FUNDAMENTAL GROUPS OF SCHEMES

0BQ6

Contents

1. Introduction

0BQ7 In this chapter we discuss Grothendieck's fundamental group of a scheme and applications. A foundational reference is [Gro71]. A nice introduction is [Len]. Other references [Mur67] and [GM71].

2. Schemes étale over a point

04JI In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

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04JJ **Definition 2.1.** Let G be a topological group. A *G-set*, sometime called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 6.5) provided we endow G with the discrete topology.

Recall that if L/K is an infinite Galois extension then the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 22.

03QR **Lemma 2.2.** *Let K be a field. Let K^{sep} a separable closure of K. Consider the profinite group $G = Gal(K^{sep}/K)$. The functor*

$$\begin{array}{ccc} \textit{schemes étale over } K & \longrightarrow & G\textit{-Sets} \\ X/K & \longmapsto & \text{Mor}_{\text{Spec}(K)}(\text{Spec}(K^{sep}), X) \end{array}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} \text{Spec}(K_i)$ with each K_i a finite separable extension of K (Morphisms, Lemma 34.7). The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \text{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 22.4) the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \text{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. $\square$

03QS **Remark 2.3.** Under the correspondence of Lemma 2.2, the coverings in the small étale site $\text{Spec}(K)_{étale}$ of K correspond to surjective families of maps in *G-Sets*.

3. Galois categories

0BMQ In this section we discuss some of the material the reader can find in [Gro71, Exposé V, Sections 4, 5, and 6].

Let $F : \mathcal{C} \to \textit{Sets}$ be a functor. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. There is a canonical injective map

0BS7 (3.0.1) $$\text{Aut}(F) \longrightarrow \prod\nolimits_{X \in \text{Ob}(\mathcal{C})} \text{Aut}(F(X))$$

For a set E we endow $\text{Aut}(E)$ with the compact open topology, see Topology, Example 30.2. Of course this is the discrete topology when E is finite, which is the case of interest in this section[1]. We endow $\text{Aut}(F)$ with the topology induced from

[1]When we discuss the pro-étale fundamental group the general case will be of interest.

the product topology on the right hand side of (3.0.1). In particular, the action maps

$$\mathrm{Aut}(F) \times F(X) \longrightarrow F(X)$$

are continuous when $F(X)$ is given the discrete topology because this is true for the action maps $\mathrm{Aut}(E) \times E \to E$ for any set E. The universal property of our topology on $\mathrm{Aut}(F)$ is the following: suppose that G is a topological group and $G \to \mathrm{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \mathrm{Ob}(\mathcal{C})$ where $F(X)$ has the discrete topology. Then $G \to \mathrm{Aut}(F)$ is continuous.

The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite group.

0BMR **Lemma 3.1.** *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to Sets$ be a functor. The map (3.0.1) identifies $Aut(F)$ with a closed subgroup of $\prod_{X \in \mathrm{Ob}(\mathcal{C})} Aut(F(X))$. In particular, if $F(X)$ is finite for all X, then $Aut(F)$ is a profinite group.*

Proof. Let $\xi = (\gamma_X) \in \prod \mathrm{Aut}(F(X))$ be an element not in $\mathrm{Aut}(F)$. Then there exists a morphism $f : X \to X'$ of $\mathcal{C}$ and an element $x \in F(X)$ such that $F(f)(\gamma_X(x)) \neq \gamma_{X'}(F(f)(x))$. Consider the open neighbourhood $U = \{\gamma \in \mathrm{Aut}(F(X)) \mid \gamma(x) = \gamma_X(x)\}$ of γ_X and the open neighbourhood $U' = \{\gamma' \in \mathrm{Aut}(F(X')) \mid \gamma'(F(f)(x)) = \gamma_{X'}(F(f)(x))\}$. Then $U \times U' \times \prod_{X'' \neq X, X'} \mathrm{Aut}(F(X''))$ is an open neighbourhood of ξ not meeting $\mathrm{Aut}(F)$. The final statement is follows from the fact that $\prod \mathrm{Aut}(F(X))$ is a profinite space if each $F(X)$ is finite. $\square$

0BMS **Example 3.2.** Let G be a topological group. An important example will be the forgetful functor

0BMT (3.2.1) $$\textit{Finite-G-Sets} \longrightarrow \textit{Sets}$$

where *Finite-G-Sets* is the full subcategory of *G-Sets* whose objects are the finite G-sets. The category *G-Sets* of G-sets is defined in Definition 2.1.

Let G be a topological group. The *profinite completion* of G will be the profinite group

$$G^\wedge = \lim_{U \subset G \text{ open, normal, finite index}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another. The universal property of the profinite completion is that any continuous map $G \to H$ to a profinite group H factors canonically as $G \to G^\wedge \to H$.

0BMU **Lemma 3.3.** *Let G be a topological group. The automorphism group of the functor (3.2.1) endowed with its profinite topology from Lemma 3.1 is the profinite completion of G.*

Proof. Denote F_G the functor (3.2.1). Any morphism $X \to Y$ in *Finite-G-Sets* commutes with the action of G. Thus any $g \in G$ defines an automorphism of F_G and we obtain a canonical homomorphism $G \to \mathrm{Aut}(F_G)$ of groups. Observe that any finite G-set X is a finite disjoint union of G-sets of the form G/H_i with canonical G-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_ig^{-1}$ is open, normal, and has finite index. Moreover U_i acts trivially on G/H_i hence $U = \bigcap U_i$ acts trivially on $F_G(X)$. Hence the action $G \times F_G(X) \to F_G(X)$ is continuous. By the universal property of the topology on $\mathrm{Aut}(F_G)$ the map

$G \to \text{Aut}(F_G)$ is continuous. By Lemma 3.1 and the universal property of profinite completion there is an induced continuous group homomorphism

$$G^\wedge \longrightarrow \text{Aut}(F_G)$$

Moreover, since G/U acts faithfully on G/U this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 17.8.

Let $\gamma \in \text{Aut}(F_G)$ and let $X \in \text{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that γ and g induce the same action on $F_G(X)$. This will finish the proof. As before we see that X is a finite disjoint union of G/H_i. With U_i and U as above, the finite G-set $Y = G/U$ surjects onto G/H_i for all i and hence it suffices to find $g \in G$ such that γ and g induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of *Finite-G-Sets* commutes with the action of γ. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. $\square$

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

0BMV **Lemma 3.4.** *Let G be a topological group. Let F : Finite-G-Sets $\to$ Sets be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the functor (3.2.1).*

Proof. Let X be a nonempty object of *Finite-G-Sets*. The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod\nolimits_{gU \in G/U} G/U$$

where the summand corresponding to gU corresponds to the orbit of (eU, gU) on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod\nolimits_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim\nolimits_{U \subset G \text{ open, normal, finite index}} F(G/U)$$

is nonempty (Categories, Lemma 21.7). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote F_G the functor (3.2.1). We can identify F_G with the functor

$$X \longmapsto \text{colim}_U \text{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU) \in X = F_G(X)$ (details omitted). Hence the element γ determines a well defined map

$$t : F_G \longrightarrow F$$

Namely, given $x \in X$ choose U and $f : G/U \to X$ sending eU to x and then set $t_X(x) = F(f)(\gamma_U)$. We will show that t induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any U. This implies in a straightforward manner that t is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(\mathrm{id}_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because F commutes with equalizers. It follows that if $g_1, \ldots, g_{|G/U|}$ is a system of representatives for G/U, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_iU) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. □

0BMW **Example 3.5.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathit{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$. By Lemma 3.1 we see that $G = \mathrm{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

0BMX (3.5.1) $$\mathcal{C} \longrightarrow \mathit{Finite\text{-}G\text{-}Sets}, \quad X \longmapsto F(X)$$

where $F(X)$ is endowed with the induced action of G. This action is continuous by our construction of the topology on $\mathrm{Aut}(F)$.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (3.5.1) is an equivalence. Our definition of a Galois category is as follows.

0BMY **Definition 3.6.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathit{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a *Galois category* if

Different from the definition in [Gro71, Exposé V, Definition 5.1]. Compare with [BS13, Definition 7.2.1].

(1) $\mathcal{C}$ has finite limits and finite colimits,
0BMZ (2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
(3) $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$, and
(4) F reflects isomorphisms and is exact.

Here we say $X \in \mathrm{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either Y is initial or $Y \to X$ is an isomorphism.

Warning: This definition is not the same (although eventually we'll see it is equivalent) as the definition given in most references. Namely, in [Gro71, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to *Finite-G-Sets* for some profinite group G. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor F. The price we'll pay for this later is that we'll have to work a bit harder to apply the results of this section.

0BN0 **Lemma 3.7.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in \mathit{Arrows}(\mathcal{C})$. Then*

(1) *F is faithful,*

(2) *$X \to Y$ is a monomorphism $\Leftrightarrow F(X) \to F(Y)$ is injective,*
(3) *$X \to Y$ is an epimorphism $\Leftrightarrow F(X) \to F(Y)$ is surjective,*
(4) *an object A of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,*
(5) *an object Z of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,*
(6) *if X and Y are connected, then $X \to Y$ is an epimorphism,*
0BN1 (7) *if X is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,*
(8) *if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where X_i, Y_j are connected, then there is map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.*

Proof. Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let E be the equalizer of a and b. Then $F(E) = F(X)$ and we see that $E = X$ because F reflects isomorphisms.

Proof of (2). This is true because F turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and F reflects isomorphisms.

Proof of (3). This is true because F turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and F reflects isomorphisms.

Proof of (4). There exists an initial object A and certainly $F(A) = \emptyset$. On the other hand, if X is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

Proof of (5). There exists a final object Z and certainly $F(Z)$ is a singleton. On the other hand, if X is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer E of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through E and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer E of a and b comes with a monomorphism $E \to X$ and $F(E) \subset F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either E is initial or $E = X$.

Proof of (8). For each i, j we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to X_i. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if s maps to an element of $F(Y_j) \subset F(Y)$, hence this happens for a unique $j = \alpha(i)$. $\square$

By the lemma above we see that, given a connected object X of a Galois category $(\mathcal{C}, F)$, the automorphism group $\text{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \text{Aut}(X)$ we see that $g(s) = s$ if and only if $g = \text{id}_X$ by (7). We say X is *Galois* if equality holds. Equivalently, X is Galois if it is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

0BN2 **Lemma 3.8.** *Let $(\mathcal{C}, F)$ be a Galois category. For any connected object X of $\mathcal{C}$ there exists a Galois object Y and a morphism $Y \to X$.*

Proof. We will use the results of Lemma 3.7 without further mention. Let $n = |F(X)|$. Consider X^n endowed with its natural action of S_n. Let

$$X^n = \coprod_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a t such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with s_i pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim s'_i are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then Z_t is the image of an connected component of X^{n-1} under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \longrightarrow X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying F it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of S_n on

$$F(X)^n = F(X^n) = \coprod\nolimits_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that s'_i are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of G on $F(Z_t)$ is transitive and the proof is complete. □

Here is a key lemma.

0BN3 **Lemma 3.9.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 3.5. For any connected X in $\mathcal{C}$ the action of G on $F(X)$ is transitive.*

Compare with [BS13, Definition 7.2.4].

Proof. We will use the results of Lemma 3.7 without further mention. Let I be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let X_i be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on I by setting $i \geq i'$ if and only if there is a morphism $f_{ii'} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{ii'})(\gamma_i) = \gamma_{i'}$. With this normalization the morphism $f_{ii'}$ is unique. Observe that I is a directed partially ordered set: (Categories, Definition 21.1) if $i_1, i_2 \in I$ there exists a Galois object Y and a morphism $Y \to X_{i_1} \times X_{i_2}$ by Lemma 3.8 applied to a connected component of $X_{i_1} \times X_{i_2}$. Then $Y \cong X_i$ for some $i \in I$ and $i \geq i_1$, $i \geq I_2$.

We claim that the functor F is isomorphic to the functor F' which sends X to

$$F'(X) = \text{colim}_I \operatorname{Mor}_\mathcal{C}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using (7) we find that t_X is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where X is connected by the definition of a Galois category. Then we may assume X is Galois by Lemma 3.8. In this case X is isomorphic to X_i for some i and we can choose the isomorphism $X_i \to X$ such that γ_i maps to γ (by definition of Galois objects). We conclude that t is an isomorphism.

Set $A_i = \text{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{ii'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$\begin{tikzcd}
X_i \arrow[r, "f_{ii'}"'] \arrow[d, "a"'] & X_{i'} \arrow[d, "h_{ii'}(a)"] \\
X_i \arrow[r, "f_{ii'}"] & X_{i'}
\end{tikzcd}$$

commutes. Namely, just let $h_{ii'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i'}) = F(f_{ii'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i'i''} \circ h_{ii'} = h_{ii''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_I A_i$$

This is a profinite group since the automorphism groups are finite. The map $A \to A_i$ is surjective for all i by Categories, Lemma 21.7.

Since elements of A act on the inverse system X_i we get an action of A (on the right) on F' by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that G acts transitively on $F(X_i)$ for all i. Since every connected object is dominated by one of the X_i we conclude the lemma is true. □

0BN4 **Proposition 3.10.** *Let $(\mathcal{C}, F)$ be a Galois category. Let $G = Aut(F)$ be as in Example 3.5. The functor $F : \mathcal{C} \to$ Finite-G-Sets (3.5.1) an equivalence.*

This is a weak version of [Gro71, Exposé V]. The proof is borrowed from [BS13, Theorem 7.2.5].

Proof. We will use the results of Lemma 3.7 without further mention. In particular we know the functor is faithful. By Lemma 3.9 we know that for any connected X the action of G on $F(X)$ is transitive. Hence F preserves the decomposition into connected components (existence of which is an axiom of a Galois category). Let X and Y be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of s is a union of connected components. Hence there exists a union of connected components Z of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence F is fully faithful.

To finish the proof we show that F is essentially surjective. It suffices to show that G/H is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on G there exists a finite collection of objects X_i such that

$$\mathrm{Ker}(G \longrightarrow \prod\nolimits_i \mathrm{Aut}(F(X_i)))$$

is contained in H. We may assume X_i is connected for all i. We can choose a Galois object Y mapping to a connected component of $\prod X_i$ using Lemma 3.8. Choose an isomorphism $F(Y) = G/U$ in *G-sets* for some open subgroup $U \subset G$. As Y is Galois, the group $\mathrm{Aut}(Y) = \mathrm{Aut}_{G\text{-}Sets}(G/U)$ acts transitively on $F(Y) = G/U$. This implies that U is normal. Since $F(Y)$ surjects onto $F(X_i)$ for each i we see that $U \subset H$. Let $M \subset \mathrm{Aut}(Y)$ be the finite subgroup corresponding to

$$(H/U)^{opp} \subset (G/U)^{opp} = \mathrm{Aut}_{G\text{-}Sets}(G/U) = \mathrm{Aut}(Y).$$

Set $X = Y/M$, i.e., X is the coequalizer of the arrows $m : Y \to Y$, $m \in M$. Since F is exact we see that $F(X) = G/H$ and the proof is complete. □

0BN5 **Lemma 3.11.** *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories. Let $H : \mathcal{C} \to \mathcal{C}'$ be an exact functor. There exists an isomorphism $t : F' \circ H \to F$. The choice of t determines a continuous homomorphism $h : G' = Aut(F') \to Aut(F) = G$ and a*

2-commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d] & \mathcal{C}' \arrow[d] \\ \textit{Finite-}G\textit{-Sets} \arrow[r, "h"] & \textit{Finite-}G'\textit{-Sets} \end{tikzcd}$$

The map h is independent of t up to an inner automorphism of G. Conversely, given a continuous homomorphism $h : G' \to G$ there is an exact functor $H : \mathcal{C} \to \mathcal{C}'$ and an isomorphism t recovering h as above.

Proof. By Proposition 3.10 and Lemma 3.3 we may assume $\mathcal{C} =$ *Finite-G-Sets* and F is the forgetful functor and similarly for $\mathcal{C}'$. Thus the existence of t follows from Lemma 3.4. The map h comes from transport of structure via t. The commutativity of the diagram is obvious. Uniqueness of h up to inner conjugation by an element of G comes from the fact that the choice of t is unique up to an element of G. The final statement is straightforward. □

4. Functors and homomorphisms

0BTQ Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$, $(\mathcal{C}'', F'')$ be Galois categories. Set $G = \text{Aut}(F)$, $G' = \text{Aut}(F')$, and $G'' = \text{Aut}(F'')$. Let $H : \mathcal{C} \to \mathcal{C}'$ and $H' : \mathcal{C}' \to \mathcal{C}''$ be exact functors. Let $h : G' \to G$ and $h' : G'' \to G'$ be the corresponding continuous homomorphism as in Lemma 3.11. In this section we consider the corresponding 2-commutative diagram

0BTR (4.0.1)
$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d] & \mathcal{C}' \arrow[r, "H'"] \arrow[d] & \mathcal{C}'' \arrow[d] \\ \textit{Finite-}G\textit{-Sets} \arrow[r, "h"] & \textit{Finite-}G'\textit{-Sets} \arrow[r, "h'"] & \textit{Finite-}G''\textit{-Sets} \end{tikzcd}$$

and we relate exactness properties of the sequence $1 \to G'' \to G' \to G \to 1$ to properties of the functors H and H'.

0BN6 **Lemma 4.1.** *In diagram (4.0.1) the following are equivalent*

1. *$h : G' \to G$ is surjective,*
2. *$H : \mathcal{C} \to \mathcal{C}'$ is fully faithful,*
3. *if $X \in \text{Ob}(\mathcal{C})$ is connected, then $H(X)$ is connected,*
4. *if $X \in \text{Ob}(\mathcal{C})$ is connected and there is a morphism $*' \to H(X)$ in $\mathcal{C}'$, then there is a morphism $* \to X$, and*
5. *for any object X of $\mathcal{C}$ the map $\text{Mor}_\mathcal{C}(*, X) \to \text{Mor}_{\mathcal{C}'}(*', H(X))$ is bijective.*

Here $*$ and $*'$ are final objects of $\mathcal{C}$ and $\mathcal{C}'$.

Proof. The implications (5) ⇒ (4) and (2) ⇒ (5) are clear.

Assume (3). Let X be a connected object of $\mathcal{C}$ and let $*' \to H(X)$ be a morphism. Since $H(X)$ is connected by (3) we see that $*' \to H(X)$ is an isomorphism. Hence the G'-set corresponding to $H(X)$ has exactly one element, which means the G-set corresponding to X has one element which means X is isomorphic to the final object of $\mathcal{C}$, in particular there is a map $* \to X$. In this way we see that (3) ⇒ (4).

If (1) is true, then the functor *Finite-G-Sets* $\to$ *Finite-G'-Sets* is fully faithful: in this case a map of G-sets commutes with the action of G if and only if it commutes with the action of G'. Thus (1) ⇒ (2).

If (1) is true, then for a G-set X the G-orbits and G'-orbits agree. Thus (1) $\Rightarrow$ (3).

To finish the proof it suffices to show that (4) implies (1). If (1) is false, i.e., if h is not surjective, then there is an open subgroup $U \subset G$ containing $h(G')$ which is not equal to G. Then the finite G-set $M = G/U$ has a transitive action but G' has a fixed point. The object X of $\mathcal{C}$ corresponding to M would contradict (3). In this way we see that (3) $\Rightarrow$ (1) and the proof is complete. $\square$

0BS8 **Lemma 4.2.** *In diagram (4.0.1) the following are equivalent*

(1) *$h \circ h'$ is trivial, and*
(2) *the image of $H' \circ H$ consists of objects isomorphic to finite coproducts of final objects.*

Proof. We may replace H and H' by the canonical functors *Finite-G-Sets* $\to$ *Finite-G'-Sets* $\to$ *Finite-G''-Sets* determined by h and h'. Then we are saying that the action of G'' on every G-set is trivial if and only if the homomorphism $G'' \to G$ is trivial. This is clear. $\square$

0BS9 **Lemma 4.3.** *In diagram (4.0.1) the following are equivalent*

(1) *the sequence $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ is exact in the following sense: h is surjective, $h \circ h'$ is trivial, and $\mathrm{Ker}(h)$ is the smallest closed normal subgroup containing $\mathrm{Im}(h')$,*
(2) *H is fully faithful and an object X' of $\mathcal{C}'$ is in the essential image of H if and only if $H'(X')$ is isomorphic to a finite coproduct of final objects, and*
(3) *H is fully faithful, $H \circ H'$ sends every object to a finite coproduct of final objects, and for an object X' of $\mathcal{C}'$ such that $H'(X')$ is a finite coproduct of final objects there exists an object X of $\mathcal{C}$ and an epimorphism $H(X) \to X'$.*

Proof. By Lemmas 4.1 and 4.2 we may assume that H is fully faithful, h is surjective, $H' \circ H$ maps objects to disjoint unions of the final object, and $h \circ h'$ is trivial. Let $N \subset G'$ be the smallest closed normal subgroup containing the image of h'. It is clear that $N \subset \mathrm{Ker}(h)$. We may assume the functors H and H' are the canonical functors *Finite-G-Sets* $\to$ *Finite-G'-Sets* $\to$ *Finite-G''-Sets* determined by h and h'.

Suppose that (2) holds. This means that for a finite G'-set X' such that G'' acts trivially, the action of G' factors through G. Apply this to $X' = G'/U'N$ where U' is a small open subgroup of G'. Then we see that $\mathrm{Ker}(h) \subset U'N$ for all U'. Since N is closed this implies $\mathrm{Ker}(h) \subset N$, i.e., (1) holds.

Suppose that (1) holds. This means that $N = \mathrm{Ker}(h)$. Let X' be a finite G'-set such that G'' acts trivially. This means that $\mathrm{Ker}(G' \to \mathrm{Aut}(X'))$ is a closed normal subgroup containing $\mathrm{Im}(h')$. Hence $N = \mathrm{Ker}(h)$ is contained in it and the G'-action on X' factors through G, i.e., (2) holds.

