

We still have to prove that \( K = \text{hocolim} K_n \). This follows by an application of Derived Categories, Lemma 32.8 to the functors \( H^i(-) : D(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X) \) and our choice of \( K_n \).

**Lemma 18.2.** Let \( X \) be a quasi-compact and quasi-separated scheme. Let \( T \subset X \) be a closed subset such that \( X \setminus T \) is quasi-compact. Let \( K \in D(\mathcal{O}_X) \) supported on \( T \). The following are equivalent

1. \( K \) is pseudo-coherent, and
2. \( K = \text{hocolim} K_n \) where \( K_n \) is perfect, supported on \( T \), and \( \tau_{\geq -n} K_n \to \tau_{\geq -n} K \) is an isomorphism for all \( n \).

**Proof.** The proof of this lemma is exactly the same as the proof of Lemma 18.1 except that in the choice of the approximations we use the triples \((T, K, m)\). \( \square \)

### 19. An example equivalence

In Section 15 we proved that the derived category of projective space \( \mathbf{P}^n_A \) over a ring \( A \) is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf. In this section we prove this determines an equivalence of \( D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A}) \) with the derived category of an \( A \)-algebra.

Before we can state the result we need some notation. Let \( A \) be a ring. Let \( X = \mathbf{P}^n_A = \text{Proj}(S) \) where \( S = A[X_0, \ldots, X_n] \). By Lemma 15.3 we know that

\[ P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n) \]

is a perfect generator of \( D_{QCoh}(\mathcal{O}_X) \). Consider the (noncommutative) \( A \)-algebra

\[ R = \text{Hom}_{\mathcal{O}_X}(P, P) = \begin{pmatrix}
S_0 & S_1 & S_2 & \ldots & \ldots \\
0 & S_0 & S_1 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & S_0
\end{pmatrix} \]
with obvious multiplication and addition. If we view $P$ as a complex of $\mathcal{O}_X$-modules in the usual way (i.e., with $P$ in degree 0 and zero in every other degree), then we have

$$R = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(P, P)$$

where on the right hand side we view $R$ as a differential graded algebra over $A$ with zero differential (i.e., with $R$ in degree 0 and zero in every other degree). According to the discussion in Differential Graded Algebra, Section 26 we obtain a derived functor

$$- \otimes^L_R P : D(R) \rightarrow D(\mathcal{O}_X),$$

see especially Differential Graded Algebra, Lemma 26.3. By Lemma 17.1 we see that the essential image of this functor is contained in $D_{\text{QCoh}}(\mathcal{O}_X)$.

Lemma 19.1. Let $A$ be a ring. Let $X = P^n_A = \text{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. With $P$ as in (19.0.1) and $R$ as in (19.0.2) the functor

$$- \otimes^L_R P : D(R) \rightarrow D_{\text{QCoh}}(\mathcal{O}_X)$$

is an $A$-linear equivalence of triangulated categories sending $R$ to $P$.

In words: the derived category of quasi-coherent modules on projective space is equivalent to the derived category of modules over a (noncommutative) algebra. This property of projective space appears to be quite unusual among all projective schemes over $A$.

Proof. To prove that our functor is fully faithful it suffices to prove that $\text{Ext}^i_X(P, P)$ is zero for $i \neq 0$ and equal to $R$ for $i = 0$, see Differential Graded Algebra, Lemma 26.6. As in the proof of Lemma 17.2 we see that

$$\text{Ext}^i_X(P, P) = H^i(X, P^\wedge \otimes P) = \bigoplus_{0 \leq a, b \leq n} H^i(X, \mathcal{O}_X(a - b))$$

By the computation of cohomology of projective space (Cohomology of Schemes, Lemma 8.1) we find that these Ext-groups are zero unless $i = 0$. For $i = 0$ we recover $R$ because this is how we defined $R$ in (19.0.2). By Differential Graded Algebra, Lemma 26.5 our functor has a right adjoint, namely $R \text{Hom}(P, -) : D_{\text{QCoh}}(\mathcal{O}_X) \rightarrow D(R)$. Since $P$ is a generator for $D_{\text{QCoh}}(\mathcal{O}_X)$ by Lemma 15.3 we see that the kernel of $R \text{Hom}(P, -)$ is zero. Hence our functor is an equivalence of triangulated categories by Derived Categories, Lemma 7.2.

□

20. The coherator revisited

In Section 0 we constructed and studied the right adjoint $RQ_X$ to the canonical functor $D(Q\text{Coh}(\mathcal{O}_X)) \rightarrow D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : \text{Mod}(\mathcal{O}_X) \rightarrow Q\text{Coh}(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{\text{QCoh}}(\mathcal{O}_X) \rightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote it

$$DQ_X : D(\mathcal{O}_X) \rightarrow D_{\text{QCoh}}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated schemes have such a right adjoint.

\[\text{This is probably nonstandard notation. However, we have already used } Q_X \text{ for the coherator and } RQ_X \text{ for its derived extension.}\]
0CR0 Lemma 20.1. Let $X$ be a quasi-compact and quasi-separated scheme. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint $DQ_X$.

First proof. We will use the induction principle as in Cohomology of Schemes, Lemma 4.1 to prove this. If $D(\text{QCoh}(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor $RQ_X$ of Section 3.1 is a right adjoint to the functor $D(\text{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine schemes, see Lemma 6.3. Thus we see that it suffices to show: if $X = U \cup V$ is a union of two quasi-compact opens and the lemma holds for $U$, $V$, and $U \cap V$, then the lemma holds for $X$.

The adjoint exists if and only if for every object $K$ of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that $E'$ is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\text{Hom}(M, K) = 0$ for all $M$ in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 37.3. Consider the distinguished triangle

$$E \to Rj_\ast E|_U \oplus Rj_\ast E|_V \to Rj_{U \cap V, \ast} E|_{U \cap V} \to E[1]$$

in $D(\mathcal{O}_X)$ of Cohomology, Lemma 33.2. By Derived Categories, Lemma 37.2 it suffices to construct the desired distinguished triangles for $Rj_\ast E|_U$, $Rj_\ast E|_V$, and $Rj_{U \cap V, \ast} E|_{U \cap V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an open immersion corresponding with $U$ a quasi-compact open for which the lemma is true. Let $L$ be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_\ast L \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that $E'$ is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\text{Hom}(M, K) = 0$ for all $M$ in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that $L'$ is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\text{Hom}(N, Q) = 0$ for all $N$ in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 37.3 is an if and only if. We obtain a distinguished triangle

$$Rj_\ast L' \to Rj_\ast L \to Rj_\ast Q \to Rj_\ast L'[1]$$

in $D(\mathcal{O}_X)$. Observe that $Rj_\ast L'$ is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 4.1. On the other hand, if $M$ in $D_{QCoh}(\mathcal{O}_X)$, then

$$\text{Hom}(M, Rj_\ast Q) = \text{Hom}(Lj^* M, Q) = 0$$

because $Lj^* M$ is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 3.8. This finishes the proof.

Second proof. The adjoint exists by Derived Categories, Proposition 36.2 The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 3.1). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 14.3 and because perfect objects are compact by Proposition 16.1.

0CR1 Lemma 20.2. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If the right adjoints $DQ_X$ and $DQ_Y$ of the inclusion functors $D_{QCoh} \to D$ exist for $X$ and $Y$, then

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$
Proof. The statement makes sense because $Rf_* \circ D_{QCoh}(O_X)$ by Lemma $4.1$. The statement is true because $Lf^*$ similarly maps $D_{QCoh}(O_Y)$ into $D_{QCoh}(O_X)$ (Lemma $3.8$) and hence both $Rf_* \circ D_X$ and $D_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(O_Y) \to D(O_X)$. □

0CR2 Remark 20.3. Let $X$ be a quasi-compact and quasi-separated scheme. Let $X = U \cup V$ with $U$ and $V$ quasi-compact open. By Lemma $20.1$ the functors $D^X, D^U, D^V, D^U \cap V$ exist. Moreover, there is a canonical distinguished triangle

$$D^X(K) \to Rj_{U*}D^U(K|_U) \oplus Rj_{V*}D^V(K|_V) \to Rj_{U \cap V*}D^U \cap V(K|_{U \cap V}) \to$$

for any $K \in D(O_X)$. This follows by applying the exact functor $D^X$ to the distinguished triangle of Cohomology, Lemma $20.2$ and using Lemma $20.2$ three times.

0CSA Lemma 20.4. Let $X$ be a quasi-compact and quasi-separated scheme. The functor $D^X$ of Lemma $20.1$ has the following boundedness property: there exists an integer $N = N(X)$ such that, if $K$ in $D(O_X)$ with $H^i(U, K) = 0$ for $U$ affine open in $X$ and $i \notin [a, b]$, then the cohomology sheaves $H^i(D^X(K))$ are zero for $i \notin [a, b + N]$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma $4.1$.

If $X$ is affine, then the lemma is true with $N = 0$ because then $RQ_X = D^X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemmas $3.5$ and $6.3$.

Assume $U, V$ are quasi-compact open in $X$ and the lemma holds for $U, V,$ and $U \cap V$. Say with integers $N(U), N(V), N(U \cap V)$. Now suppose $K$ is in $D(O_X)$ with $H^i(W, K) = 0$ for all affine open $W \subset X$ and all $i \notin [a, b]$. Then $K|_U, K|_V, K|_{U \cap V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U \cap V}(K|_{U \cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + \max(N(U), N(V), N(U \cap V))]$. Since the functors $Rj_{U*}, Rj_{V*}, Rj_{U \cap V*}$ have finite cohomological dimension on $D_{QCoh}$ by Lemma $4.1$, we see that there exists an $N$ such that $Rj_{U*}D^U(K|_U), Rj_{V*}D^V(K|_V)$, and $Rj_{U \cap V*}D^U \cap V(K|_{U \cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark $20.3$. □

0CSB Example 20.5. Let $X$ be a quasi-compact and quasi-separated scheme. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves. Since $D^X$ is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$D^X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(O_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(O_X)$. In fact, let’s write $K = R\lim \mathcal{F}_n$ for this. For any affine open $U \subset X$ we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(O_X)$, which is $D^X(K)$ by the above, is in $D^b_{QCoh}(O_X)$ by Lemma $20.4$. 

21. Cohomology and base change, IV

08ET This section continues the discussion of Cohomology of Schemes, Section 22. First, we have a very general version of the projection formula for quasi-compact and quasi-separated morphisms of schemes and complexes with quasi-coherent cohomology sheaves.

08EU Lemma 21.1. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For $E$ in $D_{QCoh}(\mathcal{O}_X)$ and $K$ in $D_{QCoh}(\mathcal{O}_Y)$ the map

$$Rf_*(E) \otimes^{L}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{L}_{\mathcal{O}_X} Lf^*K)$$

defined in Cohomology, Equation (47.2.1) is an isomorphism.

Proof. To check the map is an isomorphism we may work locally on $Y$. Hence we reduce to the case that $Y$ is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each $K_i$, then it holds for $K$. Namely, the functors $Lf^*$ and $\otimes^L$ preserve direct sums by construction and $Rf_*$ commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 4.2. Moreover, we can use Cohomology, Equation (47.2.1) is an isomorphism.

57.13 shows that it suffices to prove $T$ holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

08A Definition 21.2. Let $S$ be a scheme. Let $X, Y$ be schemes over $S$. We say $X$ and $Y$ are Tor independent over $S$ if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 59.1).

08B Lemma 21.3. Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \longrightarrow & S \\
\downarrow{g} & & \downarrow{g} \\
\end{array}
$$

If $X$ and $S'$ are Tor independent over $S$, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*(L(g')^*E) = Lg^*Rf_*(E)$.

Proof. For any object $E$ of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 28.3 to get a canonical base change map $Lg^*Rf_*(E) \to Rf'_*(L(g')^*E)$. To check this is an isomorphism we may work locally on $S'$. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, $g$ is affine and it suffices to show that

$$Rg_*Lg^*Rf_*(E) \to Rg_*Rf'_*(L(g')^*E) = Rf_*(Rg_*L(g')^*E)$$

is an isomorphism, see Lemma 5.1 (and use Lemmas 3.3, 3.9 and 4.1 to see that the objects $Rf'_*(L(g')^*E)$ and $Lg^*Rf_*(E)$ have quasi-coherent cohomology sheaves).
Note that $g'$ is affine as well (Morphisms, Lemma 11.8). By Lemma 5.2, the map becomes a map

$$Rf_*E \otimes_{L_S} g_*\mathcal{O}_{S'} \to Rf_*(E \otimes_{L_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f'_*g_*\mathcal{O}_{S'}$. Thus by Lemma 21.1 it suffices to prove that $Lf'_*g_*\mathcal{O}_{S'} = f'_*g_*\mathcal{O}_{S'}$. This follows from our assumption that $X$ and $S'$ are Tor independent over $S$. Namely, to check it we may work locally on $X$, hence we may also assume $X$ is affine. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and $S' = \text{Spec}(R')$. Our assumption implies that $A$ and $R'$ are Tor independent over $R$ (More on Algebra, Lemma 59.5), i.e., $\text{Tor}^R_i(A, R') = 0$ for $i > 0$. In other words $A \otimes^L_R R' = A \otimes_R R'$ which exactly means that $Lf'_*g_*\mathcal{O}_{S'} = f'_*g_*\mathcal{O}_{S'}$ (use Lemma 3.8).

The following lemma will be used in the chapter on dualizing complexes.

**Lemma 21.4.** Consider a cartesian square

$$
\begin{array}{ccc}
X' & \to & X \\
\downarrow f' & & \downarrow f \\
S' & \to & S
\end{array}
$$

of quasi-compact and quasi-separated schemes. Assume $g$ and $f$ Tor independent and $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$ affine. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\text{Hom}_X(M, K) \otimes^L_R R' \to R\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

1. $M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or
2. $M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^-_{QCoh}(X)$, and $R'$ has finite Tor dimension over $R$.

**Proof.** There is a canonical map $R\text{Hom}_X(M, K) \to R\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology, Section 40. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $- \otimes^L_R R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\text{Hom}_X(M, R(g')_*L(g')^*K) = R\text{Hom}_X(M, K \otimes_{L_X} g'_*\mathcal{O}_{X'})$$

by Lemma 5.2. In both cases the complex $R\text{Hom}(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 9.8. There is a natural map

$$R\text{Hom}(M, K) \otimes_{L_X} g'_*\mathcal{O}_{X'} \to R\text{Hom}(M, K \otimes_{L_X} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases by Lemma 9.9. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf'_*g_*\mathcal{O}_X$ the second equality by Lemma 21.3. Using Lemma 9.4 and Cohomology, Lemma 44.4 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing $K$ by $R\text{Hom}(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes_{L_A} \Lambda' \to R\Gamma(X, K \otimes_{L_X} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 5.2. Hence the result follows from Lemma 21.3. \qed
Consider a cartesian square of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow f & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
\]

Assume \(g\) and \(f\) are Tor independent.

1. If \(E \in D(\mathcal{O}_X)\) has tor amplitude in \([a, b]\) as a complex of \(f^{-1}\mathcal{O}_S\)-modules, then \(L(g')^*E\) has tor amplitude in \([a, b]\) as a complex of \(f^{-1}\mathcal{O}_{S'}\)-modules.
2. If \(G\) is an \(\mathcal{O}_X\)-module flat over \(S\), then \(L(g')^*G = (g')^*G\).

Proof. We can compute tor dimension at stalks, see Cohomology, Lemma 44.5. If \(x' \in X'\) with image \(x \in X\), then

\[
(L(g')^*E)_{x'} = E_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}
\]

Let \(s' \in S'\) and \(s \in S\) be the image of \(x'\) and \(x\). Since \(X\) and \(S'\) are tor independent over \(S\), we can apply More on Algebra, Lemma 59.2 to see that the right hand side of the displayed formula is equal to \(E_x \otimes_{\mathcal{O}_{S',s'}} \mathcal{O}_{X',s'}\) in \(D(\mathcal{O}_{S',s'})\). Thus (1) follows from More on Algebra, Lemma 63.13. To see (2) observe that flatness of \(G\) is equivalent to the condition that \(G[0]\) has tor amplitude in \([0, 0]\). Applying (1) we conclude. \(\square\)
22. Küneth formula

For the case where the base is a field, please see Varieties, Section 29. Consider a cartesian diagram of schemes

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{p} & X \\
\downarrow & & \downarrow^f \\
\downarrow & & \downarrow^a \\
S & \xrightarrow{q} & Y
\end{array}
\]

Let \( K \in D_{QCoh}(\mathcal{O}_X) \) and \( M \in D_{QCoh}(\mathcal{O}_Y) \). There is a canonical map

\[
R_a^*(K \otimes_{\mathcal{O}_X} L) \to R_f^*(L_p^* K \otimes_{\mathcal{O}_{X \times_S Y}} Lq^* M)
\]

Namely, we can use the maps \( R_a^* \to R_a^* R_p^* K = R_f^* L_p^* K \) and \( R_b^* \to R_b^* R_q^* M = R_f^* Lq^* M \) and then we can use the relative cup product (Cohomology, Remark 28.6).

**Lemma 22.1.** In the situation above, if \( a \) and \( b \) are quasi-compact and quasi-separated and \( X \) and \( Y \) are tor-independent over \( S \), then (22.0.1) is an isomorphism.

**Proof.** This follows from the following sequence of isomorphisms

\[
R_f^*(L_p^* K \otimes_{\mathcal{O}_{X \times_S Y}} Lq^* M) = R_a^*(K \otimes_{\mathcal{O}_X} R_p^* Lq^* M) \\
= R_a^*(K \otimes_{\mathcal{O}_X} L a^* R_b^* M) \\
= R_a^*(K \otimes_{\mathcal{O}_X} L a^* R_b^* M)
\]

The first equality holds because \( f = a \circ p \). The second equality by Lemma 21.1. The third equality by Lemma 21.3. The fourth equality by Lemma 21.1. We omit the verification that the composition of these isomorphisms is the same as the map (22.0.1).

