FUNDAMENTAL GROUPS OF SCHEMES

Contents
1. Introduction 1
2. Schemes étale over a point 2
3. Galois categories 3
4. Functors and homomorphisms 9
5. Finite étale morphisms 11
6. Fundamental groups 14
7. Galois covers of connected schemes 16
8. Topological invariance of the fundamental group 16
9. Finite étale covers of proper schemes 18
10. Local connectedness 20
11. Fundamental groups of normal schemes 25
12. Group actions and integral closure 27
13. Ramification theory 30
14. Geometric and arithmetic fundamental groups 36
15. Homotopy exact sequence 37
16. Specialization maps 39
17. Restriction to a closed subscheme 42
18. Pushouts and fundamental groups 46
19. Finite étale covers of punctured spectra, I 46
20. Purity in local case, I 49
21. Purity of branch locus 51
22. Finite étale covers of punctured spectra, II 55
23. Finite étale covers of punctured spectra, III 56
24. Finite étale covers of punctured spectra, IV 57
25. Purity in local case, II 57
26. Purity in local case, III 58
27. Lefschetz for the fundamental group 61
28. Purity of ramification locus 62
29. Affineness of complement of ramification locus 65
30. Specialization maps in the smooth proper case 66
31. Tame ramification 69
32. Other chapters 76
References 77

1. Introduction

This is a chapter of the Stacks Project, version 3bc21583, compiled on Mar 29, 2019.
In this chapter we discuss Grothendieck’s fundamental group of a scheme and applications. A foundational reference is [Gro71]. A nice introduction is [Len]. Other references [Mur67] and [GM71].

2. Schemes étale over a point

Let $G$ be a topological group. A $G$-set, sometime called a discrete $G$-set, is a set $X$ endowed with a left action $a : G \times X \to X$ such that $a$ is continuous when $X$ is given the discrete topology and $G \times X$ the product topology. A morphism of $G$-sets $f : X \to Y$ is simply any $G$-equivariant map from $X$ to $Y$. The category of $G$-sets is denoted $G\text{-Sets}$.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in $G$. If $G$ is an abstract group $G$ (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 6.5) provided we endow $G$ with the discrete topology.

Recall that if $L/K$ is an infinite Galois extension then the Galois group $G = \text{Gal}(L/K)$ comes endowed with a canonical topology, see Fields, Section 22.

Lemma 2.2. Let $K$ be a field. Let $K^{\text{sep}}$ be a separable closure of $K$. Consider the profinite group $G = \text{Gal}(K^{\text{sep}}/K)$. The functor

$$
\text{ schemes étale over } K \quad \longrightarrow \quad G\text{-Sets} \\
X/K \quad \longmapsto \quad \text{Mor}_{\text{Spec}(K)}(\text{Spec}(K^{\text{sep}}), X)
$$

is an equivalence of categories.

Proof. A scheme $X$ over $K$ is étale over $K$ if and only if $X \cong \coprod_{i \in I} \text{Spec}(K_i)$ with each $K_i$ a finite separable extension of $K$ (Morphisms, Lemma 34.7). The functor of the lemma associates to $X$ the $G$-set

$$
\prod_i \text{Hom}_K(K_i, K^{\text{sep}})
$$

with its natural left $G$-action. Each element has an open stabilizer by definition of the topology on $G$. Conversely, any $G$-set $S$ is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory (Fields, Theorem 22.4) the fields $(K^{\text{sep}})^{G_i}$ are finite separable field extensions of $K$, and hence the scheme

$$
\coprod \text{Spec}((K^{\text{sep}})^{G_i})
$$

is étale over $K$. This gives an inverse to the functor of the lemma. Some details omitted. □

Remark 2.3. Under the correspondence of Lemma 2.2, the coverings in the small étale site $\text{Spec}(K)_{\text{étale}}$ of $K$ correspond to surjective families of maps in $G\text{-Sets}$. 


3. Galois categories

In this section we discuss some of the material the reader can find in [Gro71, Exposé V, Sections 4, 5, and 6].

Let $F : C \to \text{Sets}$ be a functor. Recall that by our conventions categories have a set of objects and for any pair of objects a set of morphisms. There is a canonical injective map

$$\text{Aut}(F) \to \prod_{X \in \text{Ob}(C)} \text{Aut}(F(X))$$

For a set $E$ we endow $\text{Aut}(E)$ with the compact open topology, see Topology, Example [30.2]. Of course this is the discrete topology when $E$ is finite, which is the case of interest in this section. We endow $\text{Aut}(F)$ with the topology induced from the product topology on the right hand side of (3.0.1). In particular, the action maps $\text{Aut}(F) \times F(X) \to F(X)$ are continuous when $F(X)$ is given the discrete topology because this is true for the action maps $\text{Aut}(E) \times E \to E$ for any set $E$. The universal property of our topology on $\text{Aut}(F)$ is the following: suppose that $G$ is a topological group and $G \to \text{Aut}(F)$ is a group homomorphism such that the induced actions $G \times F(X) \to F(X)$ are continuous for all $X \in \text{Ob}(C)$ where $F(X)$ has the discrete topology. Then $G \to \text{Aut}(F)$ is continuous.

The following lemma tells us that the group of automorphisms of a functor to the category of finite sets is automatically a profinite group.

**Lemma 3.1.** Let $C$ be a category and let $F : C \to \text{Sets}$ be a functor. The map (3.0.1) identifies $\text{Aut}(F)$ with a closed subgroup of $\prod_{X \in \text{Ob}(C)} \text{Aut}(F(X))$. In particular, if $F(X)$ is finite for all $X$, then $\text{Aut}(F)$ is a profinite group.

**Proof.** Let $\xi = (\gamma_X) \in \prod \text{Aut}(F(X))$ be an element not in $\text{Aut}(F)$. Then there exists a morphism $f : X \to X'$ of $C$ and an element $x \in F(X)$ such that $F(f)(\gamma_X(x)) \neq \gamma_{X'}(F(f)(x))$. Consider the open neighbourhood $U = \{\gamma \in \text{Aut}(F(X)) \mid \gamma(x) = \gamma_X(x)\}$ of $\gamma_X$ and the open neighbourhood $U' = \{\gamma' \in \text{Aut}(F(X')) \mid \gamma'(F(f)(x)) = \gamma_{X'}(F(f)(x))\}$. Then $U \times U' \times \prod_{X'' \neq X, X'} \text{Aut}(F(X''))$ is an open neighbourhood of $\xi$ not meeting $\text{Aut}(F)$. The final statement is follows from the fact that $\prod \text{Aut}(F(X))$ is a profinite space if each $F(X)$ is finite. \[\square\]

**Example 3.2.** Let $G$ be a topological group. An important example will be the forgetful functor

$$\text{Finite-G- Sets} \to \text{Sets}$$

where $\text{Finite-G- Sets}$ is the full subcategory of $\text{G- Sets}$ whose objects are the finite $G$-sets. The category $\text{G- Sets}$ of $G$-sets is defined in Definition 2.1.

Let $G$ be a topological group. The profinite completion of $G$ will be the profinite group

$$G^\wedge = \varinjlim_{U \subset G \text{ open, normal, finite index}} G/U$$

with its profinite topology. Observe that the limit is cofiltered as a finite intersection of open, normal subgroups of finite index is another. The universal property of the profinite completion is

1. When we discuss the pro-étale fundamental group the general case will be of interest.
profinite completion is that any continuous map $G \to H$ to a profinite group $H$ factors canonically as $G \to G^\wedge \to H$.

**Lemma 3.3.** Let $G$ be a topological group. The automorphism group of the functor $(3.2.1)$ endowed with its profinite topology from Lemma 3.1 is the profinite completion of $G$.

**Proof.** Denote $F_G$ the functor $(3.2.1)$. Any morphism $X \to Y$ in $\text{Finite-}G\text{-Sets}$ commutes with the action of $G$. Thus any $g \in G$ defines an automorphism of $F_G$ and we obtain a canonical homomorphism $G \to \text{Aut}(F_G)$ of groups. Observe that any finite $G$-set $X$ is a finite disjoint union of $G$-sets of the form $G/H_i$ with canonical $G$-action where $H_i \subset G$ is an open subgroup of finite index. Then $U_i = \bigcap gH_i g^{-1}$ is open, normal, and has finite index. Moreover $U_i$ acts trivially on $G/H_i$ hence $U = \bigcap U_i$ acts trivially on $F_G(X)$. Hence the action $G \times F_G(X) \to F_G(X)$ is continuous. By the universal property of the topology on $\text{Aut}(F_G)$ the map $G \to \text{Aut}(F_G)$ is continuous. By Lemma 3.1 and the universal property of profinite completion there is an induced continuous group homomorphism $G^\wedge \longrightarrow \text{Aut}(F_G)$.

Moreover, since $G/U$ acts faithfully on $G/U$ this map is injective. If the image is dense, then the map is surjective and hence a homeomorphism by Topology, Lemma 17.8.

Let $\gamma \in \text{Aut}(F_G)$ and let $X \in \text{Ob}(\mathcal{C})$. We will show there is a $g \in G$ such that $\gamma$ and $g$ induce the same action on $F_G(X)$. This will finish the proof. As before we see that $X$ is a finite disjoint union of $G/H_i$. With $U_i$ and $U$ as above, the finite $G$-set $Y = G/U$ surjects onto $G/H_i$ for all $i$ and hence it suffices to find $g \in G$ such that $\gamma$ and $g$ induce the same action on $F_G(G/U) = G/U$. Let $e \in G$ be the neutral element and say that $\gamma(eU) = g_0U$ for some $g_0 \in G$. For any $g_1 \in G$ the morphism

$$R_{g_1} : G/U \longrightarrow G/U, \quad gU \longmapsto gg_1U$$

of $\text{Finite-}G\text{-Sets}$ commutes with the action of $\gamma$. Hence

$$\gamma(g_1U) = \gamma(R_{g_1}(eU)) = R_{g_1}(\gamma(eU)) = R_{g_1}(g_0U) = g_0g_1U$$

Thus we see that $g = g_0$ works. □

Recall that an exact functor is one which commutes with all finite limits and finite colimits. In particular such a functor commutes with equalizers, coequalizers, fibred products, pushouts, etc.

**Lemma 3.4.** Let $G$ be a topological group. Let $F : \text{Finite-}G\text{-Sets} \to \text{Sets}$ be an exact functor with $F(X)$ finite for all $X$. Then $F$ is isomorphic to the functor $(3.2.1)$.

**Proof.** Let $X$ be a nonempty object of $\text{Finite-}G\text{-Sets}$. The diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \{*\} \\
\downarrow & & \downarrow \\
\{*\} & \longrightarrow & \{*\}
\end{array}
$$
is cocartesian. Hence we conclude that $F(X)$ is nonempty. Let $U \subset G$ be an open, normal subgroup with finite index. Observe that

$$G/U \times G/U = \coprod_{gU \in G/U} G/U$$

where the summand corresponding to $gU$ corresponds to the orbit of $(eU, gU)$ on the left hand side. Then we see that

$$F(G/U) \times F(G/U) = F(G/U \times G/U) = \coprod_{gU \in G/U} F(G/U)$$

Hence $|F(G/U)| = |G/U|$ as $F(G/U)$ is nonempty. Thus we see that

$$\lim_{U \subset G \text{ open, normal, finite index}} F(G/U)$$

is nonempty (Categories, Lemma 21.7). Pick $\gamma = (\gamma_U)$ an element in this limit. Denote $F_G$ the functor (3.2.1). We can identify $F_G$ with the functor $X \mapsto \colim_U \text{Mor}(G/U, X)$ where $f : G/U \to X$ sending $eU$ to $x$ and then set $t_{X}(x) = F(f)(\gamma_U)$. We will show that $t$ induces a bijective map $t_{G/U} : F_G(G/U) \to F(G/U)$ for any $U$. This implies in a straightforward manner that $t$ is an isomorphism (details omitted). Since $|F_G(G/U)| = |F(G/U)|$ it suffices to show that $t_{G/U}$ is surjective. The image contains at least one element, namely $t_{G/U}(eU) = F(id_{G/U})(\gamma_U) = \gamma_U$. For $g \in G$ denote $R_g : G/U \to G/U$ right multiplication. Then set of fixed points of $F(R_g) : F(G/U) \to F(G/U)$ is equal to $F(\emptyset) = \emptyset$ if $g \notin U$ because $F$ commutes with equalizers. It follows that if $g_1, \ldots, g_{|G/U|}$ is a system of representatives for $G/U$, then the elements $F(R_{g_i})(\gamma_U)$ are pairwise distinct and hence fill out $F(G/U)$. Then

$$t_{G/U}(g_iU) = F(R_{g_i})(\gamma_U)$$

and the proof is complete. \(\square\)

**Example 3.5.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \text{Sets}$ be a functor such that $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$. By Lemma 3.1 we see that $G = \text{Aut}(F)$ comes endowed with the structure of a profinite topological group in a canonical manner. We obtain a functor

$$\mathcal{C} \to \text{Finite-}\_G\text{-}\text{Sets}, \quad X \mapsto F(X)$$

where $F(X)$ is endowed with the induced action of $G$. This action is continuous by our construction of the topology on $\text{Aut}(F)$.

The purpose of defining Galois categories is to single out those pairs $(\mathcal{C}, F)$ for which the functor (3.5.1) is an equivalence. Our definition of a Galois category is as follows.

**Definition 3.6.** Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \text{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is a Galois category if

1. $\mathcal{C}$ has finite limits and finite colimits,
2. every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects,
3. $F(X)$ is finite for all $X \in \text{Ob}(\mathcal{C})$, and

Different from the definition in [Gro71 Exposé V, Definition 5.1]. Compare with [BS13 Definition 7.2.1].
(4) $F$ reflects isomorphisms and is exact.

Here we say $X \in \text{Ob}(\mathcal{C})$ is connected if it is not initial and for any monomorphism $Y \to X$ either $Y$ is initial or $Y \to X$ is an isomorphism.

**Warning:** This definition is not the same (although eventually we’ll see it is equivalent) as the definition given in most references. Namely, in [Gro71, Exposé V, Definition 5.1] a Galois category is defined to be a category equivalent to $\text{Finite-G-Sets}$ for some profinite group $G$. Then Grothendieck characterizes Galois categories by a list of axioms (G1) – (G6) which are weaker than our axioms above. The motivation for our choice is to stress the existence of finite limits and finite colimits and exactness of the functor $F$. The price we’ll pay for this later is that we’ll have to work a bit harder to apply the results of this section.

**Lemma 3.7.** Let $(\mathcal{C}, F)$ be a Galois category. Let $X \to Y \in \text{Arrows}(\mathcal{C})$. Then

1. $F$ is faithful,
2. $X \to Y$ is a monomorphism $\iff F(X) \to F(Y)$ is injective,
3. $X \to Y$ is an epimorphism $\iff F(X) \to F(Y)$ is surjective,
4. an object $A$ of $\mathcal{C}$ is initial if and only if $F(A) = \emptyset$,
5. an object $Z$ of $\mathcal{C}$ is final if and only if $F(Z)$ is a singleton,
6. if $X$ and $Y$ are connected, then $X \to Y$ is an epimorphism,
7. if $X$ is connected and $a, b : X \to Y$ are two morphisms then $a = b$ as soon as $F(a)$ and $F(b)$ agree on one element of $F(X)$,
8. if $X = \coprod_{i=1,\ldots,n} X_i$ and $Y = \coprod_{j=1,\ldots,m} Y_j$ where $X_i, Y_j$ are connected, then there is map $\alpha : \{1,\ldots,n\} \to \{1,\ldots,m\}$ such that $X \to Y$ comes from a collection of morphisms $X_i \to Y_{\alpha(i)}$.

**Proof.**

Proof of (1). Suppose $a, b : X \to Y$ with $F(a) = F(b)$. Let $E$ be the equalizer of $a$ and $b$. Then $F(E) = F(X)$ and we see that $E = X$ because $F$ reflects isomorphisms.

Proof of (2). This is true because $F$ turns the morphism $X \to X \times_Y X$ into the map $F(X) \to F(X) \times_{F(Y)} F(X)$ and $F$ reflects isomorphisms.