Suppose that (3) holds. This means that for a finite G'-set X' such that G'' acts trivially, there is a surjection of G'-sets $X \to X'$ where X is a G-set. Clearly this means the action of G' on X' factors through G, i.e., (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate. This finishes the proof. $\square$

0BN7 **Lemma 4.4.** *In diagram (4.0.1) the following are equivalent*

(1) *h' is injective, and*

(2) *for every connected object X'' of $\mathcal{C}''$ there exists an object X' of $\mathcal{C}'$ and a diagram*

$$X'' \leftarrow Y'' \rightarrow H(X')$$

in $\mathcal{C}''$ where $Y'' \to X''$ is an epimorphism and $Y'' \to H(X')$ is a monomorphism.

Proof. We may replace H' by the corresponding functor between the categories of finite G'-sets and finite G''-sets.

Assume $h' : G'' \to G'$ is injective. Let $H'' \subset G''$ be an open subgroup. Since the topology on G'' is the induced topology from G' there exists an open subgroup $H' \subset G'$ such that $(h')^{-1}(H') \subset H''$. Then the desired diagram is

$$G''/H'' \leftarrow G''/(h')^{-1}(H') \rightarrow G'/H'$$

Conversely, assume (2) holds for the functor *Finite-G'-Sets* $\to$ *Finite-G''-Sets*. Let $g'' \in \mathrm{Ker}(h')$. Pick any open subgroup $H'' \subset G''$. By assumption there exists a finite G'-set X' and a diagram

$$G''/H'' \leftarrow Y'' \rightarrow X'$$

of G''-sets with the left arrow surjective and the right arrow injective. Since g'' is in the kernel of h' we see that g'' acts trivially on X'. Hence g'' acts trivially on Y'' and hence trivially on G''/H''. Thus $g'' \in H''$. As this holds for all open subgroups we conclude that g'' is the identity element as desired. □

0BTS **Lemma 4.5.** *In diagram (4.0.1) the following are equivalent*

(1) *the image of h' is normal, and*
(2) *for every connected object X' of $\mathcal{C}'$ such that there is a morphism from the final object of $\mathcal{C}''$ to $H'(X')$ we have that $H'(X')$ is isomorphic to a finite coproduct of final objects.*

Proof. This translates into the following statement for the continuous group homomorphism $h' : G'' \to G'$: the image of h' is normal if and only if every open subgroup $U' \subset G'$ which contains $h'(G'')$ also contains every conjugate of $h'(G'')$. The result follows easily from this; some details omitted. □

5. Finite étale morphisms

0BL6 In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let X be a scheme. We will use the notation $F\acute{E}t_X$ to denote the category of schemes finite and étale over X. Thus

(1) an object of $F\acute{E}t_X$ is a finite étale morphism $Y \to X$ with target X, and
(2) a morphism in $F\acute{E}t_X$ from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

commute.

We will often call an object of $\textit{FÉt}_X$ a *finite étale cover* of X (even if Y is empty). It turns out that there is a stack $p : \textit{FÉt} \to \textit{Sch}$ over the category of schemes whose fibre over X is the category $\textit{FÉt}_X$ just defined. See Examples of Stacks, Section 6.

0BN8 **Example 5.1.** Let k be an algebraically closed field and $X = \text{Spec}(k)$. In this case $\textit{FÉt}_X$ is equivalent to the category of finite sets. This works more generally when k is separably algebraically closed. The reason is that a scheme étale over k is the disjoint union of spectra of fields finite separable over k, see Morphisms, Lemma 34.7.

0BN9 **Lemma 5.2.** *Let X be a scheme. The category $\textit{FÉt}_X$ has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $\textit{FÉt}_X \to \textit{FÉt}_{X'}$ is exact.*

Proof. Finite limits and left exactness. By Categories, Lemma 18.4 it suffices to show that $\textit{FÉt}_X$ has a final object and fibred products. This is clear because the category of all schemes over X has a final object (namely X) and fibred products and fibred products of schemes finite étale over X are finite étale over X. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 23.2).

Finite colimits and right exactness. By Categories, Lemma 18.7 it suffices to show that $\textit{FÉt}_X$ has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of $\textit{FÉt}_X$. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \underline{\text{Spec}}(\mathcal{C})$ and $Y = \underline{\text{Spec}}(\mathcal{B})$ for some finite locally free $\mathcal{O}_X$-algebras $\mathcal{C}$ and $\mathcal{B}$. The morphisms a, b induce two maps $a^\sharp, b^\sharp : \mathcal{B} \to \mathcal{C}$. Let $\mathcal{A} = \text{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$\underline{\text{Spec}}(\mathcal{A}) \longrightarrow X$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of $\textit{FÉt}_X$ as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing X by the members of an étale covering (Descent, Lemmas 20.23 and 20.6). Thus by Étale Morphisms, Lemma 18.3 we may assume that $Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{j=1,\ldots,m} X$. Then

$$\mathcal{C} = \prod\nolimits_{1 \leq j \leq m} \mathcal{O}_X \quad \text{and} \quad \mathcal{B} = \prod\nolimits_{1 \leq i \leq n} \mathcal{O}_X$$

After a further replacement by the members of an open covering we may assume that a, b correspond to maps $a_s, b_s : \{1, \ldots, m\} \to \{1, \ldots, n\}$, i.e., the summand X of Z corresponding to the index j maps into the summand X of Y corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism a, resp. b. Let $\{1, \ldots, n\} \to T$ be the coequalizer of a_s, b_s. Then we see that

$$\mathcal{A} = \prod\nolimits_{t \in T} \mathcal{O}_X$$

whose spectrum is certainly finite étale over X. We omit the verification that this is compatible with base change. Thus base change is a right exact functor. $\square$

0BNA **Remark 5.3.** Let X be a scheme. Consider the natural functors $F_1 : \textit{FÉt}_X \to \textit{Sch}$ and $F_2 : \textit{FÉt}_X \to \textit{Sch}/X$. Then

(1) The functors F_1 and F_2 commute with finite colimits.

(2) The functor F_2 commutes with finite limits,
(3) The functor F_1 commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 5.2 and Categories, Lemma 16.2. It is clear that F_1 and F_2 commute with finite coproducts. By the dual of Categories, Lemma 23.2 we need to show that F_1 and F_2 commute with coequalizers. In the proof of Lemma 5.2 we saw that coequalizers in $F\acute{E}t_X$ look étale locally like this

$$\begin{tikzcd} \coprod_{j\in J} U \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \coprod_{i\in I} U \arrow[r] & \coprod_{t\in \mathrm{Coeq}(a,b)} U \end{tikzcd}$$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulate precisely in Descent, Lemma 10.4.

0BL7 **Lemma 5.4.** *Let X be a scheme. Given U, V finite étale over X there exists a scheme W finite étale over X such that*

$$\mathrm{Mor}_X(X, W) = \mathrm{Mor}_X(U, V)$$

and such that the same remains true after any base change.

Proof. By More on Morphisms, Lemma 56.4 there exists a scheme W representing $Mor_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 34.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing X by the members of an étale covering (Descent, Lemmas 20.23 and 20.6). Thus by Étale Morphisms, Lemma 18.3 we may assume that $U = \coprod_{i=1,\ldots,n} X$ and $V = \coprod_{j=1,\ldots,m} X$. In this case $W = \coprod_{\alpha:\{1,\ldots,n\}\to\{1,\ldots,m\}} X$ by inspection (details omitted) and the proof is complete. $\square$

Let X be a scheme. A *geometric point* of X is a morphism $\mathrm{Spec}(k) \to X$ where k is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{x})$ to denote k. We say $\overline{x}$ *lies over* x to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$|U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x}$$

Since $U_{\overline{x}}$ as a scheme over $\overline{x}$ is a disjoint union of copies of $\overline{x}$ (Morphisms, Lemma 34.7) we can also describe this set as

$$|U_{\overline{x}}| = \left\{ \begin{array}{c} \text{commutative} \\ \text{diagrams} \end{array} \begin{tikzcd} \overline{x} \arrow[r, "\overline{u}"] \arrow[rd, "\overline{x}"'] & U \arrow[d] \\ & X \end{tikzcd} \right\}$$

The assignment $U \mapsto |U_{\overline{x}}|$ is a functor which is often denoted $F_{\overline{x}}$.

0BNB **Lemma 5.5.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point. The functor*

$$F_{\overline{x}} : F\acute{E}t_X \longrightarrow Sets, \quad Y \longmapsto |Y_{\overline{x}}|$$

defines a Galois category (Definition 3.6).

Proof. After identifying $F\acute{E}t_{\overline{x}}$ with the category of finite sets (Example 5.1) we see that our functor $F_{\overline{x}}$ is nothing but the base change functor for the morphism $\overline{x} \to X$. Thus we see that $F\acute{E}t_X$ has finite limits and finite colimits and that $F_{\overline{x}}$ is exact by Lemma 5.2. We will also use that finite limits in $F\acute{E}t_X$ agree with the corresponding finite limits in the category of schemes over X, see Remark 5.3.

If $Y' \to Y$ is a monomorphism in $F\acute{E}t_X$ then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Étale Morphisms, Theorem 14.1). Since Y' is finite over X and Y separated over X, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 42.13), hence closed (Morphisms, Lemma 42.10), hence it is the inclusion of an open and closed subscheme of Y. It follows that Y is a connected objects of the category $F\acute{E}t_X$ (as in Definition 3.6) if and only if Y is connected as a scheme. Then it follows from Topology, Lemma 7.6 that Y is a finite coproduct of its connected components both as a scheme and in the sense of Definition 3.6.

Let $Y \to Z$ be a morphism in $F\acute{E}t_X$ which induces a bijection $F_{\overline{x}}(Y) \to F_{\overline{x}}(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume Z is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of Z lying over $\overline{x}$ and since Z is connected and the degree is locally constant we conclude. $\square$

6. Fundamental groups

0BQ8 In this section we define Grothendieck's algebraic fundamental group. The following definition makes sense thanks to Lemma 5.5.

0BNC **Definition 6.1.** Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X. The *fundamental group* of X with *base point* $\overline{x}$ is the group

$$\pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$$

of automorphisms of the fibre functor $F_{\overline{x}} : F\acute{E}t_X \to \textit{Sets}$ endowed with its canonical profinite topology from Lemma 3.1.

Combining the above with the material from Section 3 we obtain the following theorem.

0BND **Theorem 6.2.** *Let X be a connected scheme. Let $\overline{x}$ be a geometric point of X.*

(1) *The fibre functor $F_{\overline{x}}$ defines an equivalence of categories*

$$F\acute{E}t_X \longrightarrow \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$$

(2) *Given a second geometric point $\overline{x}'$ of X there exists an isomorphism $t : F_{\overline{x}} \to F_{\overline{x}'}$. This gives an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalences in (1). This isomorphism is independent of t up to inner conjugation.*

(3) *Given a morphism $f : X \to Y$ of connected schemes denote $\overline{y} = f \circ \overline{x}$. There is a canonical continuous homomorphism*

$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$$

such that the diagram

$$\begin{tikzcd} \textit{FÉt}_Y \arrow[rr, "\textit{base change}"'] \arrow[d, "F_{\overline{y}}"'] & & \textit{FÉt}_X \arrow[d, "F_{\overline{x}}"] \\ \textit{Finite-}\pi_1(Y, \overline{y})\textit{-Sets} \arrow[rr, "f_*"] & & \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets} \end{tikzcd}$$

is commutative.

Proof. Part (1) follows from Lemma 5.5 and Proposition 3.10. Part (2) is a special case of Lemma 3.11. For part (3) observe that the diagram

$$\begin{tikzcd} \textit{FÉt}_Y \arrow[r] \arrow[d, "F_{\overline{y}}"'] & \textit{FÉt}_X \arrow[d, "F_{\overline{x}}"] \\ \textit{Sets} \arrow[r, equal] & \textit{Sets} \end{tikzcd}$$

is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \overline{x}$. Hence we can apply Lemma 3.11 with the implied transformation of functors to get (3). $\square$

03SF **Remark 6.3.** Let X be a connected scheme with geometric point $\overline{x}$. Since $F_{\overline{x}} : \textit{FÉt}_X \to \textit{Sets}$ is a Galois category (Lemma 5.5) the material in Section 3 applies. We will say a finite étale morphism $Y \to X$ is a *Galois cover* if Y defines a Galois object of $\textit{FÉt}_X$. Recall that this means that Y is connected and that $G = \text{Aut}(Y/X)$ acts transitively (or equivalently simply transitively) on $F_{\overline{x}}(Y)$. For any finite étale morphism $f : Y \to X$ with Y connected, there is a Galois cover $Y' \to X$ which dominates Y (Lemma 3.8). The Galois objects of $\textit{FÉt}_X$ correspond, via the equivalence $F_{\overline{x}} : \textit{FÉt}_X \to \textit{Finite-}\pi_1(X, \overline{x})\textit{-Sets}$ of Theorem 6.2, with the finite $\pi_1(X, \overline{x})$*-Sets* of the form $G = \pi_1(X, \overline{x})/H$ where H is a normal open subgroup. Equivalently, if G is a finite group and $\pi_1(X, \overline{x}) \to G$ is a continuous surjection, then G viewed as a $\pi_1(X, \overline{x})$-set corresponds to a Galois covering.

0BNE **Lemma 6.4.** *Let K be a field and set $X = \text{Spec}(K)$. Let $\overline{K}$ be an algebraic closure and denote $\overline{x} : \text{Spec}(\overline{K}) \to X$ the corresponding geometric point. Let $K^{sep} \subset \overline{K}$ be the separable algebraic closure.*

1. *The functor of Lemma 2.2 induces an equivalence*
$$\textit{FÉt}_X \longrightarrow \textit{Finite-Gal}(K^{sep}/K)\textit{-Sets}.$$
compatible with $F_{\overline{x}}$ and the functor Finite-Gal(K^{sep}/K)-Sets $\to$ Sets.
2. *This induces a canonical isomorphism*
$$\textit{Gal}(K^{sep}/K) \longrightarrow \pi_1(X, \overline{x})$$
of profinite topological groups.

Proof. The functor of Lemma 2.2 is the same as the functor $F_{\overline{x}}$ because for any Y étale over X we have

$$\text{Mor}_X(\text{Spec}(\overline{K}), Y) = \text{Mor}_X(\text{Spec}(K^{sep}), Y)$$

Namely, as seen in the proof of Lemma 2.2 we have $Y = \coprod_{i \in I} \text{Spec}(L_i)$ with L_i/K finite separable over K. Hence any K-algebra homomorphism $L_i \to \overline{K}$ factors through K^{sep}. Also, note that $F_{\overline{x}}(Y)$ is finite if and only if I is finite if and only if $Y \to X$ is finite étale. This proves (1).

Part (2) is a formal consequence of (1), Lemma 3.11, and Lemma 3.3. (Please also see the remark below.) $\square$

0BQ9 **Remark 6.5.** In the situation of Lemma 6.4 let us give a more explicit construction of the isomorphism $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{x}) = \mathrm{Aut}(F_{\overline{x}})$. Observe that $\mathrm{Gal}(K^{sep}/K) = \mathrm{Aut}(\overline{K}/K)$ as $\overline{K}$ is the perfection of K^{sep}. Since $F_{\overline{x}}(Y) = \mathrm{Mor}_X(\mathrm{Spec}(\overline{K}), Y)$ we may consider the map

$$\mathrm{Aut}(\overline{K}/K) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y), \quad (\sigma, \overline{y}) \mapsto \sigma \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma)$$

This is an action because

$$\sigma\tau \cdot \overline{y} = \overline{y} \circ \mathrm{Spec}(\sigma\tau) = \overline{y} \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma) = \sigma \cdot (\tau \cdot \overline{y})$$

The action is functorial in $Y \in F\acute{E}t_X$ and we obtain the desired map.

7. Topological invariance of the fundamental group

0BTT The main result of this section is that a universal homeomorphism of connected schemes induces an isomorphism on fundamental groups. See Proposition 7.4.

Instead of directly proving two schemes have the same fundamental group, we often prove that their categories of finite étale coverings are the same. This of course implies that their fundamental groups are equal provided they are connected.

0BQA **Lemma 7.1.** *Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes such that the base change functor $F\acute{E}t_Y \to F\acute{E}t_X$ is an equivalence of categories. In this case*

1. *f induces a homeomorphism $\pi_0(X) \to \pi_0(Y)$,*
2. *if X or equivalently Y is connected, then $\pi_1(X, \overline{x}) = \pi_1(Y, \overline{y})$.*

Proof. Let $Y = Y_0 \amalg Y_1$ be a decomposition into nonempty open and closed subschemes. We claim that $f(X)$ meets both Y_i. Namely, if not, say $f(X) \subset Y_1$, then we can consider the finite étale morphism $V = Y_1 \to Y$. This is not an isomorphism but $V \times_Y X \to X$ is an isomorphism, which is a contradiction.

Suppose that $X = X_0 \amalg X_1$ is a decomposition into open and closed subschemes. Consider the finite étale morphism $U = X_1 \to X$. Then $U = X \times_Y V$ for some finite étale morphism $V \to Y$. The degree of the morphism $V \to Y$ is locally constant, hence we obtain a decomposition $Y = \coprod_{d \geq 0} Y_d$ into open and closed subschemes such that $V \to Y$ has degree d over Y_d. Since $f^{-1}(Y_d) = \emptyset$ for $d > 1$ we conclude that $Y_d = \emptyset$ for $d > 1$ by the above. And we conclude that $f^{-1}(Y_i) = X_i$ for $i = 0, 1$.

It follows that f^{-1} induces a bijection between the set of open and closed subsets of Y and the set of open and closed subsets of X. Note that X and Y are spectral spaces, see Properties, Lemma 2.4. By Topology, Lemma 12.10 the lattice of open and closed subsets of a spectral space determines the set of connected components. Hence $\pi_0(X) \to \pi_0(Y)$ is bijective. Since $\pi_0(X)$ and $\pi_0(Y)$ are profinite spaces (Topology, Lemma 22.4) we conclude that $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism by Topology, Lemma 17.8. This proves (1). Part (2) is immediate. $\square$

The following lemma tells us that the fundamental group of a henselian pair is the fundamental group of the closed subset.

09ZS **Lemma 7.2.** *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. The functor*

$$F\acute{E}t_X \longrightarrow F\acute{E}t_Z, \quad U \longmapsto U \times_X Z$$

is an equivalence of categories.

Proof. This is a translation of More on Algebra, Lemma 11.2. □

The following lemma tells us that the fundamental group of a thickening is the same as the fundamental group of the original. We will use this in the proof of the strong proposition concerning universal homeomorphisms below.

0BQB **Lemma 7.3.** *Let $X \subset X'$ be a thickening of schemes. The functor*

$$F\acute{E}t_{X'} \longrightarrow F\acute{E}t_X, \quad U' \longmapsto U' \times_{X'} X$$

is an equivalence of categories.

Proof. For a discussion of thickenings see More on Morphisms, Section 2. Let $U' \to X'$ be an étale morphism such that $U = U' \times_{X'} X \to X$ is finite étale. Then $U' \to X'$ is finite étale as well. This follows for example from More on Morphisms, Lemma 3.4. Now, if $X \subset X'$ is a finite order thickening then this remark combined with Étale Morphisms, Theorem 15.2 proves the lemma. Below we will prove the lemma for general thickenings, but we suggest the reader skip the proof.

Let $X' = \bigcup X'_i$ be an affine open covering. Set $X_i = X \times_{X'} X'_i$, $X'_{ij} = X'_i \cap X'_j$, $X_{ij} = X \times_{X'} X'_{ij}$, $X'_{ijk} = X'_i \cap X'_j \cap X'_k$, $X_{ijk} = X \times_{X'} X'_{ijk}$. Suppose that we can prove the theorem for each of the thickenings $X_i \subset X'_i$, $X_{ij} \subset X'_{ij}$, and $X_{ijk} \subset X'_{ijk}$. Then the result follows for $X \subset X'$ by relative glueing of schemes, see Constructions, Section 2. Observe that the schemes X'_i, X'_{ij}, X'_{ijk} are each separated as open subschemes of affine schemes. Repeating the argument one more time we reduce to the case where the schemes X'_i, X'_{ij}, X'_{ijk} are affine.

In the affine case we have $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A'/I')$ where I' is a locally nilpotent ideal. Then (A', I') is a henselian pair (More on Algebra, Lemma 10.2) and the result follows from Lemma 7.2 (which is much easier in this case). □

The "correct" way to prove the following proposition would be to deduce it from the invariance of the étale site, see Étale Cohomology, Theorem 45.2.

0BQN **Proposition 7.4.** *Let $f : X \to Y$ be a universal homeomorphism of schemes. Then*

$$F\acute{E}t_Y \longrightarrow F\acute{E}t_X, \quad V \longmapsto V \times_Y X$$

is an equivalence. Thus if X and Y are connected, then f induces an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ of fundamental groups.

Proof. Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 43.5. In particular, the diagonal $\Delta : X \to X \times_Y X$ is a thickening by Morphisms, Lemma 10.2. Thus by Lemma 7.3 we see that given a finite étale morphism $U \to X$ there is a unique isomorphism

$$\varphi : U \times_Y X \to X \times_Y U$$

of schemes finite étale over $X \times_Y X$ which pulls back under Δ to $\text{id} : U \to U$ over X. Since $X \to X \times_Y X \times_Y X$ is a thickening as well (it is bijective and a closed immersion) we conclude that (U, φ) is a descent datum relative to X/Y. By Étale Morphisms, Proposition 20.6 we conclude that $U = X \times_Y V$ for some $V \to Y$ quasi-compact, separated, and étale. We omit the proof that $V \to Y$ is finite (hints: the morphism $U \to V$ is surjective and $U \to Y$ is integral). We conclude that $\textit{FÉt}_Y \to \textit{FÉt}_X$ is essentially surjective.