In the situation above assume \( S = \text{Spec}(A) \) affine. Let \( \mathcal{F}^* \) be a bounded below complex of \( a^{-1}\mathcal{O}_S \)-modules. Let \( \mathcal{G}^* \) be a bounded below complex of \( b^{-1}\mathcal{O}_S \)-modules. Then there exists a “cup product” map

\[
R\Gamma(X, \mathcal{F}^*) \otimes_{\mathcal{K}^*} R\Gamma(Y, \mathcal{G}^*) \to R\Gamma(X \times_S Y, \text{Tot}(p^{-1}\mathcal{F}^* \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^*))
\]

in \( D(A) \). This is constructed using the pullback maps \( R\Gamma(X, \mathcal{F}^*) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^*) \) and \( R\Gamma(Y, \mathcal{G}^*) \to R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^*) \) and the cup product constructed in Cohomology, Section 31.
In the situation above assume we have an $O_X$-module $F$ and an $O_Y$-module $G$. Then we have
\[
p^{-1}F \otimes_{f^{-1}O_S} q^{-1}G = p^{-1}(F \otimes_{O_X} O_X) \otimes_{f^{-1}O_S} q^{-1}(O_Y \otimes_{O_Y} G)
\]
\[
= p^{-1}F \otimes_{p^{-1}O_X} p^{-1}O_X \otimes_{f^{-1}O_S} q^{-1}O_Y \otimes_{q^{-1}O_Y} q^{-1}G
\]
\[
= p^{-1}F \otimes_{q^{-1}O_X} O_{X \times S} \otimes_{q^{-1}O_Y} q^{-1}G
\]
\[
= p^{-1}F \otimes_{q^{-1}O_X} O_{X \times S} \otimes_{O_{X \times S}} O_{X \times S} \otimes_{q^{-1}O_Y} q^{-1}G
\]
\[
= p^*F \otimes_{O_{X \times S}} q^*G
\]

**Lemma 22.3.** In the situation above assume

1. $X$ and $Y$ quasi-compact with affine diagonal,
2. $F^n$ is a flat $p^{-1}O_S$-module and $F^n$ has the structure of a quasi-coherent $O_X$-module compatible with the given $p^{-1}O_S$-module structure (but the differentials in the complex $F^*$ need not be $O_X$-linear),
3. $G^m$ is a flat $q^{-1}O_S$-module and $G^m$ has the structure of a quasi-coherent $O_Y$-module compatible with the given $q^{-1}O_S$-module structure (but the differentials in the complex $G^*$ need not be $O_Y$-linear).

Then the cup product \([22.1.1]\) is an isomorphism in $D(A)$.

This lemma is true without the assumption on the diagonals of $X$ and $Y$. The assumption on the diagonals can be removed by replacing open coverings in the proof below by hypercoverings.

**Proof.** We will prove the result on the cup product in two steps: (1) we reduce to the case where each of the two complexes consist of a single sheaf placed in degree 0 and (2) the case of a single sheaf is proved as in Varieties, Lemma \([29.1]\).

Suppose that we have maps of complexes of $p^{-1}O_S$-modules

\[
F_1^* \to F_2^* \to F_3^* \to F_1^*[1]
\]

Then by the functoriality of the cup product we obtain a commutative diagram

\[
\begin{array}{c}
\text{RG}(X, F_1^*) \otimes_A \text{RG}(Y, G^*) \ar[d] \ar[r] & \text{RG}(X \times_S Y, \text{Tot}(p^{-1}F_1^* \otimes_{f^{-1}O_S} q^{-1}G^*)) \ar[d] \\
\text{RG}(X, F_2^*) \otimes_A \text{RG}(Y, G^*) \ar[d] \ar[r] & \text{RG}(X \times_S Y, \text{Tot}(p^{-1}F_2^* \otimes_{f^{-1}O_S} q^{-1}G^*)) \ar[d] \\
\text{RG}(X, F_3^*) \otimes_A \text{RG}(Y, G^*) \ar[d] \ar[r] & \text{RG}(X \times_S Y, \text{Tot}(p^{-1}F_3^* \otimes_{f^{-1}O_S} q^{-1}G^*)) \ar[d] \\
\text{RG}(X, F_1^*[1]) \otimes_A \text{RG}(Y, G^*) \ar[r] & \text{RG}(X \times_S Y, \text{Tot}(p^{-1}F_1^*[1] \otimes_{f^{-1}O_S} q^{-1}G^*))
\end{array}
\]

If the original maps form a distinguished triangle in the homotopy category of complexes of $p^{-1}O_S$-modules, then the columns of this diagram form distinguished triangles in $D(A)$. 
Suppose that $F^n = 0$ for $n < a$. Then we may consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} F^\bullet \to F^\bullet \to F^a[-a] \to 0$$

where the truncation is as in Homology, Section 14. This produces the distinguished triangle

$$\sigma_{\geq a+1} F^\bullet \to F^\bullet \to F^a[-a] \to (\sigma_{\geq a+1} F^\bullet)[1]$$

where the final arrow is given by the boundary map $F^a \to F^{a+1}$. It follows from the discussion above that it suffices to prove the lemma for $F^a[-a]$ and $\sigma_{\geq a+1} F^\bullet$. Hence if we can prove the lemma if $F^\bullet$ is nonzero only in single degree, then the lemma follows. Namely, $R\Gamma(X, \sigma_{\geq N} F^\bullet)$ has no nonzero cohomology in degree $< N$ and both $R\Gamma(X, \sigma_{\geq N} F^\bullet) \otimes_A^L R\Gamma(Y, G^\bullet)$ and $R\Gamma(X \times_S Y, \text{Tot}(p^{-1}\sigma_{\geq N} F^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1} G^\bullet))$ have no nonzero cohomology in degree $< N + b$ if $G^m = 0$ for $m < b$. Thus we may assume $F^\bullet$ is nonzero only in one degree. Repeating the argument with $G^\bullet$ we reduce to the case discussed in the next paragraph.

Assume $F^\bullet = F$ and $G^\bullet = G$ are both reduced to a single quasi-coherent module placed in degree 0 and both $F$ and $G$ are flat over $S$. Choose finite affine open coverings $U : X = \bigcup_{i \in I} U_i$ and $V : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $W : X \times_S Y = \bigcup_{(i,j) \in I \times J} U_i \times_S V_j$. Note that $W$ is a refinement of $pr_1^* U$ and of $pr_2^{-1} V$. Thus by the discussion in Cohomology, Section 25 we obtain maps

$$\check{\mathcal{C}}^\bullet(U,F) \to \check{\mathcal{C}}^\bullet(W, p^* F) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(V,G) \to \check{\mathcal{C}}^\bullet(W, q^* G)$$

well defined up to homotopy and compatible with pullback maps on cohomology. In Cohomology, Equation (25.3.2) we have constructed a map of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(W, p^* F) \otimes_A \check{\mathcal{C}}^\bullet(W, q^* G)) \to \check{\mathcal{C}}^\bullet(W, p^* F \otimes_{\mathcal{O}_X \times_S Y} q^* G)$$

which is compatible with the cup product by Cohomology, Lemma 31.1. Combining the above we obtain a map of complexes

$$(22.3.1) \quad \text{Tot}(\check{\mathcal{C}}^\bullet(U,F) \otimes_A \check{\mathcal{C}}^\bullet(V,G)) \to \text{Tot}(\check{\mathcal{C}}^\bullet(W, p^* F \otimes_{\mathcal{O}_X \times_S Y} q^* G))$$

We claim this is the map in the statement of the lemma, i.e., the source and target of this arrow are the same as the source and target of (22.1.1). Namely, by Cohomology of Schemes, Lemma 22.2 and Cohomology, Lemma 25.2 the canonical maps

$$\check{\mathcal{C}}^\bullet(U,F) \to R\Gamma(X,F), \quad \check{\mathcal{C}}^\bullet(V,G) \to R\Gamma(Y,G)$$

and

$$\check{\mathcal{C}}^\bullet(W, p^* F \otimes_{\mathcal{O}_X \times_S Y} q^* G) \to R\Gamma(X \times_S Y, p^* F \otimes_{\mathcal{O}_X \times_S Y} q^* G)$$

are isomorphisms. On the other hand, the complex $\check{\mathcal{C}}^\bullet(U,F)$ is homotopy equivalent to the alternating Čech complex $\check{\mathcal{C}}^\bullet_{\text{alt}}(U,F)$ by Cohomology, Lemma 23.6. As the modules $F^n(U_{i_0 \ldots i_s})$ are $A$-flat, we see that the complex $\check{\mathcal{C}}^\bullet_{\text{alt}}(U,F)$ is K-flat (More on Algebra, Lemma 57.9). Hence $\check{\mathcal{C}}^\bullet(U,F)$ is a K-flat complex of $A$-modules too and we conclude that $\text{Tot}(\check{\mathcal{C}}^\bullet(U,F) \otimes_A \check{\mathcal{C}}^\bullet(V,G))$ represents the derived tensor product $R\Gamma(X,F) \otimes_A^L R\Gamma(Y,G)$ as claimed. Finally, we have $p^* F \otimes_{\mathcal{O}_X \times_S Y} q^* G = p^{-1} F \otimes_{f^{-1}\mathcal{O}_S} q^{-1} G$ by Remark 22.2.

We still have to show that (22.3.1) is a quasi-isomorphism. We will do this using dimension shifting. Set $d(F) = \max\{d \mid H^d(X,F) \neq 0\}$. Assume $d(F) > 0$. Set $U = \coprod_{i \in I} U_i$. This is an affine scheme as $I$ is finite. Denote $j : U \to X$ the
morphism which is the inclusion $U_i \to X$ on each $U_i$. Since the diagonal of $X$ is affine, the morphism $j$ is affine, see Morphisms, Lemma [11.11]. It follows that $F' = j_* j^* F$ is $S$-flat, see Morphisms, Lemma [24.4]. It also follows that $d(F') = 0$ by combining Cohomology of Schemes, Lemmas [2.4] and [2.2]. For all $x \in X$ we have $F_x \to F'_x$ is the inclusion of a direct summand: if $x \in U_i$, then $F' \to (U_i \to X)_* F|_{U_i}$ gives a splitting. We conclude that $F \to F'$ is injective and $F'' = F'/F$ is $S$-flat as well. Also $d(F'') < d(F)$. In this way we reduce to the case $d(F) = 0$.

Arguing in the same fashion for $G$ we find that we may assume that both $F$ and $G$ have nonzero cohomology only in degree 0. Observe that this means that $\Gamma(X, F)$ is quasi-isomorphic to the finite complex $\check{C}_d(U, F)$ of flat $A$-modules sitting in degrees $0, \ldots, |I|$. It follows that $\Gamma(X, F)$ is a flat $A$-module. Let $V \subset Y$ be an affine open. Consider the affine open covering $U_V : X \times_S V = \bigcup_{i \in I} U_i \times_S V$. It is immediate that

$$\check{C}^\bullet(U, F) \otimes_A G(V) = \check{C}^\bullet(U_V, p^* F \otimes \mathcal{O}_{X \times Y} q^* G)$$

(equality of complexes). By the flatness of $G(V)$ over $A$ we see that $\Gamma(X, F) \otimes_A G(V) \to \check{C}^\bullet(U, F) \otimes_A G(V)$ is a quasi-isomorphism. Since the sheafification of $V \to \check{C}^\bullet(U_V, p^* F \otimes \mathcal{O}_{X \times Y} q^* G)$ represents $Rq_* (p^* F \otimes \mathcal{O}_{X \times Y} q^* G)$ by Cohomology of Schemes, Lemma [7.1] we conclude that

$$Rq_* (p^* F \otimes \mathcal{O}_{X \times Y} q^* G) \cong \Gamma(X, F) \otimes_A G$$

on $Y$ with obvious notation. Using this and Lemma [21.1] (which applies as $\Gamma(X, F)$ is $A$-flat) we conclude that $H^n(X \times_S Y, p^* F \otimes \mathcal{O}_{X \times Y} q^* G)$ is zero for $n > 0$ and isomorphic to $H^0(X, F) \otimes_A H^0(Y, G)$ for $n = 0$. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification).

23. Cohomology and base change, V

0DJ6 In Section [21] we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section [59]. In this section we analyze when one can get a base change result “one complex at a time”.

To make this work, suppose we have a commutative diagram

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S \\
\end{array}$$

of schemes (usually we will assume it is cartesian). Let $K \in D_{Qcoh}(\mathcal{O}_X)$ and let $L(g')^* K \to K'$ be a map in $D_{Qcoh}(\mathcal{O}_{X'})$. For a point $x' \in X'$ set $x = g'(x') \in X$, $s' = f'(x') \in S'$ and $s = f(x) = g(s')$. Then we can consider the maps

$$K_x \otimes \mathcal{O}_{S, s} \to K_x \otimes \mathcal{O}_{X, x} \otimes \mathcal{O}_{X', x'} \to K'_{x'}$$

where the first arrow is More on Algebra, Equation [59.0.1] and the second comes from $(L(g')^* K)_{x'} = K_x \otimes \mathcal{O}_{X, x} \otimes \mathcal{O}_{X', x'}$ and the given map $L(g')^* K \to K'$. For each
Let $i \in \mathbb{Z}$ we obtain a $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S',x'}} \mathcal{O}_{S',x'}$-module structure on $H^i(K_x \otimes_{\mathcal{O}_{S',x'}} \mathcal{O}_{S',x'})$. Putting everything together we obtain canonical maps

$$H^i(K_x \otimes_{\mathcal{O}_{S',x'}} \mathcal{O}_{S',x'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S',x'}} \mathcal{O}_{S',x'})} \mathcal{O}_{X',x'} \to H^i(K_x')$$

of $\mathcal{O}_{X',x'}$-modules.

Lemma 23.1. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
\]

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

1. for any $x' \in X'$ and $i \in \mathbb{Z}$ the map $H^i(L(g')^*K) \to H^i(K')$ is an isomorphism,
2. for $U \subset X$, $V' \subset S'$ affine open both mapping into the affine open $V \subset S$ with $U' = V' \times_V U$ the composition

$$R\Gamma(U, K) \otimes_{\mathcal{O}_{S'(U)}} \mathcal{O}_{S'}(V') \to R\Gamma(U, K) \otimes_{\mathcal{O}_{O_X(U)}} \mathcal{O}_{X'}(U') \to R\Gamma(U', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(V'))$, and
3. there is a set $I$ of quadruples $U_i, V'_i, V_i, U'_i$, $i \in I$ as in (2) with $X' = \bigcup U'_i$.

Proof. The second arrow in (2) comes from the equality

$$R\Gamma(U, K) \otimes_{\mathcal{O}_{O_X(U)}} \mathcal{O}_{X'}(U') = R\Gamma(U', L(g')^*K)$$

of Lemma 3.8 and the given arrow $L(g')^*K \to K'$. The first arrow of (2) is More on Algebra, Equation (59.0.1). It is clear that (2) implies (3). Observe that (1) is local on $X'$. Therefore it suffices to show that if $X$, $S$, $S'$, $X'$ are affine, then (1) is equivalent to the condition that

$$R\Gamma(X, K) \otimes_{\mathcal{O}_{S(S)}} \mathcal{O}_{S'}(S') \to R\Gamma(X, K) \otimes_{\mathcal{O}_{O_X(X)}} \mathcal{O}_{X'}(X') \to R\Gamma(X', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(S'))$. Say $S = \text{Spec}(R)$, $X = \text{Spec}(A)$, $S' = \text{Spec}(R')$, $X' = \text{Spec}(A')$, $K$ corresponds to the complex $M^\bullet$ of $A$-modules, and $K'$ corresponds to the complex $N^\bullet$ of $A'$-modules. Note that $A' = A \otimes_R R'$. The condition above is that the composition

$$M^\bullet \otimes_R R' \to M^\bullet \otimes_A A' \to N^\bullet$$

is an isomorphism in $D(R')$. Equivalently, it is that for all $i \in \mathbb{Z}$ the map

$$H^i(M^\bullet \otimes_r R') \to H^i(M^\bullet \otimes_A A') \to H^i(N^\bullet)$$

is an isomorphism. Observe that this is a map of $A \otimes_R R'$-modules, i.e., of $A'$-modules. On the other hand, (1) is the requirement that for compatible primes $q' \subset A'$, $q \subset A$, $p' \subset R'$, $p \subset R$ the composition

$$H^i(M_q^\bullet \otimes_{R_p} R_{p'}) \otimes (A_q \otimes_{R_p} R_{p'}) A'_q \to H^i(M_q^\bullet \otimes_{A_q} A'_q) \to H^i(N_q^\bullet)$$

is an isomorphism. Since

$$H^i(M_q^\bullet \otimes_{R_p} R_{p'}) \otimes (A_q \otimes_{R_p} R_{p'}) A'_q = H^i(M^\bullet \otimes_R R') \otimes_{A'} A'_q$$

is the localization at $q'$, we see that these two conditions are equivalent by Algebra, Lemma 22.1. \qed
Lemma 23.2. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S \\
\end{array}
\]

be a cartesian diagram of schemes. Let \( K \in D_{QCoh}(\mathcal{O}_X) \) and let \( L(g')^*K \to K' \) be a map in \( D_{QCoh}(\mathcal{O}_{X'}) \). If

1. the equivalent conditions of Lemma 23.1 hold, and
2. \( f \) is quasi-compact and quasi-separated,

then the composition \( Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K' \) is an isomorphism.

Proof. We could prove this using the same method as in the proof of Lemma 21.3 but instead we will prove it using the induction principle and relative Mayer-Vietoris.

To check the map is an isomorphism we may work locally on \( S' \). Hence we may assume \( g : S' \to S \) is a morphism of affine schemes. In particular \( X \) is a quasi-compact and quasi-separated scheme. We will use the induction principle of Cohomology of Schemes, Lemma 4.1 to prove that for any quasi-compact open \( U \subset X \) the similarly constructed map \( Lg^*R(U \to S)_*K|_U \to R(U' \to S')_*K'|_{U'} \) is an isomorphism. Here \( U' = (g')^{-1}(U) \).

If \( U \subset X \) is an affine open, then we find that the result is true by assumption, see Lemma 23.1 part (2) and the translation into algebra afforded to us by Lemmas 3.5 and 3.8.

The induction step. Suppose that \( X = U \cup V \) is an open covering with \( U, V, U \cap V \) quasi-compact such that the result holds for \( U, V, \) and \( U \cap V \). Denote \( a = f|_U \), \( b = f|_V \) and \( c = f|_{U \cap V} \). Let \( a' : U' \to S' \), \( b' : V' \to S' \) and \( c' : U' \cap V' \to S' \) be the base changes of \( a, b, \) and \( c \). Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 33.5) we obtain a commutative diagram

\[
\begin{array}{ccc}
Lg^*Rf_*K & \longrightarrow & Rf'_*K' \\
\downarrow & & \downarrow \\
Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V & \longrightarrow & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \\
\downarrow & & \downarrow \\
Lg^*Rc_*K|_{U \cap V} & \longrightarrow & Rc'_*K'|_{U' \cap V'} \\
\downarrow & & \downarrow \\
Lg^*Rf_*K[1] & \longrightarrow & Rf'_*K'[1] \\
\end{array}
\]

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 4.3) and the proof of the lemma is finished. \(\square\)
Lemma 23.3. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
\]

be a cartesian diagram of schemes. Let \( K \in D_{\text{QCoh}}(\mathcal{O}_X) \) and let \( \mathbf{L}(g')^* K \to K' \) be a map in \( D_{\text{QCoh}}(\mathcal{O}_X') \). If the equivalent conditions of Lemma 23.1 hold, then

1. for \( E \in D_{\text{QCoh}}(\mathcal{O}_X) \) the equivalent conditions of Lemma 23.1 hold for \( \mathbf{L}(g')^*(E \otimes^L K) \to \mathbf{L}(g')^* E \otimes^L K' \),
2. if \( E \in D(\mathcal{O}_X) \) is perfect the equivalent conditions of Lemma 23.1 hold for \( \mathbf{L}(g')^* \mathbf{R}\text{Hom}(E, K) \to \mathbf{R}\text{Hom}(\mathbf{L}(g')^* E, K') \), and
3. if \( K \) is bounded below and \( E \) in \( D(\mathcal{O}_X) \) pseudo-coherent the equivalent conditions of Lemma 23.1 hold for \( \mathbf{L}(g')^* \mathbf{R}\text{Hom}(E, K) \to \mathbf{R}\text{Hom}(\mathbf{L}(g')^* E, K') \).