Proof of (3). This is true because $F$ turns the morphism $Y \amalg_X Y \to Y$ into the map $F(Y) \amalg_{F(X)} F(Y) \to F(Y)$ and $F$ reflects isomorphisms.

Proof of (4). There exists an initial object $A$ and certainly $F(A) = \emptyset$. On the other hand, if $X$ is an object with $F(X) = \emptyset$, then the unique map $A \to X$ induces a bijection $F(A) \to F(X)$ and hence $A \to X$ is an isomorphism.

Proof of (5). There exists a final object $Z$ and certainly $F(Z)$ is a singleton. On the other hand, if $X$ is an object with $F(X)$ a singleton, then the unique map $X \to Z$ induces a bijection $F(X) \to F(Z)$ and hence $X \to Z$ is an isomorphism.

Proof of (6). The equalizer $E$ of the two maps $Y \to Y \amalg_X Y$ is not an initial object of $\mathcal{C}$ because $X \to Y$ factors through $E$ and $F(X) \neq \emptyset$. Hence $E = Y$ and we conclude.

Proof of (7). The equalizer $E$ of $a$ and $b$ comes with a monomorphism $E \to X$ and $F(E) \subseteq F(X)$ is the set of elements where $F(a)$ and $F(b)$ agree. To finish use that either $E$ is initial or $E = X$.

Proof of (8). For each $i, j$ we see that $E_{ij} = X_i \times_Y Y_j$ is either initial or equal to $X_i$. Picking $s \in F(X_i)$ we see that $E_{ij} = X_i$ if and only if $s$ maps to an element of $F(Y_j) \subseteq F(Y)$, hence this happens for a unique $j = \alpha(i)$. $\square$
By the lemma above we see that, given a connected object $X$ of a Galois category $(\mathcal{C}, F)$, the automorphism group $\text{Aut}(X)$ has order at most $|F(X)|$. Namely, given $s \in F(X)$ and $g \in \text{Aut}(X)$ we see that $g(s) = s$ if and only if $g = \text{id}_X$ by $[\text{7}]$. We say $X$ is Galois if equality holds. Equivalently, $X$ is Galois if it is connected and $\text{Aut}(X)$ acts transitively on $F(X)$.

**Lemma 3.8.** Let $(\mathcal{C}, F)$ be a Galois category. For any connected object $X$ of $\mathcal{C}$ there exists a Galois object $Y$ and a morphism $Y \to X$.

**Proof.** We will use the results of Lemma 3.7 without further mention. Let $n = |F(X)|$. Consider $X^n$ endowed with its natural action of $S_n$. Let

$$X^n = \coprod_{t \in T} Z_t$$

be the decomposition into connected objects. Pick a $t$ such that $F(Z_t)$ contains $(s_1, \ldots, s_n)$ with $s_i$ pairwise distinct. If $(s'_1, \ldots, s'_n) \in F(Z_t)$ is another element, then we claim $s'_i$ are pairwise distinct as well. Namely, if not, say $s'_i = s'_j$, then $Z_t$ is the image of an connected component of $X^{n-1}$ under the diagonal morphism

$$\Delta_{ij} : X^{n-1} \to X^n$$

Since morphisms of connected objects are epimorphisms and induce surjections after applying $F$ it would follow that $s_i = s_j$ which is not the case.

Let $G \subset S_n$ be the subgroup of elements with $g(Z_t) = Z_t$. Looking at the action of $S_n$ on

$$F(X)^n = F(X^n) = \coprod_{t' \in T} F(Z_{t'})$$

we see that $G = \{g \in S_n \mid g(s_1, \ldots, s_n) \in F(Z_t)\}$. Now pick a second element $(s'_1, \ldots, s'_n) \in F(Z_t)$. Above we have seen that $s'_i$ are pairwise distinct. Thus we can find a $g \in S_n$ with $g(s_1, \ldots, s_n) = (s'_1, \ldots, s'_n)$. In other words, the action of $G$ on $F(Z_t)$ is transitive and the proof is complete. \hfill \Box

Here is a key lemma.

**Lemma 3.9.** Let $(\mathcal{C}, F)$ be a Galois category. Let $G = \text{Aut}(F)$ be as in Example 3.3. For any connected $X$ in $\mathcal{C}$ the action of $G$ on $F(X)$ is transitive.

**Proof.** We will use the results of Lemma 3.7 without further mention. Let $I$ be the set of isomorphism classes of Galois objects in $\mathcal{C}$. For each $i \in I$ let $X_i$ be a representative of the isomorphism class. Choose $\gamma_i \in F(X_i)$ for each $i \in I$. We define a partial ordering on $I$ by setting $i \geq i'$ if and only if there is a morphism $f_{i'i} : X_i \to X_{i'}$. Given such a morphism we can post-compose by an automorphism $X_{i'} \to X_{i'}$ to assure that $F(f_{i'i})(\gamma_i) = \gamma_{i'}$. With this normalization the morphism $f_{i'i}$ is unique. Observe that $I$ is a directed partially ordered set: (Categories, Definition 21.1) if $i_1, i_2 \in I$ there exists a Galois object $Y$ and a morphism $Y \to X_{i_1} \times X_{i_2}$ by Lemma 3.8 applied to a connected component of $X_{i_1} \times X_{i_2}$. Then $Y \cong X_i$ for some $i \in I$ and $i \geq i_1, i \geq i_2$.

We claim that the functor $F$ is isomorphic to the functor $F'$ which sends $X$ to

$$F'(X) = \text{colim}_I \text{Mor}_\mathcal{C}(X_i, X)$$

via the transformation of functors $t : F' \to F$ defined as follows: given $f : X_i \to X$ we set $t_X(f) = F(f)(\gamma_i)$. Using $[\text{7}]$ we find that $t_X$ is injective. To show surjectivity, let $\gamma \in F(X)$. Then we can immediately reduce to the case where $X$ is connected by
the definition of a Galois category. Then we may assume $X$ is Galois by Lemma 3.8. In this case $X$ is isomorphic to $X_i$ for some $i$ and we can choose the isomorphism $X_i \to X$ such that $\gamma_i$ maps to $\gamma$ (by definition of Galois objects). We conclude that $t$ is an isomorphism.

Set $A_i = \text{Aut}(X_i)$. We claim that for $i \geq i'$ there is a canonical map $h_{i,i'} : A_i \to A_{i'}$ such that for all $a \in A_i$ the diagram

$$
\begin{array}{ccc}
X_i & \xrightarrow{f_{i,i'}} & X_{i'} \\
\downarrow & & \downarrow \gamma_{i,i'}(a) \\
X_i & \xrightarrow{f_{i,i'}} & X_{i'}
\end{array}
$$

commutes. Namely, just let $h_{i,i'}(a) = a' : X_{i'} \to X_{i'}$ be the unique automorphism such that $F(a')(\gamma_{i,i'}) = F(f_{i,i'} \circ a)(\gamma_i)$. As before this makes the diagram commute and moreover the choice is unique. It follows that $h_{i,i'} \circ h_{i,i'} = h_{i,i''}$ if $i \geq i' \geq i''$. Since $F(X_i) \to F(X_{i'})$ is surjective we see that $A_i \to A_{i'}$ is surjective. Taking the inverse limit we obtain a group

$$A = \lim_{\longrightarrow} A_i$$

This is a profinite group since the automorphism groups are finite. The map $A \to A_i$ is surjective for all $i$ by Categories, Lemma 21.7. Since elements of $A$ act on the inverse system $X_i$ we get an action of $A$ (on the right) on $X'$ by pre-composing. In other words, we get a homomorphism $A^{opp} \to G$. Since $A \to A_i$ is surjective we conclude that $G$ acts transitively on $F(X_i)$ for all $i$. Since every connected object is dominated by one of the $X_i$ we conclude the lemma is true. □

**Proposition 3.10.** Let $(C, F)$ be a Galois category. Let $G = \text{Aut}(F)$ be as in Example 3.7. The functor $F : C \to \text{Finite-G-Sets}$ is an equivalence.

**Proof.** We will use the results of Lemma 3.7 without further mention. In particular we know the functor is faithful. By Lemma 3.9 we know that for any connected $X$ the action of $G$ on $F(X)$ is transitive. Hence $F$ preserves the decomposition into connected components (existence of which is an axiom of a Galois category). Let $X$ and $Y$ be objects and let $s : F(X) \to F(Y)$ be a map. Then the graph $\Gamma_s \subset F(X) \times F(Y)$ of $s$ is a union of connected components. Hence there exists a union of connected components $Z$ of $X \times Y$, which comes equipped with a monomorphism $Z \to X \times Y$, with $F(Z) = \Gamma_s$. Since $F(Z) \to F(X)$ is bijective we see that $Z \to X$ is an isomorphism and we conclude that $s = F(f)$ where $f : X \cong Z \to Y$ is the composition. Hence $F$ is fully faithful.

To finish the proof we show that $F$ is essentially surjective. It suffices to show that $G/H$ is in the essential image for any open subgroup $H \subset G$ of finite index. By definition of the topology on $G$ there exists a finite collection of objects $X_i$ such that

$$\text{Ker}(G \to \prod_i \text{Aut}(F(X_i)))$$

is contained in $H$. We may assume $X_i$ is connected for all $i$. We can choose a Galois object $Y$ mapping to a connected component of $\prod X_i$ using Lemma 3.8. Choose an isomorphism $F(Y) = G/U$ in $G$-sets for some open subgroup $U \subset G$. As $Y$ is Galois, the group $\text{Aut}(Y) = \text{Aut}_{G\text{-Sets}}(G/U)$ acts transitively on $F(Y) = G/U$. This is a weak version of [Grothendieck, Exposé V]. The proof is borrowed from [Bourbaki].
This implies that \( U \) is normal. Since \( F(Y) \) surjects onto \( F(X_i) \) for each \( i \) we see that \( U \subset H \). Let \( M \subset \text{Aut}(Y) \) be the finite subgroup corresponding to \( (H/U)^{opp} \subset (G/U)^{opp} = \text{Aut}_{G\text{-Sets}}(G/U) = \text{Aut}(Y) \).

Set \( X = Y/M \), i.e., \( X \) is the coequalizer of the arrows \( m : Y \to Y \), \( m \in M \). Since \( F \) is exact we see that \( F(X) = G/H \) and the proof is complete.

**Lemma 3.11.** Let \( (C, F) \) and \( (C', F') \) be Galois categories. Let \( H : C \to C' \) be an exact functor. There exists an isomorphism \( t : F' \circ H \to F \). The choice of \( t \) determines a continuous homomorphism \( h : G' = \text{Aut}(F') \to \text{Aut}(F) = G \) and a 2-commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{H} & C' \\
\downarrow & & \downarrow \\
\text{Finite-}G\text{-Sets} & \xrightarrow{h} & \text{Finite-}G'\text{-Sets}
\end{array}
\]

The map \( h \) is independent of \( t \) up to an inner automorphism of \( G \). Conversely, given a continuous homomorphism \( h : G' \to G \) there is an exact functor \( H : C \to C' \) and an isomorphism \( t \) recovering \( h \) as above.

**Proof.** By Proposition 3.10 and Lemma 3.3, we may assume \( C = \text{Finite-}G\text{-Sets} \) and \( F \) is the forgetful functor and similarly for \( C' \). Thus the existence of \( t \) follows from Lemma 3.4. The map \( h \) comes from transport of structure via \( t \). The commutativity of the diagram is obvious. Uniqueness of \( h \) up to inner conjugation by an element of \( G \) comes from the fact that the choice of \( t \) is unique up to an element of \( G \). The final statement is straightforward.

### 4. Functors and homomorphisms

**Lemma 4.1.** In diagram (4.0.1) the following are equivalent

1. \( h : G' \to G \) is surjective,
2. \( H : C \to C' \) is fully faithful,
3. if \( X \in \text{Ob}(C) \) is connected, then \( H(X) \) is connected,
4. if \( X \in \text{Ob}(C) \) is connected and there is a morphism \( \ast' \to H(X) \) in \( C' \), then there is a morphism \( \ast \to X \), and
5. for any object \( X \) of \( C \) the map \( \text{Mor}_C(\ast, X) \to \text{Mor}_{C'}(\ast', H(X)) \) is bijective.

Here \( \ast \) and \( \ast' \) are final objects of \( C \) and \( C' \).
Proof. The implications (5) ⇒ (4) and (2) ⇒ (5) are clear.

Assume (3). Let \( X \) be a connected object of \( \mathcal{C} \) and let \( \ast' \to H(X) \) be a morphism. Since \( H(X) \) is connected by (3) we see that \( \ast' \to H(X) \) is an isomorphism. Hence the \( G' \)-set corresponding to \( H(X) \) has exactly one element, which means the \( G \)-set corresponding to \( X \) has one element which means \( X \) is isomorphic to the final object of \( \mathcal{C} \), in particular there is a map \( * \to X \). In this way we see that (3) ⇒ (4).

If (1) is true, then the functor \( \text{Finite-G-Sets} \to \text{Finite-G'-Sets} \) is fully faithful: in this case a map of \( G \)-sets commutes with the action of \( G \) if and only if it commutes with the action of \( G' \). Thus (1) ⇒ (2).

If (1) is true, then for a \( G \)-set \( X \) the \( G \)-orbits and \( G' \)-orbits agree. Thus (1) ⇒ (3).

To finish the proof it suffices to show that (4) implies (1). If (1) is false, i.e., if \( h \) is not surjective, then there is an open subgroup \( U \subset G \) containing \( h(G') \) which is not equal to \( G \). Then the finite \( G \)-set \( M = G/U \) has a transitive action but \( G' \) has a fixed point. The object \( X \) of \( \mathcal{C} \) corresponding to \( M \) would contradict (3). In this way we see that (3) ⇒ (1) and the proof is complete. \( \square \)

**Lemma 4.2.** In diagram \( (4.0.1) \) the following are equivalent

1. \( h \circ h' \) is trivial, and
2. the image of \( H' \circ H \) consists of objects isomorphic to finite coproducts of final objects.

**Proof.** We may replace \( H \) and \( H' \) by the canonical functors \( \text{Finite-G-Sets} \to \text{Finite-G'-Sets} \) determined by \( h \) and \( h' \). Then we are saying that the action of \( G'' \) on every \( G \)-set is trivial if and only if the homomorphism \( G'' \to G \) is trivial. This is clear. \( \square \)

**Lemma 4.3.** In diagram \( (4.0.1) \) the following are equivalent

1. the sequence \( G'' \to G' \to G \to 1 \) is exact in the following sense: \( h \) is surjective, \( h \circ h' \) is trivial, and \( \text{Ker}(h) \) is the smallest closed normal subgroup containing \( \text{Im}(h') \),
2. \( H \) is fully faithful and an object \( X' \) of \( \mathcal{C}' \) is in the essential image of \( H \) if and only if \( H'(X') \) is isomorphic to a finite coproduct of final objects, and
3. \( H \) is fully faithful, \( H \circ H' \) sends every object to a finite coproduct of final objects, and for an object \( X' \) of \( \mathcal{C}' \) such that \( H'(X') \) is a finite coproduct of final objects there exists an object \( X \) of \( \mathcal{C} \) and an epimorphism \( H(X) \to X' \).

**Proof.** By Lemmas 4.1 and 4.2 we may assume that \( H \) is fully faithful, \( h \) is surjective, \( H' \circ H \) maps objects to disjoint unions of the final object, and \( h \circ h' \) is trivial. Let \( N \subset G' \) be the smallest closed normal subgroup containing the image of \( h' \). It is clear that \( N \subset \text{Ker}(h) \). We may assume the functors \( H \) and \( H' \) are the canonical functors \( \text{Finite-G-Sets} \to \text{Finite-G'-Sets} \to \text{Finite-G''-Sets} \) determined by \( h \) and \( h' \).

Suppose that (2) holds. This means that for a finite \( G' \)-set \( X' \) such that \( G'' \) acts trivially, the action of \( G' \) factors through \( G \). Apply this to \( X' = G'/U'N \) where \( U' \) is a small open subgroup of \( G' \). Then we see that \( \text{Ker}(h) \subset U'N \) for all \( U' \). Since \( N \) is closed this implies \( \text{Ker}(h) \subset N \), i.e., (1) holds.