Arguing in the same manner as above we see that given $V_1 \to Y$ and $V_2 \to Y$ in $\textit{FÉt}_Y$ any morphism $a : X \times_Y V_1 \to X \times_Y V_2$ over X is compatible with the canonical descent data. Thus a descends to a morphism $V_1 \to V_2$ over Y by Étale Morphisms, Lemma 20.3. $\square$

8. Finite étale covers of proper schemes

0BQC In this section we show that the fundamental group of a connected proper scheme over a henselian local ring is the same as the fundamental group of its special fibre. We also show that the fundamental group of a connected proper scheme over an algebraically closed field k does not change if we replace k by an algebraically closed extension. Instead of stating and proving the results in the connected case we prove the results in general and we leave it to the reader to deduce the result for fundamental groups using Lemma 7.1.

0A48 **Lemma 8.1.** *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Then the functor*

$$\textit{FÉt}_X \to \textit{FÉt}_{X_0}, \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Proof. The proof given here is an example of applying algebraization and approximation. We proceed in a number of stages.

Essential surjectivity when A is a complete local Noetherian ring. Let $X_n = X \times_{\text{Spec}(A)} \text{Spec}(A/\mathfrak{m}^{n+1})$. By Étale Morphisms, Theorem 15.2 the inclusions

$$X_0 \to X_1 \to X_2 \to \ldots$$

induce equivalence of categories between the category of schemes étale over X_0 and the category of schemes étale over X_n. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 3.3. In this case the morphism $U_0 \to \text{Spec}(A/\mathfrak{m})$ is proper as X_0 is proper over $A/\mathfrak{m}$. Thus we may apply Grothendieck's algebraization theorem (in the form of Cohomology of Schemes, Lemma 27.2) to see that there is a finite morphism $U \to X$ whose restriction to X_0 recovers U_0. By More on Morphisms, Lemma 12.3 we see that $U \to X$ is étale at every point of U_0. However, since every point of U specializes to a point of U_0 (as U is proper over A), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when A is a complete local Noetherian ring. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over X_0. Look at the morphism

$$\Gamma_{\varphi_0} : U_0 \longrightarrow U_0 \times_{X_0} V_0$$

This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to Γ_{φ_0}. Consider the projection $W \to U$. It is finite étale and an isomorphism over U_0 by construction. By Étale Morphisms, Lemma 14.2 $W \to U$ is an isomorphism in an open neighbourhood of U_0. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of φ_0.

Essential surjectivity when A is a henselian local Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of A with respect to the maximal ideal. Let $X^\wedge$ be the base change of X to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is U_0. Write $A^\wedge = \operatorname{colim} A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is V. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Writing

$$A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Theorem 13.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in A. Translating back we find an A-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to U_0.

Fully faithfulness when A is a henselian local Noetherian G-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that A is complete Noetherian.

General case. Let $(A, \mathfrak{m})$ be a henselian local ring. Set $S = \operatorname{Spec}(A)$ and denote $s \in S$ the closed point. By Limits, Lemma 13.3 we can write $X \to \operatorname{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with S_i of finite type over $\mathbf{Z}$. For each i let $s_i \in S_i$ be the image of s. Since $S = \lim S_i$ and $A = \mathcal{O}_{S,s}$ we have $A = \operatorname{colim} \mathcal{O}_{S_i,s_i}$. The ring $A_i = \mathcal{O}_{S_i,s_i}$ is a Noetherian local G-ring (More on Algebra, Proposition 47.12). By More on Algebra, Lemma 10.18 we see that $A = \operatorname{colim} A_i^h$. By More on Algebra, Lemma 47.8 the rings A_i^h are G-rings. Thus we see that $A = \operatorname{colim} A_i^h$ and

$$X = \lim(X_i \times_{S_i} \operatorname{Spec}(A_i^h))$$

as schemes. The category of schemes finite étale over X is the limit of the category of schemes finite étale over $X_i \times_{S_i} \operatorname{Spec}(A_i^h)$ (by Limits, Lemmas 10.1, 8.3, and 8.10) The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} s_i)$. Thus we formally deduce the result for $X/\operatorname{Spec}(A)$ from the result for the $(X_i \times_{S_i} \operatorname{Spec}(A_i^h))/\operatorname{Spec}(A_i^h)$ which we dealt with above. ☐

0A49 **Lemma 8.2.** *Let $k \subset k'$ be an extension of algebraically closed fields. Let X be a proper scheme over k. Then the functor*

$$U \longmapsto U_{k'}$$

is an equivalence of categories between schemes finite étale over X and schemes finite étale over $X_{k'}$.

Proof. Let us prove the functor is essentially surjective. Let $U' \to X_{k'}$ be a finite étale morphism. Write $k' = \operatorname{colim} A_i$ as a filtered colimit of finite type k-algebras. By Limits, Lemma 10.1 there exists an i and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X_{k'}$ is U'. After increasing i we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Since k is algebraically closed we can find a k-valued point t in $\text{Spec}(A_i)$. Let $U = (U_i)_t$ be the fibre of U_i over t. Let A_i^h be the henselization of $(A_i)_\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal corresponding to the point t. By Lemma 8.1 we see that $(U_i)_{A_i^h} = U \times \text{Spec}(A_i^h)$ as schemes over $X_{A_i^h}$. Now since A_i^h is algebraic over A_i (see for example discussion in Smoothing Ring Maps, Example 13.3) and since k' is algebraically closed we can find a ring map $A_i^h \to k'$ extending the given inclusion $A_i \subset k'$. Hence we conclude that U' is isomorphic to the base change of U. The proof of fully faithfulness is exactly the same. $\square$

9. Local connectedness

0BQD In this section we ask when $\pi_1(U) \to \pi_1(X)$ is surjective for U a dense open of a scheme X. We will see that this is the case (roughly) when $U \cap B$ is connected for any small "ball" B around a point $x \in X \setminus U$.

0BQE **Lemma 9.1.** *Let $f : X \to Y$ be a morphism of schemes. If $f(X)$ is dense in Y then the base change functor $\textit{FÉt}_Y \to \textit{FÉt}_X$ is faithful.*

Proof. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given W finite étale over Y and a morphism $s : X \to W$ over X there is at most one section $t : Y \to W$ such that $s = t \circ f$. Consider two sections $t_1, t_2 : Y \to W$ such that $s = t_1 \circ f = t_2 \circ f$. Since the equalizer of t_1 and t_2 is closed in Y (Schemes, Lemma 21.5) and since $f(X)$ is dense in Y we see that t_1 and t_2 agree on Y_{red}. Then it follows that t_1 and t_2 have the same image which is an open and closed subscheme of W mapping isomorphically to Y (Étale Morphisms, Proposition 6.1) hence they are equal. $\square$

The condition in the following lemma that the punctured spectrum of the strict henselization is connected follows for example from the assumption that the local ring is geometrically unibranch, see More on Algebra, Lemma 89.5. There is a partial converse in Properties, Lemma 15.3.

0BLQ **Lemma 9.2.** *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. If the punctured spectrum of the strict henselization of A is connected, then*

$$\textit{FÉt}_X \longrightarrow \textit{FÉt}_U, \quad Y \longmapsto Y \times_X U$$

is a fully faithful functor.

Proof. Assume A is strictly henselian. In this case any finite étale cover Y of X is isomorphic to a finite disjoint union of copies of X. Thus it suffices to prove that any morphism $U \to U \amalg \ldots \amalg U$ over U, extends uniquely to a morphism $X \to X \amalg \ldots \amalg X$ over X. If U is connected (in particular nonempty), then this is true.

The general case. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given Y finite étale over X any morphism $s : U \to Y$ over X extends to a morphism $t : X \to Y$ over Y. Let A^{sh}

be the strict henselization of A and denote $X^{sh} = \text{Spec}(A^{sh})$, $U^{sh} = U \times_X X^{sh}$, $Y^{sh} = Y \times_X X^{sh}$. By the first paragraph and our assumption on A, we can extend the base change $s^{sh} : U^{sh} \to Y^{sh}$ of s to $t^{sh} : X^{sh} \to Y^{sh}$. Set $A' = A^{sh} \otimes_A A^{sh}$. Then the two pullbacks t'_1, t'_2 of t^{sh} to $X' = \text{Spec}(A')$ are extensions of the pullback s' of s to $U' = U \times_X X'$. As $A \to A'$ is flat we see that $U' \subset X'$ is (topologically) dense by going down for $A \to A'$ (Algebra, Lemma 38.18). Thus $t'_1 = t'_2$ by Lemma 9.1. Hence t^{sh} descends to a morphism $t : X \to Y$ for example by Descent, Lemma 10.3. $\square$

In view of Lemma 9.2 it is interesting to know when the punctured spectrum of a ring (and of its strict henselization) is connected. The following famous lemma due to Hartshorne gives a sufficient condition.

0BLR **Lemma 9.3.** *Let A be a Noetherian local ring of depth ≥ 2. Then the punctured spectra of A, A^h, and A^{sh} are connected.* [Har62, Proposition 2.1]

Proof. Let U be the punctured spectrum of A. If U is disconnected then we see that $\Gamma(U, \mathcal{O}_U)$ has a nontrivial idempotent. But A, being local, does not have a nontrivial idempotent. Hence $A \to \Gamma(U, \mathcal{O}_U)$ is not an isomorphism. By Local Cohomology, Lemma 4.2 we conclude that either $H^0_{\mathfrak{m}}(A)$ or $H^1_{\mathfrak{m}}(A)$ is nonzero. Thus $\text{depth}(A) \leq 1$ by Dualizing Complexes, Lemma 11.1. To see the result for A^h and A^{sh} use More on Algebra, Lemma 42.8. $\square$

0BQF **Lemma 9.4.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

1. *the underlying topological space of X is Noetherian, and*
2. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 9.1.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 9.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 17.2 we find an open $U \subset U'$ containing x and an extension $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ of φ. Since the underlying topological space of X is Noetherian this finishes the proof by Noetherian induction on the complement of the open over which φ is defined. $\square$

0BSA **Lemma 9.5.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

1. *$U \to X$ is quasi-compact,*
2. *every point of $X \setminus U$ is closed, and*
3. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 9.1.

Let $x \in X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. Since every point of $X \setminus U$ is closed V is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 9.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 17.2 (this uses that U is retrocompact in X) we find an open $U \subset U'_x$ containing x and an extension $\varphi'_x : (Y_1)_{U'_x} \to (Y_2)_{U'_x}$ of φ. Note that given two points $x, x' \in X \setminus U$ the morphisms φ'_x and $\varphi'_{x'}$ agree over $U'_x \cap U'_{x'}$ as U is dense in that open (Lemma 9.1). Thus we can extend φ to $\bigcup U'_x = X$ as desired. □

0BQG **Lemma 9.6.** *Let X be a scheme. Let $U \subset X$ be a dense open. Assume*

1. *every quasi-compact open of X has finitely many irreducible components,*
2. *for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.*

Then $F\acute{E}t_X \to F\acute{e}t_U$ is fully faithful.

Proof. Let Y_1, Y_2 be finite étale over X and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over U. We have to show that φ lifts uniquely to a morphism $Y_1 \to Y_2$ over X. Uniqueness follows from Lemma 9.1. We will prove existence by showing that we can enlarge U if $U \neq X$ and using Zorn's lemma to finish the proof.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By our choice of x this is the punctured spectrum of $\mathrm{Spec}(\mathcal{O}_{X,x})$. By Lemma 9.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ (uniquely) to a morphism $(Y_1)_{\mathrm{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\mathrm{Spec}(\mathcal{O}_{X,x})}$. Choose an affine neighbourhood $W \subset X$ of x. Since $U \cap W$ is dense in W it contains the generic points $\eta_1, \ldots, \eta_n$ of W. Choose an affine open $W' \subset W \cap U$ containing $\eta_1, \ldots, \eta_n$. Set $V' = W' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$. By Limits, Lemma 17.2 applied to $x \in W \supset W'$ we find an open $W' \subset W'' \subset W$ with $x \in W''$ and a morphism $\varphi'' : (Y_1)_{W''} \to (Y_2)_{W''}$ agreeing with φ over W'. Since W' is dense in $W'' \cap U$, we see by Lemma 9.1 that φ and φ'' agree over $U \cap W'$. Thus φ and φ'' glue to a morphism φ' over $U' = U \cup W''$ agreeing with φ over U. Observe that $x \in U'$ so that we've extended φ to a strictly larger open.

Consider the set $\mathcal{S}$ of pairs (U', φ') where $U \subset U'$ and φ' is an extension of φ. We endow $\mathcal{S}$ with a partial ordering in the obvious manner. If (U'_i, φ'_i) is a totally ordered subset, then it has a maximum (U', φ'). Just take $U' = \bigcup U'_i$ and let $\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}$ be the morphism agreeing with φ'_i over U'_i. Thus Zorn's lemma applies and $\mathcal{S}$ has a maximal element. By the argument above we see that this maximal element is an extension of φ over all of X. □

0BSB **Lemma 9.7.** *Let $(A, \mathfrak{m})$ be a local ring. Set $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Let U^{sh} be the punctured spectrum of the strict henselization A^{sh} of A. Assume U is quasi-compact and U^{sh} is connected. Then the sequence*

$$\pi_1(U^{sh}, \overline{u}) \to \pi_1(U, \overline{u}) \to \pi_1(X, \overline{u}) \to 1$$

is exact in the sense of Lemma 4.3 part (1).

Proof. The map $\pi_1(U) \to \pi_1(X)$ is surjective by Lemmas 9.2 and 4.1.

Write $X^{sh} = \mathrm{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X U^{sh}$. It follows that the composition $\pi_1(U^{sh}) \to \pi_1(U) \to \pi_1(X)$ is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 4.3 to show the following: Given a finite étale morphism $V \to U$ such that $V \times_U U^{sh}$ is a disjoint union of copies of U^{sh}, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X U$ over U. The assumption implies that there exists a finite étale morphism $Y^{sh} \to X^{sh}$ and an isomorphism $V \times_U U^{sh} \cong Y^{sh} \times_{X^{sh}} U^{sh}$. Consider the following diagram

$$\begin{tikzcd} U \arrow[d] & U^{sh} \arrow[l] \arrow[d] & U^{sh} \times_U U^{sh} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & U^{sh} \times_U U^{sh} \times_U U^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\ X & X^{sh} \arrow[l] & X^{sh} \times_X X^{sh} \arrow[l, shift left] \arrow[l, shift right] & X^{sh} \times_X X^{sh} \times_X X^{sh} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

Since $U \subset X$ is quasi-compact by assumption, all the downward arrows are quasi-compact open immersions. Let $\xi \in X^{sh} \times_X X^{sh}$ be a point not in $U^{sh} \times_U U^{sh}$. Then ξ lies over the closed point x^{sh} of X^{sh}. Consider the local ring homomorphism

$$A^{sh} = \mathcal{O}_{X^{sh}, x^{sh}} \to \mathcal{O}_{X^{sh} \times_X X^{sh}, \xi}$$

determined by the first projection $X^{sh} \times_X X^{sh}$. This is a filtered colimit of local homomorphisms which are localizations étale ring maps. Since A^{sh} is strictly henselian, we conclude that it is an isomorphism. Since this holds for every ξ in the complement it follows there are no specializations among these points and hence every such ξ is a closed point (you can also prove this directly). As the local ring at ξ is isomorphic to A^{sh}, it is strictly henselian and has connected punctured spectrum. Similarly for points ξ of $X^{sh} \times_X X^{sh} \times_X X^{sh}$ not in $U^{sh} \times_U U^{sh} \times_U U^{sh}$. It follows from Lemma 9.5 that pullback along the vertical arrows induce fully faithful functors on the categories of finite étale schemes. Thus the canonical descent datum on $V \times_U U^{sh}$ relative to the fpqc covering $\{U^{sh} \to U\}$ translates into a descent datum for Y^{sh} relative to the fpqc covering $\{X^{sh} \to X\}$. Since $Y^{sh} \to X^{sh}$ is finite hence affine, this descent datum is effective (Descent, Lemma 34.1). Thus we get an affine morphism $Y \to X$ and an isomorphism $Y \times_X X^{sh} \to Y^{sh}$ compatible with descent data. By fully faithfulness of descent data (as in Descent, Lemma 32.11) we get an isomorphism $V \to U \times_X Y$. Finally, $Y \to X$ is finite étale as $Y^{sh} \to X^{sh}$ is, see Descent, Lemmas 20.29 and 20.23. $\square$

Let X be an irreducible scheme. Let $\eta \in X$ be the geometric point. The canonical morphism $\eta \to X$ induces a canonical map

0BQH (9.7.1) $$\mathrm{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

The identification on the left hand side is Lemma 6.4.

0BQI **Lemma 9.8.** *Let X be an irreducible, geometrically unibranch scheme. For any nonempty open $U \subset X$ the canonical map*

$$\pi_1(U, \overline{u}) \longrightarrow \pi_1(X, \overline{u})$$

is surjective. The map (9.7.1) $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective as well.

Proof. By Lemma 7.3 we may replace X by its reduction. Thus we may assume that X is an integral scheme. By Lemma 4.1 the assertion of the lemma translates into the statement that the functors $F\acute{E}t_X \to F\acute{E}t_U$ and $F\acute{E}t_X \to F\acute{E}t_\eta$ are fully faithful.

The result for $F\acute{E}t_X \to F\acute{E}t_U$ follows from Lemma 9.6 and the fact that for a local ring A which is geometrically unibranch its strict henselization has an irreducible spectrum. See More on Algebra, Lemma 89.5.

Observe that the residue field $\kappa(\eta) = \mathcal{O}_{X,\eta}$ is the filtered colimit of $\mathcal{O}_X(U)$ over $U \subset X$ nonempty open affine. Hence $F\acute{E}t_\eta$ is the colimit of the categories $F\acute{E}t_U$ over such U, see Limits, Lemmas 10.1, 8.3, and 8.10. A formal argument then shows that fully faithfulness for $F\acute{E}t_X \to F\acute{E}t_\eta$ follows from the fully faithfulness of the functors $F\acute{E}t_X \to F\acute{E}t_U$. $\square$

0BSC **Lemma 9.9.** *Let X be a scheme. Let $x_1, \ldots, x_n \in X$ be a finite number of closed points such that*

(1) *$U = X \setminus \{x_1, \ldots, x_n\}$ is connected and is a retrocompact open of X, and*
(2) *for each i the punctured spectrum U_i^{sh} of the strict henselization of $\mathcal{O}_{X,x_i}$ is connected.*

Then the map $\pi_1(U) \to \pi_1(X)$ is surjective and the kernel is the smallest closed normal subgroup of $\pi_1(U)$ containing the image of $\pi_1(U_i^{sh}) \to \pi_1(U)$ for $i = 1, \ldots, n$.

Proof. Surjectivity follows from Lemmas 9.5 and 4.1. We can consider the sequence of maps

$$\pi_1(U) \to \ldots \to \pi_1(X \setminus \{x_1, x_2\}) \to \pi_1(X \setminus \{x_1\}) \to \pi_1(X)$$

A group theory argument then shows it suffices to prove the statement on the kernel in the case $n = 1$ (details omitted). Write $x = x_1$, $U^{sh} = U_1^{sh}$, set $A = \mathcal{O}_{X,x}$, and let A^{sh} be the strict henselization. Consider the diagram

$$\begin{tikzcd} U \arrow[d] & \operatorname{Spec}(A) \setminus \{\mathfrak{m}\} \arrow[l] \arrow[d] & U^{sh} \arrow[l] \arrow[d] \\ X & \operatorname{Spec}(A) \arrow[l] & \operatorname{Spec}(A^{sh}) \arrow[l] \end{tikzcd}$$

By Lemma 4.3 we have to show finite étale morphisms $V \to U$ which pull back to trivial coverings of U^{sh} extend to finite étale schemes over X. By Lemma 9.7 we know the corresponding statement for finite étale schemes over the punctured spectrum of A. However, by Limits, Lemma 17.1 schemes of finite presentation over X are the same thing as schemes of finite presentation over U and A glued over the punctured spectrum of A. This finishes the proof. $\square$

10. Fundamental groups of normal schemes

0BQJ Let X be an integral, geometrically unibranch scheme. In the previous section we have seen that the fundamental group of X is a quotient of the Galois group of the function field K of X. Since the map is continuous the kernel is a normal closed subgroup of the Galois group. Hence this kernel corresponds to a Galois extension M/K by Galois theory (Fields, Theorem 22.4). In this section we will determine M when X is a normal integral scheme.

Let X be an integral normal scheme with function field K. Let $K \subset L$ be a finite extension. Consider the normalization $Y \to X$ of X in the morphism $\operatorname{Spec}(L) \to X$ as defined in Morphisms, Section 50. We will say (in this setting) that X *is unramified in L* if $Y \to X$ is an unramified morphism of schemes. In Lemma 12.4 we will elucidate this condition. Observe that the scheme theoretic fibre of

$Y \to X$ over $\mathrm{Spec}(K)$ is $\mathrm{Spec}(L)$. Hence the field extension L/K is separable if X is unramified in L, see Morphisms, Lemmas 33.11.

0BQK **Lemma 10.1.** *In the situation above the following are equivalent*

(1) *X is unramified in L,*
(2) *$Y \to X$ is étale, and*
(3) *$Y \to X$ is finite étale.*

Proof. Observe that $Y \to X$ is an integral morphism. In each case the morphism $Y \to X$ is locally of finite type by definition. Hence we find that in each case the lemma is finite by Morphisms, Lemma 42.4. In particular we see that (2) is equivalent to (3). An étale morphism is unramified, hence (2) implies (1).