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 3.8, 3.9, and 9.8 and Cohomology, Lemmas 43.3 and 45.6. Having said this, we can check the maps (23.0.1) are isomorphisms in case (1) by computing the source and target of (23.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 57.17. The map in (2) and (3) is the composition

\[ \mathbf{L}(g')^* \mathbf{R}\text{Hom}(E, K) \to \mathbf{R}\text{Hom}(\mathbf{L}(g')^* E, L(g')^* K) \to \mathbf{R}\text{Hom}(\mathbf{L}(g')^* E, K') \]

where the first arrow is Cohomology, Remark 38.11 and the second arrow comes from the given map (23.0.1) isomorphisms. One represents \( E_x \) by a bounded complex of finite projective \( \mathcal{O}_{X,x} \)-modules in case (2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. \( \square \)

Lemma 23.4. Let \( f : X \to S \) be a quasi-compact and quasi-separated morphism of schemes. Let \( E \in D_{\text{QCoh}}(\mathcal{O}_X) \). Let \( G^\bullet \) be a bounded above complex of quasi-coherent \( \mathcal{O}_X \)-modules flat over \( S \). Then formation of

\[ \mathbf{R}f_*(E \otimes_{\mathcal{O}_X} G^\bullet) \]

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let \( g : S' \to S \) be a morphism of schemes and consider the base change diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
\]

in other words \( X' = S' \times_S X \). The lemma asserts that

\[ Lg^* \mathbf{R}f_*(E \otimes_{\mathcal{O}_X} G^\bullet) \to \mathbf{R}f'_* \left( L(g')^* E \otimes_{\mathcal{O}_{X'}} (g')^* G^\bullet \right) \]

is an isomorphism. Observe that on the right hand side we do \textbf{not} use the derived pullback on \( G^\bullet \). To prove this, we apply Lemmas 23.2 and 23.3 to see that it suffices to prove the canonical map

\[ L(g')^* G^\bullet \to (g')^* G^\bullet \]
satisfies the equivalent conditions of Lemma 23.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $G_x \otimes_{O_{S,x'}} O_{S',x'}$ computes the derived tensor product by our assumptions on the complex $G^\bullet$. □

Lemma 23.5. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E$ be an object of $D(O_X)$. Let $G^\bullet$ be a complex of quasi-coherent $O_X$-modules. If

1. $E$ is perfect, $G^\bullet$ is a bounded above, and $G^n$ is flat over $S$, or
2. $E$ is pseudo-coherent, $G^\bullet$ is bounded, and $G^n$ is flat over $S$,

then formation of $Rf_*R\text{Hom}(E, G^\bullet)$ commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
g' \downarrow & & \downarrow f \\
S' & \longrightarrow & S
\end{array}
\]

in other words $X' = S' \times_S X$. The lemma asserts that

\[
Lg^*Rf_*R\text{Hom}(E, G^\bullet) \longrightarrow R(f')_*R\text{Hom}(L(g')^*E, (g')^*G^\bullet)
\]

is an isomorphism. Observe that on the right hand side we do not use the derived pullback on $G^\bullet$. To prove this, we apply Lemmas 23.2 and 23.3 to see that it suffices to prove the canonical map

\[
L(g')^*G^\bullet \to (g')^*G^\bullet
\]

satisfies the equivalent conditions of Lemma 23.1. This was shown in the proof of Lemma 23.4. □

24. Producing perfect complexes

The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 27.

Lemma 24.1. Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(O_X)$ such that

1. $E \in D^b_{\text{Coh}}(O_X)$,
2. the support of $H^i(E)$ is proper over $S$ for all $i$, and
3. $E$ has finite tor dimension as an object of $D(f^{-1}O_S)$.

Then $Rf_*E$ is a perfect object of $D(O_S)$.

Proof. By Lemma 10.3 we see that $Rf_*E$ is an object of $D^b_{\text{Coh}}(O_S)$. Hence $Rf_*E$ is pseudo-coherent (Lemma 9.3). Hence it suffices to show that $Rf_*E$ has finite tor dimension, see Cohomology, Lemma 45.5. By Lemma 9.6 it suffices to check that $Rf_*(E) \otimes_{O_S} F$ has universally bounded cohomology for all quasi-coherent sheaves $F$ on $S$. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all $i$. Then $T$ is proper over $S$ by assumptions (1) and (2), see Cohomology of Schemes, Lemma 26.6. In particular
there exists a quasi-compact open \( X' \subset X \) containing \( T \). Setting \( f' = f|_{X'} \), we have \( Rf'_*(E) = Rf'_*(E|_{X'}) \) because \( E \) restricts to zero on \( X \setminus T \). Thus we may replace \( X \) by \( X' \) and assume \( f \) is quasi-compact. Moreover, \( f \) is quasi-separated by Morphisms, Lemma \[14.7\]. Now

\[
Rf_*(E) \otimes_{\mathcal{O}_S}^L F = Rf_* \left( E \otimes_{\mathcal{O}_X}^L Lf^*F \right) = Rf_* \left( E \otimes_{f^{-1}\mathcal{O}_S}^L f^{-1}F \right)
\]

by Lemma \[21.1\] and Cohomology, Lemma \[27.4\]. By assumption (3) the complex \( E \otimes_{f^{-1}\mathcal{O}_S}^L f^{-1}F \) has cohomology sheaves in a given finite range, say \([a,b]\). Then \( Rf_* \) of it has cohomology in the range \([a,\infty)\) and we win. \( \square \)

**Lemma 24.2.** Let \( S \) be a Noetherian scheme. Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. Let \( E \in D(\mathcal{O}_X) \) be perfect. Let \( \mathcal{G}^\bullet \) be a bounded complex of coherent \( \mathcal{O}_X \)-modules flat over \( S \) with support proper over \( S \). Then \( K = Rf_* (E \otimes_{\mathcal{O}_X}^L \mathcal{G}^\bullet) \) is a perfect object of \( D(\mathcal{O}_S) \).

**Proof.** The object \( K \) is perfect by Lemma \[24.1\]. We check the lemma applies: Locally \( E \) is isomorphic to a finite complex of finite free \( \mathcal{O}_X \)-modules. Hence locally \( E \otimes_{\mathcal{O}_X}^L \mathcal{G}^\bullet \) is isomorphic to a finite complex whose terms are of the form

\[
\bigoplus_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}
\]

for some integers \( a, b, r_a, \ldots, r_b \). This immediately implies the cohomology sheaves \( H^i(\mathcal{E} \otimes_{\mathcal{O}_X}^L \mathcal{G}) \) are coherent. The hypothesis on the tor dimension also follows as \( \mathcal{G}^i \) is flat over \( f^{-1}\mathcal{O}_S \). \( \square \)

**Lemma 24.3.** Let \( S \) be a Noetherian scheme. Let \( f : X \to S \) be a morphism of schemes which is locally of finite type. Let \( E \in D(\mathcal{O}_X) \) be perfect. Let \( \mathcal{G}^\bullet \) be a bounded complex of coherent \( \mathcal{O}_X \)-modules flat over \( S \) with support proper over \( S \). Then \( K = Rf_* R\mathcal{H}om(E, \mathcal{G}^\bullet) \) is a perfect object of \( D(\mathcal{O}_S) \).

**Proof.** Since \( E \) is a perfect complex there exists a dual perfect complex \( E^\vee \), see Cohomology, Lemma \[45.11\]. Observe that \( R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet \). Thus the perfectness of \( K \) follows from Lemma \[24.2\]. \( \square \)

We will generalize the following lemma to flat and proper morphisms over general bases in Lemma \[27.4\] and to perfect proper morphisms in More on Morphisms, Lemma \[53.12\].

**Lemma 24.4.** Let \( S \) be a Noetherian scheme. Let \( f : X \to S \) be a flat proper morphism of schemes. Let \( E \in D(\mathcal{O}_X) \) be perfect. Then \( Rf_* E \) is a perfect object of \( D(\mathcal{O}_S) \).

**Proof.** We claim that Lemma \[24.1\] applies. Conditions (1) and (2) are immediate. Condition (3) is local on \( X \). Thus we may assume \( X \) and \( S \) affine and \( E \) represented by a strictly perfect complex of \( \mathcal{O}_X \)-modules. Since \( \mathcal{O}_X \) is flat as a sheaf of \( f^{-1}\mathcal{O}_S \)-modules we find that condition (3) is satisfied. \( \square \)

**25. A projection formula for Ext**

Lemma \[25.3\] (or similar results in the literature) is sometimes used to verify one of Artin’s criteria for Quot functors, Hilbert schemes, and other moduli problems.
Suppose that $f : X \to S$ is a proper, flat, finitely presented morphism of schemes and $E \in D(O_X)$ is perfect. Here the lemma says

$$\text{Ext}^i_X(E, f^*F) = \text{Ext}^i_S((Rf_*)E^\vee, F)$$

for $F$ quasi-coherent on $S$. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 21.1.

**Lemma 25.1.** Assumptions and notation as in Lemma 24.3. Then there are functorial isomorphisms

$$H^i(S, K \otimes_{O_S} F) \to H^i(X, E \otimes_{O_X} (G^* \otimes_{O_X} f^*F))$$

for $F$ quasi-coherent on $S$ compatible with boundary maps (see proof).

**Proof.** We have

$$G^* \otimes_{O_X} Lf^*F = G^* \otimes_{f^{-1}O_S} f^{-1}F = G^* \otimes_{f^{-1}O_S} f^{-1}F = G^* \otimes_{O_X} f^*F$$

the first equality by Cohomology, Lemma 27.4, the second as $G^n$ is a flat $f^{-1}O_S$-module, and the third by definition of pullbacks. Hence we obtain

$$H^i(X, E \otimes_{O_X} (G^* \otimes_{O_X} f^*F)) = H^i(X, E \otimes_{O_X} G^* \otimes_{O_X} Lf^*F)$$

$$= H^i(S, Rf_*(E \otimes_{O_X} G^* \otimes_{O_X} Lf^*F))$$

$$= H^i(S, Rf_*(E \otimes_{O_X} G^*) \otimes_{O_S} F)$$

$$= H^i(S, K \otimes_{O_S} F)$$

The first equality by the above, the second by Leray (Cohomology, Lemma 13.1), and the third equality by Lemma 21.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ of quasi-coherent $O_S$-modules, the isomorphisms fit into commutative diagrams

$$
\begin{array}{ccc}
H^i(S, K \otimes_{O_S} F_3) & \to & H^i(X, E \otimes_{O_X} (G^* \otimes_{O_X} f^*F_3)) \\
\downarrow \delta & & \downarrow \delta \\
H^{i+1}(S, K \otimes_{O_S} F_1) & \to & H^{i+1}(X, E \otimes_{O_X} (G^* \otimes_{O_X} f^*F_1))
\end{array}
$$

where the boundary maps come from the distinguished triangle

$$K \otimes_{O_S} F_1 \to K \otimes_{O_S} F_2 \to K \otimes_{O_S} F_3 \to K \otimes_{O_S} F_1[1]$$

and the distinguished triangle in $D(O_X)$ associated to the short exact sequence

$$0 \to G^* \otimes_{O_X} f^*F_1 \to G^* \otimes_{O_X} f^*F_2 \to G^* \otimes_{O_X} f^*F_3 \to 0$$

of complexes of $O_X$-modules. This sequence is exact because $G^n$ is flat over $S$. We omit the verification of the commutativity of the displayed diagram. \hfill \Box

**Lemma 25.2.** Assumptions and notation as in Lemma 24.3. Then there are functorial isomorphisms

$$H^i(S, K \otimes_{O_S} F) \to \text{Ext}^i_{O_X}(E, G^* \otimes_{O_X} f^*F)$$

for $F$ quasi-coherent on $S$ compatible with boundary maps (see proof).
Let recall that applied to $E$ by construction of Given a short exact sequence As in the proof of Lemma 24.3 let Proof. into commutative diagrams where the boundary maps come from the distinguished triangle and the distinguished triangle in Then the following two statements are true Lemma 25.3. for every $m \in \mathbb{Z}$ there exists a perfect object $K$ of $D(O_S)$ and functorial maps $\alpha_F^i : \text{Ext}^i_{O_X}(E, G^* \otimes O_X f^* \mathcal{F}) \to H^i(S, K \otimes_{O_S}^L \mathcal{F})$ for $\mathcal{F}$ quasi-coherent on $S$ compatible with boundary maps (see proof) such that $\alpha_F^i$ is an isomorphism for $i \leq m$ There exists a pseudo-coherent $L \in D(O_S)$ and functorial isomorphisms $\text{Ext}^i_{O_S}(L, \mathcal{F}) \to \text{Ext}^i_{O_X}(E, G^* \otimes O_X f^* \mathcal{F})$ for $\mathcal{F}$ quasi-coherent on $S$ compatible with boundary maps. 

**Proof.** Proof of (A). Suppose $G^i$ is nonzero only for $i \in [a, b]$. We may replace $X$ by a quasi-compact open neighbourhood of the union of the supports of $G^i$. Hence we may assume $X$ is Noetherian. In this case $X$ and $f$ are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex $P$ of $(X, E, -m - 1 + a)$ (possible by Theorem 13.0). Then the induced map $\text{Ext}^i_{O_X}(E, G^* \otimes O_X f^* \mathcal{F}) \to \text{Ext}^i_{O_X}(P, G^* \otimes O_X f^* \mathcal{F})$
is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\text{Ext}^i_{O_X}(C, G^\bullet \otimes_{O_X} f^* \mathcal{F}) \quad \text{resp.} \quad \text{Ext}^{i+1}_{O_X}(C, G^\bullet \otimes_{O_X} f^* \mathcal{F})$$

where $C$ is the cone of $P \to E$. Since $C$ has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 27.3. This reduces us to the case that $E$ is a perfect complex which is Lemma 25.2.

The statement on boundaries is explained in the proof of Lemma 25.2.

Proof of (B). As in the proof of (A) we may assume $X$ is Noetherian. Observe that $E$ is pseudo-coherent by Lemma 25.3. By Lemma 18.1, we can write $E = \text{hocolim} E_n$ with $E_n$ perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n$ be the dual perfect complex (Cohomology, Lemma 45.11). We obtain an inverse system $\ldots \to E_3 \to E_2 \to E_1$ of perfect objects. This in turn gives rise to an inverse system

$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes_{O_X} G^\bullet)$$

perfect on $S$, see Lemma 24.2. By Lemma 25.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) we have functorial canonical maps

\[
\text{Ext}^i_{O_X}(E, G^\bullet \otimes_{O_X} f^* \mathcal{F}) \quad \xrightarrow{\sim} \quad H^i(S, K_{n+1} \otimes_{O_S} \mathcal{F}) \quad \xrightarrow{\sim} \quad H^i(S, K_n \otimes_{O_S} \mathcal{F})
\]

which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(O_S)$ such that for any quasi-coherent $\mathcal{F}$ on $S$ the maps

$$\text{Ext}^i_{O_S}(L_{n+1}, \mathcal{F}) \to \text{Ext}^i_{O_S}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 18.1. The mapping property of homotopy colimits gives that $\text{Ext}^i_{O_S}(L, \mathcal{F}) = \text{Ext}^i_{O_S}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. \hfill $\square$

0DJS. The pseudo-coherent complex $L$ of part (B) of Lemma 25.3 is canonically associated to the situation. For example, formation of $L$ as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}$$

of schemes we have canonical functorial isomorphisms

$$\text{Ext}^i_{O_{X'}}(Lg^*L, \mathcal{F}') \quad \to \quad \text{Ext}^i_{O_X}(L(g')^*E, (g')^*G^\bullet \otimes_{O_{X'}} (f')^* \mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on $S'$. Observe that we do not use derived pullback on $G^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.
26. Limits and derived categories

In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

**Situation 26.1.** Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{ij} : S_j \to S_i$. We assume that $S_i$ is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

**Lemma 26.2.** In Situation 26.1 Let $E_0$ and $K_0$ be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \to \text{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

1. $E_0$ is perfect and $K_0 \in D_{QQCoh}(\mathcal{O}_{S_0})$, or
2. $E_0$ is pseudo-coherent and $K_0 \in D_{QQCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.

**Proof.** For every open $U_0 \subset S_0$ consider the condition $P$ that the canonical map

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{U_i})}(E|_{U_i}, K|_{U_i}) \to \text{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_i^{-1}(U_0)$. We will prove $P$ holds for all quasi-compact opens $U_0$ by the induction principle of Cohomology of Schemes, Lemma 4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 33.3. Thus it suffices to prove the lemma when $S_0$ is affine.

Assume $S_0$ is affine. Say $S_0 = \text{Spec}(A_0)$, $S_i = \text{Spec}(A_i)$, and $S = \text{Spec}(A)$. We will use Lemma 3.5 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective $A_0$-modules, see Lemma 9.7. We may represent the object $K_0$ by a K-flat complex $K_0^\bullet$ of $A_0$-modules. In this situation we are trying to prove

$$\text{colim}_{i \geq 0} \text{Hom}_{D(A_0)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \to \text{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\text{colim}_{i \geq 0} \text{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \to \text{Hom}_{K(A)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules $E_0^n$ and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object $E_0$ corresponds to a bounded above complex $E_0^\bullet$ of finite free $A_0$-modules, see Lemma 9.2. We may represent $K_0$ by a finite complex $K_0^\bullet$ of flat $A_0$-modules, see Lemma 9.4 and More on Algebra, Lemma 63.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\text{colim}_{i \geq 0} \text{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \to \text{Hom}_{K(A)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). $\square$

**Lemma 26.3.** In Situation 26.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$. 
Proof. For every open $U_0 \subset S_0$ consider the condition $P$ that the functor
\[
\text{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)
\]
is an equivalence where $\text{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove $P$ holds for all quasi-compact opens $U_0$ by the induction principle of Cohomology of Schemes, Lemma 14. Let us observe that we already know the functor is fully faithful by Lemma 26.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that $P$ holds for $U_0$, $V_0$, and $U_0 \cap V_0$. Let $E$ be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on $U_i$ and $E_{V,i}$ perfect on $V_i$ whose pullback to $U$ and $V$ are isomorphic to $E|_U$ and $E|_V$. Denote
\[
a : E_{U,i} \to (Rf_{i*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i*}E)|_{V_i}
\]
the maps adjoint to the isomorphisms $Lf_{i*}E_{U,i} \to E|_U$ and $Lf_{i*}E_{V,i} \to E|_V$. By fully faithfulness, after increasing $i$, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications
\[
Lf_{i*}E_{U,i}|_{U_i \cap V_i} \to E|_{U_i \cap V_i} \quad \text{and} \quad Lf_{i*}E_{V,i}|_{U_i \cap V_i}.
\]
Apply Cohomology, Lemma 41.1 to get an object $E_i$ on $S_i$ and a map $d : E_i \to Rf_{i*}E$ which restricts to the maps $a$ and $b$ over $U_i$ and $V_i$. Then it is clear that $E_i$ is perfect and that $d$ is adjoint to an isomorphism $Lf_{i*}E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when $S_0$ is affine. Say $S_0 = \text{Spec}(A_0)$, $S_i = \text{Spec}(A_i)$, and $S = \text{Spec}(A)$. Using Lemmas 3.5 and 9.7 we see that we have to show that
\[
D_{perf}(A) = \text{colim} D_{perf}(A_i)
\]
This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. See More on Algebra, Lemma 69.16 for details. \qed

27. Cohomology and base change, VI

0A1G A final section on cohomology and base change continuing the discussion of Sections 21 and 23. An easy to grok special case is given in Remark 27.2.