Suppose that (1) holds. This means that \( N = \text{Ker}(h) \). Let \( X' \) be a finite \( G' \)-set such that \( G'' \) acts trivially. This means that \( \text{Ker}(G' \to \text{Aut}(X')) \) is a closed normal
subgroup containing $\text{Im}(h')$. Hence $N = \text{Ker}(h)$ is contained in it and the $G'$-action on $X'$ factors through $G$, i.e., (2) holds.

Suppose that (3) holds. This means that for a finite $G'$-set $X'$ such that $G''$ acts trivially, there is a surjection of $G'$-sets $X \to X'$ where $X$ is a $G$-set. Clearly this means the action of $G'$ on $X'$ factors through $G$, i.e., (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate. This finishes the proof. \hfill $\square$

**Lemma 4.4.** In diagram (4.0.1) the following are equivalent

1. $h'$ is injective, and
2. for every connected object $X''$ of $C''$ there exists an object $X'$ of $C'$ and a diagram

$$X'' \leftarrow Y'' \to H(X')$$

in $C''$ where $Y'' \to X''$ is an epimorphism and $Y'' \to H(X')$ is a monomorphism.

**Proof.** We may replace $H'$ by the corresponding functor between the categories of finite $G'$-sets and finite $G''$-sets.

Assume $h' : G'' \to G'$ is injective. Let $H'' \subset G''$ be an open subgroup. Since the topology on $G''$ is the induced topology from $G'$ there exists an open subgroup $H' \subset G'$ such that $(h')^{-1}(H') \subset H''$. Then the desired diagram is

$$G''/H'' \leftarrow G''/(h')^{-1}(H') \to G'/H'$$

Conversely, assume (2) holds for the functor $\text{Finite-}G''\text{-Sets} \to \text{Finite-}G''\text{-Sets}$. Let $g'' \in \text{Ker}(h')$. Pick any open subgroup $H'' \subset G''$. By assumption there exists a finite $G'$-set $X'$ and a diagram

$$G''/H'' \leftarrow Y'' \to X'$$

of $G''$-sets with the left arrow surjective and the right arrow injective. Since $g''$ is in the kernel of $h'$ we see that $g''$ acts trivially on $X'$. Hence $g''$ acts trivially on $Y''$ and hence trivially on $G''/H''$. Thus $g'' \in H''$. As this holds for all open subgroups we conclude that $g''$ is the identity element as desired. \hfill $\square$

**Lemma 4.5.** In diagram (4.0.1) the following are equivalent

1. the image of $h'$ is normal, and
2. for every connected object $X'$ of $C'$ such that there is a morphism from the final object of $C''$ to $H'(X')$ we have that $H'(X')$ is isomorphic to a finite coproduct of final objects.

**Proof.** This translates into the following statement for the continuous group homomorphism $h' : G'' \to G'$: the image of $h'$ is normal if and only if every open subgroup $U' \subset G'$ which contains $h'(G'')$ also contains every conjugate of $h'(G')$. The result follows easily from this; some details omitted. \hfill $\square$

## 5. Finite étale morphisms

In this section we prove enough basic results on finite étale morphisms to be able to construct the étale fundamental group.

Let $X$ be a scheme. We will use the notation $\text{FÉt}_X$ to denote the category of schemes finite and étale over $X$. Thus

1. an object of $\text{FÉt}_X$ is a finite étale morphism $Y \to X$ with target $X$, and
(2) a morphism in $\text{FÉt}_X$ from $Y \to X$ to $Y' \to X$ is a morphism $Y \to Y'$ making the diagram

$$
\begin{array}{ccc}
Y & \to & Y' \\
\downarrow & & \downarrow \\
X & \to & X
\end{array}
$$

commute.

We will often call an object of $\text{FÉt}_X$ a \textsl{finite étale cover} of $X$ (even if $Y$ is empty). It turns out that there is a stack $p : \text{FÉt} \to \text{Sch}$ over the category of schemes whose fibre over $X$ is the category $\text{FÉt}_X$ just defined. See Examples of Stacks, Section 6.

\textbf{Example 5.1.} Let $k$ be an algebraically closed field and $X = \text{Spec}(k)$. In this case $\text{FÉt}_X$ is equivalent to the category of finite sets. This works more generally when $k$ is separably algebraically closed. The reason is that a scheme étale over $k$ is the disjoint union of spectra of fields finite separable over $k$, see Morphisms, Lemma 34.7.

\textbf{Lemma 5.2.} Let $X$ be a scheme. The category $\text{FÉt}_X$ has finite limits and finite colimits and for any morphism $X' \to X$ the base change functor $\text{FÉt}_X \to \text{FÉt}_{X'}$ is exact.

\textbf{Proof.} Finite limits and left exactness. By Categories, Lemma 18.4 it suffices to show that $\text{FÉt}_X$ has a final object and fibred products. This is clear because the category of all schemes over $X$ has a final object (namely $X$) and fibred products and fibred products of schemes finite étale over $X$ are finite étale over $X$. Moreover, it is clear that base change commutes with these operations and hence base change is left exact (Categories, Lemma 23.2).

Finite colimits and right exactness. By Categories, Lemma 18.7 it suffices to show that $\text{FÉt}_X$ has finite coproducts and coequalizers. Finite coproducts are given by disjoint unions (the empty coproduct is the empty scheme). Let $a, b : Z \to Y$ be two morphisms of $\text{FÉt}_X$. Since $Z \to X$ and $Y \to X$ are finite étale we can write $Z = \text{Spec}(C)$ and $Y = \text{Spec}(B)$ for some finite locally free $\mathcal{O}_X$-algebras $C$ and $B$. The morphisms $a, b$ induce two maps $a^\sharp, b^\sharp : B \to C$. Let $A = \text{Eq}(a^\sharp, b^\sharp)$ be their equalizer. If

$$
\text{Spec}(A) \to X
$$

is finite étale, then it is clear that this is the coequalizer (after all we can write any object of $\text{FÉt}_X$ as the relative spectrum of a sheaf of $\mathcal{O}_X$-algebras). This we may do after replacing $X$ by the members of an étale covering (Descent, Lemmas 20.23 and 20.6). Thus by Étale Morphisms, Lemma 18.3 we may assume that $Y = \coprod_{i=1,\ldots,n} X$ and $Z = \coprod_{i=1,\ldots,m} X$. Then

$$
C = \prod_{1 \leq j \leq m} \mathcal{O}_X \quad \text{and} \quad B = \prod_{1 \leq i \leq n} \mathcal{O}_X
$$

After a further replacement by the members of an open covering we may assume that $a, b$ correspond to maps $a_s, b_s : \{1,\ldots,m\} \to \{1,\ldots,n\}$, i.e., the summand $X$ of $Z$ corresponding to the index $j$ maps into the summand $X$ of $Y$ corresponding to the index $a_s(j)$, resp. $b_s(j)$ under the morphism $a$, resp. $b$. Let $\{1,\ldots,n\} \to T$ be the coequalizer of $a_s, b_s$. Then we see that

$$
A = \prod_{t \in T} \mathcal{O}_X
$$
whose spectrum is certainly finite étale over $X$. We omit the verification that this is compatible with base change. Thus base change is a right exact functor.

\[ \square \]

**Remark 5.3.** Let $X$ be a scheme. Consider the natural functors $F_1 : \text{FÉt}_X \to \text{Sch}$ and $F_2 : \text{FÉt}_X \to \text{Sch}/X$. Then

1. The functors $F_1$ and $F_2$ commute with finite colimits.
2. The functor $F_2$ commutes with finite limits,
3. The functor $F_1$ commutes with connected finite limits, i.e., with equalizers and fibre products.

The results on limits are immediate from the discussion in the proof of Lemma 5.2 and Categories, Lemma 16.2. It is clear that $F_1$ and $F_2$ commute with finite coproducts. By the dual of Categories, Lemma 23.2 we need to show that $F_1$ and $F_2$ commute with coequalizers. In the proof of Lemma 5.2 we saw that coequalizers in $\text{FÉt}_X$ look étale locally like this

$$ \bigcup_{j \in J} U_j \to \bigcup_{i \in I} U_i \to \bigcup_{t \in \text{Coeq}(a,b)} U $$

which is certainly a coequalizer in the category of schemes. Hence the statement follows from the fact that being a coequalizer is fpqc local as formulate precisely in Descent, Lemma 10.8.

**Lemma 5.4.** Let $X$ be a scheme. Given $U, V$ finite étale over $X$ there exists a scheme $W$ finite étale over $X$ such that

$$ \text{Mor}_X(X, W) = \text{Mor}_X(U, V) $$

and such that the same remains true after any base change.

**Proof.** By More on Morphisms, Lemma 60.4 there exists a scheme $W$ representing $\text{Mor}_X(U, V)$. (Use that an étale morphism is locally quasi-finite by Morphisms, Lemmas 34.6 and that a finite morphism is separated.) This scheme clearly satisfies the formula after any base change. To finish the proof we have to show that $W \to X$ is finite étale. This we may do after replacing $X$ by the members of an étale covering (Descent, Lemmas 20.23 and 20.6). Thus by Étale Morphisms, Lemma 18.3 we may assume that $U = \bigcup_{i=1}^n X$ and $V = \bigcup_{j=1}^m X$. In this case $W = \bigcup_{a:(1,...,n)\to(1,...,m)} X$ by inspection (details omitted) and the proof is complete.

\[ \square \]

Let $X$ be a scheme. A geometric point of $X$ is a morphism $\text{Spec}(k) \to X$ where $k$ is algebraically closed. Such a point is usually denoted $\overline{x}$, i.e., by an overlined small case letter. We often use $\overline{x}$ to denote the scheme $\text{Spec}(k)$ as well as the morphism, and we use $k(\overline{x})$ to denote $k$. We say $\overline{x}$ lies over $x$ to indicate that $x \in X$ is the image of $\overline{x}$. We will discuss this further in Étale Cohomology, Section 29. Given $\overline{x}$ and an étale morphism $U \to X$ we can consider

$$ |U_{\overline{x}}| : \text{the underlying set of points of the scheme } U_{\overline{x}} = U \times_X \overline{x} $$
Since $U_x$ as a scheme over $\pi$ is a disjoint union of copies of $\pi$ (Morphisms, Lemma 34.7) we can also describe this set as

$$|U_x| = \left\{ \begin{array}{c}
\text{commutative diagrams} \\
\pi \downarrow \downarrow \rightarrow \rightarrow U \\
\tau \downarrow \downarrow \rightarrow \rightarrow X
\end{array} \right\}$$

The assignment $U \mapsto |U_x|$ is a functor which is often denoted $F_\pi$.

**Lemma 5.5.** Let $X$ be a connected scheme. Let $\pi$ be a geometric point. The functor $F_\pi : F\text{Ét}_X \to \text{Sets}$, $Y \mapsto |Y_x|$ defines a Galois category (Definition 3.6).

**Proof.** After identifying $F\text{Ét}_X$ with the category of finite sets (Example 5.1) we see that our functor $F_\pi$ is nothing but the base change functor for the morphism $\pi \to X$. Thus we see that $F\text{Ét}_X$ has finite limits and finite colimits and that $F_\pi$ is exact by Lemma 5.2. We will also use that finite limits in $F\text{Ét}_X$ agree with the corresponding finite limits in the category of schemes over $X$, see Remark 5.3.

If $Y' \to Y$ is a monomorphism in $F\text{Ét}_X$ then we see that $Y' \to Y' \times_Y Y'$ is an isomorphism, and hence $Y' \to Y$ is a monomorphism of schemes. It follows that $Y' \to Y$ is an open immersion (Étale Morphisms, Theorem 14.1). Since $Y'$ is finite over $X$ and $Y$ separated over $X$, the morphism $Y' \to Y$ is finite (Morphisms, Lemma 42.14), hence closed (Morphisms, Lemma 42.11), hence it is the inclusion of an open and closed subscheme of $Y$. It follows that $Y$ is a connected objects of the category $F\text{Ét}_X$ (as in Definition 3.6) if and only if $Y$ is connected as a scheme. Then it follows from Topology, Lemma 7.6 that $Y$ is a finite coproduct of its connected components both as a scheme and in the sense of Definition 3.6.

Let $Y \to Z$ be a morphism in $F\text{Ét}_X$ which induces a bijection $F_\pi(Y) \to F_\pi(Z)$. We have to show that $Y \to Z$ is an isomorphism. By the above we may assume $Z$ is connected. Since $Y \to Z$ is finite étale and hence finite locally free it suffices to show that $Y \to Z$ is finite locally free of degree 1. This is true in a neighbourhood of any point of $Z$ lying over $\pi$ and since $Z$ is connected and the degree is locally constant we conclude. \qed

6. **Fundamental groups**

In this section we define Grothendieck’s algebraic fundamental group. The following definition makes sense thanks to Lemma 5.5.

**Definition 6.1.** Let $X$ be a connected scheme. Let $\pi$ be a geometric point of $X$. The **fundamental group** of $X$ with base point $\pi$ is the group $\pi_1(X, \pi) = \text{Aut}(F_\pi)$ of automorphisms of the fibre functor $F_\pi : F\text{Ét}_X \to \text{Sets}$ endowed with its canonical profinite topology from Lemma 3.1.

Combining the above with the material from Section 3 we obtain the following theorem.

**Theorem 6.2.** Let $X$ be a connected scheme. Let $\pi$ be a geometric point of $X$. 
(1) The fibre functor $F_{\pi}$ defines an equivalence of categories
$$F\text{Ét}_{X} \longrightarrow \text{Finite}-\pi_{1}(X, \pi)-\text{Sets}$$

(2) Given a second geometric point $\pi'$ of $X$ there exists an isomorphism $t : F_{\pi} \rightarrow F_{\pi'}$. This gives an isomorphism $\pi_{1}(X, \pi) \rightarrow \pi_{1}(X, \pi')$ compatible with the equivalences in (1). This isomorphism is independent of $t$ up to inner conjugation.

(3) Given a morphism $f : X \rightarrow Y$ of connected schemes denote $\overline{y} = f \circ \pi$. There is a canonical continuous homomorphism
$$f_{*} : \pi_{1}(X, \pi) \rightarrow \pi_{1}(Y, \overline{y})$$
such that the diagram
$$\begin{array}{ccc}
F\text{Ét}_{Y} & \longrightarrow & F\text{Ét}_{X} \\
\downarrow F_{\pi} & & \downarrow F_{\pi} \\
\text{Finite}-\pi_{1}(Y, \overline{y})-\text{Sets} & \longrightarrow & \text{Finite}-\pi_{1}(X, \pi)-\text{Sets}
\end{array}$$
is commutative.

Proof. Part (1) follows from Lemma 5.5 and Proposition 3.10. Part (2) is a special case of Lemma 3.11. For part (3) observe that the diagram
$$\begin{array}{ccc}
F\text{Ét}_{Y} & \longrightarrow & F\text{Ét}_{X} \\
\downarrow F_{\pi} & & \downarrow F_{\pi} \\
\text{Sets} & \longrightarrow & \text{Sets}
\end{array}$$
is commutative (actually commutative, not just 2-commutative) because $\overline{y} = f \circ \pi$. Hence we can apply Lemma 3.11 with the implied transformation of functors to get (3).

Lemma 6.3. Let $K$ be a field and set $X = \text{Spec}(K)$. Let $\overline{K}$ be an algebraic closure and denote $\pi : \text{Spec}(\overline{K}) \rightarrow X$ the corresponding geometric point. Let $K^{\text{sep}} \subset \overline{K}$ be the separable algebraic closure.

(1) The functor of Lemma 2.2 induces an equivalence
$$F\text{Ét}_{X} \rightarrow \text{Finite}-\text{Gal}(K^{\text{sep}}/K)-\text{Sets}.$$

compatible with $F_{\pi}$ and the functor $\text{Finite}-\text{Gal}(K^{\text{sep}}/K)-\text{Sets} \rightarrow \text{Sets}$.

(2) This induces a canonical isomorphism
$$\text{Gal}(K^{\text{sep}}/K) \rightarrow \pi_{1}(X, \pi)$$
of profinite topological groups.