Conversely, assume $Y \to X$ is unramified. Let $x \in X$. We can choose an étale neighbourhood $(U, u) \to (X, x)$ such that

$$Y \times_X U = \coprod V_j \longrightarrow U$$

is a disjoint union of closed immersions, see Étale Morphisms, Lemma 17.3. Shrinking we may assume U is quasi-compact. Then U has finitely many irreducible components (Descent, Lemma 13.3). Since U is normal (Descent, Lemma 15.2) the irreducible components of U are open and closed (Properties, Lemma 7.5) and we may assume U is irreducible. Then U is an integral scheme whose generic point ξ maps to the generic point of X. On the other hand, we know that $Y \times_X U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$ by More on Morphisms, Lemma 17.2. Every point of $\mathrm{Spec}(L) \times_X U$ maps to ξ. Thus every V_j contains a point mapping to ξ by Morphisms, Lemma 50.9. Thus $V_j \to U$ is an isomorphism as $U = \overline{\{\xi\}}$. Thus $Y \times_X U \to U$ is étale. By Descent, Lemma 20.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x). $\square$

0BQL **Lemma 10.2.** *Let X be a normal integral scheme with function field K. Let $Y \to X$ be a finite étale morphism. If Y is connected, then Y is an integral normal scheme and Y is the normalization of X in the function field of Y.*

Proof. The scheme Y is normal by Descent, Lemma 15.2. Since $Y \to X$ is flat every generic point of Y maps to the generic point of X by Morphisms, Lemma 24.8. Since $Y \to X$ is finite we see that Y has a finite number of irreducible components. Thus Y is the disjoint union of a finite number of integral normal schemes by Properties, Lemma 7.5. Thus if Y is connected, then Y is an integral normal scheme.

Let L be the function field of Y and let $Y' \to X$ be the normalization of X in L. By Morphisms, Lemma 50.4 we obtain a factorization $Y' \to Y \to X$ and $Y' \to Y$ is the normalization of Y in L. Since Y is normal it is clear that $Y' = Y$ (this can also be deduced from Morphisms, Lemma 51.8). $\square$

0BQM **Proposition 10.3.** *Let X be a normal integral scheme with function field K. Then the canonical map (9.7.1)*

$$Gal(K^{sep}/K) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \overline{\eta})$$

is identified with the quotient map $Gal(K^{sep}/K) \to Gal(M/K)$ where $M \subset K^{sep}$ is the union of the finite subextensions L such that X is unramified in L.

Proof. The normal scheme X is geometrically unibranch (Properties, Lemma 15.2). Hence Lemma 9.8 applies to X. Thus $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective and top horizontal arrow of the commutative diagram

$$\begin{tikzcd} F\acute{E}t_X \arrow[rr] \arrow[d] \arrow[drr, "c"] & & F\acute{E}t_\eta \arrow[d] \\ \textit{Finite-}\pi_1(X, \overline{\eta})\textit{-sets} \arrow[r] & & \textit{Finite-}\mathrm{Gal}(K^{sep}/K)\textit{-sets} \end{tikzcd}$$

is fully faithful. The left vertical arrow is the equivalence of Theorem 6.2 and the right vertical arrow is the equivalence of Lemma 6.4. The lower horizontal arrow is induced by the map of the proposition. By Lemmas 10.1 and 10.2 we see that the essential image of c consists of $\mathrm{Gal}(K^{sep}/K)$-*Sets* isomorphic to sets of the form

$$S = \mathrm{Hom}_K(\prod\nolimits_{i=1,\ldots,n} L_i, K^{sep}) = \coprod\nolimits_{i=1,\ldots,n} \mathrm{Hom}_K(L_i, K^{sep})$$

with L_i/K finite separable such that X is unramified in L_i. Thus if $M \subset K^{sep}$ is as in the statement of the lemma, then $\mathrm{Gal}(K^{sep}/M)$ is exactly the subgroup of $\mathrm{Gal}(K^{sep}/K)$ acting trivially on every object in the essential image of c. On the other hand, the essential image of c is exactly the category of S such that the $\mathrm{Gal}(K^{sep}/K)$-action factors through the surjection $\mathrm{Gal}(K^{sep}/K) \to \pi_1(X, \overline{\eta})$. We conclude that $\mathrm{Gal}(K^{sep}/M)$ is the kernel. Hence $\mathrm{Gal}(K^{sep}/M)$ is a normal subgroup, M/K is Galois, and we have a short exact sequence

$$1 \to \mathrm{Gal}(K^{sep}/M) \to \mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(M/K) \to 1$$

by Galois theory (Fields, Theorem 22.4 and Lemma 22.5). The proof is done. □

0BSM **Lemma 10.4.** *Let $(A, \mathfrak{m})$ be a normal local ring. Set $X = \mathrm{Spec}(A)$. Let A^{sh} be the strict henselization of A. Let K and K^{sh} be the fraction fields of A and A^{sh}. Then the sequence*

$$\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(\mathrm{Spec}(K)) \to \pi_1(X) \to 1$$

is exact in the sense of Lemma 4.3 part (1).

Proof. Note that A^{sh} is a normal domain, see More on Algebra, Lemma 42.6. The map $\pi_1(\mathrm{Spec}(K)) \to \pi_1(X)$ is surjective by Proposition 10.3.

Write $X^{sh} = \mathrm{Spec}(A^{sh})$. Let $Y \to X$ be a finite étale morphism. Then $Y^{sh} = Y \times_X X^{sh} \to X^{sh}$ is a finite étale morphism. Since A^{sh} is strictly henselian we see that Y^{sh} is isomorphic to a disjoint union of copies of X^{sh}. Thus the same is true for $Y \times_X \mathrm{Spec}(K^{sh})$. It follows that the composition $\pi_1(\mathrm{Spec}(K^{sh})) \to \pi_1(X)$ is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 4.3 to show the following: Given a finite étale morphism $V \to \mathrm{Spec}(K)$ such that $V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{sh})$ is a disjoint union of copies of $\mathrm{Spec}(K^{sh})$, we can find a finite étale morphism $Y \to X$ with $V \cong Y \times_X \mathrm{Spec}(K)$ over $\mathrm{Spec}(K)$. Write $V = \mathrm{Spec}(L)$, so L is a finite product of finite separable extensions of K. Let $B \subset L$ be the integral closure of A in L. If $A \to B$ is étale, then we can take $Y = \mathrm{Spec}(B)$ and the proof is complete. By Algebra, Lemma 143.4 (and a limit argument we omit) we see that $B \otimes_A A^{sh}$ is the integral closure of A^{sh} in $L^{sh} = L \otimes_K K^{sh}$. Our assumption is that L^{sh} is a product of copies of K^{sh} and hence B^{sh} is a product of copies of A^{sh}. Thus $A^{sh} \to B^{sh}$ is étale. As $A \to A^{sh}$ is faithfully flat it follows that $A \to B$ is étale (Descent, Lemma 20.29) as desired. $\square$

11. Group actions and integral closure

0BSN In this section we continue the discussion of More on Algebra, Section 92. Recall that a normal local ring is a domain by definition.

0BSP **Lemma 11.1.** *Let A be a normal domain whose fraction field is separably algebraically closed. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal. Then the residue field $\kappa(\mathfrak{p})$ is algebraically closed.*

Proof. Assume the lemma is not true to get a contradiction. Then there exists a monic irreducible polynomial $P(T) \in \kappa(\mathfrak{p})[T]$ of degree $d > 1$. After replacing P by $a^d P(a^{-1}T)$ for suitable $a \in A$ (to clear denominators) we may assume that P is the image of a monic polynomial Q in $A[T]$. Observe that Q is irreducible in $f.f.(A)[T]$. Namely a factorization over $f.f.(A)$ leads to a factorization over A by Algebra, Lemma 37.5 which we could reduce modulo $\mathfrak{p}$ to get a factorization of P. As $f.f.(A)$ is separably closed, Q is not a separable polynomial (Fields, Definition 12.2). Then the characteristic of $f.f.(A)$ is $p > 0$ and Q has vanishing linear term (Fields, Definition 12.2). However, then we can replace Q by $Q + aT$ where $a \in \mathfrak{p}$ is nonzero to get a contradiction. $\square$

0BSQ **Lemma 11.2.** *A normal local ring with separably closed fraction field is strictly henselian.*

Proof. Let $(A, \mathfrak{m}, \kappa)$ be normal local with separably closed fraction field K. If $A = K$, then we are done. If not, then the residue field κ is algebraically closed by Lemma 11.1 and it suffices to check that A is henselian. Let $f \in A[T]$ be monic and let $a_0 \in \kappa$ be a root of multiplicity 1 of the reduction $\overline{f} \in \kappa[T]$. Let $f = \prod f_i$ be the factorization in $K[T]$. By Algebra, Lemma 37.5 we have $f_i \in A[T]$. Thus a_0 is a root of f_i for some i. After replacing f by f_i we may assume f is irreducible. Then, since the derivative f' cannot be zero in $A[T]$ as a_0 is a single root, we conclude that f is linear due to the fact that K is separably algebraically closed. Thus A is henselian, see Algebra, Definition 148.1. $\square$

0BSS **Lemma 11.3.** *Let G be a finite group acting on a ring R. Let $R^G \to A$ be a ring map. Let $\mathfrak{q}' \subset A \otimes_{R^G} R$ be a prime lying over the prime $\mathfrak{q} \subset R$. Then*

$$I_\mathfrak{q} = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_{\kappa(\mathfrak{q})}\}$$

is equal to

$$I_{\mathfrak{q}'} = \{\sigma \in G \mid \sigma(\mathfrak{q}') = \mathfrak{q}' \text{ and } \sigma \bmod \mathfrak{q}' = id_{\kappa(\mathfrak{q}')}\}$$

Proof. Since $\mathfrak{q}$ is the inverse image of $\mathfrak{q}'$ and since $\kappa(\mathfrak{q}) \subset \kappa(\mathfrak{q}')$, we get $I_{\mathfrak{q}'} \subset I_\mathfrak{q}$. Conversely, if $\sigma \in I_\mathfrak{q}$, the σ acts trivially on the fibre ring $A \otimes_{R^G} \kappa(\mathfrak{q})$. Thus σ fixes all the primes lying over $\mathfrak{q}$ and induces the identity on their residue fields. $\square$

0BST **Lemma 11.4.** *Let G be a finite group acting on a ring R. Let $\mathfrak{q} \subset R$ be a prime. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q} \text{ and } \sigma \bmod \mathfrak{q} = id_\mathfrak{q}\}$$

Then $R^G \to R^I$ is étale at $R^I \cap \mathfrak{q}$.

Proof. The strategy of the proof is to use étale localization to reduce to the case where $R \to R^I$ is a local isomorphism at $R^I \cap \mathfrak{p}$. Let $R^G \to A$ be an étale ring map. We claim that if the result holds for the action of G on $A \otimes_{R^G} R$ and some prime $\mathfrak{q}'$ of $A \otimes_{R^G} R$ lying over $\mathfrak{q}$, then the result is true.

To check this, note that since $R^G \to A$ is flat we have $A = (A \otimes_{R^G} R)^G$, see More on Algebra, Lemma 92.4. By Lemma 11.3 the group I does not change. Then a second application of More on Algebra, Lemma 92.4 shows that $A \otimes_{R^G} R^I = (A \otimes_{R^G} R)^I$ (because $R^I \to A \otimes_{R^G} R^I$ is flat). Thus

$$\begin{tikzcd} \mathrm{Spec}((A \otimes_{R^G} R)^I) \arrow[r] \arrow[d] & \mathrm{Spec}(R^I) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(R^G) \end{tikzcd}$$

is cartesian and the horizontal arrows are étale. Thus if the left vertical arrow is étale in some open neighbourhood W of $(A \otimes_{R^G} R)^I \cap \mathfrak{q}'$, then the right vertical arrow is étale at the points of the (open) image of W in $\mathrm{Spec}(R^I)$, see Descent, Lemma 11.5. In particular the morphism $\mathrm{Spec}(R^I) \to \mathrm{Spec}(R^G)$ is étale at $R^I \cap \mathfrak{q}$.

Let $\mathfrak{p} = R^G \cap \mathfrak{q}$. By More on Algebra, Lemma 92.5 the fibre of $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ over $\mathfrak{p}$ is finite. Moreover the residue field extensions at these points are algebraic, normal, with finite automorphism groups by More on Algebra, Lemma 92.6. Thus we may apply More on Morphisms, Lemma 37.1 to the integral ring map $R^G \to R$ and the prime $\mathfrak{p}$. Combined with the claim above we reduce to the case where $R = A_1 \times \ldots \times A_n$ with each A_i having a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ such that the residue field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Of course $\mathfrak{q}$ is one of these primes, say $\mathfrak{q} = \mathfrak{q}_1$.

It may not be the case that G permutes the factors A_i (this would be true if the spectrum of A_i were connected, for example if R^G was local). This we can fix as follows; we suggest the reader think this through for themselves, perhaps using idempotents instead of topology. Recall that the product decomposition gives a corresponding disjoint union decomposition of $\mathrm{Spec}(R)$ by open and closed subsets U_i. Since G is finite, we can refine this covering by a finite disjoint union decomposition $\mathrm{Spec}(R) = \coprod_{j \in J} W_j$ by open and closed subsets W_j, such that for all $j \in J$ there exists a $j' \in J$ with $\sigma(W_j) = W_{j'}$. The union of the W_j not meeting $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ is a closed subset not meeting the fibre over $\mathfrak{p}$ hence maps to a closed subset of $\mathrm{Spec}(R^G)$ not meeting $\mathfrak{p}$ as $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is closed. Hence after replacing R^G by a principal localization (permissible by the claim) we may assume each W_j meets one of the points $\mathfrak{q}_i$. Then we set $U_i = W_j$ if $\mathfrak{q}_i \in W_j$. The corresponding product decomposition $R = A_1 \times \ldots \times A_n$ is one where G permutes the factors A_i.

Thus we may assume we have a product decomposition $R = A_1 \times \ldots \times A_n$ compatible with G-action, where each A_i has a single prime $\mathfrak{q}_i$ lying over $\mathfrak{p}$ and the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ are purely inseparable. Write $A' = A_2 \times \ldots \times A_n$ so that

$$R = A_1 \times A'$$

Since $\mathfrak{q} = \mathfrak{q}_1$ we find that every $\sigma \in I$ preserves the product decomposition above. Hence

$$R^I = (A_1)^I \times (A')^I$$

Observe that $I = D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\}$ because $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is purely inseparable. Since the action of G on primes over $\mathfrak p$ is transitive (More on Algebra, Lemma 92.5) we conclude that, the index of I in G is n and we can write $G = eI \amalg \sigma_2 I \amalg \ldots \amalg \sigma_n I$ so that $A_i = \sigma_i(A_1)$ for $i = 2, \ldots, n$. It follows that

$$R^G = (A_1)^I.$$

Thus the map $R^G \to R^I$ is étale at $R^I \cap \mathfrak q$ and the proof is complete. □

The following lemma generalizes More on Algebra, Lemma 93.15.

0BSU **Lemma 11.5.** *Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = Gal(L/K)$ and let B be the integral closure of A in L. Let $\mathfrak q \subset B$. Set*

$$I = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q \text{ and } \sigma \bmod \mathfrak q = id_{\kappa(\mathfrak q)}\}$$

Then $(B^I)_{B^I \cap \mathfrak q}$ is a filtered colimit of étale A-algebras.

Proof. We can write L as the filtered colimit of finite Galois extensions of K. Hence it suffices to prove this lemma in case L/K is a finite Galois extension, see Algebra, Lemma 149.3. Since $A = B^G$ as A is integrally closed in $K = L^G$ the result follows from Lemma 11.4. □

12. Ramification theory

0BSD In this section we continue the discussion of More on Algebra, Section 93 and we relate it to our discussion of the fundamental groups of schemes.

Let $(A, \mathfrak m, \kappa)$ be a normal local ring with fraction field K. Choose a separable algebraic closure K^{sep}. Let A^{sep} be the integral closure of A in K^{sep}. Choose maximal ideal $\mathfrak m^{sep} \subset A^{sep}$. Let $A \subset A^h \subset A^{sh}$ be the henselization and strict henselization. Observe that A^h and A^{sh} are normal rings as well (More on Algebra, Lemma 42.6). Denote K^h and K^{sh} their fraction fields. Since $(A^{sep})_{\mathfrak m^{sep}}$ is strictly henselian by Lemma 11.2 we can choose an A-algebra map $A^{sh} \to (A^{sep})_{\mathfrak m^{sep}}$. Namely, first choose a κ-embedding[2] $\kappa(\mathfrak m^{sh}) \to \kappa(\mathfrak m^{sep})$ and then extend (uniquely) to an A-algebra homomorphism by Algebra, Lemma 150.12. We get the following diagram

$$\begin{tikzcd} K^{sep} & K^{sh} \arrow[l] & K^h \arrow[l] & K \arrow[l] \\ (A^{sep})_{\mathfrak m^{sep}} \arrow[u] & A^{sh} \arrow[l] \arrow[u] & A^h \arrow[l] \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

We can take the fundamental groups of the spectra of these rings. Of course, since K^{sep}, $(A^{sep})_{\mathfrak m^{sep}}$, and A^{sh} are strictly henselian, for them we obtain trivial groups. Thus the interesting part is the following

0BSV (12.0.1)
$$\begin{tikzcd} \pi_1(U^{sh}) \arrow[r] \arrow[dr, "1"'] & \pi_1(U^h) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ & \pi_1(X^h) \arrow[r] & \pi_1(X) \end{tikzcd}$$

[2]This is possible because $\kappa(\mathfrak m^{sh})$ is a separable algebraic closure of κ and $\kappa(\mathfrak m^{sep})$ is an algebraic closure of κ by Lemma 11.1.

Here X^h and X are the spectra of A^h and A and U^{sh}, U^h, U are the spectra of K^{sh}, K^h, and K. The label 1 means that the map is trivial; this follows as it factors through the trivial group $\pi_1(X^{sh})$. On the other hand, the profinite group $G = \text{Gal}(K^{sep}/K)$ acts on A^{sep} and we can make the following definitions

$$D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\} \supset I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = \text{id}_{\kappa(\mathfrak{m}^{sep})}\}$$

These groups are sometimes called the *decomposition group* and the *inertia group* especially when A is a discrete valuation ring.

0BSW **Lemma 12.1.** *In the situation described above, via the isomorphism $\pi_1(U) = Gal(K^{sep}/K)$ the diagram (12.0.1) translates into the diagram*

$$\begin{tikzcd} I \arrow[r] \arrow[dr, "1"'] & D \arrow[r] \arrow[d] & Gal(K^{sep}/K) \arrow[d] \\ & Gal(\kappa(\mathfrak{m}^{sh})/\kappa) \arrow[r] & Gal(M/K) \end{tikzcd}$$

where $K^{sep}/M/K$ is the maximal subextension unramified with respect to A. Moreover, the vertical arrows are surjective, the kernel of the left vertical arrow is I and the kernel of the right vertical arrow is the smallest closed normal subgroup of $Gal(K^{sep}/K)$ containing I.

Proof. By construction the group D acts on $(A^{sep})_{\mathfrak{m}^{sep}}$ over A. By the uniqueness of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ given the map on residue fields (Algebra, Lemma 150.12) we see that the image of $A^{sh} \to (A^{sep})_{\mathfrak{m}^{sep}}$ is contained in $((A^{sep})_{\mathfrak{m}^{sep}})^I$. On the other hand, Lemma 11.5 shows that $((A^{sep})_{\mathfrak{m}^{sep}})^I$ is a filtered colimit of étale extensions of A. Since A^{sh} is the maximal such extension, we conclude that $A^{sh} = ((A^{sep})_{\mathfrak{m}^{sep}})^I$. Hence $K^{sh} = (K^{sep})^I$.

Recall that I is the kernel of a surjective map $D \to \text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa)$, see More on Algebra, Lemma 92.7. We have $\text{Aut}(\kappa(\mathfrak{m}^{sep})/\kappa) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$ as we have seen above that these fields are the algebraic and separable algebraic closures of κ. On the other hand, any automorphism of A^{sh} over A is an automorphism of A^{sh} over A^h by the uniqueness in Algebra, Lemma 150.6. Furthermore, A^{sh} is the colimit of finite étale extensions $A^h \subset A'$ which correspond 1-to-1 with finite separable extension κ'/κ, see Algebra, Remark 150.4. Thus

$$\text{Aut}(A^{sh}/A) = \text{Aut}(A^{sh}/A^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$$

Let $\kappa \subset \kappa'$ be a finite Galois extension with Galois group G. Let $A^h \subset A'$ be the finite étale extension corresponding to $\kappa \subset \kappa'$ by Algebra, Lemma 148.7. Then it follows that $(A')^G = A^h$ by looking at fraction fields and degrees (small detail omitted). Taking the colimit we conclude that $(A^{sh})^{\text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)} = A^h$. Combining all of the above, we find $A^h = ((A^{sep})_{\mathfrak{m}^{sep}})^D$. Hence $K^h = (K^{sep})^D$.

Since U, U^h, U^{sh} are the spectra of the fields K, K^h, K^{sh} we see that the top lines of the diagrams correspond via Lemma 6.4. By Lemma 7.2 we have $\pi_1(X^h) = \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa)$. The exactness of the sequence $1 \to I \to D \to \text{Gal}(\kappa(\mathfrak{m}^{sh})/\kappa) \to 1$ was pointed out above. By Proposition 10.3 we see that $\pi_1(X) = \text{Gal}(M/K)$. Finally, the statement on the kernel of $\text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) = \pi_1(X)$ follows from Lemma 10.4. This finishes the proof. $\square$

Let X be a normal integral scheme with function field K. Let K^{sep} be a separable algebraic closure of K. Let $X^{sep} \to X$ be the normalization of X in K^{sep}. Since $G = \text{Gal}(K^{sep}/K)$ acts on K^{sep} we obtain a right action of G on X^{sep}. For $y \in X^{sep}$ define

$$D_y = \{\sigma \in G \mid \sigma(y) = y\} \supset I_y = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}_y = \text{id}_{\kappa(y)}\}$$

similarly to the above. On the other hand, for $x \in X$ let $\mathcal{O}^{sh}_{X,x}$ be a strict henselization, let K^{sh}_x be the fraction field of $\mathcal{O}^{sh}_{X,x}$ and choose a K-embedding $K^{sh}_x \to K^{sep}$.