0A1H Lemma 27.1. Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $G^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over $S$, with support proper over $S$. Then
\[
K = Rf_*(E \otimes_{\mathcal{O}_X} G^\bullet)
\]
is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 23.4. Thus it suffices to show that $K$ is a perfect object. If $S$ is Noetherian, then this follows from Lemma 24.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on $S$, hence we may assume $S$ is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim} R_i$ as a filtered colimit of Noetherian rings $R_i$. By Limits, Lemma 10.1 there exists an $i$ and a scheme $X_i$ of finite presentation over $R_i$ whose base change to $R$ is $X$. By Limits, Lemma 10.2 we may assume after increasing $i$,
that there exists a bounded complex of finitely presented \( \mathcal{O}_X \)-modules \( \mathcal{G}_i^* \) whose pullback to \( X \) is \( \mathcal{G}^* \). After increasing \( i \) we may assume \( \mathcal{G}_i^\text{n} \) is flat over \( R_i \), see Limits, Lemma \([10.4] \). After increasing \( i \) we may assume the support of \( \mathcal{G}_i^\text{n} \) is proper over \( R_i \), see Limits, Lemma \([13.3] \) and Cohomology of Schemes, Lemma \([26.7] \). Finally, by Lemma \([26.3] \) we may, after increasing \( i \), assume there exists a perfect object \( E_i \) of \( D(\mathcal{O}_X) \) whose pullback to \( X \) is \( E \). Applying Lemma \([24.2] \) to \( X_i \to \text{Spec}(R_i) \), \( E_i \), \( \mathcal{G}_i^* \) and using the base change property already shown we obtain the result. \( \square \)

0AII \textbf{Remark 27.2.} Let \( R \) be a ring. Let \( X \) be a scheme of finite presentation over \( R \). Let \( \mathcal{G} \) be a finitely presented \( \mathcal{O}_X \)-module flat over \( R \) with support proper over \( R \). By Lemma \([27.1] \) there exists a finite complex of finite projective \( R \)-modules \( M^* \) such that we have

\[
R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^* \otimes_R R'
\]

functionally in the \( R \)-algebra \( R' \).

0CSC \textbf{Lemma 27.3.} Let \( f : X \to S \) be a morphism of finite presentation. Let \( E \in D(\mathcal{O}_X) \) be a pseudo-coherent object. Let \( \mathcal{G}^* \) be a bounded above complex of finitely presented \( \mathcal{O}_X \)-modules, flat over \( S \), with support proper over \( S \). Then

\[
K = Rf_*(E \otimes_{\mathcal{O}_X} \mathcal{G}^*)
\]

is a pseudo-coherent object of \( D(\mathcal{O}_S) \) and its formation commutes with arbitrary base change.

\textbf{Proof.} The statement on base change is Lemma \([23.4] \). Thus it suffices to show that \( K \) is a pseudo-coherent object. This will follow from Lemma \([27.1] \) by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is local on \( S \), hence we may assume \( S \) is affine. Then \( X \) is quasi-compact and quasi-separated. Moreover, there exists an integer \( N \) such that total direct image \( Rf_* : D_{\mathcal{Q}Coh}(\mathcal{O}_X) \to D_{\mathcal{Q}Coh}(\mathcal{O}_S) \) has cohomological dimension \( N \) as explained in Lemma \([4.1] \). Choose an integer \( b \) such that \( \mathcal{G}^i = 0 \) for \( i > b \). It suffices to show that \( K \) is \( m \)-pseudo-coherent for every \( m \). Choose an approximation \( P \to E \) by a perfect complex \( P \) of \( (X, E, m - N - 1 - b) \). This is possible by Theorem \([13.6] \).

Choose a distinguished triangle

\[
P \to E \to C \to P[1]
\]

in \( D_{\mathcal{Q}Coh}(\mathcal{O}_X) \). The cohomology sheaves of \( C \) are zero in degrees \( \geq m - N - 1 - b \). Hence the cohomology sheaves of \( C \otimes^L \mathcal{G}^* \) are zero in degrees \( \geq m - N - 1 \). Thus the cohomology sheaves of \( Rf_*(C \otimes^L \mathcal{G}^*) \) are zero in degrees \( \geq m - 1 \). Hence

\[
Rf_*(P \otimes^L \mathcal{G}^*) \to Rf_*(E \otimes^L \mathcal{G}^*)
\]

is an isomorphism on cohomology sheaves in degrees \( \geq m \). Next, suppose that \( H^i(P) = 0 \) for \( i > a \). Then \( P \otimes^L \sigma_{\geq m - N - 1 - a} \mathcal{G}^* \to P \otimes^L \mathcal{G}^* \) is an isomorphism on cohomology sheaves in degrees \( \geq m - N - 1 \). Thus again we find that

\[
Rf_*(P \otimes^L \sigma_{\geq m - N - 1 - a} \mathcal{G}^*) \to Rf_*(P \otimes^L \mathcal{G}^*)
\]

is an isomorphism on cohomology sheaves in degrees \( \geq m \). By Lemma \([27.1] \) the source is a perfect complex. We conclude that \( K \) is \( m \)-pseudo-coherent as desired. \( \square \)

0B91 \textbf{Lemma 27.4.} Let \( S \) be a scheme. Let \( f : X \to S \) be a proper morphism of finite presentation.
(1) Let $E \in D(\mathcal{O}_X)$ be perfect and $f$ flat. Then $Rf_*E$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

(2) Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over $S$. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

**Proof.** Special cases of Lemma 27.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. \qed

0CSD **Lemma 27.5.** Let $S$ be a scheme. Let $f : X \to S$ be a flat proper morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then $Rf_*E$ is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

More generally, if $f : X \to S$ is proper and $E$ on $X$ is pseudo-coherent relative to $S$ (More on Morphisms, Definition 51.2), then $Rf_*E$ is pseudo-coherent (but formation does not commute with base change in this generality). See [Kle72].

**Proof.** Special case of Lemma 27.3 applied with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0. \qed

0D2Q **Lemma 27.6.** Let $R$ be a ring. Let $X$ be a scheme and let $f : X \to \text{Spec}(R)$ be proper, flat, and of finite presentation. Let $(M_n)$ be an inverse system of $R$-modules with surjective transition maps. Then the canonical map

$$\mathcal{O}_X \otimes_R (\lim M_n) \to \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to $D\mathcal{Q}_X$ applied to the target.

**Proof.** The statement means that for any object $E$ of $D_{\mathcal{Q}\mathcal{Coh}}(\mathcal{O}_X)$ the induced map

$$\text{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \to \text{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{\mathcal{Q}\mathcal{Coh}}(\mathcal{O}_X)$ has a perfect generator (Theorem 14.3) it suffices to check this for perfect $E$. By Lemma 3.2 we have $\lim \mathcal{O}_X \otimes_R M_n = R \lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\text{Hom}_X(E, -) : D_{\mathcal{Q}\mathcal{Coh}}(\mathcal{O}_X) \to D(R)$ of Cohomology, Section 40 commutes with products and hence with derived limits, whence

$$R\text{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R \lim R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology, Lemma 45.11. We have

$$R\text{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes_{\mathcal{O}_X} Lf^* M_n) = R\Gamma(X, E^\vee) \otimes_R M_n$$

by Lemma 21.1. From Lemma 27.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of $R$-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 91.5 we obtain

$$R \lim R\Gamma(X, E^\vee) \otimes_R M_n = R\Gamma(X, E^\vee) \otimes_R \lim M_n$$

as desired. \qed

0A1J **Lemma 27.7.** Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over $S$, with support proper over $S$. Then

$$K = Rf_* R\text{Hom}(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

**Proof.** Special cases of Lemma 27.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. \qed
Proof. The statement on base change is Lemma 23.5. Thus it suffices to show that $K$ is a perfect object. If $S$ is Noetherian, then this follows from Lemma 24.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on $S$, hence we may assume $S$ is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim} R_i$ as a filtered colimit of Noetherian rings $R_i$. By Limits, Lemma 10.1 there exists an $i$ and a scheme $X_i$ of finite presentation over $R_i$ whose base change to $S$ is $X$. By Limits, Lemma 10.2 we may assume after increasing $i$, that there exists a bounded complex of finitely presented $O_{X_i}$-modules $G_i^\bullet$ whose pullback to $X$ is $G_i^\bullet$. After increasing $i$ we may assume the support of $G_i^n$ is proper over $R_i$, see Limits, Lemma 13.5 and Cohomology of Schemes, Lemma 26.7. Finally, by Lemma 26.3 we may, after increasing $i$, assume there exists a perfect object $E_i$ of $D(O_{X_i})$ whose pullback to $X$ is $E_i$. Applying Lemma 24.3 to $X_i \to \text{Spec}(R_i), E_i, G_i^\bullet$ and using the base change property already shown we obtain the result. □

28. Perfect complexes

0BDH We first talk about jumping loci for betti numbers of perfect complexes. Given a complex $E$ on a scheme $X$ and a point $x$ of $X$ we often write $E \otimes L_{O_X} \kappa(x)$ instead of the more correct $Li_x^*E$, where $i_x : x \to X$ is the canonical morphism.

0BDI Lemma 28.1. Let $X$ be a scheme. Let $E \in D(O_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbb{Z}$ consider the function

$$\beta_i : X \to \{0, 1, 2, \ldots\}, \quad x \mapsto \dim_{\kappa(x)} H^i(E \otimes_{O_X} \kappa(x))$$

Then we have

(1) formation of $\beta_i$ commutes with arbitrary base change,

(2) the functions $\beta_i$ are upper semi-continuous, and

(3) the level sets of $\beta_i$ are locally constructible in $X$.

Proof. Consider a morphism of schemes $f : Y \to X$ and a point $y \in Y$. Let $x$ be the image of $y$ and consider the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{j} & X \\
\downarrow{g} & & \downarrow{f} \\
X & \xrightarrow{i} & X
\end{array}
$$

Then we see that $Lg^* \circ Li^* = Lj^* \circ Lf^*$. This implies that the function $\beta'_i$ associated to the perfect complex $Lf^*K$ is the pullback of the function $\beta_i$, in a formula: $\beta'_i = \beta_i \circ f$. This is the meaning of (1).

Fix $i$ and let $x \in X$. It is enough to prove (2) and (3) holds in an open neighbourhood of $x$, hence we may assume $X$ affine. Then we can represent $E$ by a bounded above complex $\mathcal{F}^\bullet$ of finite free modules (Lemma 12.2). Then $P = \sigma_{\geq i+1}F^\bullet$ is a perfect object and $P \to E$ induces an isomorphism

$$H^i(P \otimes_{O_X} \kappa(x')) \to H^i(E \otimes_{O_X} \kappa(x'))$$

for all $x' \in X$. Thus we may assume $E$ is perfect. In this case by More on Algebra, Lemma 70.6 there exists an affine open neighbourhood $U$ of $x$ and $a \leq b$ such that
$K|_U$ is represented by a complex
\[ \ldots \to 0 \to \mathcal{O}_U^{\oplus \beta_{\leq i}(x)} \to \mathcal{O}_U^{\oplus \beta_{i+1}(x)} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{r-1}(x)} \to \mathcal{O}_U^{\oplus \beta_0(x)} \to 0 \to \ldots \]
(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 9.7.) It follows immediately that $\beta_i(x') \leq \beta_i(x)$ for all $x' \in U$. This proves that $\beta_i$ is upper semi-continuous.

To prove (3) we may assume that $X$ is affine and $K$ is given by a complex of finite free $\mathcal{O}_X$-modules (for example by arguing as in the previous paragraph, or by using Cohomology, Lemma 45.3). Thus we have to show that given a complex $\mathcal{O}_X^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X^{\oplus c}$ the function associated to a point $x \in X$ the dimension of the cohomology of $\kappa_x^{\oplus a} \to \kappa_x^{\oplus b} \to \kappa_x^{\oplus c}$ in the middle has constructible level sets. Let $A \in \text{Mat}(a \times b, \Gamma(X, \mathcal{O}_X))$ be the matrix of the first arrow. The rank of the image of $A$ in $\text{Mat}(a \times b, \kappa(x))$ is equal to $r$ if all $(r+1) \times (r+1)$-minors of $A$ vanish at $x$ and there is some $r \times r$-minor of $A$ which does not vanish at $x$. Thus the set of points where the rank is $r$ is a constructible locally closed set. Arguing similarly for the second arrow and putting everything together we obtain the desired result. \hfill \square

\textbf{Lemma 28.2.} Let $X$ be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. The function
\[ \chi_E : X \to \mathbb{Z}, \quad x \mapsto \sum (-1)^i \dim_{\kappa(x)} H^i(E \otimes \mathcal{O}_X \kappa(x)) \]
is locally constant on $X$.

\textbf{Proof.} By Cohomology, Lemma 45.3 we see that we can, locally on $X$, represent $E$ by a finite complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules. On such an open the function $\chi_E$ is constant with value $\sum (-1)^i \text{rank}(\mathcal{E}^i)$. \hfill \square

\textbf{Lemma 28.3.} Let $X$ be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbb{Z}$, there exists an open subscheme $U \subset X$ characterized by the following
\begin{enumerate}
  \item $E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank $r$,
  \item a morphism $f : Y \to X$ factors through $U$ if and only if $L f^* E$ is isomorphic to a locally free module of rank $r$ placed in degree $i$.
\end{enumerate}

\textbf{Proof.} Let $\beta_j : X \to \{0, 1, 2, \ldots\}$ for $j \in \mathbb{Z}$ be the functions of Lemma 28.1. Then the set
\[ W = \{ x \in X \mid \beta_j(x) \leq 0 \text{ for all } j \neq i \} \]
is open in $X$ and its formation commutes with pullback to any $Y$ over $X$. This follows from the lemma using that apriori in a neighbourhood of any point only a finite number of the $\beta_j$ are nonzero. Thus we may replace $X$ by $W$ and assume that $\beta_j(x) = 0$ for all $x \in X$ and all $j \neq i$. In this case $H^i(E)$ is a finite locally free module and $E \cong H^i(E)[-i]$, see for example More on Algebra, Lemma 70.6. Thus $X$ is the disjoint union of the open subschemes where the rank of $H^i(E)$ is fixed and we win. \hfill \square

\textbf{Lemma 28.4.} Let $X$ be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbb{Z}$. Let $r \geq 0$. Then there exists a locally closed subscheme $j : Z \to X$ characterized by the following
\begin{enumerate}
  \item $H^a(Lj^* E)$ is a locally free $\mathcal{O}_Z$-module of rank $r$, and
(2) a morphism \( f : Y \to X \) factors through \( Z \) if and only if for all morphisms \( g : Y' \to Y \) the \( \mathcal{O}_Y \)-module \( H^a(L(f \circ g)^*E) \) is locally free of rank \( r \). Moreover, \( j : Z \to X \) is of finite presentation and we have

(3) if \( f : Y \to X \) factors as \( Y \to Z \to X \), then \( H^a(Lf^*E) = g^*H^a(Lj^*E) \),

(4) if \( \beta_a(x) \leq r \) for all \( x \in X \), then \( j \) is a closed immersion and given \( f : Y \to X \) the following are equivalent

(a) \( f : Y \to X \) factors through \( Z \),
(b) \( H^0(Lf^*E) \) is a locally free \( \mathcal{O}_Y \)-module of rank \( r \),
and if \( r = 1 \) these are also equivalent to
(c) \( \mathcal{O}_Y \to \text{Hom}_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E)) \) is injective.

**Proof.** First, let \( U \subset X \) be the locally constructible open subscheme where the function \( \beta_a \) of Lemma 28.1 has values \( \leq r \). Let \( f : Y \to X \) be as in (2). Then for any \( y \in Y \) we have \( \beta_a(Lf^*E) = r \) hence \( y \) maps into \( U \) by Lemma 28.1. Hence \( f \) as in (2) factors through \( U \). Thus we may replace \( X \) by \( U \) and assume that \( \beta_a(x) \in \{0, 1, \ldots, r\} \) for all \( x \in X \). We will show that in this case there is a closed subscheme \( Z \subset X \) cut out by a finite type quasi-coherent ideal characterized by the equivalence of (4) (a), (b) and (4)(c) if \( r = 1 \) and that (3) holds. This will finish the proof because it will a fortiori show that morphisms as in (2) factor through \( Z \).

If \( x \in X \) and \( \beta_a(x) < r \), then there is an open neighbourhood of \( x \) where \( \beta_a < r \) (Lemma 28.1). In this way we see that set theoretically at least \( Z \) is a closed subset.

To get a scheme theoretic structure, consider a point \( x \in X \) with \( \beta_a(x) = r \). Set \( \beta = \beta_{a+1}(x) \). By More on Algebra, Lemma 70.6 there exists an affine open neighbourhood \( U \) of \( x \) such that \( K|_U \) is represented by a complex

\[
\cdots \to 0 \to \mathcal{O}^\beta_U \to \mathcal{O}^\beta_U (f_{ij}) \to \cdots \to \mathcal{O}^\beta_{U} (\beta_{a-1}(x)) \to \mathcal{O}^\beta_{U} (\beta_a(x)) \to 0 \to \cdots
\]

(This also uses earlier results to turn the problem into algebra, for example Lemmas 3.5 and 9.7) Now, if \( g : Y \to U \) is any morphism of schemes such that \( g^i(f_{ij}) \) is nonzero for some pair \( i, j \), then \( H^0(Lg^*E) \) is not a locally free \( \mathcal{O}_Y \)-module of rank \( r \). See More on Algebra, Lemma 15.7. Trivially \( H^0(Lg^*E) \) is a locally free \( \mathcal{O}_Y \)-module if \( g^i(f_{ij}) = 0 \) for all \( i, j \). Thus we see that over \( U \) the closed subscheme cut out by all \( f_{ij} \) satisfies (3) and we have the equivalence of (4)(a) and (b). The characterization of \( Z \) shows that the locally constructed patches glue (details omitted). Finally, if \( r = 1 \) then (4)(c) is equivalent to (4)(b) because in this case locally \( H^0(Lg^*E) \subset \mathcal{O}_Y \) is the annihilator of the ideal generated by the elements \( g^i(f_{ij}) \).

\[
\square
\]

29. Applications

0BDM Mostly applications of cohomology and base change. In the future we may generalize these results to the situation discussed in Lemma 27.1.