Proof. The functor of Lemma 2.2 is the same as the functor $F_{\pi}$ because for any $Y$ étale over $X$ we have
$$\text{Mor}_{X}(\text{Spec}(\overline{K}), Y) = \text{Mor}_{X}(\text{Spec}(K^{\text{sep}}), Y)$$
Namely, as seen in the proof of Lemma 2.2 we have $Y = \coprod_{i \in I} \text{Spec}(L_{i})$ with $L_{i}/K$ finite separable over $K$. Hence any $K$-algebra homomorphism $L_{i} \rightarrow \overline{K}$ factors through $K^{\text{sep}}$. Also, note that $F_{\pi}(Y)$ is finite if and only if $I$ is finite if and only if $Y \rightarrow X$ is finite étale. This proves (1).
Part (2) is a formal consequence of (1), Lemma 3.11, and Lemma 3.3. (Please also see the remark below.) □

Remark 6.4. In the situation of Lemma 6.3 let us give a more explicit construction of the isomorphism $\text{Gal}(K^{\text{sep}}/K) \to \pi_1(X, \bar{x}) = \text{Aut}(F_{\bar{x}})$. Observe that $\text{Gal}(K^{\text{sep}}/K) = \text{Aut}(K/K)$ as $K$ is the perfection of $K^{\text{sep}}$. Since $F_{\bar{x}}(Y) = \text{Mor}_X(\text{Spec}(K), Y)$ we may consider the map

$$\text{Aut}(K/K) \times F_{\bar{x}}(Y) \to F_{\bar{x}}(Y), \quad (\sigma, \bar{y}) \mapsto \sigma \cdot \bar{y} = \bar{y} \circ \text{Spec}(\sigma)$$

This is an action because

$$\sigma \tau \cdot \bar{y} = \bar{y} \circ \text{Spec}(\sigma \tau) = \bar{y} \circ \text{Spec}(\tau) \circ \text{Spec}(\sigma) = \sigma \cdot (\tau \cdot \bar{y})$$

The action is functorial in $Y \in F\text{Ét}_X$ and we obtain the desired map.

7. Galois covers of connected schemes

Let $X$ be a connected scheme with geometric point $\bar{x}$. Since $F_{\bar{x}} : F\text{Ét}_X \to \text{Sets}$ is a Galois category (Lemma 5.5), the material in Section 3 applies. In this section we explicitly transfer some of the terminology and results to the setting of schemes and finite étale morphisms.

We will say a finite étale morphism $Y \to X$ is a Galois cover if $Y$ defines a Galois object of $F\text{Ét}_X$. For a finite étale morphism $Y \to X$ with $G = \text{Aut}(X/Y)$ the following are equivalent

1. $Y$ is a Galois cover of $X$,
2. $Y$ is connected and $|G|$ is equal to the degree of $Y \to X$,
3. $Y$ is connected and $G$ acts transitively on $F_{\bar{x}}(Y)$, and
4. $Y$ is connected and $G$ acts simply transitively on $F_{\bar{x}}(Y)$.

This follows immediately from the discussion in Section 3.

For any finite étale morphism $f : Y \to X$ with $Y$ connected, there is a finite étale Galois cover $Y' \to X$ which dominates $Y$ (Lemma 3.8).

The Galois objects of $F\text{Ét}_X$ correspond, via the equivalence

$$F_{\bar{x}} : F\text{Ét}_X \to \text{Finite-}\pi_1(X, \bar{x})\text{-Sets}$$

of Theorem 6.2, with the finite $\pi_1(X, \bar{x})\text{-Sets}$ of the form $G = \pi_1(X, \bar{x})/H$ where $H$ is a normal open subgroup. Equivalently, if $G$ is a finite group and $\pi_1(X, \bar{x}) \to G$ is a continuous surjection, then $G$ viewed as a $\pi_1(X, \bar{x})$-set corresponds to a Galois covering.

If $Y_i \to X, i = 1, 2$ are finite étale Galois covers with Galois groups $G_i$, then there exists a finite étale Galois cover $Y \to X$ whose Galois group is a subgroup of $G_1 \times G_2$.

Namely, take the corresponding continuous homomorphisms $\pi_1(X, \bar{x}) \to G_i$ and let $G$ be the image of the induced continuous homomorphism $\pi_1(X, \bar{x}) \to G_1 \times G_2$.

8. Topological invariance of the fundamental group

The main result of this section is that a universal homeomorphism of connected schemes induces an isomorphism on fundamental groups. See Proposition 8.4.

Instead of directly proving two schemes have the same fundamental group, we often prove that their categories of finite étale coverings are the same. This of course implies that their fundamental groups are equal provided they are connected.
Lemma 8.1. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes such that the base change functor $\text{FÉt}_Y \to \text{FÉt}_X$ is an equivalence of categories. In this case

1. $f$ induces a homeomorphism $\pi_0(X) \to \pi_0(Y)$,
2. if $X$ or equivalently $Y$ is connected, then $\pi_1(X, x) = \pi_1(Y, y)$.

Proof. Let $Y = Y_0 \amalg Y_1$ be a decomposition into nonempty open and closed subschemes. We claim that $f(X)$ meets both $Y_i$. Namely, if not, say $f(X) \subset Y_1$, then we can consider the finite étale morphism $V = Y_1 \to Y$. This is not an isomorphism but $V \times_Y X \to X$ is an isomorphism, which is a contradiction.

Suppose that $X = X_0 \amalg X_1$ is a decomposition into open and closed subschemes. Consider the finite étale morphism $U = X_1 \to X$. Then $U = X \times_Y V$ for some finite étale morphism $V \to Y$. The degree of the morphism $V \to Y$ is locally constant, hence we obtain a decomposition $Y = \bigsqcup_{d \geq 0} Y_d$ into open and closed subschemes such that $V \to Y$ has degree $d$ over $Y_d$. Since $f^{-1}(Y_d) = \emptyset$ for $d > 1$ we conclude that $Y_d = \emptyset$ for $d > 1$ by the above. And we conclude that $f^{-1}(Y_i) = X_i$ for $i = 0, 1$.

It follows that $f^{-1}$ induces a bijection between the set of open and closed subsets of $Y$ and the set of open and closed subsets of $X$. Note that $X$ and $Y$ are spectral spaces, see Properties, Lemma \ref{properties-lemma}. By Topology, Lemma \ref{topology-lemma} the lattice of open and closed subsets of a spectral space determines the set of connected components. Hence $\pi_0(X) \to \pi_0(Y)$ is bijective. Since $\pi_0(X)$ and $\pi_0(Y)$ are profinite spaces (Topology, Lemma \ref{topology-lemma}) we conclude that $\pi_0(X) \to \pi_0(Y)$ is a homeomorphism by Topology, Lemma \ref{topology-lemma}. This proves (1). Part (2) is immediate.

The following lemma tells us that the fundamental group of a henselian pair is the fundamental group of the closed subset.

Lemma 8.2. Let $(A, I)$ be a henselian pair. Set $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$. The functor

$$\text{FÉt}_X \to \text{FÉt}_Z, \quad U \mapsto U \times_X Z$$

is an equivalence of categories.

Proof. This is a translation of More on Algebra, Lemma \ref{more-on-algebra-lemma}.

The following lemma tells us that the fundamental group of a thickening is the same as the fundamental group of the original. We will use this in the proof of the strong proposition concerning universal homeomorphisms below.

Lemma 8.3. Let $X \subset X'$ be a thickening of schemes. The functor

$$\text{FÉt}_{X'} \to \text{FÉt}_X, \quad U' \mapsto U' \times_{X'} X$$

is an equivalence of categories.

Proof. For a discussion of thickenings see More on Morphisms, Section \ref{more-on-morphisms-section}. Let $U' \to X'$ be an étale morphism such that $U = U' \times_{X'} X \to X$ is finite étale. Then $U' \to X'$ is finite étale as well. This follows for example from More on Morphisms, Lemma \ref{more-on-morphisms-lemma}. Now, if $X \subset X'$ is a finite order thickening then this remark combined with Étale Morphisms, Theorem \ref{etale-morphisms-theorem} proves the lemma. Below we will prove the lemma for general thickenings, but we suggest the reader skip the proof.
Let $X' = \bigcup X'_i$ be an affine open covering. Set $X_i = X \times_X \eta_i$, $X'_i = X_i \cap X'_i$, $X_{ij} = X \times_X \eta_j$, $X'_{ij} = X_i \cap X'_j$, $X_{ijk} = X \times_X \eta_k$ and $X'_{ijk} = X_i \cap X'_j \cap X'_k$. Suppose that we can prove the theorem for each of the thickenings $X_i \subset X'_i$, $X_{ij} \subset X'_{ij}$, and $X_{ijk} \subset X'_{ijk}$. Then the result follows for $X \subset X'$ by relative glueing of schemes, see Constructions, Section 2. Observe that the schemes $X'_i$, $X'_{ij}$, $X'_{ijk}$ are each separated as open subschemes of affine schemes. Repeating the argument one more time we reduce to the case where the schemes $X'_i$, $X'_{ij}$, $X'_{ijk}$ are affine.

In the affine case we have $X' = \text{Spec}(A')$ and $X = \text{Spec}(A'/I')$ where $I'$ is a locally nilpotent ideal. Then $(A', I')$ is a henselian pair (More on Algebra, Lemma 11.2) and the result follows from Lemma 8.2 (which is much easier in this case).

The “correct” way to prove the following proposition would be to deduce it from the invariance of the étale site, see Étale Cohomology, Theorem 45.2.

**Proposition 8.4.** Let $f : X \to Y$ be a universal homeomorphism of schemes. Then

$$F\text{Ét}_Y \to F\text{Ét}_X, \quad V \mapsto V \times_Y X$$

is an equivalence. Thus if $X$ and $Y$ are connected, then $f$ induces an isomorphism $\pi_1(X, \overline{\pi}) \to \pi_1(Y, \overline{\gamma})$ of fundamental groups.

**Proof.** Recall that a universal homeomorphism is the same thing as an integral, universally injective, surjective morphism, see Morphisms, Lemma 43.5. In particular, the diagonal $\Delta : X \to X \times_X X$ is a thickening by Morphisms, Lemma 10.2. Thus by Lemma 8.3 we see that given a finite étale morphism $U \to X$ there is a unique isomorphism

$$\varphi : U \times_Y X \to X \times_Y U$$

of schemes finite étale over $X \times_Y X$ which pulls back under $\Delta$ to $\text{id} : U \to U$ over $X$. Since $X \to X \times_X X \times_X X$ is a thickening as well (it is bijective and a closed immersion) we conclude that $(U, \varphi)$ is a descent datum relative to $X/Y$. By Étale Morphisms, Proposition 20.6 we conclude that $U = X \times_Y V$ for some $V \to Y$ quasi-compact, separated, and étale. We omit the proof that $V \to Y$ is finite (hints: the morphism $U \to V$ is surjective and $U \to Y$ is integral). We conclude that $F\text{Ét}_Y \to F\text{Ét}_X$ is essentially surjective.

Arguing in the same manner as above we see that given $V_1 \to Y$ and $V_2 \to Y$ in $F\text{Ét}_Y$ any morphism $a : X \times_Y V_1 \to X \times_Y V_2$ over $X$ is compatible with the canonical descent data. Thus $a$ descends to a morphism $V_1 \to V_2$ over $Y$ by Étale Morphisms, Lemma 20.3.

**9. Finite étale covers of proper schemes**

In this section we show that the fundamental group of a connected proper scheme over a henselian local ring is the same as the fundamental group of its special fibre. We also show that the fundamental group of a connected proper scheme over an algebraically closed field $k$ does not change if we replace $k$ by an algebraically closed extension. Instead of stating and proving the results in the connected case we prove the results in general and we leave it to the reader to deduce the result for fundamental groups using Lemma 8.1.
Lemma 9.1. Let $A$ be a henselian local ring. Let $X$ be a proper scheme over $A$ with closed fibre $X_0$. Then the functor
\[ FE\text{ét}_X \to FE\text{ét}_{X_0}, \quad U \mapsto U_0 = U \times_X X_0 \]
is an equivalence of categories.

Proof. The proof given here is an example of applying algebraization and approximation. We proceed in a number of stages.

Essential surjectivity when $A$ is a complete local Noetherian ring. Let $X_n = X \times_{\text{Spec}(A)} \text{Spec}(A/m^{n+1})$. By Étale Morphisms, Theorem 13.2 the inclusions
\[ X_0 \to X_1 \to X_2 \to \ldots \]
induce equivalence of categories between the category of schemes étale over $X_0$ and the category of schemes étale over $X_n$. Moreover, if $U_n \to X_n$ corresponds to a finite étale morphism $U_0 \to X_0$, then $U_n \to X_n$ is finite too, for example by More on Morphisms, Lemma 13.3. In this case the morphism $U_0 \to \text{Spec}(A/m)$ is proper as $X_0$ is proper over $A/m$. Thus we may apply Grothendieck’s algebraization theorem (in the form of Cohomology of Schemes, Lemma 28.2) to see that there is a finite morphism $U \to X$ whose restriction to $X_0$ recovers $U_0$. By More on Morphisms, Lemma 12.3 we see that $U \to X$ is étale at every point of $U_0$. However, since every point of $U$ specializes to a point of $U_0$ (as $U$ is proper over $A$), we conclude that $U \to X$ is étale. In this way we conclude the functor is essentially surjective.

Fully faithfulness when $A$ is a complete local Noetherian ring. Let $U \to X$ and $V \to X$ be finite étale morphisms and let $\varphi_0 : U_0 \to V_0$ be a morphism over $X_0$. Look at the morphism
\[ \Gamma_{\varphi_0} : U_0 \to U_0 \times_{X_0} V_0 \]
This morphism is both finite étale and a closed immersion. By essential surjectivity applied to $X = U \times_X V$ we find a finite étale morphism $W \to U \times_X V$ whose special fibre is isomorphic to $\Gamma_{\varphi_0}$. Consider the projection $W \to U$. It is finite étale and an isomorphism over $U_0$ by construction. By Étale Morphisms, Lemma 14.2 $W \to U$ is an isomorphism in an open neighbourhood of $U_0$. Thus it is an isomorphism and the composition $\varphi : U \cong W \to V$ is the desired lift of $\varphi_0$.

Essential surjectivity when $A$ is a henselian local Noetherian G-ring. Let $U_0 \to X_0$ be a finite étale morphism. Let $A^\wedge$ be the completion of $A$ with respect to the maximal ideal. Let $X^\wedge$ be the base change of $X$ to $A^\wedge$. By the result above there exists a finite étale morphism $V \to X^\wedge$ whose special fibre is $U_0$. Write $A^\wedge = \text{colim} A_i$ with $A \to A_i$ of finite type. By Limits, Lemma 10.1 there exists an $i$ and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X^\wedge$ is $V$. After increasing $i$ we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Writing
\[ A_i = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \]
the ring map $A_i \to A^\wedge$ can be reinterpreted as a solution $(a_1, \ldots, a_n)$ in $A^\wedge$ for the system of equations $f_j = 0$. By Smoothing Ring Maps, Theorem 13.1 we can approximate this solution (to order 11 for example) by a solution $(b_1, \ldots, b_n)$ in $A$. Translating back we find an $A$-algebra map $A_i \to A$ which gives the same closed point as the original map $A_i \to A^\wedge$ (as $11 > 1$). The base change $U \to X$ of $V \to X_{A_i}$ by this ring map will therefore be a finite étale morphism whose special fibre is isomorphic to $U_0$. 


Fully faithfulness when $A$ is a henselian local Noetherian $G$-ring. This can be deduced from essential surjectivity in exactly the same manner as was done in the case that $A$ is complete Noetherian.

General case. Let $(A, m)$ be a henselian local ring. Set $S = \text{Spec}(A)$ and denote $s \in S$ the closed point. By Limits, Lemma 13.3 we can write $X \to \text{Spec}(A)$ as a cofiltered limit of proper morphisms $X_i \to S_i$ with $S_i$ of finite type over $\mathbb{Z}$. For each $i$ let $s_i \in S_i$ be the image of $s$. Since $S = \lim S_i$ and $A = \mathcal{O}_{S,s}$ we have $A = \colim \mathcal{O}_{S_i,s_i}$. The ring $A_i = \mathcal{O}_{S_i,s_i}$ is a Noetherian local $G$-ring (More on Algebra, Proposition 49.12). By More on Algebra, Lemma 12.5 we see that $A = \colim A_i$. By More on Algebra, Lemma 49.8 the rings $A_i$ are $G$-rings. Thus we see that $A = \colim A_i$ and

$$X = \lim(X_i \times_{S_i} \text{Spec}(A_i))$$

as schemes. The category of schemes finite étale over $X$ is the limit of the category of schemes finite étale over $X_i \times_{S_i} \text{Spec}(A_i)$ (by Limits, Lemmas 10.1, 8.3 and 8.10). The same thing is true for schemes finite étale over $X_0 = \lim(X_i \times_{S_i} s_i)$. Thus we formally deduce the result for $X/\text{Spec}(A)$ from the result for the $(X_i \times_{S_i} \text{Spec}(A_i))/\text{Spec}(A_i)$ which we dealt with above.