0BTD **Lemma 12.2.** *Let X be a normal integral scheme with function field K. With notation as above, the following three subgroups of Gal$(K^{sep}/K) = \pi_1(\text{Spec}(K))$ are equal*

1. *the kernel of the surjection Gal$(K^{sep}/K) \longrightarrow \pi_1(X)$,*
2. *the smallest normal closed subgroup containing I_y for all $y \in X^{sep}$, and*
3. *the smallest normal closed subgroup containing Gal(K^{sep}/K^{sh}_x) for all $x \in X$.*

Proof. The equivalence of (2) and (3) follows from Lemma 12.1 which tells us that I_y is conjugate to $\text{Gal}(K^{sep}/K^{sh}_x)$ if y lies over x. By Lemma 10.4 we see that $\text{Gal}(K^{sep}/K^{sh}_x)$ maps trivially to $\pi_1(\text{Spec}(\mathcal{O}_{X,x}))$ and therefore the subgroup $N \subset G = \text{Gal}(K^{sep}/K)$ of (2) and (3) is contained in the kernel of $G \longrightarrow \pi_1(X)$.

To prove the other inclusion, since N is normal, it suffices to prove: given $N \subset U \subset G$ with U open normal, the quotient map $G \to G/U$ factors through $\pi_1(X)$. In other words, if L/K is the Galois extension corresponding to U, then we have to show that X is unramified in L (Section 10, especially Proposition 10.3). It suffices to do this when X is affine (we do this so we can refer to algebra results in the rest of the proof). Let $Y \to X$ be the normalization of X in L. The inclusion $L \subset K^{sep}$ induces a morphism $\pi : X^{sep} \to Y$. For $y \in X^{sep}$ the inertia group of $\pi(y)$ in $\text{Gal}(L/K)$ is the image of I_y in $\text{Gal}(L/K)$; this follows from More on Algebra, Lemma 92.8. Since $N \subset U$ all these inertia groups are trivial. We conclude that $Y \to X$ is étale by applying Lemma 11.4. (Alternative: you can use Lemma 10.4 to see that the pullback of Y to $\text{Spec}(\mathcal{O}_{X,x})$ is étale for all $x \in X$ and then conclude from there with a bit more work.) $\square$

0BTE **Example 12.3.** Let X be a normal integral Noetherian scheme with function field K. Purity of branch locus (see below) tells us that if X is regular, then it suffices in Lemma 12.2 to consider the inertia groups $I = \pi_1(\text{Spec}(K^{sh}_x))$ for points x of codimension 1 in X. In general this is not enough however. Namely, let $Y = \mathbf{A}^n_k = \text{Spec}(k[t_1, \ldots, t_n])$ where k is a field not of characteristic 2. Let $G = \{\pm 1\}$ be the group of order 2 acting on Y by multiplication on the coordinates. Set

$$X = \text{Spec}(k[t_it_j, i, j \in \{1, \ldots, n\}])$$

The embedding $k[t_it_j] \subset k[t_1, \ldots, t_n]$ defines a degree 2 morphism $Y \to X$ which is unramified everywhere except over the maximal ideal $\mathfrak{m} = (t_it_j)$ which is a point of codimension n in X.

0BTF **Lemma 12.4.** *Let X be an integral normal scheme with function field K. Let L/K be a finite extension. Let $Y \to X$ be the normalization of X in L. The following are equivalent*

1. *X is unramified in L as defined in Section 10,*

(2) *$Y \to X$ is an unramified morphism of schemes,*
(3) *$Y \to X$ is an étale morphism of schemes,*
(4) *$Y \to X$ is a finite étale morphism of schemes,*
(5) *for $x \in X$ the projection $Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ is unramified,*
(6) *same as in (5) but with $\mathcal{O}^h_{X,x}$,*
(7) *same as in (5) but with $\mathcal{O}^{sh}_{X,x}$,*
(8) *for $x \in X$ the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$.*

If L/K is Galois with Galois group G, then these are also equivalent to

(9) *for $y \in Y$ the group $I_y = \{g \in G \mid g(y) = y \text{ and } g \bmod \mathfrak{m}_y = id_{\kappa(y)}\}$ is trivial.*

Proof. The equivalence of (1) and (2) is the definition of (1). The equivalence of (2), (3), and (4) is Lemma 10.1. It is straightforward to prove that (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6), (6) $\Rightarrow$ (7).

Assume (7). Observe that $\mathcal{O}^{sh}_{X,x}$ is a normal local domain (More on Algebra, Lemma 42.6). Let $L^{sh} = L \otimes_K K^{sh}_x$ where K^{sh}_x is the fraction field of $\mathcal{O}^{sh}_{X,x}$. Then $L^{sh} = \prod_{i=1,\ldots,n} L_i$ with L_i/K^{sh}_x finite separable. By Algebra, Lemma 143.4 (and a limit argument we omit) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ is the integral closure of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ in L^{sh}. Hence by Lemma 10.1 (applied to the factors L_i of L^{sh}) we see that $Y \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ is finite étale. Looking at the generic point we see that the degree is equal to $[L : K]$ and hence we see that (8) is true.

Assume (8). Assume that $x \in X$ and that the scheme theoretic fibre Y_x is étale over x of degree $\geq [L : K]$. Observe that this means that Y has $\geq [L : K]$ geometric points lying over x. We will show that $Y \to X$ is finite étale over a neighbourhood of x. This will prove (1) holds. To prove this we may assume $X = \mathrm{Spec}(R)$, the point x corresponds to the prime $\mathfrak{p} \subset R$, and $Y = \mathrm{Spec}(S)$. We apply More on Morphisms, Lemma 37.1 and we find an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U = V_1 \amalg \ldots \amalg V_m$ such that V_i has a unique point v_i lying over u with $\kappa(v_i)/\kappa(u)$ purely inseparable. Shrinking U if necessary we may assume U is a normal integral scheme with generic point ξ (use Descent, Lemmas 13.3 and 15.2 and Properties, Lemma 7.5). By our remark on geometric points we see that $m \geq [L : K]$. On the other hand, by More on Morphisms, Lemma 17.2 we see that $\coprod V_i \to U$ is the normalization of U in $\mathrm{Spec}(L) \times_X U$. As $K \subset \kappa(\xi)$ is finite separable, we can write $\mathrm{Spec}(L) \times_X U = \mathrm{Spec}(\prod_{i=1,\ldots,n} L_i)$ with $L_i/\kappa(\xi)$ finite and $[L : K] = \sum [L_i : \kappa(\xi)]$. Since V_j is nonempty for each j and $m \geq [L : K]$ we conclude that $m = n$ and $[L_i : \kappa(\xi)] = 1$ for all i. Then $V_j \to U$ is an isomorphism in particular étale, hence $Y \times_X U \to U$ is étale. By Descent, Lemma 20.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of x).

Assume L/K is Galois and (9) holds. Then $Y \to X$ is étale by Lemma 11.5. We omit the proof that (1) implies (9). $\square$

In the case of infinite Galois extensions of discrete valuation rings we can say a tiny bit more. To do so we introduce the following notation. A subset $S \subset \mathbf{N}$ of integers is *multiplicativity directed* if $1 \in S$ and for $n, m \in S$ there exists $k \in S$ with $n|k$ and $m|k$. Define a partial ordering on S by the rule $n \geq_S m$ if and only if $m|n$. Given a field κ we obtain an inverse system of finite groups $\{\mu_n(\kappa)\}_{n \in S}$ with transition maps

$$\mu_n(\kappa) \longrightarrow \mu_m(\kappa), \quad \zeta \longmapsto \zeta^{n/m}$$

for $n \geq_S m$. Then we can form the profinite group

$$\lim_{n \in S} \mu_n(\kappa)$$

Observe that the limit is cofiltered (as S is directed). The construction is functorial in κ. In particular $\text{Aut}(\kappa)$ acts on this profinite group. For example, if $S = \{1, n\}$, then this gives $\mu_n(\kappa)$. If $S = \{1, \ell, \ell^2, \ell^3, \ldots\}$ for some prime ℓ different from the characteristic of κ this produces $\lim_n \mu_{\ell^n}(\kappa)$ which is sometimes called the ℓ-adic Tate module of the multiplicative group of κ (compare with More on Algebra, Example 81.4).

0BUA **Lemma 12.5.** *Let A be a discrete valuation ring with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let B be the integral closure of A in L. Let $\mathfrak m$ be a maximal ideal of B. Let $G = Gal(L/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak m) = \mathfrak m\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak m = id_{\kappa(\mathfrak m)}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Aut(\kappa(\mathfrak m)/\kappa_A) \to 1$$

The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

1. *P is a normal subgroup of D,*
2. *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
3. *there is a multiplicatively directed $S \subset \mathbf{N}$ such that $\kappa(\mathfrak m)$ contains a primitive nth root of unity for each $n \in S$ (elements of S are prime to p),*
4. *there exists a canonical surjective map*

   $$\theta_{can} : I \to \lim_{n \in S} \mu_n(\kappa(\mathfrak m))$$

   whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(\mathfrak m))$.

Proof. This is mostly a reformulation of the results on finite Galois extensions proved in More on Algebra, Section 93. The surjectivity of the map $D \to \text{Aut}(\kappa(\mathfrak m)/\kappa)$ is More on Algebra, Lemma 92.7. This gives the first exact sequence.

To construct the second short exact sequence let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$. Set $G_\lambda = \text{Gal}(L_\lambda/K)$. Recall that G_λ is an inverse system of finite groups with surjective transition maps and that $G = \lim_{\lambda \in \Lambda} G_\lambda$, see Fields, Lemma 22.3. We let B_λ be the integral closure of A in L_λ. Then we set $\mathfrak m_\lambda = \mathfrak m \cap B_\lambda$ and we denote $P_\lambda, I_\lambda, D_\lambda$ the wild inertia, inertia, and decomposition group of $\mathfrak m_\lambda$, see More on Algebra, Lemma 93.12. For $\lambda \geq \lambda'$ the restriction defines a commutative diagram

$$\begin{tikzcd}
P_\lambda \arrow[r] \arrow[d] & I_\lambda \arrow[r] \arrow[d] & D_\lambda \arrow[r] \arrow[d] & G_\lambda \arrow[d] \\
P_{\lambda'} \arrow[r] & I_{\lambda'} \arrow[r] & D_{\lambda'} \arrow[r] & G_{\lambda'}
\end{tikzcd}$$

with surjective vertical maps, see More on Algebra, Lemma 93.17.

From the definitions it follows immediately that $I = \lim I_\lambda$ and $D = \lim D_\lambda$ under the isomorphism $G = \lim G_\lambda$ above. Since $L = \mathrm{colim}\, L_\lambda$ we have $B = \mathrm{colim}\, B_\lambda$ and $\kappa(\mathfrak m) = \mathrm{colim}\, \kappa(\mathfrak m_\lambda)$. Since the transition maps of the system D_λ are compatible with the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak m_\lambda)/\kappa)$ (see More on Algebra, Lemma 93.17) we see that the map $D \to \mathrm{Aut}(\kappa(\mathfrak m)/\kappa)$ is the limit of the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak m_\lambda)/\kappa)$.

There exist canonical maps

$$\theta_{\lambda,can} : I_\lambda \longrightarrow \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$$

where $n_\lambda = |I_\lambda|/|P_\lambda|$, where $\mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$ has order n_λ, such that $\theta_{\lambda,can}(\tau\sigma\tau^{-1}) = \tau(\theta_{\lambda,can}(\sigma))$ for $\tau \in D_\lambda$ and $\sigma \in I_\lambda$, and such that we get commutative diagrams

$$\begin{tikzcd}
I_\lambda \arrow[r, "\theta_{\lambda,can}"'] \arrow[d] & \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda)) \arrow[d, "(-)^{n_\lambda/n_{\lambda'}}"] \\
I_{\lambda'} \arrow[r, "\theta_{\lambda',can}"] & \mu_{n_{\lambda'}}(\kappa(\mathfrak m_{\lambda'}))
\end{tikzcd}$$

see More on Algebra, Remark 93.18.

Let $S \subset \mathbf{N}$ be the collection of integers n_λ. Since Λ is directed, we see that S is multiplicatively directed. By the displayed commutative diagrams above we can take the limits of the maps $\theta_{\lambda,can}$ to obtain

$$\theta_{can} : I \to \lim\nolimits_{n\in S} \mu_n(\kappa(\mathfrak m)).$$

This map is continuous (small detail omitted). Since the transition maps of the system of I_λ are surjective and Λ is directed, the projections $I \to I_\lambda$ are surjective. For every λ the diagram

$$\begin{tikzcd}
I \arrow[r, "\theta_{can}"'] \arrow[d] & \lim_{n\in S} \mu_n(\kappa(\mathfrak m)) \arrow[d] \\
I_\lambda \arrow[r, "\theta_{\lambda,can}"] & \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))
\end{tikzcd}$$

commutes. Hence the image of θ_{can} surjects onto the finite group $\mu_{n_\lambda}(\kappa(\mathfrak m)) = \mu_{n_\lambda}(\kappa(\mathfrak m_\lambda))$ of order n_λ (see above). It follows that the image of θ_{can} is dense. On the other hand θ_{can} is continuous and the source is a profinite group. Hence θ_{can} is surjective by a topological argument.

The property $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$ follows from the corresponding properties of the maps $\theta_{\lambda,can}$ and the compatibility of the map $D \to \mathrm{Aut}(\kappa(\mathfrak m))$ with the maps $D_\lambda \to \mathrm{Aut}(\kappa(\mathfrak m_\lambda))$. Setting $P = \mathrm{Ker}(\theta_{can})$ this implies that P is a normal subgroup of D. Setting $I_t = I/P$ we obtain the isomorphism $I_t \to \lim_{n\in S} \mu_n(\kappa(\mathfrak m))$ from the surjectivity of θ_{can}.

To finish the proof we show that $P = \lim P_\lambda$ which proves that P is a pro-p-group. Recall that the tame inertia group $I_{\lambda,t} = I_\lambda/P_\lambda$ has order n_λ. Since the transition maps $P_\lambda \to P_{\lambda'}$ are surjective and Λ is directed, we obtain a short exact sequence

$$1 \to \lim P_\lambda \to I \to \lim I_{\lambda,t} \to 1$$

(details omitted). Since for each λ the map $\theta_{\lambda,can}$ induces an isomorphism $I_{\lambda,t} \cong \mu_{n_\lambda}(\kappa(\mathfrak m))$ the desired result follows. $\square$

0BUB **Lemma 12.6.** *Let A be a discrete valuation ring with fraction field K. Let K^{sep} be a separable closure of K. Let A^{sep} be the integral closure of A in K^{sep}. Let $\mathfrak{m}^{sep}$ be a maximal ideal of A^{sep}. Let $\mathfrak{m} = \mathfrak{m}^{sep} \cap A$, let $\kappa = A/\mathfrak{m}$, and let $\overline{\kappa} = A^{sep}/\mathfrak{m}^{sep}$. Then $\overline{\kappa}$ is an algebraic closure of κ. Let $G = Gal(K^{sep}/K)$, $D = \{\sigma \in G \mid \sigma(\mathfrak{m}^{sep}) = \mathfrak{m}^{sep}\}$, and $I = \{\sigma \in D \mid \sigma \bmod \mathfrak{m}^{sep} = id_{\kappa(\mathfrak{m}^{sep})}\}$. The decomposition group D fits into a canonical exact sequence*

$$1 \to I \to D \to Gal(\kappa^{sep}/\kappa) \to 1$$

where $\kappa^{sep} \subset \overline{\kappa}$ is the separable closure of κ. The inertia group I fits into a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

1. *P is a normal subgroup of D,*
2. *P is a pro-p-group if the characteristic of κ_A is $p > 1$ and $P = \{1\}$ if the characteristic of κ_A is zero,*
3. *there exists a canonical surjective map*

$$\theta_{can} : I \to \lim_{n \textit{ prime to } p} \mu_n(\kappa^{sep})$$

whose kernel is P, which satisfies $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \textit{ prime to } p} \mu_n(\kappa^{sep})$.

Proof. The field $\overline{\kappa}$ is the algebraic closure of κ by Lemma 11.1. Most of the statements immediately follow from the corresponding parts of Lemma 12.5. For example because $\text{Aut}(\overline{\kappa}/\kappa) = \text{Gal}(\kappa^{sep}/\kappa)$ we obtain the first sequence. Then the only other assertion that needs a proof is the fact that with S as in Lemma 12.5 the limit $\lim_{n \in S} \mu_n(\overline{\kappa})$ is equal to $\lim_{n \text{ prime to } p} \mu_n(\kappa^{sep})$. To see this it suffices to show that every integer n prime to p divides an element of S. Let $\pi \in A$ be a uniformizer and consider the splitting field L of the polynomial $X^n - \pi$. Since the polynomial is separable we see that L is a finite Galois extension of K. Choose an embedding $L \to K^{sep}$. Observe that if B is the integral closure of A in L, then the ramification index of $A \to B_{\mathfrak{m}^{sep} \cap B}$ is divisible by n (because π has an nth root in B; in fact the ramification index equals n but we do not need this). Then it follows from the construction of the S in the proof of Lemma 12.5 that n divides an element of S. □

13. Geometric and arithmetic fundamental groups

0BTU In this section we work out what happens when comparing the fundamental group of a scheme X over a field k with the fundamental group of $X_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of k.

0BTV **Lemma 13.1.** *Let I be a directed set. Let X_i be an inverse system of quasi-compact and quasi-separated schemes over I with affine transition morphisms. Let $X = \lim X_i$ as in Limits, Section 2. Then there is an equivalence of categories*

$$\text{colim}\, F\acute{E}t_{X_i} = F\acute{E}t_X$$

If X_i is connected for all sufficiently large i and $\overline{x}$ is a geometric point of X, then

$$\pi_1(X, \overline{x}) = \lim \pi_1(X_i, \overline{x})$$

Proof. The equivalence of categories follows from Limits, Lemmas 10.1, 8.3, and 8.10. The second statement is formal given the statement on categories. □

0BTW **Lemma 13.2.** *Let k be a field with perfection k^{perf}. Let X be a connected scheme over k. Then $X_{k^{perf}}$ is connected and $\pi_1(X_{k^{perf}}) \to \pi_1(X)$ is an isomorphism.*

Proof. Special case of topological invariance of the fundamental group. See Proposition 7.4. To see that $\mathrm{Spec}(k^{perf}) \to \mathrm{Spec}(k)$ is a universal homeomorphism you can use Algebra, Lemma 45.9. $\square$

0BTX **Lemma 13.3.** *Let k be a field with algebraic closure $\overline{k}$. Let X be a quasi-compact and quasi-separated scheme over k. If the base change $X_{\overline{k}}$ is connected, then there is a short exact sequence*

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k)) \to 1$$

of profinite topological groups.

Proof. Connected objects of $\textit{FÉt}_{\mathrm{Spec}(k)}$ are of the form $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ with k'/k a finite separable extension. Then $X_{\mathrm{Spec}\, k'}$ is connected, as the morphism $X_{\overline{k}} \to X_{\mathrm{Spec}(k')}$ is surjective and $X_{\overline{k}}$ is connected by assumption. Thus $\pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is surjective by Lemma 4.1.

Before we go on, note that we may assume that k is a perfect field. Namely, we have $\pi_1(X_{k^{perf}}) = \pi_1(X)$ and $\pi_1(\mathrm{Spec}(k^{perf})) = \pi_1(\mathrm{Spec}(k))$ by Lemma 13.2.

It is clear that the composition of the functors $\textit{FÉt}_{\mathrm{Spec}(k)} \to \textit{FÉt}_X \to \textit{FÉt}_{X_{\overline{k}}}$ sends objects to disjoint unions of copies of $X_{\mathrm{Spec}(\overline{k})}$. Therefore the composition $\pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ is the trivial homomorphism by Lemma 4.2.

Let $U \to X$ be a finite étale morphism with U connected. Observe that $U \times_X X_{\overline{k}} = U_{\overline{k}}$. Suppose that $U_{\overline{k}} \to X_{\overline{k}}$ has a section $s : X_{\overline{k}} \to U_{\overline{k}}$. Then $s(X_{\overline{k}})$ is an open connected component of $U_{\overline{k}}$. For $\sigma \in \mathrm{Gal}(\overline{k}/k)$ denote s^σ the base change of s by $\mathrm{Spec}(\sigma)$. Since $U_{\overline{k}} \to X_{\overline{k}}$ is finite étale it has only a finite number of sections. Thus

$$\overline{T} = \bigcup s^\sigma(X_{\overline{k}})$$

is a finite union and we see that $\overline{T}$ is a $\mathrm{Gal}(\overline{k}/k)$-stable open and closed subset. By Varieties, Lemma 7.10 we see that $\overline{T}$ is the inverse image of a closed subset $T \subset U$. Since $U_{\overline{k}} \to U$ is open (Morphisms, Lemma 22.4) we conclude that T is open as well. As U is connected we see that $T = U$. Hence $U_{\overline{k}}$ is a (finite) disjoint union of copies of $X_{\overline{k}}$. By Lemma 4.5 we conclude that the image of $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is normal.

Let $V \to X_{\overline{k}}$ be a finite étale cover. Recall that $\overline{k}$ is the union of finite separable extensions of k. By Lemma 13.1 we find a finite separable extension k'/k and a finite étale morphism $U \to X_{k'}$ such that $V = X_{\overline{k}} \times_{X_{k'}} U = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$. Then the composition $U \to X_{k'} \to X$ is finite étale and $U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ contains $V = U \times_{\mathrm{Spec}(k')} \mathrm{Spec}(\overline{k})$ as an open and closed subscheme. (Because $\mathrm{Spec}(\overline{k})$ is an open and closed subscheme of $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ via the multiplication map $k' \otimes_k \overline{k} \to \overline{k}$.) By Lemma 4.4 we conclude that $\pi_1(X_{\overline{k}}) \to \pi_1(X)$ is injective.