0BDN \textbf{Lemma 29.1.} Let \( f : X \to S \) be a flat, proper morphism of finite presentation. Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module of finite presentation, flat over \( S \). For fixed \( i \in \mathbb{Z} \) consider the function

\[
\beta_i : S \to \{0, 1, 2, \ldots\}, \quad s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)
\]

Then we have

(1) formation of \( \beta_i \) commutes with arbitrary base change,

(2) the functions \( \beta_i \) are upper semi-continuous, and

(3) the level sets of \( \beta_i \) are locally constructible in \( S \).
Proof. By cohomology and base change (more precisely by Lemma \ref{lem:cohomology-base-change}) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of $S$ whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_s, \mathcal{F}_s) = H^i(K \otimes_{\mathcal{O}_S} \kappa(s))$$

Thus the lemma follows from Lemma \ref{lem:local-constant-function}. □

Lemma 29.2. Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over $S$. The function

$$s \mapsto \chi(X_s, \mathcal{F}_s)$$

is locally constant on $S$. Formation of this function commutes with base change.

Proof. By cohomology and base change (more precisely by Lemma \ref{lem:cohomology-base-change}) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of $S$ whose formation commutes with arbitrary base change. Thus we have to show the map

$$s \mapsto \sum (-1)^i \dim_{\kappa(s)} H^i(K \otimes_{\mathcal{O}_S} \kappa(s))$$

is locally constant on $S$. This is Lemma \ref{lem:local-constant-function}. □

Lemma 29.3. Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over $S$. Fix $i, r \in \mathbb{Z}$. Then there exists an open subscheme $U \subset S$ with the following property: A morphism $T \to S$ factors through $U$ if and only if $Rf_{T*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank $r$ placed in degree $i$.

Proof. By cohomology and base change (more precisely by Lemma \ref{lem:cohomology-base-change}) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of $S$ whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma \ref{lem:local-constant-function}. □

Lemma 29.4. Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over $S$ with support proper over $S$. If $R^i f_*\mathcal{F} = 0$ for $i > 0$, then $f_*\mathcal{F}$ is locally free and its formation commutes with arbitrary base change (see proof for explanation).

Proof. By Lemma \ref{lem:cohomology-base-change} the object $E = Rf_*\mathcal{F}$ of $D(\mathcal{O}_S)$ is perfect and its formation commutes with arbitrary base change, in the sense that $Rf_i(g')^*E = Lg^*E$ for any cartesian diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S \\
\end{array}
$$

of schemes. Since there is never any cohomology in degrees $< 0$, we see that $E$ (locally) has tor-amplitude in $[0, b]$ for some $b$. If $H^i(E) = R^i f_*\mathcal{F} = 0$ for $i > 0$, then $E$ has tor amplitude in $[0, 0]$. Whence $E = H^0(E)[0]$. We conclude $H^0(E) = f_*\mathcal{F}$ is finite locally free by More on Algebra, Lemma \ref{lem:finitely-presented} (and the characterization of finite projective modules in Algebra, Lemma \ref{lem:finite-projective}). Commutation with base change means that $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$ for a diagram as above and it follows from the already established commutation of base change for $E$. □

Lemma 29.5. Let $f : X \to S$ be a morphism of schemes. Assume
In this section we state and prove some results that can be deduced from the theory

Let $f : X \to S$ be a morphism of schemes. Assume

1. $f$ is proper, flat, and of finite presentation, and
2. for all $s \in S$ we have $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$.

Then we have

(a) $f_* \mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change,
(b) locally on $S$ we have

$$Rf_* \mathcal{O}_X = \mathcal{O}_S \oplus P$$

in $D(\mathcal{O}_S)$ where $P$ is perfect of tor amplitude in $[1, \infty)$.

**Proof.** By cohomology and base change (Lemma 27.4) the complex $E = Rf_* \mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. In particular, for $s \in S$ we see that $H^0(E \otimes^L \kappa(s)) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Thus $\beta_0(s) \leq 1$ for all $s \in S$ with notation as in Lemma 28.1. Apply Lemma 28.4 with $a = 0$ and $r = 1$. We obtain a universal closed subscheme $j : Z \to S$ with $H^0(Lj^* E)$ invertible characterized by the equivalence of (4)(a), (b), and (c) of the lemma.

Since formation of $E$ commutes with base change, we have

$$Lf^* E = R\text{pr}_{1, *} \mathcal{O}_{X \times_S X}$$

The morphism $\text{pr}_1 : X \times_S X$ has a section namely the diagonal morphism $\Delta$ for $X$ over $S$. We obtain maps

$$\mathcal{O}_X \to R\text{pr}_{1, * \mathcal{O}_{X \times_S X}} \to \mathcal{O}_X$$

in $D(\mathcal{O}_X)$ whose composition is the identity. Thus $R\text{pr}_{1, * \mathcal{O}_{X \times_S X}} = \mathcal{O}_X \oplus E'$ in $D(\mathcal{O}_X)$. Thus $\mathcal{O}_X$ is a direct summand of $H^0(Lf^* E)$ and we conclude that $X \to S$ factors through $Z$ by the equivalence of (4)(c) and (4)(a) of the lemma cited above. Since $\{X \to S\}$ is an fpqc covering, we have $Z = S$. Thus $f_* \mathcal{O}_X$ is an invertible $\mathcal{O}_S$-module. We conclude $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an isomorphism because a ring map $A \to B$ such that $B$ is invertible as an $A$-module is an isomorphism. Since the assumptions are preserved under base change, we see that (a) is true.

Proof of (b). Above we have seen that for every $s \in S$ the map $\mathcal{O}_S \to H^0(E \otimes^L \kappa(s))$ is surjective. Thus we may apply More on Algebra, Lemma 71.2 to see that in an open neighbourhood of $s$ we have a decomposition $Rf_* \mathcal{O}_X = \mathcal{O}_S \oplus P$ as stated in the lemma. \hfill $\square$

**Lemma 29.6.** Let $f : X \to S$ be a morphism of schemes. Assume

1. $f$ is proper, flat, and of finite presentation, and
2. the geometric fibres of $f$ are reduced and connected.

Then $f_* \mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change.

**Proof.** By Lemma 29.5 it suffices to show that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$. This follows from Varieties, Lemma 9.3 and the fact that $X_s$ is geometrically connected and geometrically reduced. \hfill $\square$

**30. Other applications**

In this section we state and prove some results that can be deduced from the theory worked out above.

**Lemma 30.1.** Let $R$ be a coherent ring. Let $X$ be a scheme of finite presentation over $R$. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over $R$, with support proper over $R$. Then $H^i(X, \mathcal{G})$ is a coherent $R$-module.
Proof. Combine Lemma 27.1 with More on Algebra, Lemmas 62.19 and 69.2.

0CRP Lemma 30.2. Let $X$ be a quasi-compact and quasi-separated scheme. Let $K$ be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens. Then for any quasi-compact open $U \subset X$ and any perfect object $E$ in $D(\mathcal{O}_X)$ the sets

$$H^i(U, K \otimes^L E), \quad \operatorname{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology, Lemma 45.11 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^L E)$. We will use the induction principle to prove the lemma, see Cohomology of Schemes, Lemma 4.1.

First we show that it holds when $U = \text{Spec}(A)$ is affine. Namely, we can represent $K$ by a complex of $A$-modules $K^\bullet$ and $E$ by a finite complex of finite projective $A$-modules $P^\bullet$. See Lemmas 3.5 and 9.7 and our definition of perfect complexes of $A$-modules (More on Algebra, Definition 69.1). Then $(E \otimes^L K)|_U$ is represented by the total complex associated to the double complex $P^\bullet \otimes A K^\bullet$ (Lemma 3.9). Using induction on the length of the complex $P^\bullet$ (or using a suitable spectral sequence) we see that it suffices to show that $H^i(P^a \otimes_A K^\bullet)$ is countable for each $a$. Since $P^a$ is a direct summand of $A^{\oplus n}$ for some $n$ this follows from the assumption that the cohomology group $H^i(K^\bullet)$ is countable.

To finish the proof it suffices to show: if $U = V \cup W$ and the result holds for $V$, $W$, and $V \cap W$, then the result holds for $U$. This is an immediate consequence of the Mayer-Vietoris sequence, see Cohomology, Lemma 33.4.

0CRQ Lemma 30.3. Let $X$ be a quasi-compact and quasi-separated scheme such that the sets of sections of $\mathcal{O}_X$ over affine opens are countable. Let $K$ be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

1. $K = \text{hocolim} E_n$ with $E_n$ a perfect object of $D(\mathcal{O}_X)$, and
2. the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens.

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 32.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on $X$.

Assume (2). Choose a $K$-injective complex $\mathcal{K}^\bullet$ representing $K$. Choose a perfect generator $E$ of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a $K$-injective complex $\mathcal{I}^\bullet$. According to Theorem 17.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A,d)$ where $(A,d)$ is the differential graded algebra

$$(A,d) = \operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps $K$ to the differential graded module

$$M = \operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \operatorname{Ext}^i(E,E)$ and $H^i(M) = \operatorname{Ext}^i(E,K)$. Moreover, since $F$ is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write $M$ as a homotopy colimit of compact objects of $D(A,d)$.
By Differential Graded Algebra, Lemma \ref{lemma-dga} it suffices to show that \( \text{Ext}^i(E,E) \) and \( \text{Ext}^i(E,K) \) are countable for each \( i \). This follows from Lemma \ref{lemma-characterize}

0CRR

**Lemma 30.4.** Let \( A \) be a ring. Let \( X \) be a scheme of finite presentation over \( A \). Let \( f : U \to X \) be a flat morphism of finite presentation. Then

1. there exists an inverse system of perfect objects \( L_n \) of \( D(\mathcal{O}_X) \) such that
   \[
   R\Gamma(U, Lf^* K) = \text{hocolim} \ R\text{Hom}_X(L_n, K)
   \]
   in \( D(A) \) functorially in \( K \) in \( D_{QCoh}(\mathcal{O}_X) \), and
2. there exists a system of perfect objects \( E_n \) of \( D(\mathcal{O}_X) \) such that
   \[
   R\Gamma(U, Lf^* K) = \text{hocolim} \ R\Gamma(X, E_n \otimes^L K)
   \]
   in \( D(A) \) functorially in \( K \) in \( D_{QCoh}(\mathcal{O}_X) \).

**Proof.** By Lemma \ref{lemma-characterize} we have
\[
R\Gamma(U, Lf^* K) = R\Gamma(X, Rf_* \mathcal{O}_U \otimes^L K)
\]
functorially in \( K \). Observe that \( R\Gamma(X, -) \) commutes with homotopy colimits because it commutes with direct sums by Lemma \ref{lemma-homotopy-colimits}. Similarly, \( - \otimes^L K \) commutes with derived colimits because \( - \otimes^L K \) commutes with direct sums (because direct sums in \( D(\mathcal{O}_X) \) are given by direct sums of representing complexes). Hence to prove (2) it suffices to write \( Rf_* \mathcal{O}_U = \text{hocolim} E_n \) for a system of perfect objects \( E_n \) of \( D(\mathcal{O}_X) \). Once this is done we obtain (1) by setting \( L_n = E_n' \), see Cohomology, Lemma \ref{lemma-characterize}.

Write \( A = \text{colim} A_i \) with \( A_i \) of finite type over \( \mathbb{Z} \). By Limits, Lemma \ref{lemma-characterize} we can find an \( i \) and morphisms \( U_i \to X_i \to \text{Spec}(A_i) \) of finite presentation whose base change to \( \text{Spec}(A) \) recovers \( U \to X \to \text{Spec}(A) \). After increasing \( i \) we may assume that \( f_i : U_i \to X_i \) is flat, see Limits, Lemma \ref{lemma-characterize}. By Lemma \ref{lemma-characterize} the derived pullback of \( Rf_{i,*} \mathcal{O}_{U_i} \) by \( g : X \to X_i \) is equal to \( Rf_* \mathcal{O}_U \). Since \( Lg^* \) commutes with derived colimits, it suffices to prove what we want for \( f_i \). Hence we may assume that \( U \) and \( X \) are of finite type over \( \mathbb{Z} \).

Assume \( f : U \to X \) is a morphism of schemes of finite type over \( \mathbb{Z} \). To finish the proof we will show that \( Rf_* \mathcal{O}_U \) is a homotopy colimit of perfect complexes. To see this we apply Lemma \ref{lemma-characterize} Thus it suffices to show that \( Rf_* \mathcal{O}_U \) has countable sets of sections over affine opens. This follows from Lemma \ref{lemma-characterize} applied to the structure sheaf. \( \square \)

31. Characterizing pseudo-coherent complexes, II

0CSE This section is a continuation of Section \ref{section-characterize}. In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. More results of this nature can be found in More on Morphisms, Section \ref{section-characterize}.

0CSF **Lemma 31.1.** Let \( A \) be a ring. Let \( R \) be a (possibly noncommutative) \( A \)-algebra which is finite free as an \( A \)-module. Then any object \( M \) of \( D(R) \) which is pseudo-coherent in \( D(A) \) can be represented by a bounded above complex of finite free (right) \( R \)-modules.

**Proof.** Choose a complex \( M^\bullet \) of right \( R \)-modules representing \( M \). Since \( M \) is pseudo-coherent we have \( H^i(M) = 0 \) for large enough \( i \). Let \( m \) be the smallest index such that \( H^m(M) \) is nonzero. Then \( H^m(M) \) is a finite \( A \)-module by More on
Algebra, Lemma \(^{62.3}\). Thus we can choose a finite free \(R\)-module \(F^m\) and a map \(F^m \to M^m\) such that \(F^m \to M^m \to M^{m+1}\) is zero and such that \(F^m \to H^m(M)\) is surjective. Picture:

\[
\begin{array}{c}
F^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \\
M^{i-1} & \longrightarrow & M^i & \longrightarrow & \cdots
\end{array}
\]

By descending induction on \(n \leq m\) we are going to construct finite free \(R\)-modules \(F^i\) for \(i \geq n\), differentials \(d^i : F^i \to F^{i+1}\) for \(i \geq n\), maps \(\alpha : F^i \to K^i\) compatible with differentials, such that (1) \(H^i(\alpha)\) is an isomorphism for \(i > n\) and surjective for \(i = n\), and (2) \(F^i = 0\) for \(i > b\). Picture

\[
\begin{array}{c}
F^n & \longrightarrow & F^{n+1} & \longrightarrow & \cdots & \longrightarrow & F^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & & & & & & & \\
M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \cdots & \longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & \cdots
\end{array}
\]

The base case is \(n = m\) which we’ve done above. Induction step. Let \(C^\bullet\) be an object of \(\text{QCoh}(\mathcal{O}_A)\). By More on Algebra, Lemma \(^{62.2}\) we see that \(C^\bullet\) is \((n-1)\)-pseudo-coherent as a complex of \(A\)-modules. By More on Algebra, Lemma \(^{62.3}\) we see that \(H^{n-1}(C^\bullet)\) is a finite \(A\)-module. Choose a finite free \(R\)-module \(F^{n-1}\) and a map \(\beta : F^{n-1} \to C^{n-1}\) such that the composition \(F^{n-1} \to C^{n-1} \to C^n\) is zero and such that \(F^{n-1}\) surjects onto \(H^{n-1}(C^\bullet)\). Since \(C^{n-1} = M^{n-1} \oplus F^n\) we can write \(\beta = (a^{n-1}, -d^{n-1})\). The vanishing of the composition \(F^n \to C^{n-1} \to C^n\) implies these maps fit into a morphism of complexes

\[
\begin{array}{c}
F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \cdots \\
\downarrow & & & & & & \\
\cdots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \cdots
\end{array}
\]

Moreover, these maps define a morphism of distinguished triangles

\[
\begin{array}{c}
(F^n \to \cdots) & \longrightarrow & (F^{n-1} \to \cdots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \to \cdots)[1] \\
\downarrow & & & & & & \\
(F^n \to \cdots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \to \cdots)[1]
\end{array}
\]

Hence our choice of \(\beta\) implies that the map of complexes \((F^{n-1} \to \cdots) \to M^\bullet\) induces an isomorphism on cohomology in degrees \(\geq n\) and a surjection in degree \(n-1\). This finishes the proof of the lemma. \(\square\)

**Lemma 31.2.** Let \(A\) be a ring. Let \(n \geq 0\). Let \(K \in D_{\text{QCoh}}(\mathcal{O}_{P^n_A})\). The following are equivalent

1. \(K\) is pseudo-coherent,
2. \(\check{R}^n(P^n_A, E \otimes^L K)\) is a pseudo-coherent object of \(D(A)\) for each pseudo-coherent object \(E\) of \(D(\mathcal{O}_{P^n_A})\),
(3) $R\Gamma(P^n_A, E \otimes^L K)$ is a pseudo-coherent object of $D(A)$ for each perfect object $E$ of $D(O_{p^n_A})$.

(4) $R\text{Hom}_{p^n_A}(E, K)$ is a pseudo-coherent object of $D(A)$ for each perfect object $E$ of $D(O_{p^n_A})$.

(5) $R\Gamma(P^n_A, K \otimes^L O_{p^n_A}(d))$ is pseudo-coherent object of $D(A)$ for $d = 0, 1, \ldots, n$.

**Proof.** Recall that if $(1)$ holds, then $E$ is pseudo-coherent. By Cohomology, Lemma 45.11, since every perfect complex is pseudo-coherent, it is clear that $(2)$ implies $(3)$. Assume $(1)$ holds. Then $E \otimes^L K$ is pseudo-coherent for every pseudo-coherent $E$, see Cohomology, Lemma 43.5. By Lemma 27.5 the direct image of such a pseudo-coherent complex is pseudo-coherent and we see that $(2)$ is true.

Part $(3)$ implies $(5)$ because we can take $E = O_{p^n_A}(d)$ for $d = 0, 1, \ldots, n$.

To finish the proof we have to show that $(5)$ implies $(1)$. Let $P$ be as in (19.0.1) and $R$ as in (19.0.2). By Lemma 19.1 we have an equivalence

$$- \otimes^L \mathcal{R}P : D(R) \rightarrow D_{\text{QCoh}}(O_{p^n_A})$$

Let $M \in D(R)$ be an object such that $M \otimes^L P = K$. By Differential Graded Algebra, Lemma 26.4 there is an isomorphism

$$R\text{Hom}(R, M) = R\text{Hom}_{p^n_A}(P, K)$$

in $D(A)$. Arguing as above we obtain

$$R\text{Hom}_{p^n_A}(P, K) = R\Gamma(P^n_A, R\text{Hom}_{O_{p^n_A}}(E, K)) = R\Gamma(P^n_A, P^\vee \otimes^L O_{p^n_A} K).$$

Using that $P^\vee$ is the direct sum of $O_{p^n_A}(d)$ for $d = 0, 1, \ldots, n$ and (5) we conclude $R\text{Hom}(R, M)$ is pseudo-coherent as a complex of $A$-modules. Of course $M = R\text{Hom}(R, M)$ in $D(A)$. Thus $M$ is pseudo-coherent as a complex of $A$-modules. By Lemma 31.1 we may represent $M$ by a bounded above complex $F^\bullet$ of finite free $R$-modules. Then $F^\bullet = \bigcup_{p \geq 0} \sigma_{\geq p} F^\bullet$ is a filtration which shows that $F^\bullet$ is a differential graded $R$-module with property (P), see Differential Graded Algebra, Section 13. Hence $K = M \otimes^L_R P$ is represented by $F^\bullet \otimes_R P$ (follows from the construction of the derived tensor functor, see for example the proof of Differential Graded Algebra, Lemma 26.3). Since $F^\bullet \otimes_R P$ is a bounded above complex whose terms are direct sums copies of $P$ we conclude that the lemma is true. □

**Lemma 31.3.** Let $A$ be a ring. Let $X$ be a scheme over $A$ which is quasi-compact and quasi-separated. Let $K \in D_{\text{QCoh}}(O_X)$. If $R\Gamma(X, E \otimes^L K)$ is pseudo-coherent in $D(A)$ for every perfect $E$ in $D(O_X)$, then $R\Gamma(X, E \otimes^L K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E$ in $D(O_X)$.