0A49 Lemma 9.2. Let $k \subset k'$ be an extension of algebraically closed fields. Let $X$ be a proper scheme over $k$. Then the functor

$$U \mapsto U_{k'}$$

is an equivalence of categories between schemes finite étale over $X$ and schemes finite étale over $X_{k'}$.

Proof. Let us prove the functor is essentially surjective. Let $U' \to X_{k'}$ be a finite étale morphism. Write $k' = \colim A_i$ as a filtered colimit of finite type $k$-algebras. By Limits, Lemma 10.7 there exists an $i$ and a finitely presented morphism $U_i \to X_{A_i}$ whose base change to $X_{k'}$ is $U'$. After increasing $i$ we may assume that $U_i \to X_{A_i}$ is finite and étale (Limits, Lemmas 8.3 and 8.10). Since $k$ is algebraically closed we can find a $k$-valued point $t$ in $\text{Spec}(A_i)$. Let $U = (U_i)_t$ be the fibre of $U_i$ over $t$. Let $A_i^{h}$ be the henselization of $(A_i)_m$ where $m$ is the maximal ideal corresponding to the point $t$. By Lemma 9.1 we see that $(U_i)_{A_i^{h}} = U \times \text{Spec}(A_i^{h})$ as schemes over $X_{A_i}$. Now since $A_i^{h}$ is algebraic over $A_i$ (see for example discussion in Smoothing Ring Maps, Example 13.3) and since $k'$ is algebraically closed we can find a ring map $A_i^{h} \to k'$ extending the given inclusion $A_i \subset k'$. Hence we conclude that $U'$ is isomorphic to the base change of $U$. The proof of fully faithfulness is exactly the same.

10. Local connectedness

0BQD In this section we ask when $\pi_1(U) \to \pi_1(X)$ is surjective for $U$ a dense open of a scheme $X$. We will see that this is the case (roughly) when $U \cap B$ is connected for any small “ball” $B$ around a point $x \in X \setminus U$.

0BQE Lemma 10.1. Let $f : X \to Y$ be a morphism of schemes. If $f(X)$ is dense in $Y$ then the base change functor $\text{FÉt}_Y \to \text{FÉt}_X$ is faithful.

Proof. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given $W$ finite étale over $Y$ and a morphism
Consider two sections \( t_1, t_2 : Y \to W \) such that \( s = t_1 \circ f = t_2 \circ f \). Since the equalizer of \( t_1 \) and \( t_2 \) is closed in \( Y \) (Schemes, Lemma 21.5) and since \( f(X) \) is dense in \( Y \) we see that \( t_1 \) and \( t_2 \) agree on \( Y_{\text{red}} \). Then it follows that \( t_1 \) and \( t_2 \) have the same image which is an open and closed subscheme of \( W \) mapping isomorphically to \( Y \) (Étale Morphisms, Proposition 6.1) hence they are equal. \( \square \)

The condition in the following lemma that the punctured spectrum of the strict henselization is connected follows for example from the assumption that the local ring is geometrically unibranch, see More on Algebra, Lemma 93.3. There is a partial converse in Properties, Lemma 15.3.

**Lemma 10.2.** Let \((A, m)\) be a local ring. Set \( X = \text{Spec}(A) \) and let \( U = X \setminus \{m\} \). If the punctured spectrum of the strict henselization of \( A \) is connected, then

\[
\text{FÉt}_X \to \text{FÉt}_U, \quad Y \mapsto Y \times_X U
\]

is a fully faithful functor.

**Proof.** Assume \( A \) is strictly henselian. In this case any finite étale cover \( Y \) of \( X \) is isomorphic to a finite disjoint union of copies of \( X \). Thus it suffices to prove that any morphism \( U \to U \coprod \ldots \coprod U \) over \( U \), extends uniquely to a morphism \( X \to X \coprod \ldots \coprod X \) over \( X \). If \( U \) is connected (in particular nonempty), then this is true.

The general case. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given \( Y \) finite étale over \( X \) any morphism \( s : U \to Y \) over \( X \) extends to a morphism \( t : X \to Y \) over \( Y \). Let \( A^{sh} \) be the strict henselization of \( A \) and denote \( X^{sh} = \text{Spec}(A^{sh}) \), \( U^{sh} = U \times_X X^{sh} \), \( Y^{sh} = Y \times_X X^{sh} \). By the first paragraph and our assumption on \( A \), we can extend the base change \( s^{sh} : U^{sh} \to Y^{sh} \) of \( s \) to \( t^{sh} : X^{sh} \to Y^{sh} \). Set \( A' = A^{sh} \otimes_A A^{sh} \).

Then the two pullbacks \( t'_1, t'_2 \) of \( t^{sh} \) to \( X' = \text{Spec}(A') \) are extensions of the pullback \( s' \) of \( s \) to \( U' = U \times_X X' \). As \( A \to A' \) is flat we see that \( U' \subset X' \) is (topologically) dense by going down for \( A \to A' \) (Algebra, Lemma 38.18). Thus \( t'_1 = t'_2 \) by Lemma 10.1.

Hence \( t^{sh} \) descends to a morphism \( t : X \to Y \) for example by Descent, Lemma 10.7. \( \square \)

In view of Lemma 10.2 it is interesting to know when the punctured spectrum of a ring (and of its strict henselization) is connected. The following famous lemma due to Hartshorne gives a sufficient condition.

**Lemma 10.3.** Let \( A \) be a Noetherian local ring of depth \( \geq 2 \). Then the punctured spectra of \( A, A^h, \) and \( A^{sh} \) are connected.

**Proof.** Let \( U \) be the punctured spectrum of \( A \). If \( U \) is disconnected then we see that \( \Gamma(U, \mathcal{O}_U) \) has a nontrivial idempotent. But \( A \), being local, does not have a nontrivial idempotent. Hence \( A \to \Gamma(U, \mathcal{O}_U) \) is not an isomorphism. By Local Cohomology, Lemma 7.2 we conclude that either \( H^0_m(A) \) or \( H^1_m(A) \) is nonzero. Thus depth(\( A \)) \( \leq 1 \) by Dualizing Complexes, Lemma 11.1. To see the result for \( A^h \) and \( A^{sh} \) use More on Algebra, Lemma 44.8. \( \square \)

**Lemma 10.4.** Let \( X \) be a scheme. Let \( U \subset X \) be a dense open. Assume

1. the underlying topological space of \( X \) is Noetherian, and
(2) for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\text{FÉt}_X \to \text{FÉt}_U$ is fully faithful.

**Proof.** Let $Y_1, Y_2$ be finite étale over $X$ and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over $U$. We have to show that $\varphi$ lifts uniquely to a morphism $Y_1 \to Y_2$ over $X$. Uniqueness follows from Lemma 10.1.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. By our choice of $x$ this is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 18.3 we find an open $U \subset U'$ containing $x$ and an extension $\varphi' : (Y_1)_U \to (Y_2)_U$ of $\varphi$. Since the underlying topological space of $X$ is Noetherian this finishes the proof by Noetherian induction on the complement of the open over which $\varphi$ is defined. □

**Lemma 10.5.** Let $X$ be a scheme. Let $U \subset X$ be a dense open. Assume

1. $U \to X$ is quasi-compact,
2. every point of $X \setminus U$ is closed, and
3. for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\text{FÉt}_X \to \text{FÉt}_U$ is fully faithful.

**Proof.** Let $Y_1, Y_2$ be finite étale over $X$ and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over $U$. We have to show that $\varphi$ lifts uniquely to a morphism $Y_1 \to Y_2$ over $X$. Uniqueness follows from Lemma 10.1.

Let $x \in X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. Since every point of $X \setminus U$ is closed $V$ is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. By Limits, Lemma 18.3 (this uses that $U$ is retrocompact in $X$) we find an open $U \subset U'$ containing $x$ and an extension $\varphi' : (Y_1)_U \to (Y_2)_U$ of $\varphi$. Note that given two points $x, x' \in X \setminus U$ the morphisms $\varphi'_x$ and $\varphi'_{x'}$ agree over $U_x \cap U_{x'}$, as $U$ is dense in that open (Lemma 10.1). Thus we can extend $\varphi$ to $U = X$ as desired. □

**Lemma 10.6.** Let $X$ be a scheme. Let $U \subset X$ be a dense open. Assume

1. every quasi-compact open of $X$ has finitely many irreducible components,
2. for every $x \in X \setminus U$ the punctured spectrum of the strict henselization of $\mathcal{O}_{X,x}$ is connected.

Then $\text{FÉt}_X \to \text{FÉt}_U$ is fully faithful.

**Proof.** Let $Y_1, Y_2$ be finite étale over $X$ and let $\varphi : (Y_1)_U \to (Y_2)_U$ be a morphism over $U$. We have to show that $\varphi$ lifts uniquely to a morphism $Y_1 \to Y_2$ over $X$. Uniqueness follows from Lemma 10.1. We will prove existence by showing that we can enlarge $U$ if $U \neq X$ and using Zorn’s lemma to finish the proof.

Let $x \in X \setminus U$ be a generic point of an irreducible component of $X \setminus U$. Set $V = U \times_X \text{Spec}(\mathcal{O}_{X,x})$. By our choice of $x$ this is the punctured spectrum of $\text{Spec}(\mathcal{O}_{X,x})$. By Lemma 10.2 we can extend the morphism $\varphi_V : (Y_1)_V \to (Y_2)_V$ uniquely to a morphism $(Y_1)_{\text{Spec}(\mathcal{O}_{X,x})} \to (Y_2)_{\text{Spec}(\mathcal{O}_{X,x})}$. Choose an affine neighbourhood $W \subset X$ of $x$. Since $U \cap W$ is dense in $W$ it contains the generic points $\eta_1, \ldots, \eta_n$ of $W$. Choose an affine open $W' \subset W \cap U$ containing $\eta_1, \ldots, \eta_n$. Set $V' = W' \times_X \text{Spec}(\mathcal{O}_{X,x})$. □
Spec(\(\mathcal{O}_{X,x}\)). By Limits, Lemma [18.3] applied to \(x \in W \supset W'\) we find an open \(W' \subset W'' \subset W\) with \(x \in W''\) and a morphism \(\varphi'' : (Y_1)_{W''} \to (Y_2)_{W''}\) agreeing with \(\varphi\) over \(W'\). Since \(W'\) is dense in \(W'' \cap U\), we see by Lemma [10.1] that \(\varphi\) and \(\varphi''\) agree over \(U \cap W'\). Thus \(\varphi\) and \(\varphi''\) glue to a morphism \(\varphi'\) over \(U' = U \cup W''\) agreeing with \(\varphi\) over \(U\). Observe that \(x \in U'\) so that we’ve extended \(\varphi\) to a strictly larger open.

Consider the set \(\mathcal{S}\) of pairs \((U', \varphi')\) where \(U \subset U'\) and \(\varphi'\) is an extension of \(\varphi\). We endow \(\mathcal{S}\) with a partial ordering in the obvious manner. If \((U_1', \varphi_1')\) is a totally ordered subset, then it has a maximum \((U', \varphi')\). Just take \(U' = \bigcup U_i'\) and let \(\varphi' : (Y_1)_{U'} \to (Y_2)_{U'}\) be the morphism agreeing with \(\varphi_i'\) over \(U_i'\). Thus Zorn’s lemma applies and \(\mathcal{S}\) has a maximal element. By the argument above we see that this maximal element is an extension of \(\varphi\) over all of \(X\).

Let \((A, \mathfrak{m})\) be a local ring. Set \(X = \text{Spec}(A)\) and \(U = X \setminus \{\mathfrak{m}\}\). Let \(U^{sh}\) be the punctured spectrum of the strict henselization \(A^{sh}\) of \(A\). Assume \(U\) is quasi-compact and \(U^{sh}\) is connected. Then the sequence

\[
\pi_1(U^{sh}, \mathfrak{m}) \to \pi_1(U, \mathfrak{m}) \to \pi_1(X, \mathfrak{m}) \to 1
\]

is exact in the sense of Lemma 4.3 part (1).

Proof. The map \(\pi_1(U) \to \pi_1(X)\) is surjective by Lemmas [10.2 and 4.1]

Write \(X^{sh} = \text{Spec}(A^{sh})\). Let \(Y \to X\) be a finite étale morphism. Then \(Y^{sh} = Y \times_X X^{sh} \to X^{sh}\) is a finite étale morphism. Since \(A^{sh}\) is strictly henselian we see that \(Y^{sh}\) is isomorphic to a disjoint union of copies of \(X^{sh}\). Thus the same is true for \(Y \times_X U^{sh}\). It follows that the composition \(\pi_1(U^{sh}) \to \pi_1(U) \to \pi_1(X)\) is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 4.3 to show the following: Given a finite étale morphism \(V \to U\) such that \(V \times_U U^{sh}\) is a disjoint union of copies of \(U^{sh}\), we can find a finite étale morphism \(Y \to X\) with \(V \cong Y \times_X U\) over \(U\). The assumption implies that there exists a finite étale morphism \(Y^{sh} \to X^{sh}\) and an isomorphism \(V \times_U U^{sh} \cong Y^{sh} \times_X U^{sh}\). Consider the following diagram

\[
\begin{array}{c}
U \leftarrow U^{sh} \downarrow \leftarrow U^{sh} \times_U U^{sh} \downarrow \leftarrow U^{sh} \times_U U^{sh} \downarrow \leftarrow U^{sh} \\
X \leftarrow X^{sh} \downarrow \leftarrow X^{sh} \times_X X^{sh} \downarrow \leftarrow X^{sh} \times_X X^{sh} \downarrow \leftarrow X^{sh}
\end{array}
\]

Since \(U \subset X\) is quasi-compact by assumption, all the downward arrows are quasi-compact open immersions. Let \(\xi \in X^{sh} \times_X X^{sh}\) be a point not in \(U^{sh} \times_X U^{sh}\). Then \(\xi\) lies over the closed point \(x^{sh}\) of \(X^{sh}\). Consider the local ring homomorphism

\[
A^{sh} = \mathcal{O}_{X^{sh}, x^{sh}} \to \mathcal{O}_{X^{sh} \times_X X^{sh}, \xi}
\]

determined by the first projection \(X^{sh} \times_X X^{sh}\). This is a filtered colimit of local homomorphisms which are localizations étale ring maps. Since \(A^{sh}\) is strictly henselian, we conclude that it is an isomorphism. Since this holds for every \(\xi\) in the complement it follows there are no specializations among these points and hence every such \(\xi\) is a closed point (you can also prove this directly). As the local ring at \(\xi\) is isomorphic to \(A^{sh}\), it is strictly henselian and has connected punctured spectrum. Similarly for points \(\xi\) of \(X^{sh} \times_X X^{sh} \times_X X^{sh}\) not in \(U^{sh} \times_U U^{sh} \times_U U^{sh}\). It follows
from Lemma \[10.5\] that pullback along the vertical arrows induce fully faithful functors on the categories of finite étale schemes. Thus the canonical descent datum on \(V \times_U U^{sh}\) relative to the fpqc covering \(\{U^{sh} \to U\}\) translates into a descent datum for \(Y^{sh}\) relative to the fpqc covering \(\{X^{sh} \to X\}\). Since \(Y^{sh} \to X^{sh}\) is finite hence affine, this descent datum is effective (Descent, Lemma \[34.1\]). Thus we get an affine morphism \(Y \to X\) and an isomorphism \(Y \times_X X^{sh} \to Y^{sh}\) compatible with descent data. By fully faithfulness of descent data (as in Descent, Lemma \[32.11\]) we get an isomorphism \(V \to U \times_X Y\). Finally, \(Y \to X\) is finite étale as \(Y^{sh} \to X^{sh}\) is, see Descent, Lemmas \[20.29\] and \[20.23\].