Finally, we have to show that for any finite étale morphism $U \to X$ such that $U_{\overline{k}}$ is a disjoint union of copies of $X_{\overline{k}}$ there is a finite étale morphism $V \to \mathrm{Spec}(k)$ and a surjection $V \times_{\mathrm{Spec}(k)} X \to U$. See Lemma 4.3. Arguing as above using Lemma 13.1 we find a finite separable extension k'/k such that there is an isomorphism $U_{k'} \cong \coprod_{i=1,\ldots,n} X_{k'}$. Thus setting $V = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k')$ we conclude. $\square$

14. Homotopy exact sequence

0BUM In this section we discuss the following result. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups. See Proposition 14.2.

0BUN **Lemma 14.1.** *Let $f : X \to S$ be a proper morphism of schemes. Let $X \to S' \to S$ be the Stein factorization of f, see More on Morphisms, Theorem 45.5. If f is of finite presentation, flat, with geometrically reduced fibres, then $S' \to S$ is finite étale.* [Gro71, Expose X, Proposition 1.2, p. 262].

Proof. Let $s \in S$. Set n be the number of connected components of the geometric fibre $X_{\overline{s}}$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at s is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. By More on Morphisms, Lemma 45.2 there are finitely many points $s'_1, \ldots, s'_m \in S'$ lying over s and for each i the extension $\kappa(s'_i)/\kappa(s)$ is finite. More on Morphisms, Lemma 37.1 tells us that after replacing S by an étale neighbourhood of s we may assume $S' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $s'_i \in V_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. In this case the schemes $X_{s'_i}$ are geometrically connected over $\kappa(s)$, hence $m = n$. The schemes $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are proper, flat, of finite presentation, with geometrically reduced fibres over S. It suffices to prove the lemma for each of the morphisms $X_i \to S$. This reduces us to the case where $X_{\overline{s}}$ is connected.

Assume that $X_{\overline{s}}$ is connected. By More on Morphisms, Lemma 45.8 we see that $X \to S$ has geometrically connected fibres in a neighbourhood of s. Thus we may assume the fibres of $X \to S$ are geometrically connected. Then $f_*\mathcal{O}_X = \mathcal{O}_S$ by Derived Categories of Schemes, Lemma 28.6 which finishes the proof. $\square$

0C0J **Proposition 14.2.** *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\overline{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Proof. Let $Y \to X$ be a finite étale morphism. Consider the Stein factorization

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

of $Y \to S$. By Lemma 14.1 the morphism $T \to S$ is finite étale. In this way we obtain a functor $F\acute{E}t_X \to F\acute{E}t_S$. For any finite étale morphism $U \to S$ a morphism $Y \to U \times_S X$ over X is the same thing as a morphism $Y \to U$ over S and such a morphism factors uniquely through the Stein factorization, i.e., corresponds to a unique morphism $T \to U$ (by the construction of the Stein factorization as a relative normalization in More on Morphisms, Lemma 45.1 and factorization by Morphisms, Lemma 50.4). Thus we see that the functors $F\acute{E}t_X \to F\acute{E}t_S$ and $F\acute{E}t_S \to F\acute{E}t_X$ are adjoints. Note that the Stein factorization of $U \times_S X \to S$ is U, because the fibres of $U \times_S X \to U$ are geometrically connected.

By the discussion above and Categories, Lemma 24.3 we conclude that $F\acute{E}t_S \to F\acute{E}t_X$ is fully faithful, i.e., $\pi_1(X) \to \pi_1(S)$ is surjective (Lemma 4.1).

It is immediate that the composition $F\acute{E}t_S \to F\acute{E}t_X \to F\acute{E}t_{X_{\overline{s}}}$ sends any U to a disjoint union of copies of $X_{\overline{s}}$. Hence $\pi_1(X_{\overline{s}}) \to \pi_1(X) \to \pi_1(S)$ is trivial by Lemma 4.2.

Let $Y \to X$ be a finite étale morphism with Y connected such that $Y \times_X X_{\overline{s}}$ contains a connected component Z isomorphic to $X_{\overline{s}}$. Consider the Stein factorization T as above. Let $\overline{t} \in T_{\overline{s}}$ be the point corresponding to the fibre Z. Observe that T is connected (as the image of a connected scheme) and by the surjectivity above $T \times_S X$ is connected. Now consider the factorization

$$\pi : Y \longrightarrow T \times_S X$$

Let $\overline{x} \in X_{\overline{s}}$ be any closed point. Note that $\kappa(\overline{t}) = \kappa(\overline{s}) = \kappa(\overline{x})$ is an algebraically closed field. Then the fibre of π over $(\overline{t}, \overline{x})$ consists of a unique point, namely the unique point $\overline{z} \in Z$ corresponding to $\overline{x} \in X_{\overline{s}}$ via the isomorphism $Z \to X_{\overline{s}}$. We conclude that the finite étale morphism π has degree 1 in a neighbourhood of $(\overline{t}, \overline{x})$. Since $T \times_S X$ is connected it has degree 1 everywhere and we find tat $Y \cong T \times_S X$. Thus $Y \times_X X_{\overline{s}}$ splits completely. Combining all of the above we see that Lemmas 4.3 and 4.5 both apply and the proof is complete. $\square$

15. Specialization maps

0BUP In this section we construct specialization maps. Let $f : X \to S$ be a proper morphism of schemes with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points in S. Let $\overline{s}$ and $\overline{s}'$ be geometric points lying over s and s'. Then there is a specialization map

$$sp : \pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_{\overline{s}})$$

The construction of this map is as follows. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$, see Algebra, Definition 150.3. Since $s' \rightsquigarrow s$ the point s' corresponds to a point of $\text{Spec}(\mathcal{O}_{S,s})$ and hence there is at least one point (and potentially many points) of $\text{Spec}(A)$ over s' whose residue field is a separable algebraic extension of $\kappa(s')$. Since $\kappa(\overline{s}')$ is algebraically closed we can choose a morphism $\varphi : \overline{s}' \to \text{Spec}(A)$ giving rise to a commutative diagram

$$\begin{tikzcd} \overline{s}' \arrow[r, "\varphi"'] \arrow[dr] & \text{Spec}(A) \arrow[d] & \overline{s} \arrow[l] \arrow[dl] \\ & S & \end{tikzcd}$$

The specialization map is the composition

$$\pi_1(X_{\overline{s}'}) \longrightarrow \pi_1(X_A) = \pi_1(X_{\kappa(s)^{sep}}) = \pi_1(X_{\overline{s}})$$

where the first equality is Lemma 8.1 and the second follows from Lemmas 13.2 and 8.2. By construction the specialization map fits into a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{s}'}) \arrow[rr, "sp"'] \arrow[dr] & & \pi_1(X_{\overline{s}}) \arrow[dl] \\ & \pi_1(X) & \end{tikzcd}$$

provided that X is connected. The specialization map depends on the choice of $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ above and we will write sp_φ if we want to indicate this.

0C0K **Lemma 15.1.** *Consider a commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

of schemes where f and g are proper with geometrically connected fibres. Let $t' \leadsto t$ be a specialization of points in T and consider a specialization map $sp : \pi_1(Y_{\overline{t}'}) \to \pi_1(Y_{\overline{t}})$ as above. Then there is a commutative diagram

$$\begin{tikzcd} \pi_1(Y_{\overline{t}'}) \arrow[r, "sp"'] \arrow[d] & \pi_1(Y_{\overline{t}}) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & \pi_1(X_{\overline{s}}) \end{tikzcd}$$

of specialization maps where $\overline{s}$ and $\overline{s}'$ are the images of $\overline{t}$ and $\overline{t}'$.

Proof. Let B be the strict henselization of $\mathcal{O}_{T,t}$ with respect to $\kappa(t) \subset \kappa(t)^{sep} \subset \kappa(\overline{t})$. Pick $\psi : \overline{t}' \to \mathrm{Spec}(B)$ lifting $\overline{t}' \to T$ as in the construction of the specialization map. Let s and s' denote the images of t and t' in S. Let A be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$. Since $\kappa(\overline{s}) = \kappa(\overline{t})$, by the functoriality of strict henselization (Algebra, Lemma 150.12) we obtain a ring map $A \to B$ fitting into the commutative diagram

$$\begin{tikzcd} \overline{t}' \arrow[r, "\psi"'] \arrow[d] & \mathrm{Spec}(B) \arrow[r] \arrow[d] & T \arrow[d] \\ \overline{s}' \arrow[r, "\varphi"] & \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

Here the morphism $\varphi : \overline{s}' \to \mathrm{Spec}(A)$ is simply taken to be the composition $\overline{t}' \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Applying base change we obtain a commutative diagram

$$\begin{tikzcd} Y_{\overline{t}'} \arrow[r] \arrow[d] & Y_B \arrow[d] \\ X_{\overline{s}'} \arrow[r] & X_A \end{tikzcd}$$

and from the construction of the specialization map the commutativity of this diagram implies the commutativity of the diagram of the lemma. $\square$

0C0L **Lemma 15.2.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s'' \rightsquigarrow s' \rightsquigarrow s$ be specializations of points of S. A composition of specialization maps $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is a specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct the first specialization map. Let $\mathcal{O}_{S,s'} \to A'$ be the strict henselization constructed using $\kappa(s') \subset \kappa(\overline{s}')$. By functoriality of strict henselization, there is a map $A \to A'$ such that the composition with $A' \to \kappa(\overline{s}')$ is the given map (Algebra, Lemma 149.5). Next, let $A' \to \kappa(\overline{s}'')$ be the map used to construct the second specialization map. Then it is clear that the composition of the first and second specialization maps is the specialization map $\pi_1(X_{\overline{s}''}) \to \pi_1(X_{\overline{s}})$ constructed using $A \to A' \to \kappa(\overline{s}'')$. $\square$

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian valuation ring with algebraically closed fraction field and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_\eta) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as both η and s have algebraically closed residue fields.

0C0M **Lemma 15.3.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. Then there exists a strictly henselian valuation ring R over S with algebraically closed fraction field such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a valuation ring with fraction field $\kappa(\overline{s}')$ dominating the image of A. See Algebra, Lemma 49.2. Observe that R is strictly henselian for example by Lemma 11.2 and Algebra, Lemma 49.10. Then the lemma is clear. $\square$

Let $X \to S$ be a proper morphism with geometrically connected fibres. Let R be a strictly henselian discrete valuation ring and let $\mathrm{Spec}(R) \to S$ be a morphism. Let $\eta, s \in \mathrm{Spec}(R)$ be the generic and closed point. Then we can consider the specialization map

$$sp_R : \pi_1(X_{\overline{\eta}}) \to \pi_1(X_s)$$

for the base change $X_R/\mathrm{Spec}(R)$. Note that this makes sense as s has algebraically closed residue field.

0C0N **Lemma 15.4.** *Let $f : X \to S$ be a proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization of points of S and let $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ be a specialization map. If S is Noetherian, then there exists a strictly henselian discrete valuation ring R over S such that sp is isomorphic to sp_R defined above.*

Proof. Let $\mathcal{O}_{S,s} \to A$ be the strict henselization constructed using $\kappa(s) \to \kappa(\overline{s})$. Let $A \to \kappa(\overline{s}')$ be the map used to construct sp. Let $R \subset \kappa(\overline{s}')$ be a discrete valuation ring dominating the image of A, see Algebra, Lemma 118.13. Choose a diagram of fields

$$\begin{tikzcd} \kappa(\overline{s}) \arrow[r] & k \\ A/\mathfrak{m}_A \arrow[u] \arrow[r] & R/\mathfrak{m}_R \arrow[u] \end{tikzcd}$$

with k algebraically closed. Let R^{sh} be the strict henselization of R constructed using $R \to k$. Then R^{sh} is a discrete valuation ring by More on Algebra, Lemma 42.11. Denote η, o the generic and closed point of $\mathrm{Spec}(R^{sh})$. Since the diagram of schemes

$$\begin{tikzcd} \overline{\eta} \arrow[r] \arrow[d] & \mathrm{Spec}(R^{sh}) \arrow[d] & \mathrm{Spec}(k) \arrow[l] \arrow[d] \\ \overline{s}' \arrow[r] & \mathrm{Spec}(A) & \overline{s} \arrow[l] \end{tikzcd}$$

commutes, we obtain a commutative diagram

$$\begin{tikzcd} \pi_1(X_{\overline{\eta}}) \arrow[r, "sp_{R^{sh}}"'] \arrow[d] & \pi_1(X_o) \arrow[d] \\ \pi_1(X_{\overline{s}'}) \arrow[r, "sp"] & X_{\overline{s}} \end{tikzcd}$$

of specialization maps by the construction of these maps. Since the vertical arrows are isomorphisms (Lemma 8.2), this proves the lemma. $\square$

16. Finite étale covers of punctured spectra, I

0BLE We first prove some results á la Lefschetz.

0BLF **Situation 16.1.** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA.

Recall that for a scheme X the category of schemes finite étale over X is denoted $\textit{FÉt}_X$, see Section 5. In Situation 16.1 we will study the base change functors

$$\begin{tikzcd} \textit{FÉt}_X \arrow[r] \arrow[d] & \textit{FÉt}_U \arrow[d] \\ \textit{FÉt}_{X_0} \arrow[r] & \textit{FÉt}_{U_0} \end{tikzcd}$$

In many case the right vertical arrow is faithful.

0BLG **Lemma 16.2.** *In Situation 16.1. Assume one of the following holds*

(1) $\dim(A/\mathfrak{p}) \geq 2$ *for every minimal prime* $\mathfrak{p} \subset A$ *with* $f \notin \mathfrak{p}$, *or*
(2) *every connected component of* U *meets* U_0.

Then

$$\textit{FÉt}_U \longrightarrow \textit{FÉt}_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a faithful functor.

Proof. Let $a, b : V \to W$ be two morphisms of schemes finite étale over U whose restriction to U_0 are the same. Assumption (1) means that every irreducible component of U meets U_0, see Algebra, Lemma 59.12. The image of any irreducible component of V is an irreducible component of U and hence meets U_0. Hence V_0 meets every connected component of V and we conclude that $a = b$ by Étale Morphisms, Proposition 6.3. In case (2) the argument is the same using that the image of a connected component of V is a connected component of U. □

Before we prove something more interesting, we need a couple of lemmas.

0BLH **Lemma 16.3.** *In Situation 16.1. Let $V \to U$ be a finite morphism. Let $A^\wedge$ be the $\mathfrak m$-adic completion of A, let $X' = \mathrm{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. If $Y' \to X'$ is a finite morphism such that $V' = Y' \times_{X'} U'$, then there exists a finite morphism $Y \to X$ such that $V = Y \times_X U$ and $Y' = Y \times_X X'$.*

Proof. This is a straightforward application of More on Algebra, Proposition 78.15. Namely, choose generators $f_1, \ldots, f_t$ of $\mathfrak m$. For each i write $V \times_U D(f_i) = \mathrm{Spec}(B_i)$. For $1 \leq i, j \leq n$ we obtain an isomorphism $\alpha_{ij} : (B_i)_{f_j} \to (B_j)_{f_i}$ of $A_{f_i f_j}$-algebras because the spectrum of both represent $V \times_U D(f_i f_j)$. Write $Y' = \mathrm{Spec}(B')$. Since $V \times_U U' = Y \times_{X'} U'$ we get isomorphisms $\alpha_i : B'_{f_i} \to B_i \otimes_A A^\wedge$. A straightforward argument shows that $(B', B_i, \alpha_i, \alpha_{ij})$ is an object of $\mathrm{Glue}(A \to A^\wedge, f_1, \ldots, f_t)$, see More on Algebra, Remark 78.10. Applying the proposition cited above (and using More on Algebra, Remark 78.19 to obtain the algebra structure) we find an A-algebra B such that $\mathrm{Can}(B)$ is isomorphic to $(B', B_i, \alpha_i, \alpha_{ij})$. Setting $Y = \mathrm{Spec}(B)$ we see that $Y \to X$ is a morphism which comes equipped with compatible isomorphisms $V \cong Y \times_X U$ and $Y' = Y \times_X X'$ as desired. □

0BLI **Lemma 16.4.** *In Situation 16.1 assume A is henselian or more generally that $(A, (f))$ is a henselian pair. Let $A^\wedge$ be the $\mathfrak m$-adic completion of A, let $X' = \mathrm{Spec}(A^\wedge)$ and let U' and U'_0 be the base changes of U and U_0 to X'. If $F\acute{E}t_{U'} \to F\acute{E}t_{U'_0}$ is fully faithful, then $F\acute{E}t_U \to F\acute{E}t_{U_0}$ is fully faithful.*

Proof. Assume $F\acute{E}t_{U'} \longrightarrow F\acute{E}t_{U'_0}$ is a fully faithful. Since $X' \to X$ is faithfully flat, it is immediate that the functor $V \to V_0 = V \times_U U_0$ is faithful. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given V finite étale over U we have

$$\mathrm{Mor}_U(U, V) = \mathrm{Mor}_{U_0}(U_0, V_0)$$

The we assume we have a morphism $s_0 : U_0 \to V_0$ over U_0 and we will produce a morphism $s : U \to V$ over U.

By our assumption there does exist a morphism $s' : U' \to V'$ whose restriction to V'_0 is the base change s'_0 of s_0. Since $V' \to U'$ is finite étale this means that $V' = s'(U') \amalg W'$ for some $W' \to U'$ finite and étale. Choose a finite morphism $Z' \to X'$ such that $W' = Z' \times_{X'} U'$. This is possible by Zariski's main theorem in the form stated in More on Morphisms, Lemma 38.3 (small detail omitted). Then

$$V' = s'(U') \amalg W' \longrightarrow X' \amalg Z' = Y'$$

is an open immersion such that $V' = Y' \times_{X'} U'$. By Lemma 16.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. Write $Y = \mathrm{Spec}(B)$ so that $Y' = \mathrm{Spec}(B \otimes_A A^\wedge)$. Then $B \otimes_A A^\wedge$ has an idempotent e' corresponding to the open and closed subscheme X' of $Y' = X' \amalg Z'$.

The case A is henselian (slightly easier). The image $\overline{e}$ of e' in $B \otimes_A \kappa(\mathfrak m) = B/\mathfrak m B$ lifts to an idempotent e of B as A is henselian (because B is a product of local rings by Algebra, Lemma 148.3). Then we see that e maps to e' by uniqueness of lifts of idempotents (using that $B \otimes_A A^\wedge$ is a product of local rings). Let $Y_1 \subset Y$ be the open and closed subscheme corresponding to e. Then $Y_1 \times_X X' = s'(X')$ which implies that $Y_1 \to X$ is an isomorphism (by faithfully flat descent) and gives the desired section.

The case where $(A, (f))$ is a henselian pair. Here we use that s' is a lift of s'_0. Namely, let $Y_{0,1} \subset Y_0 = Y \times_X X_0$ be the closure of $s_0(U_0) \subset V_0 = Y_0 \times_{X_0} U_0$. As $X' \to X$ is flat, the base change $Y'_{0,1} \subset Y'_0$ is the closure of $s'_0(U'_0)$ which is equal to $X'_0 \subset Y'_0$ (see Morphisms, Lemma 24.15). Since $Y'_0 \to Y_0$ is submersive (Morphisms, Lemma 24.11) we conclude that $Y_{0,1}$ is open and closed in Y_0. Let $e_0 \in B/fB$ be the corresponding idempotent. By More on Algebra, Lemma 10.8 we can lift e_0 to an idempotent $e \in B$. Then we conclude as before. $\square$

The following lemma will be superseded by Lemma 16.6 below.

0BLJ **Lemma 16.5.** *In Situation 16.1. Assume f is a nonzerodivisor, that A has depth ≥ 3, and that A is henselian or more generally $(A, (f))$ is a henselian pair. Then*

$$\textit{FÉt}_U \longrightarrow \textit{FÉt}_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a fully faithful functor.

Proof. By Lemma 16.4 we may assume A is a complete local Noetherian ring. The functor is faithful by Lemma 16.2 (to see the assumption of that lemma holds, apply Algebra, Lemma 71.9). Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given V finite étale over U we have

$$\mathrm{Mor}_U(U, V) = \mathrm{Mor}_{U_0}(U_0, V_0)$$

If we have a morphism $U_0 \to V_0$ over U_0, then we obtain an decomposition $V_0 = U_0 \amalg V'_0$ into open and closed subschemes. We will show that this implies the same thing for V thereby finishing the proof.

For $n \geq 1$ let U_n be the punctured spectrum of $A/f^{n+1}A$ and let $V_n \to U_n$ be the base change of $V \to U$. By Étale Morphisms, Theorem 15.2 we conclude that there is a unique decomposition $V_n = U_n \amalg V'_n$ into open and closed subschemes whose base change to U_0 recovers the given decomposition.

Since A has depth ≥ 3 and f is a nonzerodivisor, we see that A/fA has depth ≥ 2 (Algebra, Lemma 71.7). This implies the vanishing of $H^0_\mathfrak m(A/fA)$ and $H^1_\mathfrak m(A/fA)$, see Dualizing Complexes, Lemma 11.1. This in turn tells us that $A/fA \to \Gamma(U_0, \mathcal{O}_{U_0})$ is an isomorphism, see Local Cohomology, Lemma 4.2. As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/fA \xrightarrow{f^n} A/f^{n+1}A \to A/f^nA \to 0$$

Induction on n shows that $H^0_\mathfrak m(A/f^{n+1}A) = H^1_\mathfrak m(A/f^{n+1}A) = 0$ for all n. Hence the same reasoning shows that $A/f^{n+1}A \to \Gamma(U_n, \mathcal{O}_{U_n})$ is an isomorphism. Combined with the decompositions above this determines a map

$$\Gamma(V, \mathcal{O}_V) \to \lim \Gamma(V_n, \mathcal{O}_{V_n}) \to \lim \Gamma(U_n, \mathcal{O}_{U_n}) = A$$

Since $V \to U$ is affine, this A-algebra map corresponds to a section $U \to V$ as desired. □

In the following lemma we prove fully faithfulness under very weak assumptions. Note that the assumptions do not imply that U is a connected scheme, but the conclusion guarantees that U and U_0 have the same number of connected components.