This lemma is false if one drops the assumption that $K$ is bounded above.

**Proof.** There exists an integer $N$ such that $R\Gamma(X, -) : D_{\text{QCoh}}(O_X) \rightarrow D(A)$ has cohomological dimension $N$ as explained in Lemma 11. Let $b \in \mathbb{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let $E$ be pseudo-coherent on $X$. It suffices to show that $R\Gamma(X, E \otimes^L K)$ is $m$-pseudo-coherent for every $m$. Choose an approximation...
$P \to E$ by a perfect complex $P$ of $(X, E, m - N - 1 - b)$. This is possible by Theorem \[13.6\]. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of $C$ are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^L K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^L K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^L K) \to R\Gamma(X, E \otimes^L K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^L K)$ is $m$-pseudo-coherent as desired. \[\square\]

### 32. Relatively perfect objects

0DHZ In this section we introduce a notion from \[\text{[Lie06]}\].

**Definition 32.1.** Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. An object $E$ of $D(\mathcal{O}_X)$ is $S$-perfect if $E$ is pseudo-coherent (Cohomology, Definition \[43.1\]) and $E$ locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition \[44.1\]).

Please see Remark \[32.13\] for a discussion.

**Example 32.2.** Let $k$ be a field. Let $X$ be a scheme of finite presentation over $k$ (in particular $X$ is quasi-compact). Then an object $E$ of $D(\mathcal{O}_X)$ is $k$-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{Coh}(X)$ (by Lemma \[9.3\]). Thus being relatively perfect does not mean “perfect on the fibres”.

The corresponding algebra concept is studied in More on Algebra, Section \[76\]. We can link the notion for schemes with the algebraic notion as follows.

**Lemma 32.3.** Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $E$ be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

1. $E$ is $S$-perfect,
2. for any affine open $U \subset X$ mapping into an affine open $V \subset S$ the complex $R\Gamma(U, E)$ is $\mathcal{O}_S(V)$-perfect,
3. there exists an affine open covering $S = \bigcup V_i$ and for each $i$ an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex $R\Gamma(U_{ij}, E)$ is $\mathcal{O}_S(V_i)$-perfect.

**Proof.** Being pseudo-coherent is a local property and “locally having finite tor dimension” is a local property. Hence this lemma immediately reduces to the statement: if $X$ and $S$ are affine, then $E$ is $S$-perfect if and only if $K = R\Gamma(X, E)$ is $\mathcal{O}_S(S)$-perfect. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and $E$ corresponds to $K \in D(A)$, i.e., $K = R\Gamma(X, E)$, see Lemma \[3.5\].

Observe that $K$ is $R$-perfect if and only if $K$ is pseudo-coherent and has finite tor dimension as a complex of $R$-modules (More on Algebra, Definition \[76.1\]). By Lemma \[9.2\] we see that $E$ is pseudo-coherent if and only if $K$ is pseudo-coherent. By Lemma \[9.5\] we see that $E$ has finite tor dimension over $f^{-1}\mathcal{O}_S$ if and only if $K$ has finite tor dimension as a complex of $R$-modules. \[\square\]
Lemma 32.4. Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. The full subcategory of \( D(\mathcal{O}_X) \) consisting of \( S \)-perfect objects is a saturated\(^4\) triangulated subcategory.

**Proof.** This follows from Cohomology, Lemmas 43.6 and 44.6 and 44.8. \( \square \)

Lemma 32.5. Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. A perfect object of \( D(\mathcal{O}_X) \) is \( S \)-perfect. If \( K, M \in D(\mathcal{O}_X) \), then \( K \otimes^L_{\mathcal{O}_X} M \) is \( S \)-perfect if \( K \) is \( S \)-perfect and \( M \) is \( S \)-perfect.

**Proof.** First proof: reduce to the affine case using Lemma 32.3 and then apply Lemma 32.6. \( \square \)

Lemma 32.6. Let \( f : X \to S \) be a morphism of schemes which is flat and locally of finite presentation. Let \( g : S' \to S \) be a morphism of schemes. Set \( X' = S' \times_S X \) and denote \( g' : X' \to X \) the projection. If \( K \in D(\mathcal{O}_X) \) is \( S \)-perfect, then \( L(g')^* K \) is \( S' \)-perfect.

**Proof.** First proof: reduce to the affine case using Lemma 32.3 and then apply More on Algebra, Lemma 76.5. Second proof: \( L(g')^* K \) is pseudo-coherent by Cohomology, Lemma 43.3 and the bounded tor dimension property follows from Lemma 21.6. \( \square \)

Situation 32.7. Let \( S = \lim_{i \in I} S_i \) be a limit of a directed system of schemes with affine transition morphisms \( g_{i'} : S_{i'} \to S_i \). We assume that \( S_i \) is quasi-compact and quasi-separated for all \( i \in I \). We denote \( g_i : S \to S_i \) the projection. We fix an element \( 0 \in I \) and a flat morphism of finite presentation \( X_0 \to S_0 \). We set \( X_i = S_i \times_{S_0} X_0 \) and \( X = S \times_{S_0} X_0 \) and we denote the transition morphisms \( f_{i'} : X_{i'} \to X_i \) and \( f_i : X \to X_i \) the projections.

Lemma 32.8. In Situation 32.7. Let \( K_0 \) and \( L_0 \) be objects of \( D(\mathcal{O}_{X_0}) \). Set \( K_i = Lf_0^* K_0 \) and \( L_i = Lf_0^* L_0 \) for \( i \geq 0 \) and set \( K = Lf_0^* K_0 \) and \( L = Lf_0^* L_0 \). Then the map

\[
\colim_{i \geq 0} \Hom_{D(\mathcal{O}_{X_i})}(K_i, L_i) \to \Hom_{D(\mathcal{O}_X)}(K, L)
\]

is an isomorphism if \( K_0 \) is pseudo-coherent and \( L_0 \in D_{QCoh}(\mathcal{O}_{X_0}) \) has (locally) finite tor dimension as an object of \( D((X_0 \to S_0)^{-1}\mathcal{O}_{S_0}) \)

**Proof.** For every quasi-compact open \( U_0 \subset X_0 \) consider the condition \( P \) that

\[
\colim_{i \geq 0} \Hom_{D(\mathcal{O}_{U_i})}(K|_{U_i}, L|_{U_i}) \to \Hom_{D(\mathcal{O}_{U})}(K|_{U}, L|_{U})
\]

is an isomorphism where \( U = f_0^{-1}(U_0) \) and \( U_i = f_0^{-1}(U_0) \). If \( P \) holds for \( U_0, V_0 \) and \( U_0 \cap V_0 \), then it holds for \( U_0 \cup V_0 \) by Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 33.3.

Denote \( \pi_0 : X_0 \to S_0 \) the given morphism. Then we can first consider \( U_0 = \pi_0^{-1}(W_0) \) with \( W_0 \subset S_0 \) quasi-compact open. By the induction principle of Cohomology of Schemes, Lemma 4.1 applied to quasi-compact opens of \( S_0 \) and the remark above, we find that it is enough to prove \( P \) for \( U_0 = \pi_0^{-1}(W_0) \) with \( W_0 \) affine. In other words, we have reduced to the case where \( S_0 \) is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of \( X_0 \), to reduce to the case where \( X_0 \) is affine as well.

\(^4\)Derived Categories, Definition 6.1.
If $X_0$ and $S_0$ are affine, the result follows from More on Algebra, Lemma 76.7. Namely, by Lemmas 9.1 and 3.5, the statement is translated into computations of homs in the derived categories of modules. Then Lemma 9.2 shows that the complex of modules corresponding to $K_0$ is pseudo-coherent. And Lemma 9.3 shows that the complex of modules corresponding to $L_0$ has finite tor dimension over $\mathcal{O}_{S_0}(S_0)$. Thus the assumptions of More on Algebra, Lemma 76.7 are satisfied and we win. □

0DI8 Lemma 32.9. In Situation 32.7 the category of $S$-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of $S_i$-perfect objects of $D(\mathcal{O}_{X_i})$.

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition $P$ that the functor

$$\text{colim}_{i \geq 0} D_{S_i}\text{-perfect}(\mathcal{O}_{U_i}) \longrightarrow D_{S}\text{-perfect}(\mathcal{O}_U)$$

is an equivalence where $U = f_0^{-1}(U_0)$ and $U_i = f_0^{-1}(U_0)$. We observe that we already know this functor is fully faithful by Lemma 32.8. Thus it suffices to prove essential surjectivity.

Suppose that $P$ holds for quasi-compact opens $U_0, V_0$ of $X_0$. We claim that $P$ holds for $U_0 \cup V_0$. We will use the notation $U_i = f_0^{-1}U_0, U = f_0^{-1}U_0, V_i = f_0^{-1}V_0$, and $V = f_0^{-1}V_0$ and we will abusively use the symbol $f_i$ for all the morphisms $U \to U_i, V \to V_i, U \cap V \to U_i \cap V_i$, and $U \cup V \to U_i \cup V_i$. Suppose $E$ is an $S$-perfect object of $D(\mathcal{O}_{U,V})$. Goal: show $E$ is in the essential image of the functor. By assumption, we can find $i \geq 0$, an $S_i$-perfect object $E_{U,i}$ on $U_i$, an $S_i$-perfect object $E_{V,i}$ on $V_i$, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let $a : E_{U,i} \to (Rf_i)_*E|_{U_i}$ and $b : E_{V,i} \to (Rf_i)_*E|_{V_i}$ the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing $i$, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 41.1 to get an object $E_i$ on $U_i \cup V_i$ and a map $d : E_i \to Rf_i_*E$ which restricts to the maps $a$ and $b$ over $U_i$ and $V_i$. Then it is clear that $E_i$ is $S_i$-perfect (because being relatively perfect is a local property) and that $d$ is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 32.8 using the induction principle (Cohomology of Schemes, Lemma 4.1) we reduce to the case where both $X_0$ and $S_0$ are affine. (First work with opens in $S_0$ to reduce to $S_0$ affine, then work with opens in $X_0$ to reduce to $X_0$ affine.) In the affine case the result follows from More on Algebra, Lemma 76.7. The translation into algebra is done by Lemma 32.3. □

0DJT Lemma 32.10. Let $f : X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be $S$-perfect. Then $Rf_*E$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 23.4 (with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0). Thus it suffices to show that $Rf_*E$ is a perfect object. We will reduce to the case where $S$ is Noetherian affine by a limit argument.

The question is local on $S$, hence we may assume $S$ is affine. Say $S = \text{Spec}(R)$. We write $R = \text{colim} R_i$ as a filtered colimit of Noetherian rings $R_i$. By Limits, Lemma
there exists an \( i \) and a scheme \( X_i \) of finite presentation over \( R_i \) whose base change to \( R \) is \( X \). By Limits, Lemmas \cite[13.1]{Th} and \cite[8.7]{Th} we may assume \( X_i \) is proper and flat over \( R_i \). By Lemma \cite[32.9]{Th} we may assume there exists a \( R_i \)-perfect object \( E_i \) of \( D(\mathcal{O}_X) \) whose pullback to \( X \) is \( E \). Applying Lemma \cite[24.1]{Th} to \( X_i \to \text{Spec}(R_i) \) and \( E_i \) and using the base change property already shown we obtain the result. □

**Lemma 32.11.** Let \( f : X \to S \) be a morphism of schemes. Let \( E, K \in D(\mathcal{O}_X) \).

Assume

(1) \( S \) is quasi-compact and quasi-separated,
(2) \( f \) is proper, flat, and of finite presentation,
(3) \( E \) is \( S \)-perfect,
(4) \( K \) is pseudo-coherent.

Then there exists a pseudo-coherent \( L \in D(\mathcal{O}_S) \) such that

\( Rf_* R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_S) \)

and the same is true after arbitrary base change: given

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
\]

cartesian, then we have

\( Rf'_* R\mathcal{H}om(L(g')^* K, L(g')^* E) = R\mathcal{H}om(Lg^* L, \mathcal{O}_{S'}) \)

**Proof.** Since \( S \) is quasi-compact and quasi-separated, the same is true for \( X \). By Lemma \cite[18.1]{Th} we can write \( K = \text{hocolim} K_n \) with \( K_n \) perfect and \( K_n \to K \) inducing an isomorphism on truncations \( \tau_{\leq -n} \). Let \( K_n^\vee \) be the dual perfect complex (Cohomology, Lemma \cite[45.11]{Th}). We obtain an inverse system \( \ldots \to K'_3 \to K'_2 \to K'_1 \) of perfect objects. By Lemma \cite[32.5]{Th} we see that \( K_n^\vee \otimes_{\mathcal{O}_X} E \) is \( S \)-perfect. Thus we may apply Lemma \cite[32.10]{Th} to \( K_n^\vee \otimes_{\mathcal{O}_X} E \) and we obtain an inverse system

\( \ldots \to M_3 \to M_2 \to M_1 \)

of perfect complexes on \( S \) with

\( M_n = Rf_*(K_n^\vee \otimes_{\mathcal{O}_X} E) = Rf_* R\mathcal{H}om(K_n, E) \)

Moreover, the formation of these complexes commutes with any base change, namely \( Lg^* M_n = Rf'_* ((L(g')^* K_n)^\vee \otimes_{\mathcal{O}_X} L(g')^* E) = Rf'_* R\mathcal{H}om(L(g')^* K_n, L(g')^* E) \).

As \( K_n \to K \) induces an isomorphism on \( \tau_{\leq -n} \), we see that \( K_n \to K_{n+1} \) induces an isomorphism on \( \tau_{\leq -n} \). It follows that \( K_{n+1}^\vee \to K_n^\vee \) induces an isomorphism on \( \tau_{\leq n} \) as \( K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X) \). Suppose that \( E \) has tor amplitude in \( [a, b] \) as a complex of \( f^{-1}\mathcal{O}_Y \)-modules. Then the same is true after any base change, see Lemma \cite[21.6]{Th}.

We find that \( K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E \) induces an isomorphism on \( \tau_{\leq n+a} \) and the same is true after any base change. Applying the right derived functor \( Rf_* \) we conclude the maps \( M_{n+1} \to M_n \) induce isomorphisms on \( \tau_{\leq n+a} \) and the same is true after any base change. Choose a distinguished triangle

\[ M_{n+1} \to M_n \to C_n \to M_{n+1}[1] \]

Take \( S' \) equal to the spectrum of the residue field at a point \( s \in S \) and pull back to see that \( C_n \otimes_{\mathcal{O}_X} \kappa(s) \) has nonzero cohomology only in degrees \( \geq n + a \). By More on Algebra, Lemma \cite[70.6]{Th} we see that the perfect complex \( C_n \) has tor amplitude in \([n + a, m_n]\) for some integer \( m_n \). In particular, the dual perfect complex \( C_n^\vee \) has tor amplitude in \([-m_n, -n - a]\).
Let $L_n = M_n$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \hocolim L_n$ is pseudo-coherent, see Lemma [18.1]. Since we have

$$R \mathcal{H}om(K, E) = R \mathcal{H}om(\hocolim K_n, E) = R \lim R \mathcal{H}om(K_n, E) = R \lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology, Lemma [15.12]) and since $R \lim$ commutes with $Rf_*$ we find that

$$Rf_* R \mathcal{H}om(K, E) = \lim M_n = \lim R \mathcal{H}om(L_n, \mathcal{O}_S) = R \mathcal{H}om(L, \mathcal{O}_S)$$

This proves the formula over $S$. Since the construction of $M_n$ is compatible with base change, the formula continues to hold after any base change. □

Remark 32.12. The reader may have noticed the similarity between Lemma 32.11 and Lemma 25.3. Indeed, the pseudo-coherent complex $L$ of Lemma 32.11 may be characterized as the unique pseudo-coherent complex on $S$ such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_S}(L, F) \to \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes_{\mathcal{O}_X} L f^* F)$$

compatible with boundary maps for $F$ ranging over $\text{QCoh}(\mathcal{O}_S)$. If we ever need this we will formulate a precise result here and give a detailed proof.

Remark 32.13. Our Definition 32.1 of a relatively perfect complex is equivalent to the one given in [Lie06] whenever our definition applies. Next, suppose that $f : X \to S$ is only assumed to be locally of finite type (not necessarily flat, nor locally of finite presentation). The definition in the paper cited above is that $E$ is relatively perfect if

(A) locally on $X$ the object $E$ should be quasi-isomorphic to a finite complex of $S$-flat, finitely presented $\mathcal{O}_X$-modules.

On the other hand, the natural generalization of our Definition 32.1 is

(B) $E$ is pseudo-coherent relative to $S$ (More on Morphisms, Definition [51.2]) and $E$ locally has finite tor dimension as an object of $D(f^{-1} \mathcal{O}_S)$ (Cohomology, Definition [44.1]).

The advantage of condition (B) is that it clearly defines a triangulated subcategory of $D(\mathcal{O}_X)$, whereas we suspect this is not the case for condition (A). The advantage of condition (A) is that it is easier to work with in particular in regards to limits.

### 33. The resolution property

This notion is discussed in the paper [Tot04]. It is currently not known if a proper scheme over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com

We can make the following definition although it scarecelly makes sense to consider it for general schemes.

Definition 33.1. Let $X$ be a scheme. We say $X$ has the resolution property if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

To see this, use Lemma [32.3] and More on Algebra, Lemma [76.4].
If $X$ is a quasi-compact and quasi-separated scheme, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Properties, Lemma 22.8. If $X$ is a Noetherian scheme, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Schemes, Lemma 9.1.

0F87 **Lemma 33.2.** Let $X$ be a scheme. If $X$ has an ample invertible $\mathcal{O}_X$-module, then $X$ has the resolution property.

**Proof.** Immediate consequence of Properties, Proposition 26.13.

0FDD **Lemma 33.3.** Let $f : X \to Y$ be a morphism of schemes. Assume

1. $X$ is quasi-compact and quasi-separated and has the resolution property,
2. there exists an $f$-ample invertible module on $X$.

Then $X$ has the resolution property.