Let \(X\) be an irreducible scheme. Let \(\eta \in X\) be the generic point. The canonical morphism \(\eta \to X\) induces a canonical map

\[\text{Gal}(\kappa(\eta)^{sep}/\kappa(\eta)) = \pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})\]

The identification on the left hand side is Lemma \[0.3\].

**Lemma 10.8.** Let \(X\) be an irreducible, geometrically unibranch scheme. For any nonempty open \(U \subset X\) the canonical map

\[\pi_1(U, \overline{\pi}) \to \pi_1(X, \overline{\pi})\]

is surjective. The map \[(10.7.1)\ \pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})\] is surjective as well.

**Proof.** By Lemma \[8.3\] we may replace \(X\) by its reduction. Thus we may assume that \(X\) is an integral scheme. By Lemma \[4.1\] the assertion of the lemma translates into the statement that the functors \(\text{FÉt}_X \to \text{FÉt}_U\) and \(\text{FÉt}_X \to \text{FÉt}_\eta\) are fully faithful.

The result for \(\text{FÉt}_X \to \text{FÉt}_U\) follows from Lemma \[10.6\] and the fact that for a local ring \(A\) which is geometrically unibranch its strict henselization has an irreducible spectrum. See More on Algebra, Lemma \[93.5\].

Observe that the residue field \(\kappa(\eta) = \mathcal{O}_{X, \eta}\) is the filtered colimit of \(\mathcal{O}_X(U)\) over \(U \subset X\) nonempty open affine. Hence \(\text{FÉt}_\eta\) is the colimit of the categories \(\text{FÉt}_U\) over such \(U\), see Limits, Lemmas \[10.1\] \[8.3\] and \[8.10\]. A formal argument then shows that fully faithfulness for \(\text{FÉt}_X \to \text{FÉt}_\eta\) follows from the fully faithfulness of the functors \(\text{FÉt}_X \to \text{FÉt}_U\). \(\square\)

**Lemma 10.9.** Let \(X\) be a scheme. Let \(x_1, \ldots, x_n \in X\) be a finite number of closed points such that

\begin{itemize}
  \item[(1)] \(U = X \setminus \{x_1, \ldots, x_n\}\) is connected and is a retrocompact open of \(X\), and
  \item[(2)] for each \(i\) the punctured spectrum \(U_i^{sh}\) of the strict henselization of \(\mathcal{O}_{X, x_i}\) is connected.
\end{itemize}

Then the map \(\pi_1(U) \to \pi_1(X)\) is surjective and the kernel is the smallest closed normal subgroup of \(\pi_1(U)\) containing the image of \(\pi_1(U_i^{sh}) \to \pi_1(U)\) for \(i = 1, \ldots, n\).

**Proof.** Surjectivity follows from Lemmas \[10.5\] and \[4.1\]. We can consider the sequence of maps

\[\pi_1(U) \to \ldots \to \pi_1(X \setminus \{x_1, x_2\}) \to \pi_1(X \setminus \{x_1\}) \to \pi_1(X)\]

A group theory argument then shows it suffices to prove the statement on the kernel in the case \(n = 1\) (details omitted). Write \(x = x_1\), \(U^{sh} = U_i^{sh}\), set \(A = \mathcal{O}_{X, x}\), and
let $A^{sh}$ be the strict henselization. Consider the diagram

$$
\begin{array}{c}
U \leftarrow \text{Spec}(A) \setminus \{m\} \leftarrow U^{sh} \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
X \leftarrow \text{Spec}(A) \leftarrow \text{Spec}(A^{sh})
\end{array}
$$

By Lemma 10.7 we have to show finite étale morphisms $V \to U$ which pull back to trivial coverings of $U^{sh}$ extend to finite étale schemes over $X$. By Lemma 10.7 we know the corresponding statement for finite étale schemes over the punctured spectrum of $A$. However, by Limits, Lemma 18.1 schemes of finite presentation over $X$ are the same thing as schemes of finite presentation over $U$ and $A$ glued over the punctured spectrum of $A$. This finishes the proof. □

11. Fundamental groups of normal schemes

Let $X$ be an integral, geometrically unibranch scheme. In the previous section we have seen that the fundamental group of $X$ is a quotient of the Galois group of the function field $K$ of $X$. Since the map is continuous the kernel is a normal closed subgroup of the Galois group. Hence this kernel corresponds to a Galois extension $M/K$ by Galois theory (Fields, Theorem 22.4). In this section we will determine $M$ when $X$ is a normal integral scheme.

Let $X$ be an integral normal scheme with function field $K$. Let $K \subset L$ be a finite extension. Consider the normalization $Y \to X$ of $X$ in the morphism $\text{Spec}(L) \to X$ as defined in Morphisms, Section 51. We will say (in this setting) that $X$ is unramified in $L$ if $Y \to X$ is an unramified morphism of schemes. In Lemma 13.4 we will elucidate this condition. Observe that the scheme theoretic fibre of $Y \to X$ over $\text{Spec}(K)$ is $\text{Spec}(L)$. Hence the field extension $L/K$ is separable if $X$ is unramified in $L$, see Morphisms, Lemmas 33.11.

**Lemma 11.1.** In the situation above the following are equivalent

1. $X$ is unramified in $L$,
2. $Y \to X$ is étale, and
3. $Y \to X$ is finite étale.

**Proof.** Observe that $Y \to X$ is an integral morphism. In each case the morphism $Y \to X$ is locally of finite type by definition. Hence we find that in each case the lemma is finite by Morphisms, Lemma 42.4. In particular we see that (2) is equivalent to (3). An étale morphism is unramified, hence (2) implies (1).

Conversely, assume $Y \to X$ is unramified. Let $x \in X$. We can choose an étale neighbourhood $(U, u) \to (X, x)$ such that $Y \times_X U = \bigsqcup_{j} V_j \to U$ is a disjoint union of closed immersions, see Étale Morphisms, Lemma 17.3. Shrinking we may assume $U$ is quasi-compact. Then $U$ has finitely many irreducible components (Descent, Lemma 13.3). Since $U$ is normal (Descent, Lemma 15.2) the irreducible components of $U$ are open and closed (Properties, Lemma 7.5) and we may assume $U$ is irreducible. Then $U$ is an integral scheme whose generic point $\xi$ maps to the generic point of $X$. On the other hand, we know that $Y \times_X U$ is the normalization of $U$ in $\text{Spec}(L) \times_X U$ by More on Morphisms, Lemma 17.2. Every
point of $\text{Spec}(L) \times_X U$ maps to $\xi$. Thus every $V_j$ contains a point mapping to $\xi$ by Morphisms, Lemma 51.9. Thus $V_j \to U$ is an isomorphism as $U = \{\xi\}$. Thus $Y \times_X U \to U$ is étale. By Descent, Lemma 20.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of $x$).

0BQ Lemma 11.2. Let $X$ be a normal integral scheme with function field $K$. Let $Y \to X$ be a finite étale morphism. If $Y$ is connected, then $Y$ is an integral normal scheme and $Y$ is the normalization of $X$ in the function field of $Y$.

Proof. The scheme $Y$ is normal by Descent, Lemma 15.2. Since $Y \to X$ is flat every generic point of $Y$ maps to the generic point of $X$ by Morphisms, Lemma 24.8. Since $Y \to X$ is finite we see that $Y$ has a finite number of irreducible components. Thus $Y$ is the disjoint union of a finite number of integral normal schemes by Properties, Lemma 7.9. Thus if $Y$ is connected, then $Y$ is an integral normal scheme.

Let $L$ be the function field of $Y$ and let $Y' \to X$ be the normalization of $X$ in $L$. By Morphisms, Lemma 51.4 we obtain a factorization $Y' \to Y \to X$ and $Y' \to Y$ is the normalization of $Y$ in $L$. Since $Y'$ is normal it is clear that $Y' = Y$ (this can also be deduced from Morphisms, Lemma 52.8).

0BQM Proposition 11.3. Let $X$ be a normal integral scheme with function field $K$. Then the canonical map (10.7.1)

$$\text{Gal}(K^{sep}/K) = \pi_1(\eta, \overline{\eta}) \longrightarrow \pi_1(X, \eta)$$

is identified with the quotient map $\text{Gal}(K^{sep}/K) \to \text{Gal}(M/K)$ where $M \subset K^{sep}$ is the union of the finite subextensions $L$ such that $X$ is unramified in $L$.

Proof. The normal scheme $X$ is geometrically unibranch (Properties, Lemma 15.2). Hence Lemma 10.8 applies to $X$. Thus $\pi_1(\eta, \overline{\eta}) \to \pi_1(X, \eta)$ is surjective and top horizontal arrow of the commutative diagram

$$
\begin{array}{ccc}
F\text{Ét}_X & \longrightarrow & F\text{Ét}_Y \\
\downarrow & & \downarrow \\
\text{Finite-}\pi_1(X, \eta)\text{-sets} & \longrightarrow & \text{Finite-}\text{Gal}(K^{sep}/K)\text{-sets}
\end{array}
$$

is fully faithful. The left vertical arrow is the equivalence of Theorem 6.2 and the right vertical arrow is the equivalence of Lemma 6.3. The lower horizontal arrow is induced by the map of the proposition. By Lemmas 11.1 and 11.2 we see that the essential image of $c$ consists of $\text{Gal}(K^{sep}/K)\text{-Sets}$ isomorphic to sets of the form

$$S = \text{Hom}_K(\prod_{i=1,\ldots,n} L_i, K^{sep}) = \prod_{i=1,\ldots,n} \text{Hom}_K(L_i, K^{sep})$$

with $L_i/K$ finite separable such that $X$ is unramified in $L_i$. Thus if $M \subset K^{sep}$ is as in the statement of the lemma, then $\text{Gal}(K^{sep}/M)$ is exactly the subgroup of $\text{Gal}(K^{sep}/K)$ acting trivially on every object in the essential image of $c$. On the other hand, the essential image of $c$ is exactly the category of $S$ such that the $\text{Gal}(K^{sep}/K)$-action factors through the surjection $\text{Gal}(K^{sep}/K) \to \pi_1(X, \eta)$. We conclude that $\text{Gal}(K^{sep}/M)$ is the kernel. Hence $\text{Gal}(K^{sep}/M)$ is a normal subgroup, $M/K$ is Galois, and we have a short exact sequence

$$1 \to \text{Gal}(K^{sep}/M) \to \text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) \to 1$$
Let \((A, \mathfrak{m})\) be a normal local ring. Set \(X = \text{Spec}(A)\). Let \(A^{sh}\) be the strict henselization of \(A\). Let \(K\) and \(K^{sh}\) be the fraction fields of \(A\) and \(A^{sh}\). Then the sequence

\[
\pi_1(\text{Spec}(K^{sh})) \to \pi_1(\text{Spec}(K)) \to \pi_1(X) \to 1
\]

is exact in the sense of Lemma 4.3 part (1).

**Proof.** Note that \(A^{sh}\) is a normal domain, see More on Algebra, Lemma 14.6. The map \(\pi_1(\text{Spec}(K)) \to \pi_1(X)\) is surjective by Proposition 11.3.

Write \(X^{sh} = \text{Spec}(A^{sh})\). Let \(Y \to X\) be a finite étale morphism. Then \(Y^{sh} = Y \times_X X^{sh} \to X^{sh}\) is a finite étale morphism. Since \(A^{sh}\) is strictly henselian we see that \(Y^{sh}\) is isomorphic to a disjoint union of copies of \(X^{sh}\). Thus the same is true for \(Y \times_X \text{Spec}(K^{sh})\). It follows that the composition \(\pi_1(\text{Spec}(K^{sh})) \to \pi_1(X)\) is trivial, see Lemma 4.2.

To finish the proof, it suffices according to Lemma 14.3 to show the following: Given a finite étale morphism \(V \to \text{Spec}(K)\) such that \(V \times_{\text{Spec}(K)} \text{Spec}(K^{sh})\) is a disjoint union of copies of \(\text{Spec}(K^{sh})\), we can find a finite étale morphism \(Y \to X\) with \(V \cong Y \times_X \text{Spec}(K)\) over \(\text{Spec}(K)\). Write \(V = \text{Spec}(L)\), so \(L\) is a finite product of finite separable extensions of \(K\). Let \(B \subset L\) be the integral closure of \(A\) in \(L\). If \(A \to B\) is étale, then we can take \(Y = \text{Spec}(B)\) and the proof is complete. By Algebra, Lemma 143.4 (and a limit argument we omit) we see that \(B \otimes_A A^{sh}\) is the integral closure of \(A^{sh}\) in \(L^{sh} = L \otimes_K K^{sh}\). Our assumption is that \(L^{sh}\) is a product of copies of \(K^{sh}\) and hence \(B^{sh}\) is a product of copies of \(A^{sh}\). Thus \(A^{sh} \to B^{sh}\) is étale. As \(A \to A^{sh}\) is faithfully flat it follows that \(A \to B\) is étale (Descent, Lemma 20.29) as desired. \(\square\)

**12. Group actions and integral closure**

In this section we continue the discussion of More on Algebra, Section 0BSN. Recall that a normal local ring is a domain by definition.

**Lemma 12.1.** Let \(A\) be a normal domain whose fraction field \(K\) is separably algebraically closed. Let \(\mathfrak{p} \subset A\) be a nonzero prime ideal. Then the residue field \(\kappa(\mathfrak{p})\) is algebraically closed.

**Proof.** Assume the lemma is not true to get a contradiction. Then there exists a monic irreducible polynomial \(P(T) \in \kappa(\mathfrak{p})[T]\) of degree \(d > 1\). After replacing \(P\) by \(a^d P(a^{-1}T)\) for suitable \(a \in A\) (to clear denominators) we may assume that \(P\) is the image of a monic polynomial \(Q\) in \(A[T]\). Observe that \(Q\) is irreducible in \(K[T]\). Namely a factorization over \(K\) leads to a factorization over \(A\) by Algebra, Lemma 03.5 which we could reduce modulo \(\mathfrak{p}\) to get a factorization of \(P\). As \(K\) is separably closed, \(Q\) is not a separable polynomial (Fields, Definition 12.2). Then the characteristic of \(K\) is \(p > 0\) and \(Q\) has vanishing linear term (Fields, Definition 12.2). However, then we can replace \(Q\) by \(Q + aT\) where \(a \in \mathfrak{p}\) is nonzero to get a contradiction. \(\square\)

**Lemma 12.2.** A normal local ring with separably closed fraction field is strictly henselian.
Proof. Let $(A, m, \kappa)$ be normal local with separably closed fraction field $K$. If $A = K$, then we are done. If not, then the residue field $\kappa$ is algebraically closed by Lemma [12.4] and it suffices to check that $A$ is henselian. Let $f \in A[T]$ be monic and let $a_0 \in \kappa$ be a root of multiplicity 1 of the reduction $\overline{f} \in \kappa[T]$. Let $f = \prod f_i$ be the factorization in $K[T]$. By Algebra, Lemma [37.5] we have $f_i \in A[T]$. Thus $a_0$ is a root of $f_i$ for some $i$. After replacing $f$ by $f_i$ we may assume $f$ is irreducible. Then, since the derivative $f'$ cannot be zero in $A[T]$ as $a_0$ is a single root, we conclude that $f$ is linear due to the fact that $K$ is separably algebraically closed. Thus $A$ is henselian, see Algebra, Definition [148.1].

**Lemma 12.3.** Let $G$ be a finite group acting on a ring $R$. Let $R^G \to A$ be a ring map. Let $q' \subset A \otimes_{R^G} R$ be a prime lying over the prime $q \subset R$. Then

$$I_q = \{ \sigma \in G \mid \sigma(q) = q \text{ and } \sigma \bmod q = \text{id}_{\kappa(q)} \}$$

is equal to

$$I_q' = \{ \sigma \in G \mid \sigma(q') = q' \text{ and } \sigma \bmod q' = \text{id}_{\kappa(q')} \}$$

**Proof.** Since $q$ is the inverse image of $q'$ and since $\kappa(q) \subset \kappa(q')$, we get $I_q' \subset I_q$. Conversely, if $\sigma \in I_q$, the $\sigma$ acts trivially on the fibre ring $A \otimes_{R^G} \kappa(q)$. Thus $\sigma$ fixes all the primes lying over $q$ and induces the identity on their residue fields.