0BM6 **Lemma 16.6.** *In Situation 16.1. Assume* [BdJ14, Corollary 1.11]

(1) *f is a nonzerodivisor,*
(2) *$H^1_\mathfrak{m}(A)$ is finite,*
(3) *$H^2_\mathfrak{m}(A)$ is annihilated by a power of f, and*
(4) *A is henselian or more generally $(A, (f))$ is a henselian pair.*

Then

$$\textit{FÉt}_U \longrightarrow \textit{FÉt}_{U_0}, \quad V \longmapsto V_0 = V \times_U U_0$$

is a fully faithful functor.

Proof. By Lemma 16.4 we may assume that A is a Noetherian complete local ring. (The assumptions carry over; use Dualizing Complexes, Lemma 9.3.)

Assume A is complete in addition to the other conditions. We will show that given $\pi : V \to U$ finite étale, the set of connected components of V agrees with the set of connected components of V_0. This will prove the lemma because the category of finite étale covers has internal hom (Lemma 5.4) and images of sections are connected components (Étale Morphisms, Proposition 6.1). Some details omitted.

Set $\mathcal{B} = \pi_*\mathcal{O}_V$. This is a finite locally free $\mathcal{O}_U$-algebra. Thus $\text{Ass}(\mathcal{B}) = \text{Ass}(\mathcal{O}_U)$. Assumption (2) means that $H^0(U, \mathcal{O}_U)$ is a finite A-module and equivalently that $j_*\mathcal{O}_U$ is coherent (Local Cohomology, Lemma 4.2). By Local Cohomology, Proposition 4.7 and the agreement of Ass we see that the same holds for $\mathcal{B}$ and we conclude that $B = \Gamma(U, \mathcal{B}) = \Gamma(V, \mathcal{O}_V)$ is a finite A-algebra.

Next, using that $H^2_\mathfrak{m}(A) = H^1(U, \mathcal{O}_U)$ is annihilated by f^n for some n we see that $H^1(U, \mathcal{B}) = H^1(V, \mathcal{O}_V)$ is annihilated by f^m for some m, see Local Cohomology, Lemma 8.3.

At this point we apply Local Cohomology, Lemma 10.4 to the scheme V over $\text{Spec}(A)$ and the sheaf $\mathcal{O}_V$ with $p = 0$. Since f is a nonzerodivisor in A the f-power torsion subsheaf of $\mathcal{O}_V$ is zero. The first short exact sequence of the lemma collapses to become

$$H^0 = \lim H^0(V, \mathcal{O}_V/f^n\mathcal{O}_V) = \lim H^0(V_n, \mathcal{O}_{V_n})$$

where $V_n \subset V$ is the closed subscheme cut out by f^{n+1}. Since $H^1(V, \mathcal{O}_V)$ is annihilated by a power of f we see that the Tate module $T_f(H^1(V, \mathcal{O}_V))$ is zero. On the other hand, since A is complete and $B = H^0(V, \mathcal{O}_V)$ is a finite A-module it is complete (Algebra, Lemma 96.1) hence derived complete (More on Algebra, Proposition 80.5) and hence equal to its derived f-adic completion. Thus we see that $H^0 = B$. Since

$$V_0 \subset V_1 \subset V_2 \subset \ldots$$

are nilpotent thickenings the connected components of these schemes agree. Correspondingly the maps

$$\ldots \to H^0(V_2, \mathcal{O}_{V_2}) \to H^0(V_1, \mathcal{O}_{V_1}) \to H^0(V_0, \mathcal{O}_{V_0})$$

induce bijections between idempotents. Hence the map $B \to H^0(V_0, \mathcal{O}_{V_0})$ induces a bijection between idempotents and we conclude. $\square$

17. Purity in local case, I

0BM7 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$ be the punctured spectrum. We say *purity holds for* $(A, \mathfrak{m})$ if the restriction functor

$$F\acute{E}t_X \longrightarrow F\acute{E}t_U$$

is essentially surjective. In this section we try to understand how the question changes when one passes from X to a hypersurface X_0 in X, in other words, we study a kind of local Lefschetz property for the fundamental groups of punctured spectra. These results will be useful to proceed by induction on dimension in the proofs of our main results on local purity, namely, Lemma 18.3 and Proposition 20.3.

0BM8 **Lemma 17.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let $\pi : Y \to X$ be a finite morphism such that depth$(\mathcal{O}_{Y,y}) \geq 2$ for all closed points $y \in Y$. Then Y is the spectrum of $B = \mathcal{O}_Y(\pi^{-1}(U))$.*

Proof. Set $V = \pi^{-1}(U)$ and denote $\pi' : V \to U$ the restriction of π. Consider the $\mathcal{O}_X$-module map

$$\pi_*\mathcal{O}_Y \longrightarrow j_*\pi'_*\mathcal{O}_V$$

where $j : U \to X$ is the inclusion morphism. We claim Divisors, Lemma 2.11 applies to this map. If so, then $B = \Gamma(Y, \mathcal{O}_Y)$ and we see that the lemma holds. Let $x \in X$. If $x \in U$, then the map is an isomorphism on stalks as $V = Y \times_X U$. If x is the closed point, then $x \notin \mathrm{Ass}(j_*\pi_*\mathcal{O}_V)$ (Divisors, Lemmas 5.9 and 5.3). Thus it suffices to show that $\mathrm{depth}((\pi_*\mathcal{O}_Y)_x) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to x. By Algebra, Lemma 71.10 it suffices to show that $\mathrm{depth}(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. Since this is the assumption of the lemma the proof is complete. $\square$

0BLK **Lemma 17.2.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Assume A has depth ≥ 2. The following are equivalent*

1. *$V = Y \times_X U$ for some $Y \to X$ finite étale,*
2. *$B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.*

Proof. Denote $\pi : V \to U$ the given finite étale morphism. Assume Y as in (1) exists. Let $x \in X$ be the point corresponding to $\mathfrak{m}$. Let $y \in Y$ be a point mapping to x. We claim that $\mathrm{depth}(\mathcal{O}_{Y,y}) \geq 2$. This is true because $Y \to X$ is étale and hence $A = \mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same depth (Algebra, Lemma 157.2). Hence Lemma 17.1 applies and $Y = \mathrm{Spec}(B)$.

The implication (2) $\Rightarrow$ (1) is easier and the details are omitted. $\square$

0BM9 **Lemma 17.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \mathrm{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Assume A is normal of dimension ≥ 2. The functor*

$$F\acute{E}t_U \longrightarrow \left\{ \begin{matrix} \textit{finite normal } A\textit{-algebras } B \textit{ such} \\ \textit{that } \mathrm{Spec}(B) \to X \textit{ is étale over } U \end{matrix} \right\}, \quad V \longmapsto \Gamma(V, \mathcal{O}_V)$$

is an equivalence. Moreover, $V = Y \times_X U$ for some $Y \to X$ finite étale if and only if $B = \Gamma(V, \mathcal{O}_V)$ is finite étale over A.

Proof. Observe that $\text{depth}(A) \geq 2$ because A is normal (Serre's criterion for normality, Algebra, Lemma 151.4). Thus the final statement follows from Lemma 17.2. Given $\pi : V \to U$ finite étale, set $B = \Gamma(V, \mathcal{O}_V)$. If we can show that B is normal and finite over A, then we obtain the displayed functor. Since there is an obvious quasi-inverse functor, this is also all that we have to show.

Since A is normal, the scheme V is normal (Descent, Lemma 15.2). Hence V is a finite disjoint union of integral schemes (Properties, Lemma 7.6). Thus we may assume V is integral. In this case the function field L of V (Morphisms, Section 46) is a finite separable extension of $f.f.(A)$ (because we get it by looking at the generic fibre of $V \to U$ and using Morphisms, Lemma 34.7). By Algebra, Lemma 155.8 the integral closure $B' \subset L$ of A in L is finite over A. By More on Algebra, Lemma 21.16 we see that B' is a reflexive A-module, which in turn implies that $\text{depth}_A(B') \geq 2$ by More on Algebra, Lemma 21.14.

Let $f \in \mathfrak{m}$. Then $B_f = \Gamma(V \times_U D(f), \mathcal{O}_V)$ (Properties, Lemma 17.1). Hence $B'_f = B_f$ because B_f is normal (see above), finite over A_f with fraction field L. It follows that $V = \text{Spec}(B') \times_X U$. Then we conclude that $B = B'$ from Lemma 17.1 applied to $\text{Spec}(B') \to X$. This lemma applies because the localizations $B'_{\mathfrak{m}'}$ of B' at maximal ideals $\mathfrak{m}' \subset B'$ lying over $\mathfrak{m}$ have depth ≥ 2 by Algebra, Lemma 71.10 and the remark on depth in the preceding paragraph. $\square$

0BLL **Lemma 17.4.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{\mathfrak{m}\}$. Let V be finite étale over U. Let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A, let $X' = \text{Spec}(A^\wedge)$ and let U' and V' be the base changes of U and V to X'. The following are equivalent*

1. *$V = Y \times_X U$ for some $Y \to X$ finite étale, and*
2. *$V' = Y' \times_{X'} U'$ for some $Y' \to X'$ finite étale.*

Proof. The implication (1) $\Rightarrow$ (2) follows from taking the base change of a solution $Y \to X$. Let $Y' \to X'$ be as in (2). By Lemma 16.3 we can find $Y \to X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_X X'$. By descent we see that $Y \to X$ is finite étale (Algebra, Lemmas 82.2 and 141.3). This finishes the proof. $\square$

The following lemma will be superseded by Lemma 17.6.

0BLM **Lemma 17.5.** *In Situation 16.1. Let V be finite étale over U. Assume*

1. *f is a nonzerodivisor,*
2. *A has depth ≥ 3,*
3. *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case by Lemma 17.4. Alternatively you can use Lemma 17.2, cohomology and base change (Cohomology of Schemes, Lemma 5.2), and descent (Algebra, Lemmas 82.2 and 141.3).

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 11.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 10.4. Then we can use Lemma 16.5 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. $\square$

The point of the following lemma is that the assumptions do not force A to have depth ≥ 3. For example if A is a complete normal local domain of dimension ≥ 3 and $f \in \mathfrak{m}$ is nonzero, then the assumptions are satisfied.

0BLS **Lemma 17.6.** *In Situation 16.1. Let V be finite étale over U. Assume*

1. *f is a nonzerodivisor,*
2. *$H^1_{\mathfrak{m}}(A)$ is a finite A-module,*
3. *a power of f annihilates $H^2_{\mathfrak{m}}(A)$,*
4. *$V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.*

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

Proof. We reduce to the complete case using Lemma 17.4. (The assumptions carry over; use Dualizing Complexes, Lemma 9.3.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 11.2; observe that (A, fA) is a henselian pair by More on Algebra, Lemma 10.4. Then we can use Lemma 16.6 to see that V is isomorphic to $Y \times_X U$ and the proof is complete. ☐

18. Purity of branch locus

0BJE We will use the discriminant of a finite locally free morphism. See Discriminants, Section 3.

0BJG **Lemma 18.1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim(A) \geq 1$. Let $f \in \mathfrak{m}$. Then there exist a $\mathfrak{p} \in V(f)$ with $\dim(A_{\mathfrak{p}}) = 1$.*

Proof. By induction on $\dim(A)$. If $\dim(A) = 1$, then $\mathfrak{p} = \mathfrak{m}$ works. If $\dim(A) > 1$, then let $Z \subset \mathrm{Spec}(A)$ be an irreducible component of dimension > 1. Then $V(f) \cap Z$ has dimension > 0 (Algebra, Lemma 59.12). Pick a prime $\mathfrak{q} \in V(f) \cap Z$, $\mathfrak{q} \neq \mathfrak{m}$ corresponding to a closed point of the punctured spectrum of A; this is possible by Properties, Lemma 6.4. Then $\mathfrak{q}$ is not the generic point of Z. Hence $0 < \dim(A_{\mathfrak{q}}) < \dim(A)$ and $f \in \mathfrak{q}A_{\mathfrak{q}}$. By induction on the dimension we can find $f \in \mathfrak{p} \subset A_{\mathfrak{q}}$ with $\dim((A_{\mathfrak{q}})_{\mathfrak{p}}) = 1$. Then $\mathfrak{p} \cap A$ works. ☐

0BJH **Lemma 18.2.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$. Assume*

1. *f is flat,*
2. *f is quasi-finite at x,*
3. *x is not a generic point of an irreducible component of X,*
4. *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

Proof. Observe that the set of points where f is unramified is the same as the set of points where f is étale and that this set is open. See Morphisms, Definitions 33.1 and 34.1 and Lemma 34.16. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 20.29 and 28.1). Thus we can apply More on Morphisms, Lemma 36.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over $y = f(x)$. Then it follows that f is finite locally free (Morphisms, Lemma 45.2).

Assume f is finite locally free and that x is the unique point of X lying over $y = f(x)$. By Discriminants, Lemma 3.1 we find a locally principal closed subscheme

$D_\pi \subset Y$ such that $y' \in D_\pi$ if and only if there exists an $x' \in X$ with $f(x') = y'$ and f ramified at x'. Thus we have to prove that $y \notin D_\pi$. Assume $y \in D_\pi$ to get a contradiction.

By condition (3) we have $\dim(\mathcal{O}_{X,x}) \geq 1$. We have $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ by Algebra, Lemma 111.7. By Lemma 18.1 we can find $y' \in D_\pi$ specializing to y with $\dim(\mathcal{O}_{Y,y'}) = 1$. Choose $x' \in X$ with $f(x') = y'$ where f is ramified. Since f is finite it is closed, and hence $x' \rightsquigarrow x$. We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'}) = 1$ as before. This contradicts property (4). $\square$

0BMA **Lemma 18.3.** *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Set $X = \mathrm{Spec}(A)$ and $U = X \setminus \{\mathfrak{m}\}$. Then*

1. *the functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective,*
2. *any finite $A \to B$ with B normal which induces a finite étale morphism on punctured spectra is étale.*

Proof. Recall that a regular local ring is normal by Algebra, Lemma 151.5. Hence (1) and (2) are equivalent by Lemma 17.3. We prove the lemma by induction on d.

The case $d = 2$. In this case $A \to B$ is flat. Namely, we have going down for $A \to B$ by Algebra, Proposition 37.7. Then $\dim(B_{\mathfrak{m}'}) = 2$ for all maximal ideals $\mathfrak{m}' \subset B$ by Algebra, Lemma 111.7. Then $B_{\mathfrak{m}'}$ is Cohen-Macaulay by Algebra, Lemma 151.4. Hence and this is the important step Algebra, Lemma 127.1 applies to show $A \to B_{\mathfrak{m}'}$ is flat. Then Algebra, Lemma 38.19 shows $A \to B$ is flat. Thus we can apply Lemma 18.2 (or you can directly argue using the easier Discriminants, Lemma 3.1) to see that $A \to B$ is étale.

The case $d \geq 3$. Let $V \to U$ be finite étale. Let $f \in \mathfrak{m}_A$, $f \notin \mathfrak{m}_A^2$. Then A/fA is a regular local ring of dimension $d - 1 \geq 2$, see Algebra, Lemma 105.3. Let U_0 be the punctured spectrum of A/fA and let $V_0 = V \times_U U_0$. By Lemma 17.5 (or the more general Lemma 17.6) it suffices to show that V_0 is in the essential image of $F\acute{E}t_{\mathrm{Spec}(A/fA)} \to F\acute{E}t_{U_0}$. This follows from the induction hypothesis. $\square$

0BMB **Lemma 18.4** (Purity of branch locus)**.** *Let $f : X \to Y$ be a morphism of locally Noetherian schemes. Let $x \in X$ and set $y = f(x)$. Assume* [Nag59] and [Gro71, Exp. X, Thm. 3.1]

1. *$\mathcal{O}_{X,x}$ is normal,*
2. *$\mathcal{O}_{Y,y}$ is regular,*
3. *f is quasi-finite at x,*
4. $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) \geq 1$
5. *for specializations $x' \rightsquigarrow x$ with $\dim(\mathcal{O}_{X,x'}) = 1$ our f is unramified at x'.*

Then f is étale at x.

Proof. We will prove the lemma by induction on $d = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

An uninteresting case is when $d = 1$. In that case we are assuming that f is unramified at x and that $\mathcal{O}_{Y,y}$ is a discrete valuation ring (Algebra, Lemma 118.7). Then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ (otherwise the map would not be quasi-finite at x) and we see that f is flat at x. Since flat + unramified is étale we conclude (some details omitted).

The case $d \geq 2$. We will use induction on d to reduce to the case discussed in Lemma 18.3. To check f is étale at x we may work étale locally on the base and on the target (Descent, Lemmas 20.29 and 28.1). Thus we can apply More on

Morphisms, Lemma 36.1 and assume that $f : X \to Y$ is finite and that x is the unique point of X lying over y. Here we use that étale extensions of local rings do not change dimension, normality, and regularity, see More on Algebra, Section 41 and Étale Morphisms, Section 19.

Next, we can base change by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume that Y is the spectrum of a regular local ring. It follows that $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ as every point of X necessarily specializes to x.

The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite and necessarily injective (by equality of dimensions). We conclude we have going down for $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ by Algebra, Proposition 37.7 (and the fact that a regular ring is a normal ring by Algebra, Lemma 151.5). Pick $x' \in X$, $x' \neq x$ with image $y' = f(x')$. Then $\mathcal{O}_{X,x'}$ is normal as a localization of a normal domain. Similarly, $\mathcal{O}_{Y,y'}$ is regular (see Algebra, Lemma 109.6). We have $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{Y,y'})$ by Algebra, Lemma 111.7 (we checked going down above). Of course these dimensions are strictly less than d as $x' \neq x$ and by induction on d we conclude that f is étale at x'.

Thus we arrive at the following situation: We have a finite local homomorphism $A \to B$ of Noetherian local rings of dimension $d \geq 2$, with A regular, B normal, which induces a finite étale morphism $V \to U$ on punctured spectra. Our goal is to show that $A \to B$ is étale. This follows from Lemma 18.3 and the proof is complete. $\square$

19. Finite étale covers of punctured spectra, II

0BLU In this section we prove some variants of the material discussed in Section 16. Suppose we have a Noetherian local ring $(A, \mathfrak{m})$ and $f \in \mathfrak{m}$. We set $X = \mathrm{Spec}(A)$ and $X_0 = \mathrm{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of A and A/fA. All of this is exactly as in Situation 16.1. The difference is that we will consider the functor

$$\mathrm{colim}_{U_0 \subset U' \subset U} \textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}, \quad V' \longmapsto V_0 = V' \times_{U'} U_0$$

In other words, we will not try to lift finite étale coverings of U_0 to all of U, but just to some open neighbourhood U' of U_0 in U.

0BLN **Lemma 19.1.** *In Situation 16.1. Let $U' \subset U$ be open and contain U_0. Assume $\dim(A/\mathfrak{p}) \geq 2$ for every minimal prime $\mathfrak{p} \subset A$ corresponding to a point of U'. Then*

$$\textit{FÉt}_{U'} \longrightarrow \textit{FÉt}_{U_0}, \quad V' \longmapsto V_0 = V' \times_{U'} U_0$$

is a faithful functor. Moreover, there exists a U' satisfying these assumptions.

Proof. Let $a, b : V' \to W'$ be two morphisms of schemes finite étale over U' whose restriction to U_0 are the same. By Algebra, Lemma 59.12 we see that $V(\mathfrak{p})$ meets U_0 for every prime $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p}) \geq 2$. The assumption therefore implies that every irreducible component of U' meets U_0. The image of any irreducible component of V' is an irreducible component of U' and hence meets U_0. Hence V_0 meets every connected component of V' and we conclude that $a = b$ by Étale Morphisms, Proposition 6.3. To see the existence of such a U' note that if $\mathfrak{p} \subset A$ is a prime with $\dim(A/\mathfrak{p}) = 1$ then $\mathfrak{p}$ corresponds to a closed point of U. $\square$

0DXX **Lemma 19.2.** *In Situation 16.1 assume*

(1) *A has a dualizing complex and is f-adically complete,*

(2) *f is a nonzerodivisor,*
(3) *for $x \in X \setminus X_0$ whose closure $\overline{\{x\}}$ in X meets U_0 we have depth$(\mathcal{O}_{X,x}) \geq 1$ or depth$(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) > 2$.*

Let V', W' be finite étale over an open $U' \subset U$ which contains U_0. Let $\varphi_0 : V' \times_{U'} U_0 \to W' \times_{U'} U_0$ be a morphism over U_0. Then there exists an open $U'' \subset U'$ containing U_0 and a morphism $\varphi : V' \times_{U'} U'' \to W' \times_{U'} U''$ lifting φ_0.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given V' finite étale over U' any section $U_0 \to V' \times_{U'} U_0$ extends to a section of V' over some open $U'' \subset U'$ containing U_0. Given our section we obtain a decomposition $V' \times_{U'} U_0 = U_0 \amalg R_0$ into open and closed subschemes. We will show that this implies the same thing for $V' \times_{U'} U''$ for some $U'' \subset U'$ open containing U_0 thereby finishing the proof.

For $n \geq 1$ let U_n be the punctured spectrum of $A/f^{n+1}A$. By Étale Morphisms, Theorem 15.2 we conclude that there is a unique decomposition $V' \times_{U'} U_n = U_n \amalg R_n$ into open and closed subschemes whose base change to U_0 recovers the given decomposition.