**Proof.** Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an $f$-ample invertible module. Choose an affine open covering $Y = V_1 \cup \ldots \cup V_m$. Set $U_j = f^{-1}(V_j)$. By Properties, Proposition 26.13 for each $j$ we know there exists finitely many maps $s_{j,i} : \mathcal{L}^{\otimes n} \mid_{U_j} \to \mathcal{F} \mid_{U_j}$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$
\mathcal{H}_n = f_*(\mathcal{F} \otimes \mathcal{O}_X \mathcal{L}^{\otimes n})
$$

We may think of $s_{j,i}$ as a section over $V_j$ of the sheaf $\mathcal{H}_{n-1,i}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{H}_{n-1,i}$ such that $s_{j,i}$ is in the image. Then the corresponding maps

$$
f^*\mathcal{E}_{i,j} \otimes \mathcal{O}_X \mathcal{L}^{\otimes n} \to \mathcal{F}
$$

are going to be jointly surjective and the lemma is proved. By Properties, Lemma 22.3 for each $i,j$ we can find a finite type quasi-coherent submodule $\mathcal{H}'_{j,i} \subset \mathcal{H}_{n-1,i}$ which contains the section $s_{j,i}$ over $V_j$. Thus using the resolution property of $Y$ to get surjections $\mathcal{E}_{i,j} \to \mathcal{H}'_{j,i}$ we conclude.

0F88 **Lemma 33.4.** Let $f : X \to Y$ be an affine morphism of schemes with $Y$ quasi-compact and quasi-separated. If $Y$ has the resolution property, so does $X$.

**Proof.** Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The pushforward $f_*\mathcal{F}$ is quasi-coherent, see Schemes, Lemma 24.1. The adjunction map $f^*f_*\mathcal{F} \to \mathcal{F}$ is surjective; this follows from Schemes, Lemma 7.3 after restricting to $f^{-1}(V)$ for $V \subset Y$ affine open. Write $f_*\mathcal{F} = \text{colim} \mathcal{G}_i$ as a filtered colimit with $\mathcal{G}_i$ quasi-coherent $\mathcal{O}_Y$-modules of finite type, see Properties, Lemma 22.3. Then we see that colim $f^*\mathcal{G}_i \to \mathcal{F}$ is surjective. Since $\mathcal{F}$ is of finite type and $X$ is quasi-compact, we conclude that for some $i$ the map $f^*\mathcal{G}_i \to \mathcal{F}$ is surjective (details omitted; look at generators on affines). Hence if $\mathcal{G}_i$ is a quotient of a finite locally free $\mathcal{O}_Y$-module, then $\mathcal{F}$ is a quotient of the pullback which is a finite locally free $\mathcal{O}_X$-module.

0F89 **Lemma 33.5.** Let $X$ be a scheme. Suppose given

1. a finite affine open covering $X = U_1 \cup \ldots \cup U_m$
2. finite type quasi-coherent ideals $\mathcal{I}_j$ with $V(\mathcal{I}_j) = X \setminus U_j$

Then $X$ has the resolution property if and only if $\mathcal{I}_j$ is the quotient of a finite locally free $\mathcal{O}_X$-module for $j = 1, \ldots, m$. 
**Proof.** One direction of the lemma is trivial. For the other, say $E_j \to I_j$ is a surjection with $E_j$ finite locally free. In the next paragraph, we reduce to the Noetherian case; we suggest the reader skip it.

The first observation is that $U_j \cap U_{j'}$ is quasi-compact as the complement of the zero scheme of the quasi-coherent finite type ideal $I_j|U_j$ on the affine scheme $U_j$, see Properties, Lemma 24.1. Hence $X$ is quasi-compact and quasi-separated, see Schemes, Lemma 21.6. By Limits, Proposition 5.4 we can write $X = \lim X_i$ as the limit of a direct system of Noetherian schemes with affine transition morphisms. For each $j$ we can find an $i$ and a finite type quasi-coherent ideal sheaf $I_{i,j} \subset O_{X_i}$ pulling back to $I_j$, see Limits, Lemma 10.3. Denoting $U_{i,j} \subset X_i$ the complementary opens, we may assume these are affine for all $i, j$, see Limits, Lemma 4.13. Similarly, we may assume the maps $E_j \to I_j$ are the pullbacks of surjections $E_{i,j} \to I_{i,j}$ with $E_{i,j}$ finite locally free on $X_i$, see Limits, Lemmas 10.3 and 10.2. Using this and Lemma 33.3 we reduce to the case of a Noetherian scheme.

Assume $X$ is Noetherian. For every coherent module $F$ we can choose a finite list of sections $s_{jk} \in F(U_j)$, $k = 1, \ldots, e_j$ which generate the restriction of $F$ to $U_j$. By Cohomology of Schemes, Lemma 10.4 we can extend $s_{jk}$ to a map $s'_{jk} : I_{i,j}^{n_{jk}} \to F$ for some $n_{jk} \geq 1$. Then we can consider the compositions

$$E_j^{\otimes n_{jk}} \to I_{i,j}^{n_{jk}} \to F$$

to conclude. $\square$

---

**Lemma 33.6.** Let $X$ be a quasi-compact, regular scheme with affine diagonal. Then $X$ has the resolution property.

**Proof.** Observe that $X$ is a finite disjoint union of integral schemes (Properties, Lemmas 9.4 and 7.6). Thus we may assume that $X$ is integral as well as Noetherian, regular, and having affine diagonal. Choose an affine open covering $X = U_1 \cup \ldots \cup U_m$. We may and do assume $U_j$ nonempty for all $j$. By More on Algebra, Lemma 107.2 the local rings of $X$ are UFDs and hence by Divisors, Lemma 16.7 we can find an effective Cartier divisors $D_j \subset X$ whose complement is $U_j$. Then the ideal sheaf of $D_j$ is invertible, hence a finite locally free module and we conclude that $X$ has the resolution property by Lemma 33.5.

---

**Lemma 33.7.** Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then $X$ has the resolution property if and only if $X_i$ has the resolution properties for some $i$.

**Proof.** If $X_i$ has the resolution property, then $X$ does by Lemma 33.4. Assume $X$ has the resolution property. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$. For each $j$ choose a finite type quasi-coherent sheaf of ideals $I_j \subset O_X$ such that $X \setminus V(I_j) = U_j$, see Properties, Lemma 24.1. Choose finite locally free $O_X$-modules and surjections $E_j \to I_j$. By Limits, Lemmas 10.3 and 10.2 we can find an $i$ and locally finite free $O_{X_i}$-modules $E_{i,j}$ and maps $E_{i,j} \to O_{X_i}$ whose base changes to $X$ recover the maps $E_j \to I_j$, $j = 1, \ldots, m$. Denote $I_{i,j} \subset O_{X_i}$ the image of these maps. Set $U_{i,j} = X_i \setminus V(I_{i,j})$. After increasing $i$ we may assume $U_{i,j}$ is affine, see Limits, Lemma 4.13. Then we conclude that $X_i$ has the resolution property by Lemma 33.5.

---

**Lemma 33.8.** Let $X$ be a quasi-compact and quasi-separated scheme with the resolution property. Then $X$ has affine diagonal.

---

**Proof.** Special case of Tot04, Proposition 1.3.
Proof. Combining Limits, Proposition 5.4 and Lemma 33.7 this reduces to the case where $X$ is Noetherian (small detail omitted). Assume $X$ is Noetherian. Recall that $X \times X$ is covered by the affine opens $U \times V$ for affine opens $U, V$ of $X$, see Schemes, Section 17. Hence to show that the diagonal $\Delta : X \to X \times X$ is affine, it suffices to show that $U \cap V = \Delta^{-1}(U \times V)$ is affine for all affine opens $U, V$ of $X$, see Morphisms, Lemma 11.3. In particular, it suffices to show that the inclusion morphism $j : U \to X$ is affine if $U$ is an affine open of $X$. By Cohomology of Schemes, Lemma 3.4 it suffices to show that $R^1 j_* G = 0$ for any quasi-coherent $\mathcal{O}_U$-module $G$. By Proposition 7.3 (this is where we’ve reduced to the Noetherian case) we can represent $R j_* G$ by a complex $\mathcal{H}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Assume

$$H^1(\mathcal{H}^\bullet) = \text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)/\text{Im}(\mathcal{H}^0 \to \mathcal{H}^1)$$

is nonzero in order to get a contradiction. Then we can find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \to \text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$$

such that the composition with the projection onto $H^1(\mathcal{H}^\bullet)$ is nonzero. Namely, we can write $\text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$ as the filtered union of its coherent submodules by Properties, Lemma 22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F}$ using the resolution property of $X$. This produces a map in the derived category

$$\mathcal{E}[-1] \to R j_* \mathcal{G}$$

which is nonzero on cohomology sheaves and hence nonzero in $D(\mathcal{O}_X)$. By adjunction, this is the same thing as a map

$$j^* \mathcal{E}[-1] \to \mathcal{G}$$

nonzero in $D(\mathcal{O}_U)$. Since $\mathcal{E}$ is finite locally free this is the same thing as a nonzero element of

$$H^1(U, j^* \mathcal{E}^\vee \otimes \mathcal{O}_U \mathcal{G})$$

where $\mathcal{E}^\vee = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual finite locally free module. However, this group is zero by Cohomology of Schemes, Lemma 22 which is the desired contradiction. (If in doubt about the step using duals, please see the more general Cohomology, Lemma 45.11.)

34. The resolution property and perfect complexes

In this section we discuss the relationship between perfect complexes and strictly perfect complexes on schemes which have the resolution property.

0F8D

Lemma 34.1. Let $X$ be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{F}^\bullet$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules representing a perfect object of $D(\mathcal{O}_X)$. Then there exists a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

Proof. Let $a, b \in \mathbb{Z}$ be integers such that $\mathcal{F}^\bullet$ has tor amplitude in $[a, b]$ and such that $\mathcal{F}^n = 0$ for $n < a$. The existence of such a pair of integers follows from Cohomology, Lemma 45.5 and the fact that $X$ is quasi-compact. If $b < a$, then $\mathcal{F}^\bullet$ is zero in the derived category and the lemma holds. We will prove by induction on $b - a \geq 0$ that there exists a complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^b$ with $\mathcal{E}^i$ finite locally free and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$. 
The base case is the case $b - a = 0$. In this case $H^b(\mathcal{F}^\bullet) = H^a(\mathcal{F}^\bullet) = \text{Ker}(\mathcal{F}^a \to \mathcal{F}^{a+1})$ is finite locally free. Namely, it is a finitely presented $\mathcal{O}_X$-module of tor dimension 0 and hence finite locally free. See Cohomology, Lemmas \[45.5\] and \[43.9\] and Properties, Lemma \[20.2\] Thus we can take $\mathcal{E}^\bullet$ to be $H^b(\mathcal{F}^\bullet)$ sitting in degree $b$. The rest of the proof is dedicated to the induction step.

Assume $b > a$. Observe that

$$H^b(\mathcal{F}^\bullet) = \text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})/\text{Im}(\mathcal{F}^{b-1} \to \mathcal{F}^b)$$

is a finite type quasi-coherent $\mathcal{O}_X$-module, see Cohomology, Lemmas \[45.5\] and \[43.9\]. Then we can find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \to \text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$$

such that the composition with the projection onto $H^b(\mathcal{F}^\bullet)$ is surjective. Namely, we can write $\text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$ as the filtered union of its coherent submodules by Properties, Lemma \[22.3\] and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}^b$ and a surjection $\mathcal{E}^b \to \mathcal{F}$ using the resolution property of $X$. Consider the map of complexes

$$\alpha : \mathcal{E}^b[-b] \to \mathcal{F}^\bullet$$

and its cone $C(\alpha)^\bullet$, see Derived Categories, Definition \[9.1\]. We observe that $C(\alpha)^\bullet$ is nonzero only in degrees $\geq a$, has tor amplitude in $[a, b]$ by Cohomology, Lemma \[44.6\] and has $H^b(C(\alpha)^\bullet) = 0$ by construction. Thus we actually find that $C(\alpha)^\bullet$ has tor amplitude in $[a, b - 1]$. Hence the induction hypothesis applies to $C(\alpha)^\bullet$ and we find a map of complexes

$$(\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-1}) \to C(\alpha)^\bullet$$

with properties as stated in the induction hypothesis. Unwinding the definition of the cone this gives a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \to & \mathcal{E}^{b-2} & \to & \mathcal{E}^{b-1} & \to & 0 & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & \mathcal{F}^{b-2} & \to & \mathcal{F}^{b-1} & \oplus & \mathcal{E}^b & \to & \mathcal{F}^b & \to & \ldots
\end{array}$$

It is clear that we obtain a map of complexes $(\mathcal{E}^a \to \ldots \to \mathcal{E}^b) \to \mathcal{F}^\bullet$. We omit the verification that this map is a quasi-isomorphism. \qed

0F8F Lemma 34.2. Let $X$ be a quasi-compact and quasi-separated scheme with the resolution property. Then every perfect object of $D(\mathcal{O}_X)$ can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules.

Proof. Let $E$ be a perfect object of $D(\mathcal{O}_X)$. By Lemma \[33.8\] we see that $X$ has affine diagonal. Hence by Proposition \[6.6\] we can represent $E$ by a complex $\mathcal{F}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Observe that $E$ is in $D^b(\mathcal{O}_X)$ because $X$ is quasi-compact. Hence $\tau_{\geq n}\mathcal{F}^\bullet$ is a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules which represents $E$ if $n \ll 0$. Thus we may apply Lemma \[34.1\] to conclude. \[34.1\] \[34.1\]

0F8G Lemma 34.3. Let $X$ be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Then any $\alpha \in \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ can be represented by a diagram

$$\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet \to \mathcal{F}^\bullet$$
where \( \mathcal{G}^\bullet \) is a bounded complex of finite locally free \( \mathcal{O}_X \)-modules and where \( \mathcal{G}^\bullet \to \mathcal{E}^\bullet \) is a quasi-isomorphism.

**Proof.** By Lemma \[33.8\] we see that \( X \) has affine diagonal. Hence by Proposition \[6.6\] we can represent \( \alpha \) by a diagram
\[
\mathcal{E}^\bullet \leftarrow \mathcal{H}^\bullet \to \mathcal{F}^\bullet
\]
where \( \mathcal{H}^\bullet \) is a complex of quasi-coherent \( \mathcal{O}_X \)-modules and where \( \mathcal{H}^\bullet \to \mathcal{E}^\bullet \) is a quasi-isomorphism. For \( n \ll 0 \) the maps \( \mathcal{H}^\bullet \to \mathcal{E}^\bullet \) and \( \mathcal{H}^\bullet \to \mathcal{F}^\bullet \) factor through the quasi-isomorphism \( \mathcal{H}^\bullet \to \tau_{\geq n}\mathcal{H}^\bullet \) simply because \( \mathcal{E}^\bullet \) and \( \mathcal{F}^\bullet \) are bounded complexes. Thus we may replace \( \mathcal{H}^\bullet \) by \( \tau_{\geq n}\mathcal{H}^\bullet \) and assume that \( \mathcal{H}^\bullet \) is bounded below. Then we may apply Lemma \[34.1\] to conclude. \( \square \)

**Lemma 34.4.** Let \( X \) be a quasi-compact and quasi-separated scheme with the resolution property. Let \( \mathcal{E}^\bullet \) and \( \mathcal{F}^\bullet \) be finite complexes of finite locally free \( \mathcal{O}_X \)-modules. Let \( \alpha^\bullet, \beta^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet \) be two maps of complexes defining the same map in \( D(\mathcal{O}_X) \). Then there exists a quasi-isomorphism \( \gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet \) where \( \mathcal{G}^\bullet \) is a bounded complex of finite locally free \( \mathcal{O}_X \)-modules such that \( \alpha^\bullet \circ \gamma^\bullet \) and \( \beta^\bullet \circ \gamma^\bullet \) are homotopic maps of complexes.

**Proof.** By Lemma \[33.8\] (and the definition of the derived category) there exists a quasi-isomorphism \( \gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet \) where \( \mathcal{G}^\bullet \) is a complex of quasi-coherent \( \mathcal{O}_X \)-modules such that \( \alpha^\bullet \circ \gamma^\bullet \) and \( \beta^\bullet \circ \gamma^\bullet \) are homotopic maps of complexes. Choose a homotopy \( h^i : \mathcal{G}^i \to \mathcal{F}^{i-1} \) witnessing this fact. Choose \( n \ll 0 \). Then the map \( \gamma^i \) factors canonically over the quotient map \( \mathcal{G}^i \to \tau_{\geq n}\mathcal{G}^i \) as \( \mathcal{E}^\bullet \) is bounded below. For the exact same reason the maps \( h^i \) will factor over the surjections \( \mathcal{G}^i \to (\tau_{\geq n}\mathcal{G})^i \). Hence we see that we may replace \( \mathcal{G}^\bullet \) by \( \tau_{\geq n}\mathcal{G}^\bullet \). Then we may apply Lemma \[34.1\] to conclude. \( \square \)

**Proposition 34.5.** Let \( X \) be a quasi-compact and quasi-separated scheme with the resolution property. Denote
\begin{enumerate}
\item \( \mathcal{A} \) the additive category of finite locally free \( \mathcal{O}_X \)-modules,
\item \( K^b(\mathcal{A}) \) the homotopy category of bounded complexes in \( \mathcal{A} \), see Derived Categories, Section \[8\] and
\item \( D_{\text{perf}}(\mathcal{O}_X) \) the strictly full, saturated, triangulated subcategory of \( D(\mathcal{O}_X) \) consisting of perfect objects.
\end{enumerate}

With this notation the obvious functor
\[
K^b(\mathcal{A}) \to D_{\text{perf}}(\mathcal{O}_X)
\]
is an exact functor of triangulated categories which factors through an equivalence \( S^{-1}K^b(\mathcal{A}) \to D_{\text{perf}}(\mathcal{O}_X) \) of triangulated categories where \( S \) is the saturated multiplicative system of quasi-isomorphisms in \( K^b(\mathcal{A}) \).

**Proof.** If you can parse the statement of the proposition, then please skip this first paragraph. For some of the definitions used, please see Derived Categories, Definition \[3.4\] (triangulated subcategory), Derived Categories, Definition \[6.1\] (saturated triangulated subcategory), Derived Categories, Definition \[5.1\] (multiplicative system compatible with the triangulated structure), and Categories, Definition \[26.20\] (saturated multiplicative system). Observe that \( D_{\text{perf}}(\mathcal{O}_X) \) is a saturated triangulated subcategory of \( D(\mathcal{O}_X) \) by Cohomology, Lemmas \[45.7\] and \[45.9\]. Also, note that \( K^b(\mathcal{A}) \) is a triangulated category, see Derived Categories, Lemma \[10.5\].
It is clear that the functor sends distinguished triangles to distinguished triangles, i.e., is exact. Then $S$ is a saturated multiplicative system compatible with the triangulated structure on $K^b(A)$ by Derived Categories, Lemma 5.3. Hence the localization $S^{-1}K^b(A)$ exists and is a triangulated category by Derived Categories, Proposition 5.5. We get an exact factorization $S^{-1}K^b(A) \to D_{perf}(\mathcal{O}_X)$ by Derived Categories, Lemma 5.6. By Lemmas 34.2, 34.3, and 34.4, this functor is an equivalence. Then finally the functor $S^{-1}K^b(A) \to D_{perf}(\mathcal{O}_X)$ is an equivalence of triangulated categories (in the sense that distinguished triangles correspond) by Derived Categories, Lemma 4.17.