**Lemma 12.4.** Let $G$ be a finite group acting on a ring $R$. Let $q \subset R$ be a prime.

Set

$$I = \{ \sigma \in G \mid \sigma(q) = q \text{ and } \sigma \bmod q = \text{id}_q \}$$

Then $R^G \to R^I$ is étale at $R^I \cap q$.

**Proof.** The strategy of the proof is to use étale localization to reduce to the case where $R \to R^I$ is a local isomorphism at $R^I \cap p$. Let $R^G \to A$ be an étale ring map. We claim that if the result holds for the action of $G$ on $A \otimes_{R^G} R$ and some prime $q'$ of $A \otimes_{R^G} R$ lying over $q$, then the result is true.

To check this, note that since $R^G \to A$ is flat we have $A = (A \otimes_{R^G} R)^G$, see More on Algebra, Lemma [96.4]. By Lemma [12.3] the group $I$ does not change. Then a second application of More on Algebra, Lemma [96.4] shows that $A \otimes_{R^G} R^I = (A \otimes_{R^G} R)^I$ (because $R^I \to A \otimes_{R^G} R^I$ is flat). Thus

$$\Spec((A \otimes_{R^G} R)^I) \longrightarrow \Spec(R^I)$$

is cartesian and the horizontal arrows are étale. Thus if the left vertical arrow is étale in some open neighbourhood $W$ of $(A \otimes_{R^G} R)^I \cap q'$, then the right vertical arrow is étale at the points of the (open) image of $W$ in $\Spec(R^I)$, see Descent, Lemma [11.5]. In particular the morphism $\Spec(R^I) \to \Spec(R^G)$ is étale at $R^I \cap q$. Let $p = R^G \cap q$. By More on Algebra, Lemma [96.5] the fibre of $\Spec(R) \to \Spec(R^G)$ over $p$ is finite. Moreover the residue field extensions at these points are algebraic, normal, with finite automorphism groups by More on Algebra, Lemma [96.6]. Thus we may apply More on Morphisms, Lemma [37.1] to the integral ring map $R^G \to R$ and the prime $p$. Combined with the claim above we reduce to the case where $R = A_1 \times \ldots \times A_n$ with each $A_i$ having a single prime $q_i$ lying over $p$ such that

$$\Spec(A_1) \longrightarrow \Spec(A_i)$$
the residue field extensions \(\kappa(q_i)/\kappa(p)\) are purely inseparable. Of course \(q\) is one of these primes, say \(q = q_1\).

It may not be the case that \(G\) permutes the factors \(A_i\) (this would be true if the spectrum of \(A_i\) were connected, for example if \(R^G\) was local). This we can fix as follows; we suggest the reader think this through for themselves, perhaps using idempotents instead of topology. Recall that the product decomposition gives a corresponding disjoint union decomposition of \(\text{Spec}(R)\) by open and closed subsets \(U_i\). Since \(G\) is finite, we can refine this covering by a finite disjoint union decomposition \(\text{Spec}(R) = \coprod_{j \in J} W_j\) by open and closed subsets \(W_j\), such that for all \(j \in J\) there exists a \(j' \in J\) with \(\sigma(W_j) = W_{j'}\). The union of the \(W_j\) not meeting \(\{q_1, \ldots, q_n\}\) is a closed subset not meeting the fibre over \(p\) hence maps to a closed subset of \(\text{Spec}(R^G)\) not meeting \(p\) as \(\text{Spec}(R) \to \text{Spec}(R^G)\) is closed. Hence after replacing \(R^G\) by a principal localization (permissible by the claim) we may assume each \(W_j\) meets one of the points \(q_i\). Then we set \(U_i = W_j\) if \(q_i \in W_j\). The corresponding product decomposition \(R = A_1 \times \ldots \times A_n\) is one where \(G\) permutes the factors \(A_i\).

Thus we may assume we have a product decomposition \(R = A_1 \times \ldots \times A_n\) compatible with \(G\)-action, where each \(A_i\) has a single prime \(q_i\) lying over \(p\) and the field extensions \(\kappa(q_i)/\kappa(p)\) are purely inseparable. Write \(A' = A_2 \times \ldots \times A_n\) so that

\[ R = A_1 \times A' \]

Since \(q = q_1\) we find that every \(\sigma \in I\) preserves the product decomposition above. Hence

\[ R^I = (A_1)^I \times (A')^I \]

Observe that \(I = D = \{\sigma \in G \mid \sigma(q) = q\}\) because \(\kappa(q)/\kappa(p)\) is purely inseparable. Since the action of \(G\) on primes over \(p\) is transitive (More on Algebra, Lemma \[96.5\]) we conclude that, the index of \(I\) in \(G\) is \(n\) and we can write \(G = eI \Pi \sigma_2 I \Pi \ldots \Pi \sigma_n I\) so that \(A_i = \sigma_i(A_1)\) for \(i = 2, \ldots, n\). It follows that

\[ R^G = (A_1)^I. \]

Thus the map \(R^G \to R^I\) is étale at \(R^I \cap q\) and the proof is complete. \(\square\)

The following lemma generalizes More on Algebra, Lemma \[98.8\]

\textbf{Lemma 12.5.} Let \(A\) be a normal domain with fraction field \(K\). Let \(L/K\) be a (possibly infinite) Galois extension. Let \(G = \text{Gal}(L/K)\) and let \(B\) be the integral closure of \(A\) in \(L\). Let \(q \subset B\). Set

\[ I = \{\sigma \in G \mid \sigma(q) = q \text{ and } \sigma \mod q = id_{\kappa(q)}\} \]

Then \((B^I)_{B \cap q}\) is a filtered colimit of étale \(A\)-algebras.

\textbf{Proof.} We can write \(L\) as the filtered colimit of finite Galois extensions of \(K\). Hence it suffices to prove this lemma in case \(L/K\) is a finite Galois extension, see Algebra, Lemma \[149.3\] Since \(A = B^G\) as \(A\) is integrally closed in \(K = L^G\) the result follows from Lemma \[12.4\] \(\square\)
13. Ramification theory

In this section we continue the discussion of More on Algebra, Section 98 and we relate it to our discussion of the fundamental groups of schemes.

Let \((A, \mathfrak{m}, \kappa)\) be a normal local ring with fraction field \(K\). Choose a separable algebraic closure \(K^{sep}\). Let \(A^{sep}\) be the integral closure of \(A\) in \(K^{sep}\). Choose maximal ideal \(m^{sep} \subset A^{sep}\). Let \(A \subset A^h \subset A^{sh}\) be the henselization and strict henselization. Observe that \(A^h\) and \(A^{sh}\) are normal rings as well (More on Algebra, Lemma 44.6). Denote \(K^h\) and \(K^{sh}\) their fraction fields. Since \((A^{sep})_{m^{sep}}\) is strictly henselian by Lemma 12.2 we can choose an \(A\)-algebra map \(A^{sh} \to (A^{sep})_{m^{sep}}\).

Namely, first choose a \(\kappa\)-embedding \(\kappa(m^{sh}) \to \kappa(m^{sep})\) and then extend (uniquely) to an \(A\)-algebra homomorphism by Algebra, Lemma 150.12. We get the following diagram

\[
\begin{array}{cccc}
K^{sep} & & K^h & \longrightarrow & K \\
\uparrow & & \uparrow & & \uparrow \\
(A^{sep})_{m^{sep}} & \leftarrow & A^{sh} & \leftarrow & A
\end{array}
\]

We can take the fundamental groups of the spectra of these rings. Of course, since \(K^{sep}\), \((A^{sep})_{m^{sep}}\), and \(A^{sh}\) are strictly henselian, for them we obtain trivial groups. Thus the interesting part is the following

\[
\begin{array}{c}
\pi_1(U^{sh}) \longrightarrow \pi_1(U^h) \longrightarrow \pi_1(U) \\
\downarrow \quad \downarrow \quad \downarrow \pi_1(X^{sh}) \longrightarrow \pi_1(X)
\end{array}
\]

Here \(X^h\) and \(X\) are the spectra of \(A^h\) and \(A\) and \(U^{sh}, U^h, U\) are the spectra of \(K^{sh}, K^h,\) and \(K\). The label 1 means that the map is trivial; this follows as it factors through the trivial group \(\pi_1(X^{sh})\). On the other hand, the profinite group \(G = \text{Gal}(K^{sep}/K)\) acts on \(A^{sep}\) and we can make the following definitions

\[D = \{ \sigma \in G \mid \sigma(m^{sep}) = m^{sep} \} \supset I = \{ \sigma \in D \mid \sigma \mod m^{sep} = \text{id}_{\kappa(m^{sep})} \}\]

These groups are sometimes called the decomposition group and the inertia group especially when \(A\) is a discrete valuation ring.

**Lemma 13.1.** In the situation described above, via the isomorphism \(\pi_1(U) = \text{Gal}(K^{sep}/K)\) the diagram (13.0.1) translates into the diagram

\[
\begin{array}{c}
I \longrightarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \text{Gal}(K^{sep}/K) \\
\text{Gal}(\kappa(m^{sh})/\kappa) \quad \longrightarrow \quad \downarrow \\
\text{Gal}(M/K)
\end{array}
\]

where \(K^{sep}/M/K\) is the maximal subextension unramified with respect to \(A\). Moreover, the vertical arrows are surjective, the kernel of the left vertical arrow is \(I\) and the kernel of the right vertical arrow is the smallest closed normal subgroup of \(\text{Gal}(K^{sep}/K)\) containing \(I\).

---

\(^2\)This is possible because \(\kappa(m^{sh})\) is a separable algebraic closure of \(\kappa\) and \(\kappa(m^{sep})\) is an algebraic closure of \(\kappa\) by Lemma 12.1.
Proof. By construction the group $D$ acts on $(A^{sep})_{m^{sep}}$ over $A$. By the uniqueness of $A^{sh} \to (A^{sep})_{m^{sep}}$ given the map on residue fields (Algebra, Lemma 150.12) we see that the image of $A^{sh} \to (A^{sep})_{m^{sep}}$ is contained in $((A^{sep})_{m^{sep}})^{l}$. On the other hand, Lemma 12.5 shows that $((A^{sep})_{m^{sep}})^{l}$ is a filtered colimit of étale extensions of $A$. Since $A^{sh}$ is the maximal such extension, we conclude that $A^{sh} = (((A^{sep})_{m^{sep}})^{l})^{l}$. Hence $K^{sh} = (K^{sep})^{l}$.

Recall that $I$ is the kernel of a surjective map $D \to \text{Aut}(\kappa(m^{sep})/\kappa)$, see More on Algebra, Lemma 96.7. We have $\text{Aut}(\kappa(m^{sep})/\kappa) = \text{Gal}(\kappa(m^{sh})/\kappa)$ as we have seen above that these fields are the algebraic and separable algebraic closures of $\kappa$. On the other hand, any automorphism of $A^{sh}$ over $A$ is an automorphism of $A^{sh}$ over $A^{h}$ by the uniqueness in Algebra, Lemma 150.6. Furthermore, $A^{sh}$ is the colimit of finite étale extensions $A^{h} \subset A'$ which correspond 1-to-1 with finite separable extension $\kappa'/\kappa$, see Algebra, Remark 150.4. Thus 

$$\text{Aut}(A^{sh}/A) = \text{Aut}(A^{sh}/A^{h}) = \text{Gal}(\kappa(m^{sh})/\kappa)$$

Let $\kappa \subset \kappa'$ be a finite Galois extension with Galois group $G$. Let $A^{h} \subset A'$ be the finite étale extension corresponding to $\kappa \subset \kappa'$ by Algebra, Lemma 148.7. Then it follows that $(A')^{G} = A^{h}$ by looking at fraction fields and degrees (small detail omitted). Taking the colimit we conclude that $(A^{sh})^{\text{Gal}(\kappa(m^{sh})/\kappa)} = A^{h}$. Combining all of the above, we find $A^{h} = ((A^{sep})_{m^{sep}})^{D}$. Hence $K^{h} = (K^{sep})^{D}$.

Since $U, U^{h}, U^{sh}$ are the spectra of the fields $K, K^{h}, K^{sh}$ we see that the top lines of the diagrams correspond via Lemma 6.3. By Lemma 8.2 we have $\pi_{1}(X^{h}) = \text{Gal}(\kappa(m^{sh})/\kappa)$. The exactness of the sequence $1 \to I \to D \to \text{Gal}(\kappa(m^{sh})/\kappa) \to 1$ was pointed out above. By Proposition 11.3 we see that $\pi_{1}(X) = \text{Gal}(M/K)$. Finally, the statement on the kernel of $\text{Gal}(K^{sep}/K) \to \text{Gal}(M/K) = \pi_{1}(X)$ follows from Lemma 11.4. This finishes the proof.

Let $X$ be a normal integral scheme with function field $K$. Let $K^{sep}$ be a separable algebraic closure of $K$. Let $X^{sep} \to X$ be the normalization of $X$ in $K^{sep}$. Since $G = \text{Gal}(K^{sep}/K)$ acts on $K^{sep}$ we obtain a right action of $G$ on $X^{sep}$. For $y \in X^{sep}$ define

$$D_{y} = \{ \sigma \in G \mid \sigma(y) = y \} \supset I_{y} = \{ \sigma \in D \mid \sigma \text{ mod } m_{y} = \text{id}_{\kappa(y)} \}$$

similarly to the above. On the other hand, for $x \in X$ let $O_{X,x}^{sh}$ be a strict henselization, let $K^{sh}_{x}$ be the fraction field of $O_{X,x}^{sh}$ and choose a $K$-embedding $K^{sh}_{x} \to K^{sep}$.

0BTD Lemma 13.2. Let $X$ be a normal integral scheme with function field $K$. With notation as above, the following three subgroups of $\text{Gal}(K^{sep}/K) = \pi_{1}(\text{Spec}(K))$ are equal

(1) the kernel of the surjection $\text{Gal}(K^{sep}/K) \longrightarrow \pi_{1}(X)$,
(2) the smallest normal closed subgroup containing $I_{y}$ for all $y \in X^{sep}$, and
(3) the smallest normal closed subgroup containing $\text{Gal}(K^{sep}/K^{sh}_{x})$ for all $x \in X$.

Proof. The equivalence of (2) and (3) follows from Lemma 13.1 which tells us that $I_{y}$ is conjugate to $\text{Gal}(K^{sep}/K^{sh}_{x})$ if $y$ lies over $x$. By Lemma 11.4 we see that $\text{Gal}(K^{sep}/K^{sh}_{x})$ maps trivially to $\pi_{1}(\text{Spec}(O_{X,x}))$ and therefore the subgroup $N \subset G = \text{Gal}(K^{sep}/K)$ of (2) and (3) is contained in the kernel of $G \longrightarrow \pi_{1}(X)$.
To prove the other inclusion, since $N$ is normal, it suffices to prove: given $N \subset U \subset G$ with $U$ open normal, the quotient map $G \to G/U$ factors through $\pi_1(X)$.

In other words, if $L/K$ is the Galois extension corresponding to $U$, then we have to show that $X$ is unramified in $L$ (Section [1], especially Proposition [1.3]). It suffices to do this when $X$ is affine (we do this so we can refer to algebra results in the rest of the proof). Let $Y \to X$ be the normalization of $X$ in $L$. The inclusion $L \subset K^{sep}$ induces a morphism $\pi : X^{sep} \to Y$. For $y \in X^{sep}$ the inertia group of $\pi(y)$ in $\text{Gal}(L/K)$ is the image of $I_y$ in $\text{Gal}(L/K)$; this follows from More on Algebra, Lemma [96.8]. Since $N \subset U$ all these inertia groups are trivial. We conclude that $Y \to X$ is étale by applying Lemma [12.4]. (Alternative: you can use Lemma [11.4] to see that the pullback of $Y$ to $\text{Spec}(\mathcal{O}_{X,x})$ is étale for all $x \in X$ and then conclude from there with a bit more work.)