Via the inclusions $U_n \to V' \times_{U'} U_n \to V'$ we obtain an A-algebra map

$$\Gamma(V', \mathcal{O}_{V'}) \to B = \lim H^0(U_n, \mathcal{O}_{U_n})$$

By Local Cohomology, Theorem 13.8 applied with $s = 1$ and $\mathcal{F} = \mathcal{O}_U$ we see that $B = H^0(U'', \mathcal{O}_{U''})$ for some open $U'' \subset U$ containing U_0. Since $V \to U$ is affine, this A-algebra map corresponds to a morphism $U'' \to V'$ over U' as desired. $\square$

0BLP **Lemma 19.3.** *In Situation 16.1 assume*

(1) *A is f-adically complete,*
(2) *f is a nonzerodivisor, and*
(3) *$H^1_{\mathfrak{m}}(A/fA)$ is a finite A-module.*

Let V', W' be finite étale over an open $U' \subset U$ which contains U_0. Let $\varphi_0 : V' \times_{U'} U_0 \to W' \times_{U'} U_0$ be a morphism over U_0. Then there exists an open $U'' \subset U'$ containing U_0 and a morphism $\varphi : V' \times_{U'} U'' \to W' \times_{U'} U''$ lifting φ_0.

Proof. This lemma is a variant of Lemma 19.2 and if A is a complete local ring, then it follows from that lemma. We suggest the reader skip the proof.

Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given V' finite étale over U' any section $U_0 \to V' \times_{U'} U_0$ extends to a section of V' over some open $U'' \subset U'$ containing U_0. Given our section we obtain a decomposition $V' \times_{U'} U_0 = U_0 \amalg R_0$ into open and closed subschemes. We will show that this implies the same thing for $V' \times_{U'} U''$ for some $U'' \subset U'$ open containing U_0 thereby finishing the proof.

For $n \geq 1$ let U_n be the punctured spectrum of $A/f^{n+1}A$. By Étale Morphisms, Theorem 15.2 we conclude that there is a unique decomposition $V' \times_{U'} U_n = U_n \amalg R_n$ into open and closed subschemes whose base change to U_0 recovers the given decomposition.

The finiteness of $H^1_{\mathfrak{m}}(A/fA)$ tells us that $B_0 = \Gamma(U_0, \mathcal{O}_{U_0})$ is a finite A-module, see Local Cohomology, Lemma 4.2. Set $B_n = \Gamma(U_n, \mathcal{O}_{U_n})$. As f is a nonzerodivisor we have exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and hence short exact sequences $0 \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n+1}} \to \mathcal{O}_{U_0} \to 0$. Thus we may apply Local Cohomology, Lemma 10.1 to the inverse system $\mathcal{O}_{U_n}$ on U. We find that $B = \lim B_n$ is a finite A-algebra, such that f is a nonzerodivisor on B, and such that $B/fB \subset B_0$. Via the inclusions $U_n \to V' \times_{U'} U_n \to V'$ we obtain an A-algebra map $\Gamma(V', \mathcal{O}_{V'}) \to B$. Since $V \to U$ is affine, this A-algebra map corresponds to a morphism

$$\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U' \longrightarrow V'$$

over U'.

Let $\mathfrak{q} \in U_0$ be a prime. The kernel and cokernel of $A/fA \to B_0$ have support contained in $\{\mathfrak{m}\}$ (see above). Hence the same is true for the map $A/fA \to B/fB$. Then $A_\mathfrak{q} \to B_\mathfrak{q}$ is finite and induces an isomorphism $(A/fA)_\mathfrak{q} \to (B/fB)_\mathfrak{q}$. Since f is a nonzerodivisor on B it follows that $A_\mathfrak{q} \to B_\mathfrak{q}$ is an isomorphism. Using finiteness again we find $g \in A$, $g \notin \mathfrak{q}$ such that $A_g \to B_g$ is an isomorphism. It follows that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an isomorphism over an open $U'' \subset U'$ which produces the desired section by the above. □

0DXY **Lemma 19.4.** *In Situation 16.1 assume*

1. *A has a dualizing complex and is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *A is f-adically complete,*
4. *if $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$, then depth$((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$, and*
5. *if $\mathfrak{p} \notin V(f)$ and $V(\mathfrak{p}) \cap V(f) \neq \{\mathfrak{m}\}$, then depth$(A_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 3$.*

For any finite étale morphism $V_0 \to U_0$ there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U'$ whose base change to U_0 is $V_0 \to U_0$.

Proof. For $n \geq 1$ let U_n be the punctured spectrum of $A/f^{n+1}A$. By Étale Morphisms, Theorem 15.2 we conclude that there is a unique finite étale morphism $\pi_n : V_n \to U_n$ whose base change to U_0 recovers $V_0 \to U_0$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{V_n}$. We may and do view $\mathcal{F}_n$ as an $\mathcal{O}_U$-module on U wich is locally isomorphic to $(\mathcal{O}_U/f^{n+1}\mathcal{O}_U)^{\oplus r}$. By Local Cohomology, Lemma 14.2 there exists a coherent $\mathcal{O}_U$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/f^{n+1}\mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_U$-modules. If $x \in U_0$, then the f-adic completion of the stalk $\mathcal{F}_x$ is isomorphic to a finite free module over the f-adic completion of $\mathcal{O}_{U,u}$. Hence $\mathcal{F}$ is finite locally free in an open neighbourhood U' of U_0.

To construct an algebra structure on $\mathcal{F}$ consider the coherent $\mathcal{O}_U$-module

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{F}, \mathcal{F})$$

Observe that $\mathcal{H}|_{U'}$ is finite locally free. The multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_U} \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{V_n}$ are sheaves of algebras defines an an element in

$$\lim \Gamma(U, \mathcal{H}/f^{n+1}\mathcal{H})$$

By Local Cohomology, Theorem 13.8 this comes from a section $\mu \in \Gamma(U', \mathcal{F})$ after possibly shrinking U'. After possibly shrinking furter we may assume μ defines a commutative $\mathcal{O}_{U'}$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting

$$V' = \underline{\mathrm{Spec}}_{U'}((\mathcal{F}|_{U'}, \mu))$$

we obtain a finite locally free scheme over U' whose restriction to U_n is isomorphic to V_n. It follows that $V' \to U'$ is étale at all points lying over U_0, see More on Morphisms, Lemma 12.3. This finishes the proof. $\square$

0BLV **Lemma 19.5.** *In Situation 16.1 assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *$H^1_\mathfrak{m}(A/fA)$ and $H^2_\mathfrak{m}(A/fA)$ are finite A-modules.*

For any finite étale morphism $V_0 \to U_0$ there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U'$ whose base change to U_0 is $V_0 \to U_0$.

Proof. This lemma is a variant of Lemma 19.4 and if A is a complete local ring, then it follows from that lemma. We suggest the reader skip the proof.

For $n \geq 1$ let U_n be the punctured spectrum of $A/f^{n+1}A$. By Étale Morphisms, Theorem 15.2 we conclude that there is a unique finite étale morphism $\pi_n : V_n \to U_n$ whose base change to U_0 recovers $V_0 \to U_0$. Consider the sheaves $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{V_n}$. We may view $\mathcal{F}_n$ as an $\mathcal{O}_U$-module on U. As f is a nonzerodivisor we obtain short exact sequences

$$0 \to A/f^nA \xrightarrow{f} A/f^{n+1}A \to A/fA \to 0$$

and because $V_n \to U_n$ is finite locally free we have corresponding short exact sequences $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_0 \to 0$.

We will use Local Cohomology, Lemma 4.2 without further mention. Our assumptions imply that $H^0(U, \mathcal{O}_{U_0})$ and $H^1(U, \mathcal{O}_{U_0})$ are finite A-modules. Hence the same thing is true for $\mathcal{F}_0$, see Local Cohomology, Lemma 8.2. Thus $H^0(U, \mathcal{F}_0)$ is a finite A-module and $H^1(U, \mathcal{F}_0)$ has finite length (as a finite A-module which is $\mathfrak{m}$-power torsion). Thus Local Cohomology, Lemmas 10.1 and 10.2 apply to the system above. Set

$$B_n = \Gamma(V_n, \mathcal{O}_{V_n}) = \Gamma(U, \mathcal{F}_n)$$

We conclude that the system (B_n) satisfies the Mittag-Leffler condition, that $B = \lim B_n$ is a finite A-algebra, that f is a nonzerodivisor on B and that $B/fB \subset B_0$. To finish the proof, we will show that the finite morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (a) becomes isomorphic to $V_0 \to U_0$ after base change to U_0 and (b) is étale at all points lying over U_0.

Let $\mathfrak{q} \in U_0$ be a prime. By the Mittag-Leffler condition, we know that $B/fB \subset B_0$ is the image of $B_{n+1} \to B_0$ for some n. Since the cokernel of $B_{n+1} \to B_0$ is contained in $H^1(U, \mathcal{F}_n)$ which is $\mathfrak{m}$-power torsion, we conclude that $B/fB \to B_0$ becomes an isomorphism after localizing at $\mathfrak{q}$. This proves (a). Thus $A_\mathfrak{q} \to B_\mathfrak{q}$ is finite and $(A/fA)_\mathfrak{q} \to (B/fB)_\mathfrak{q}$ is étale. Since f is a nonzerodivisor on B it follows that $A_\mathfrak{q} \to B_\mathfrak{q}$ is flat (Algebra, Lemma 98.10). Thus $A \to B$ is étale at all primes lying over $\mathfrak{q}$ (for example by Algebra, Lemma 141.7) which proves (b). $\square$

0BLW **Remark 19.6.** Let $(A, \mathfrak{m})$ be a complete local ring and $f \in \mathfrak{m}$ a nonzerodivisor. Let U, resp. U_0 be the punctured spectrum of A, resp. A/fA. Assume

1. if $\mathfrak{p} \in V(f) \setminus \{\mathfrak{m}\}$, then $\mathrm{depth}((A/f)_\mathfrak{p}) + \dim(A/\mathfrak{p}) > 1$, and

(2) if $\mathfrak p \not\in V(f)$ and $V(\mathfrak p) \cap V(f) \neq \{\mathfrak m\}$, then $\text{depth}(A_\mathfrak p) + \dim(A/\mathfrak p) > 3$.

Combining Lemmas 19.1, 19.2, and 19.4 we see that the category

$$\text{colim}_{U' \subset U \text{ open}, U_0 \subset U} \text{ category of schemes finite étale over } U'$$

is equivalent to the category of schemes finite étale over U_0. For example it suffices if every irreducible component of $\text{Spec}(A)$ has dimension ≥ 4 and A is (S_2). For example, if A is a normal domain of dimension ≥ 4!

20. Purity in local case, II

0BPB This section is the continuation of Section 17. In the next lemma we say *purity holds* for a Noetherian local ring $(A, \mathfrak m)$ if the restriction functor $F\acute{E}t_X \to F\acute{E}t_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak m\}$ is the punctured spectrum.

0DXZ **Lemma 20.1.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $f \in \mathfrak m$. Assume*

1. *A has a dualizing complex and is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *if $\mathfrak p \in V(f) \setminus \{\mathfrak m\}$, then depth$((A/f)_\mathfrak p) + \dim(A/\mathfrak p) > 1$, and*
4. *if $\mathfrak p \not\in V(f)$ and $V(\mathfrak p) \cap V(f) \neq \{\mathfrak m\}$, then depth$(A_\mathfrak p) + \dim(A/\mathfrak p) > 3$,*
5. *for every maximal ideal $\mathfrak p \subset A_f$ purity holds for $(A_f)_\mathfrak p$, and*
6. *purity holds for A.*

Then purity holds for A/fA.

Proof. Denote $X = \text{Spec}(A)$ and $U = X \setminus \{\mathfrak m\}$ the punctured spectrum. Similarly we have $X_0 = \text{Spec}(A/fA)$ and $U_0 = X_0 \setminus \{\mathfrak m\}$. Let $V_0 \to U_0$ be a finite étale morphism. By Lemma 19.4 there exists an open $U' \subset U$ containing U_0 and a finite étale morphism $V' \to U$ whose base change to U_0 is isomorphic to $V_0 \to U_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $\mathfrak p_1, \ldots, \mathfrak p_n$ as in (4). By assumption we can find finite étale morphisms $V_i' \to \text{Spec}(A_{\mathfrak p_i})$ agreeing with $V' \to U'$ over $U' \times_U \text{Spec}(A_{\mathfrak p_i})$. By Limits, Lemma 17.1 applied n times we see that $V' \to U'$ extends to a finite étale morphism $V \to U$. By assumption (5) we find that $V \to U$ extends to a finite étale morphism $Y \to X$. Then the restriction of Y to X_0 is the desired extension of $V_0 \to U_0$. ☐

0BPC **Lemma 20.2.** *Let $(A, \mathfrak m)$ be a Noetherian local ring. Let $f \in \mathfrak m$. Assume*

1. *A is f-adically complete,*
2. *f is a nonzerodivisor,*
3. *$H^1_\mathfrak m(A/fA)$ and $H^2_\mathfrak m(A/fA)$ are finite A-modules,*
4. *for every maximal ideal $\mathfrak p \subset A_f$ purity holds for $(A_f)_\mathfrak p$,*
5. *purity holds for A.*

Then purity holds for A/fA.

Proof. The proof is identical to the proof of Lemma 20.1 using Lemma 19.5 in stead of Lemma 19.4. ☐

Now we can bootstrap the earlier results to prove that purity holds for complete intersections of dimension ≥ 3. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 8.

0BPD **Proposition 20.3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is a complete intersection of dimension ≥ 3, then purity holds for A in the sense that any finite étale cover of the punctured spectrum extends.*

Proof. By Lemma 17.4 we may assume that A is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where B is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on r. The base case is $r = 0$ which follows from Lemma 18.3 which applies to regular rings of dimension ≥ 2.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 20.2 to $f_r \in A'$. This is permissible: condition (1) holds as $f_1, \ldots, f_r$ is a regular sequence, condition (2) holds as B and hence A' is complete, condition (3) holds as $A = A'/f_rA'$ is Cohen-Macaulay of dimension $\dim(A) \geq 3$, see Dualizing Complexes, Lemma 11.1, condition (4) holds by induction hypothesis as $\dim((A'_{f_r})_\mathfrak{p}) \geq 3$ for a maximal prime $\mathfrak{p}$ of A'_{f_r} and as $(A'_{f_r})_\mathfrak{p} = B_\mathfrak{q}/(f_1, \ldots, f_{r-1})$ for some $\mathfrak{q} \subset B$, condition (5) holds by induction hypothesis. □

21. Specialization maps in the smooth proper case

0BUQ In this section we discuss the following result. Let $f : X \to S$ be a proper smooth morphism of schemes. Let $s \leadsto s'$ be a specialization of points in S. Then the specialization map

$$sp : \pi_1(X_{\overline{s}}) \longrightarrow \pi_1(X_{\overline{s}'})$$

of Section 15 is surjective and

(1) if the characteristic of $\kappa(s')$ is zero, then it is an isomorphism, or
(2) if the characteristic of $\kappa(s')$ is $p > 0$, then it induces an isomorphism on maximal prime-to-p quotients.

0C0P **Lemma 21.1.** *Let $f : X \to S$ be a flat proper morphism with geometrically connected fibres. Let $s' \leadsto s$ be a specialization. If X_s is geometrically reduced, then the specialization map $sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$ is surjective.*

Proof. Since X_s is geometrically reduced, we may assume all fibres are geometrically reduced after possibly shrinking S, see More on Morphisms, Lemma 24.7. Let $\mathcal{O}_{S,s} \to A \to \kappa(\overline{s}')$ be as in the construction of the specialization map, see Section 15. Thus it suffices to show that

$$\pi_1(X_{\overline{s}'}) \to \pi_1(X_A)$$

is surjective. This follows from Proposition 14.2 and $\pi_1(\mathrm{Spec}(A)) = \{1\}$. □

0C0Q **Proposition 21.2.** *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \leadsto s$ be a specialization. If the characteristic to $\kappa(s)$ is zero, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is an isomorphism.

Proof. The map is surjective by Lemma 21.1. Thus we have to show it is injective.

We may assume S is affine. Then S is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to S$ is the base change of $X_0 \to S_0$ where S_0

is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to S_0$ is smooth and proper. See Limits, Lemma 10.1, 8.9, and 13.1. By Lemma 15.1 we reduce to the case where the base is Noetherian.

Applying Lemma 15.4 we reduce to the case where the base S is the spectrum of a strictly henselian discrete valuation ring A and we are looking at the specialization map over A. Let K be the fraction field of A. Choose an algebraic closure $\overline{K}$ which corresponds to a geometric generic point $\overline{\eta}$ of $\mathrm{Spec}(A)$. For $\overline{K}/L/K$ finite separable, let $B \subset L$ be the integral closure of A in L. This is a discrete valuation ring by More on Algebra, Remark 93.6.

Let $X \to \mathrm{Spec}(A)$ be as in the previous paragraph. To show injectivity of the specialization map it suffices to prove that every finite étale cover V of $X_{\overline{\eta}}$ is the base change of a finite étale cover $Y \to X$. Namely, then $\pi_1(X_{\overline{\eta}}) \to \pi_1(X) = \pi_1(X_s)$ is injective by Lemma 4.4.

Given V we can first descend V to $V' \to X_{K^{sep}}$ by Lemma 13.2 and then to $V'' \to X_L$ by Lemma 13.1. Let $Z \to X_B$ be the normalization of X_B in V''. Observe that Z is normal and that $Z_L = V''$ as schemes over X_L. Hence $Z \to X_B$ is finite étale over the generic fibre. The problem is that we do not know that $Z \to X_B$ is everywhere étale. Since $X \to \mathrm{Spec}(A)$ has geometrically connected smooth fibres, we see that the special fibre X_s is geometrically irreducible. Hence the special fibre of $X_B \to \mathrm{Spec}(B)$ is irreducible; let ξ_B be its generic point. Let $\xi_1, \ldots, \xi_r$ be the points of Z mapping to ξ_B. Our first (and it will turn out only) problem is now that the extensions

$$\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$$

of discrete valuation rings may be ramified. Let e_i be the ramification index of this extension. Note that since the characteristic of $\kappa(s)$ is zero, the ramification is tame!

To get rid of the ramification we are going to choose a further finite separable extension $K^{sep}/L'/L/K$ such that the ramification index e of the induced extensions B'/B is divisible by e_i. Consider the normalized base change Z' of Z with respect to $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$, see discussion in More on Morphisms, Section 54. Let $\xi_{i,j}$ be the points of Z' mapping to $\xi_{B'}$ and to ξ_i in Z. Then the local rings

$$\mathcal{O}_{Z',\xi_{i,j}}$$

are the localizations of the integral closure of $\mathcal{O}_{Z,\xi_i}$ in $L' \otimes_L f.f.(\mathcal{O}_{Z,\xi_i})$. Hence Abhyankar's lemma (More on Algebra, Lemma 95.6) tells us that

$$\mathcal{O}_{X_{B'},\xi_{B'}} \subset \mathcal{O}_{Z',\xi_{i,j}}$$

is unramified. We conclude that the morphism $Z' \to X_{B'}$ is étale away from codimension 1. Hence by purity of branch locus (Lemma 18.4) we see that $Z' \to X_{B'}$ is finite étale!

However, since the residue field extension induced by $A \to B'$ is trivial (as the residue field of A is algebraically closed being separably closed of characteristic zero) we conclude that Z' is the base change of a finite étale cover $Y \to X$ by applying Lemma 8.1 twice (first to get Y over A, then to prove that the pullback to B is isomorphic to Z'). This finishes the proof. $\square$

Let G be a profinite group. Let p be a prime number. The *maximal prime-to-p quotient* is by definition

$$G' = \lim_{U \subset G \text{ open, normal, index prime to } p} G/U$$

If X is a connected scheme and p is given, then the maximal prime-to-p quotient of $\pi_1(X)$ is denoted $\pi_1'(X)$.

0C0R **Theorem 21.3.** *Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic of $\kappa(s)$ is p, then the specialization map*

$$sp : \pi_1(X_{\overline{s}'}) \to \pi_1(X_{\overline{s}})$$

is surjective and induces an isomorphism

$$\pi_1'(X_{\overline{s}'}) \cong \pi_1'(X_{\overline{s}})$$

of the maximal prime-to-p quotients

Proof. This is proved in exactly the same manner as Proposition 21.2 with the following differences

(1) Given X/A we no longer show that the functor $F\acute{E}t_X \to F\acute{E}t_{X_{\overline{\eta}}}$ is essentially surjective. We show only that Galois objects whose Galois group has order prime to p are in the essential image. This will be enough to conclude the injectivity of $\pi_1'(X_{\overline{s}'}) \to \pi_1'(X_{\overline{s}})$ by exactly the same argument.
(2) The extensions $\mathcal{O}_{X_B,\xi_B} \subset \mathcal{O}_{Z,\xi_i}$ are tamely ramified as the associated extension of fraction fields is Galois with group of order prime to p. See More on Algebra, Lemma 93.9.
(3) The extension $\kappa_A \subset \kappa_B$ is no longer necessarily trivial, but it is purely inseparable. Hence the morphism $X_{\kappa_B} \to X_{\kappa_A}$ is a universal homeomorphism and induces an isomorphism of fundamental groups by Proposition 7.4.

□

22. Tame ramification

0BSE Let $X \to Y$ be a finite étale morphism of schemes of finite type over $\mathbf{Z}$. There are many ways to define what it means for f to be tamely ramified at ∞. The article [KS10] discusses to what extent these notions agree.

In this section we discuss a different more elementary question which precedes the notion of tameness at infinity. Namely, given a scheme X and a dense open $U \subset X$ when is a finite morphism $f : Y \to X$ tamely ramified relative to $D = X \setminus U$? We will use the definition as given in [GM71] but only in the case that D is a divisor with normal crossings.

23. Other chapters

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