35. K-groups

A tiny bit about $K_0$ of various categories associated to schemes. Previous material can be found in Algebra, Section 54, Homology, Section 11, Derived Categories, Section 28, and More on Algebra, Lemma 106.2.

Analogous to Algebra, Section 54 we will define two $K$-groups $K'_0(X)$ and $K_0(X)$ for any Noetherian scheme $X$. The first will use coherent $\mathcal{O}_X$-modules and the second will use finite locally free $\mathcal{O}_X$-modules.

Lemma 35.1. Let $X$ be a Noetherian scheme. Then

$$K_0(Coh(\mathcal{O}_X)) = K_0(D^b(Coh(\mathcal{O}_X))) = K_0(D^b_{Coh}(\mathcal{O}_X))$$

Proof. The first equality is Derived Categories, Lemma 28.2. The second equality holds because $D^b(Coh(\mathcal{O}_X)) = D^b_{Coh}(\mathcal{O}_X)$ by Proposition 10.2. □

Here is the definition.

Definition 35.2. Let $X$ be a scheme.

1. We denote $K_0(X)$ the Grothendieck group of $X$. It is the zeroth $K$-group of the strictly full, saturated, triangulated subcategory $D_{perf}(\mathcal{O}_X)$ of $D(\mathcal{O}_X)$ consisting of perfect objects. In a formula

$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$$

2. If $X$ is locally Noetherian, then we denote $K'_0(X)$ the Grothendieck group of coherent sheaves on $X$. It is the the zeroth $K$-group of the abelian category of coherent $\mathcal{O}_X$-modules. In a formula

$$K'_0(X) = K_0(Coh(\mathcal{O}_X))$$

We will show that our definition of $K_0(X)$ agrees with the often used definition in terms of finite locally free modules if $X$ has the resolution property (for example if $X$ has an ample invertible module). See Lemma 35.5.

Lemma 35.3. Let $X = \text{Spec}(R)$ be an affine scheme. Then $K_0(X) = K_0(R)$ and if $R$ is Noetherian then $K'_0(X) = K'_0(R)$.

Proof. Recall that $K'_0(R)$ and $K_0(R)$ have been defined in Algebra, Section 54. By More on Algebra, Lemma 106.2 we have $K_0(R) = K_0(D_{perf}(R))$. By Lemmas 9.7 and 3.4 we have $D_{perf}(R) = D_{perf}(\mathcal{O}_X)$. This proves the equality $K_0(R) = K_0(X)$.

The equality $K'_0(R) = K'_0(X)$ holds because $Coh(\mathcal{O}_X)$ is equivalent to the category of finite $R$-modules by Cohomology of Schemes, Lemma 9.11. Moreover it is clear
that $K'_0(R)$ is the zeroth K-group of the category of finite $R$-modules from the definitions.

Let $X$ be a Noetherian scheme. Then both $K'_0(X)$ and $K_0(X)$ are defined. In this case there is a canonical map

$$K_0(X) = K_0(D_{\text{perf}}(\mathcal{O}_X)) \to K_0(D^b_{\text{Coh}}(\mathcal{O}_X)) = K'_0(X)$$

Namely, perfect complexes are in $D^b_{\text{Coh}}(\mathcal{O}_X)$ (by Lemma 9.3), the inclusion functor $D_{\text{perf}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_X)$ induces a map on zeroth $K$-groups (Derived Categories, Lemma 28.3), and we have the equality on the right by Lemma 35.1.

**Lemma 35.4.** Let $X$ be a Noetherian regular scheme of finite dimension. Then the map $K_0(X) \to K'_0(X)$ is an isomorphism.

**Proof.** Follows immediately from Lemma 10.7 and our construction of the map $K_0(X) \to K'_0(X)$ above.

Let $X$ be a scheme. Let us denote Vect($X$) the category of finite locally free $\mathcal{O}_X$-modules. Although Vect($X$) isn’t an abelian category in general, it is clear what a short exact sequence of Vect($X$) is. Denote $K_0(\text{Vect}(X))$ the unique abelian group with the following properties:

1. For every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ there is given an element $[\mathcal{E}]$ in $K_0(\text{Vect}(X))$,.
2. For every short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we have the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ in $K_0(\text{Vect}(X))$,
3. the group $K_0(\text{Vect}(X))$ is generated by the elements $[\mathcal{E}]$, and
4. all relations in $K_0(\text{Vect}(X))$ among the generators $[\mathcal{E}]$ are $\mathbb{Z}$-linear combinations of the relations coming from exact sequences as above.

We omit the detailed construction of $K_0(\text{Vect}(X))$. There is a natural map

$$K_0(\text{Vect}(X)) \to K_0(X)$$

Namely, given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let us denote $\mathcal{E}[0]$ the perfect complex on $X$ which has $\mathcal{E}$ sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we obtain a distinguished triangle $\mathcal{E}[0] \to \mathcal{E}'[0] \to \mathcal{E}''[0] \to \mathcal{E}[1]$, see Derived Categories, Section 12. This shows that we obtain a map $K_0(\text{Vect}(X)) \to K_0(D_{\text{perf}}(\mathcal{O}_X)) = K_0(X)$ by sending $[\mathcal{E}]$ to $[\mathcal{E}[0]]$ with apologies for the horrendous notation.

**Lemma 35.5.** Let $X$ be a quasi-compact and quasi-separated scheme with the resolution property. Then the map $K_0(\text{Vect}(X)) \to K_0(X)$ is an isomorphism.

**Proof.** This lemma will follow in a straightforward manner from Lemmas 34.2 and 34.4 whose results we will use without further mention. Let us construct an inverse map $c : K_0(X) = K_0(D_{\text{perf}}(\mathcal{O}_X)) \to K_0(\text{Vect}(X))$

Namely, any object of $D_{\text{perf}}(\mathcal{O}_X)$ can be represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules. Then we set

$$c([\mathcal{E}^\bullet]) = \sum (-1)^i [\mathcal{E}^i]$$

\text{The correct generality here would be to define $K_0$ for any exact category, see Injectives, Remark 9.6}.
Of course we have to show that this is well defined. For the moment we view \( c \) as a map defined on bounded complexes of finite locally free \( \mathcal{O}_X \)-modules.

Suppose that \( \mathcal{E}^\bullet \to \mathcal{F}^\bullet \) is a surjective map of bounded complexes of finite locally free \( \mathcal{O}_X \)-modules. Let \( \mathcal{K}^\bullet \) be the kernel. Then we obtain short exact sequences of \( \mathcal{O}_X \)-modules

\[
0 \to \mathcal{K}^n \to \mathcal{E}^n \to \mathcal{F}^n \to 0
\]

which are locally split because \( \mathcal{F}^n \) is finite locally free. Hence \( \mathcal{K}^\bullet \) is also a bounded complex of finite locally free \( \mathcal{O}_X \)-modules and we have \( c(\mathcal{E}^\bullet) = c(\mathcal{K}^\bullet) + c(\mathcal{F}^\bullet) \) in \( K_0(\text{Vect}(X)) \).

Suppose given a bounded complex \( \mathcal{E}^\bullet \) of finite locally free \( \mathcal{O}_X \)-modules which is acyclic. Say \( \mathcal{E}^n = 0 \) for \( n \not\in [a, b] \). Then we can break \( \mathcal{E}^\bullet \) into short exact sequences

\[
0 \to \mathcal{E}^a \to \mathcal{E}^{a+1} \to \mathcal{F}^{a+1} \to 0,
0 \to \mathcal{F}^{a+1} \to \mathcal{E}^{a+2} \to \mathcal{F}^{a+2} \to 0,
0 \to \mathcal{F}^{a+2} \to \mathcal{E}^{a+3} \to \mathcal{F}^{a+3} \to 0,
0 \to \mathcal{F}^{a+3} \to \mathcal{E}^{a+4} \to \mathcal{F}^{a+4} \to 0
\]

Arguing by descending induction we see that \( \mathcal{F}^{b-2}, \ldots, \mathcal{F}^{a+1} \) are finite locally free \( \mathcal{O}_X \)-modules, and

\[
c(\mathcal{E}^\bullet) = \sum (-1)^n[E^n] = \sum (-1)^n([\mathcal{F}^{a+1}]+[\mathcal{F}^n]) = 0
\]

Thus our construction gives zero on acyclic complexes.

It follows from the results of the preceding two paragraphs that \( c \) is well defined. Namely, suppose the bounded complexes \( \mathcal{E}^\bullet \) and \( \mathcal{F}^\bullet \) of finite locally free \( \mathcal{O}_X \)-modules represent the same object of \( D(\mathcal{O}_X) \). Then we can find quasi-isomorphisms \( a : \mathcal{G}^\bullet \to \mathcal{E}^\bullet \) and \( b : \mathcal{G}^\bullet \to \mathcal{F}^\bullet \) with \( \mathcal{G}^\bullet \) bounded complex of finite locally free \( \mathcal{O}_X \)-modules. We obtain a short exact sequence of complexes

\[
0 \to \mathcal{E}^\bullet \to C(a)^\bullet \to \mathcal{G}^\bullet[1] \to 0
\]

see Derived Categories, Definition \([9.1] \). Since \( a \) is a quasi-isomorphism, the cone \( C(a)^\bullet \) is acyclic (this follows for example from the discussion in Derived Categories, Section \([12] \)). Hence

\[
0 = c(C(f)^\bullet) = c(\mathcal{E}^\bullet) + c(\mathcal{G}^\bullet[1]) = c(\mathcal{E}^\bullet) - c(\mathcal{G}^\bullet)
\]

as desired. The same argument using \( b \) shows that \( 0 = c(\mathcal{F}^\bullet) - c(\mathcal{G}^\bullet) \). Hence we find that \( c(\mathcal{E}^\bullet) = c(\mathcal{F}^\bullet) \) and \( c \) is well defined.

A similar argument using the cone on a map \( \mathcal{E}^\bullet \to \mathcal{F}^\bullet \) of bounded complexes of finite locally free \( \mathcal{O}_X \)-modules shows that \( c(Y) = c(X) + c(Z) \) if \( X \to Y \to Z \) is a distinguished triangle in \( D_{\text{perf}}(\mathcal{O}_X) \). Details omitted. Thus we get the desired homomorphism of abelian groups \( c : K_0(\mathcal{X}) \to K_0(\text{Vect}(X)) \).

It is clear that the composition \( K_0(\text{Vect}(X)) \to K_0(\mathcal{X}) \to K_0(\text{Vect}(X)) \) is the identity. On the other hand, let \( \mathcal{E}^\bullet \) be a bounded complex of finite locally free \( \mathcal{O}_X \)-modules. Then the the existence of the distinguished triangles of “stupid truncations” (see Homology, Section \([15] \))

\[
\sigma_{\geq n} \mathcal{E}^\bullet \to \sigma_{\geq n-1} \mathcal{E}^\bullet \to \mathcal{E}^{n-1}[-n+1] \to (\sigma_{\geq n} \mathcal{E}^\bullet)[1]
\]
and induction show that
\[ [\mathcal{E}^*] = \sum (-1)^i [\mathcal{E}^i[0]] \]
in \( K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \) with apologies for the notation. Hence the map \( K_0(Vect(X)) \to K_0(D_{perf}(\mathcal{O}_X)) \) is surjective which finishes the proof. \( \square \)

**Remark 35.6.** Let \( X \) be a scheme. The K-group \( K_0(X) \) is canonically a commutative ring. Namely, using the derived tensor product
\[ \otimes = \bigotimes_{\mathcal{O}_X} D_{perf}(\mathcal{O}_X) \times D_{perf}(\mathcal{O}_X) \to D_{perf}(\mathcal{O}_X) \]
and Derived Categories, Lemma \( \text{[28.6]} \) we obtain a bilinear multiplication. Since \( K \otimes L \cong L \otimes K \) we see that this product is commutative. Since \( (K \otimes L) \otimes M = K \otimes (L \otimes M) \) we see that this product is associative. Finally, the unit of \( K_0(X) \) is the element \( 1 = [\mathcal{O}_X] \).

If \( Vect(X) \) and \( K_0(Vect(X)) \) are as above, then it is clearly the case that \( K_0(Vect(X)) \) also has a ring structure: if \( \mathcal{E} \) and \( \mathcal{F} \) are finite locally free \( \mathcal{O}_X \)-modules, then we set
\[ [\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}] \]
The reader easily verifies that this indeed defines a bilinear commutative, associative product. Details omitted. The map
\[ K_0(Vect(X)) \to K_0(X) \]
constructed above is a ring map with these definitions.

Now assume \( X \) is Noetherian. The derived tensor product also produces a map
\[ \otimes = \bigotimes_{\mathcal{O}_X} D_{perf}(\mathcal{O}_X) \times D_{Coh}(\mathcal{O}_X) \to D_{Coh}(\mathcal{O}_X) \]
Again using Derived Categories, Lemma \( \text{[28.6]} \) we obtain a bilinear multiplication \( K_0(X) \times K_0'(X) \to K_0'(X) \) since \( K_0(X) = K_0(D_{Coh}(\mathcal{O}_X)) \) by Lemma \( \text{[35.1]} \). The reader easily shows that this gives \( K_0'(X) \) the structure of a module over the ring \( K_0(X) \).

**Remark 35.7.** Let \( f : X \to Y \) be a proper morphism of locally Noetherian schemes. There is a map
\[ f_* : K_0'(X) \to K_0'(Y) \]
which sends \([\mathcal{F}]\) to
\[ [\bigoplus_{i \geq 0} R^i f_* \mathcal{F}] - [\bigoplus_{i \geq 0} R^{i+1} f_* \mathcal{F}] \]
This is well defined because the sheaves \( R^i f_* \mathcal{F} \) are coherent (Cohomology of Schemes, Lemma \( \text{[16.3]} \)), because locally only a finite number are nonzero, and because a short exact sequence of coherent sheaves on \( X \) produces a long exact sequence of \( R^i f_* \) on \( Y \). If \( Y \) is quasi-compact (the only case most often used in practice), then we can rewrite the above as
\[ f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}] = [Rf_* \mathcal{F}] \]
where we have used the equality \( K_0'(Y) = K_0(D_{Coh}^b(Y)) \) from Lemma \( \text{[35.1]} \).

**Lemma 35.8.** Let \( f : X \to Y \) be a proper morphism of locally Noetherian schemes. Then we have \( f_*(\alpha \cdot f^* \beta) = f_* \alpha \cdot \beta \) for \( \alpha \in K_0'(X) \) and \( \beta \in K_0(Y) \).
Proof. Follows from Lemma 21.1, the discussion in Remark 35.7, and the definition of the product $K'_0(X) \times K_0(X) \to K'_0(X)$ in Remark 35.6. □

Remark 35.9. Let $X$ be a scheme. Let $Z \subset X$ be a closed subscheme. Consider
the strictly full, saturated, triangulated subcategory
\[ D_{Z,\text{perf}}(\mathcal{O}_X) \subset D(\mathcal{O}_X) \]
consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-
theoretically supported on $Z$. The zeroth $K$-group $K_0(D_{Z,\text{perf}}(\mathcal{O}_X))$ of this trian-
gulated category is sometimes denoted $K_Z(X)$ or $K_0(Z,X)$. Using derived tensor
product exactly as in Remark 35.6 we see that $K_0(D_{Z,\text{perf}}(\mathcal{O}_X))$ has a multiplica-
tion which is associative and commutative, but in general $K_0(D_{Z,\text{perf}}(\mathcal{O}_X))$ doesn’t
have a unit.

36. Determinants of complexes

This section is the continuation of More on Algebra, Section 108. For any ringed
space $(X, \mathcal{O}_X)$ there is a functor
\[
\det : \left\{ \begin{array}{l}
\text{category of perfect complexes} \\
\text{morphisms are isomorphisms}
\end{array} \right\} \to \left\{ \begin{array}{l}
\text{category of invertible modules} \\
\text{morphisms are isomorphisms}
\end{array} \right\}
\]
Moreover, given an object $(L, F)$ of the filtered derived category $DF(\mathcal{O}_X)$ whose
filtration is finite and whose graded parts are perfect complexes, there is a canonical
isomorphism $\det(\text{gr}L) \to \det(L)$. See [KM76] for the original exposition. We will
add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case where $X$ is
a scheme and where we consider perfect objects $L$ in $D(\mathcal{O}_X)$ of tor-amplitude in
$[-1, 0]$.

Lemma 36.1. Let $X$ be a scheme. There is a functor
\[
\det : \left\{ \begin{array}{l}
\text{category of perfect complexes} \\
\text{morphisms are isomorphisms}
\end{array} \right\} \to \left\{ \begin{array}{l}
\text{category of invertible modules} \\
\text{morphisms are isomorphisms}
\end{array} \right\}
\]
In addition, given a rank 0 perfect object $L$ of $D(\mathcal{O}_X)$ with tor-amplitude in $[-1, 0]
there is a canonical element $\delta(L) \in \Gamma(X, \det(L))$ such that for any isomorphism
$a : L \to K$ in $D(\mathcal{O}_X)$ we have $\det(a)(\delta(L)) = \delta(K)$. Moreover, the construction is
affine locally given by the construction of More on Algebra, Section 108.

Proof. Let $L$ be an object of the left hand side. If $\text{Spec}(A) = U \subset X$ is an
affine open, then $L|_U$ corresponds to a perfect complex $L^\bullet$ of $A$-modules with tor-
amplitude in $[-1, 0]$, see Lemmas 3.5, 3.14 and 9.7. Then we can consider the
invertible $A$-module $\det(L^\bullet)$ constructed in More on Algebra, Lemma 108.4. If
$\text{Spec}(B) = V \subset U$ is another affine open contained in $U$, then $\det(L^\bullet) \otimes_A B =
\det(L^\bullet \otimes_A B)$ and hence this construction is compatible with restriction mappings
(see Lemma 3.8 and note $A \to B$ is flat). Thus we can glue these invertible mod-
ules to obtain an invertible module $\det(L)$ on $X$. The functoriality and canonical
sections are constructed in exactly the same manner. Details omitted. □

Remark 36.2. The construction of Lemma 36.1 is compatible with pullbacks.
More precisely, given a morphism $f : X \to Y$ of schemes and a perfect object $K$ of
$D(O_Y)$ of tor-amplitude in $[-1,0]$ then $Lf^*K$ is a perfect object $K$ of $D(O_X)$ of tor-amplitude in $[-1,0]$ and we have a canonical identification

$$f^* \det(K) \longrightarrow \det(Lf^*K)$$

Moreover, if $K$ has rank 0, then $\delta(K)$ pulls back to $\delta(Lf^*K)$ via this map. This is clear from the affine local construction of the determinant.

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