0BTE Example 13.3. Let $X$ be a normal integral Noetherian scheme with function field $K$. Purity of branch locus (see below) tells us that if $X$ is regular, then it suffices in Lemma [13.2] to consider the inertia groups $I = \pi_1(\text{Spec}(K_x^{sh}))$ for points $x$ of codimension $1$ in $X$. In general this is not enough however. Namely, let $Y = \mathbb{A}^n_k = \text{Spec}(k[t_1, \ldots, t_n])$ where $k$ is a field not of characteristic $2$. Let $G = \{\pm 1\}$ be the group of order $2$ acting on $Y$ by multiplication on the coordinates. Set

$$X = \text{Spec}(k[t_i t_j, i, j \in \{1, \ldots, n\}])$$

The embedding $k[t_i t_j] \subset k[t_1, \ldots, t_n]$ defines a degree $2$ morphism $Y \to X$ which is unramified everywhere except over the maximal ideal $m = (t_i t_j)$ which is a point of codimension $n$ in $X$.

0BTF Lemma 13.4. Let $X$ be an integral normal scheme with function field $K$. Let $L/K$ be a finite extension. Let $Y \to X$ be the normalization of $X$ in $L$. The following are equivalent

1. $X$ is unramified in $L$ as defined in Section [11],
2. $Y \to X$ is an unramified morphism of schemes,
3. $Y \to X$ is an étale morphism of schemes,
4. $Y \to X$ is a finite étale morphism of schemes,
5. for $x \in X$ the projection $Y \times_X \text{Spec}(\mathcal{O}_{X,x}) \to \text{Spec}(\mathcal{O}_{X,x})$ is unramified,
6. same as in (5) but with $\mathcal{O}^h_{X,x}$,
7. same as in (5) but with $\mathcal{O}^\text{sh}_{X,x}$,
8. for $x \in X$ the scheme theoretic fibre $Y_x$ is étale over $x$ of degree $\geq [L : K]$.

If $L/K$ is Galois with Galois group $G$, then these are also equivalent to

9. for $y \in Y$ the group $I_y = \{g \in G \mid g(y) = y$ and $g \text{ mod } m_y = id_{\kappa(y)}\}$ is trivial.

Proof. The equivalence of (1) and (2) is the definition of (1). The equivalence of (2), (3), and (4) is Lemma [11.1]. It is straightforward to prove that (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6), (6) $\Rightarrow$ (7).

Assume (7). Observe that $\mathcal{O}^\text{sh}_{X,x}$ is a normal local domain (More on Algebra, Lemma [44.6]). Let $L^\text{sh} = L \otimes_K K_x^{sh}$ where $K_x^{sh}$ is the fraction field of $\mathcal{O}^\text{sh}_{X,x}$. Then $L^\text{sh} = \prod_{i=1, \ldots, n} L_i$ with $L_i/K_x^{sh}$ finite separable. By Algebra, Lemma [143.4] (and a limit argument we omit) we see that $Y \times_X \text{Spec}(\mathcal{O}^\text{sh}_{X,x})$ is the integral closure of $\text{Spec}(\mathcal{O}^\text{sh}_{X,x})$ in $L^\text{sh}$. Hence by Lemma [11.1] (applied to the factors $L_i$ of $L^\text{sh}$) we

...
see that $Y \times X \text{Spec}(\mathcal{O}_{X,x}^h) \to \text{Spec}(\mathcal{O}_{X,x}^h)$ is finite étale. Looking at the generic point we see that the degree is equal to $[L : K]$ and hence we see that (8) is true.

Assume (8). Assume that $x \in X$ and that the scheme theoretic fibre $Y_x$ is étale over $x$ of degree $\geq [L : K]$. Observe that this means that $Y$ has $\geq [L : K]$ geometric points lying over $x$. We will show that $Y \to X$ is finite étale over a neighbourhood of $x$. This will prove (1) holds. To prove this we may assume $X = \text{Spec}(R)$, the point $x$ corresponds to the prime $p \subset R$, and $Y = \text{Spec}(S)$. We apply More on Morphisms, Lemma 37.1 and we find an étale neighbourhood $Y \times X$ of $\coprod_{U i}$ concludes that in particular étale, hence

\[ L \]

separable, we can write $\geq m \kappa$ with $\kappa(v_i)/\kappa(u)$ purely inseparable. Shrinking $U$ if necessary we may assume $U$ is a normal integral scheme with generic point $\xi$ (use Descent, Lemmas 13.3 and 15.2 and Properties, Lemma 7.5). By our remark on geometric points we see that $m \geq [L : K]$. On the other hand, by More on Morphisms, Lemma 17.2 we see that $\prod_{i=1}^n L_i$ is finite separable, we can write $\text{Spec}(L) \times X U = \text{Spec}(\prod_{i=1}^n L_i)$ with $L_i/\kappa(\xi)$ finite and $[L : K] = \sum[L_i : \kappa(\xi)]$. Since $V_j$ is nonempty for each $j$ and $m \geq [L : K]$ we conclude that $m = n$ and $[L_i : \kappa(\xi)] = 1$ for all $i$. Then $V_j \to U$ is an isomorphism in particular étale, hence $Y \times X U \to U$ is étale. By Descent, Lemma 20.29 we conclude that $Y \to X$ is étale over the image of $U \to X$ (an open neighbourhood of $x$).

Assume $L/K$ is Galois and (9) holds. Then $Y \to X$ is étale by Lemma 12.5. We omit the proof that (1) implies (9). \hfill \Box

In the case of infinite Galois extensions of discrete valuation rings we can say a tiny bit more. To do so we introduce the following notation. A subset $S \subset \mathbb{N}$ of integers is multiplicity directed if $1 \in S$ and for $n, m \in S$ there exists $k \in S$ with $n | k$ and $m | k$. Define a partial ordering on $S$ by the rule $n \geq_S m$ if and only if $m | n$. Given a field $\kappa$ we obtain an inverse system of finite groups $\{\mu_n(\kappa)\}_{n \in S}$ with transition maps

\[ \mu_n(\kappa) \longrightarrow \mu_m(\kappa), \quad \zeta \longmapsto \zeta^{n/m} \]

for $n \geq_S m$. Then we can form the profinite group

\[ \lim_{n \in S} \mu_n(\kappa) \]

Observe that the limit is cofiltered (as $S$ is directed). The construction is functorial in $\kappa$. In particular $\text{Aut}(\kappa)$ acts on this profinite group. For example, if $S = \{1, n\}$, then this gives $\mu_n(\kappa)$. If $S = \{1, \ell, \ell^2, \ell^3, \ldots\}$ for some prime $\ell$ different from the characteristic of $\kappa$ this produces $\lim_{n} \mu_{\ell^n}(\kappa)$ which is sometimes called the $\ell$-adic Tate module of the multiplicative group of $\kappa$ (compare with More on Algebra, Example 83.4).

0BUA. Lemma 13.5. Let $A$ be a discrete valuation ring with fraction field $K$. Let $L/K$ be a (possibly infinite) Galois extension. Let $B$ be the integral closure of $A$ in $L$. Let $m$ be a maximal ideal of $B$. Let $G = \text{Gal}(L/K)$, $D = \{ \sigma \in G \mid \sigma(m) = m \}$, and $I = \{ \sigma \in D \mid \sigma \text{ mod } m = id_{\kappa(m)} \}$. The decomposition group $D$ fits into a canonical exact sequence

\[ 1 \to I \to D \to \text{Aut}(\kappa(m)/\kappa) \to 1 \]

The inertia group $I$ fits into a canonical exact sequence

\[ 1 \to P \to I \to I_1 \to 1 \]
such that

1. $P$ is a normal subgroup of $D$,
2. $P$ is a pro-$p$-group if the characteristic of $\kappa_A$ is $p > 1$ and $P = \{1\}$ if the characteristic of $\kappa_A$ is zero,
3. there is a multiplicatively directed $S \subset \mathbb{N}$ such that $\kappa(m)$ contains a primitive $n$th root of unity for each $n \in S$ (elements of $S$ are prime to $p$),
4. there exists a canonical surjective map

$$\theta_{can}: I \to \lim_{n \in S} \mu_n(\kappa(m))$$

whose kernel is $P$, which satisfies $\theta_{can}(\tau \sigma \tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(m))$.

Proof. This is mostly a reformulation of the results on finite Galois extensions proved in More on Algebra, Section 98. The surjectivity of the map $D \to \text{Aut}(\kappa(m)/\kappa)$ is More on Algebra, Lemma 96.7. This gives the first exact sequence.

To construct the second short exact sequence let $\Lambda$ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$. Set $G_\lambda = \text{Gal}(L_\lambda/K)$. Recall that $G_\lambda$ is an inverse system of finite groups with surjective transition maps and that $G = \lim_{\lambda \in \Lambda} G_\lambda$, see Fields, Lemma 22.3. We let $B_\lambda$ be the integral closure of $\Lambda$ in $L_\lambda$. Then we set $m_\lambda = m \cap B_\lambda$ and we denote $P_\lambda, I_\lambda, D_\lambda$ the wild inertia, inertia, and decomposition group of $m_\lambda$, see More on Algebra, Lemma 98.5. For $\lambda \geq \lambda'$ the restriction defines a commutative diagram

$$
\begin{array}{cccc}
P_\lambda & \longrightarrow & I_\lambda & \longrightarrow & D_\lambda & \longrightarrow & G_\lambda \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_{\lambda'} & \longrightarrow & I_{\lambda'} & \longrightarrow & D_{\lambda'} & \longrightarrow & G_{\lambda'}
\end{array}
$$

with surjective vertical maps, see More on Algebra, Lemma 98.10.

From the definitions it follows immediately that $I = \lim I_\lambda$ and $D = \lim D_\lambda$ under the isomorphism $G = \lim G_\lambda$ above. Since $L = \text{colim} L_\lambda$ we have $B = \text{colim} B_\lambda$ and $\kappa(m) = \text{colim} \kappa(m_\lambda)$. Since the transition maps of the system $D_\lambda$ are compatible with the maps $D_\lambda \to \text{Aut}(\kappa(m_\lambda)/\kappa)$ (see More on Algebra, Lemma 98.10) we see that the map $D \to \text{Aut}(\kappa(m)/\kappa)$ is the limit of the maps $D_\lambda \to \text{Aut}(\kappa(m_\lambda)/\kappa)$.

There exist canonical maps

$$\theta_{\lambda,can}: I_\lambda \longrightarrow \mu_{n_\lambda}(\kappa(m_\lambda))$$

where $n_\lambda = [I_\lambda]/[P_\lambda]$, where $\mu_{n_\lambda}(\kappa(m_\lambda))$ has order $n_\lambda$, such that $\theta_{\lambda,can}(\tau \sigma \tau^{-1}) = \tau(\theta_{\lambda,can}(\sigma))$ for $\tau \in D_\lambda$ and $\sigma \in I_\lambda$, and such that we get commutative diagrams

$$
\begin{array}{ccc}
I_\lambda & \longrightarrow & \mu_{n_\lambda}(\kappa(m_\lambda)) \\
\downarrow & & \downarrow \cong \\
I_{\lambda'} & \longrightarrow & \mu_{n_{\lambda'}}(\kappa(m_{\lambda'}))
\end{array}
$$

see More on Algebra, Remark 98.11.

Let $S \subset \mathbb{N}$ be the collection of integers $n_\lambda$. Since $\Lambda$ is directed, we see that $S$ is multiplicatively directed. By the displayed commutative diagrams above we can
take the limits of the maps $\theta_{\lambda, \text{can}}$ to obtain
\[ \theta_{\text{can}} : I \to \lim_{n \in S} \mu_n(\kappa(m)). \]

This map is continuous (small detail omitted). Since the transition maps of the system of $I_\lambda$ are surjective and $\Lambda$ is directed, the projections $I \to I_\lambda$ are surjective. For every $\lambda$ the diagram
\[
\begin{array}{ccc}
I & \xrightarrow{\theta_{\text{can}}} & \lim_{n \in S} \mu_n(\kappa(m)) \\
\downarrow & & \downarrow \\
I_\lambda & \xrightarrow{\theta_{\lambda, \text{can}}} & \mu_{n, \lambda}(\kappa(m))
\end{array}
\]
commutes. Hence the image of $\theta_{\text{can}}$ surjects onto the finite group $\mu_{n, \lambda}(\kappa(m)) = \mu_{n, \lambda}(\kappa(m))$ of order $n_\lambda$ (see above). It follows that the image of $\theta_{\text{can}}$ is dense. On the other hand $\theta_{\text{can}}$ is continuous and the source is a profinite group. Hence $\theta_{\text{can}}$ is surjective by a topological argument.

The property $\theta_{\text{can}}(\tau \sigma \tau^{-1}) = \tau(\theta_{\text{can}}(\sigma))$ for $\tau \in D$, $\sigma \in I$ follows from the corresponding properties of the maps $\theta_{\lambda, \text{can}}$ and the compatibility of the map $D \to \text{Aut}(\kappa(m))$ with the maps $D_\lambda \to \text{Aut}(\kappa(m_\lambda))$. Setting $P = \text{Ker}(\theta_{\text{can}})$ this implies that $P$ is a normal subgroup of $D$. Setting $I_t = I/P$ we obtain the isomorphism $I_t \to \lim_{n \in S} \mu_n(\kappa(m))$ from the surjectivity of $\theta_{\text{can}}$.

To finish the proof we show that $P = \lim P_\lambda$ which proves that $P$ is a pro-$p$-group. Recall that the tame inertia group $I_{\lambda,t} = I_\lambda/P_\lambda$ has order $n_\lambda$. Since the transition maps $P_\lambda \to P_{\lambda'}$ are surjective and $\Lambda$ is directed, we obtain a short exact sequence
\[ 1 \to \lim P_\lambda \to I \to \lim I_{\lambda,t} \to 1 \]
(details omitted). Since for each $\lambda$ the map $\theta_{\lambda, \text{can}}$ induces an isomorphism $I_{\lambda,t} \cong \mu_{n, \lambda}(\kappa(m))$ the desired result follows.

\begin{lemma}
Let $A$ be a discrete valuation ring with fraction field $K$. Let $K^{sep}$ be a separable closure of $K$. Let $A^{sep}$ be the integral closure of $A$ in $K^{sep}$. Let $m^{sep}$ be a maximal ideal of $A^{sep}$. Let $m = m^{sep} \cap A$, let $\kappa = A/m$, and let $\overline{\kappa} = A^{sep}/m^{sep}$. Then $\overline{\kappa}$ is an algebraic closure of $\kappa$. Let $G = \text{Gal}(K^{sep}/K)$, $D = \{ \sigma \in G \mid \sigma(m^{sep}) = m^{sep} \}$, and $I = \{ \sigma \in D \mid \sigma \mod m^{sep} = \text{id}_{\kappa(m^{sep})} \}$. The decomposition group $D$ fits into a canonical exact sequence
\[ 1 \to I \to D \to \text{Gal}(K^{sep}/K) \to 1 \]
where $\kappa^{sep} \subset \overline{\kappa}$ is the separable closure of $\kappa$. The inertia group $I$ fits into a canonical exact sequence
\[ 1 \to P \to I \to I_t \to 1 \]
such that
\begin{enumerate}
\item $P$ is a normal subgroup of $D$,
\item $P$ is a pro-$p$-group if the characteristic of $\kappa_A$ is $p > 1$ and $P = \{ 1 \}$ if the characteristic of $\kappa_A$ is zero,
\item there exists a canonical surjective map
\[ \theta_{\text{can}} : I \to \lim_{n \; \text{prime to} \; p} \mu_n(\kappa^{sep}) \]
whose kernel is $P$, which satisfies $\theta_{\text{can}}(\tau \sigma \tau^{-1}) = \tau(\theta_{\text{can}}(\sigma))$ for $\tau \in D$, $\sigma \in I$, and which induces an isomorphism $I_t \to \lim_{n \; \text{prime to} \; p} \mu_n(\kappa^{sep})$.
\end{enumerate}
\end{lemma}
Proof. The field $\pi$ is the algebraic closure of $\kappa$ by Lemma 12.1. Most of the statements immediately follow from the corresponding parts of Lemma 13.5. For example because $\text{Aut}(\pi/\kappa) = \text{Gal}(\kappa^{\text{sep}}/\kappa)$ we obtain the first sequence. Then the only other assertion that needs a proof is the fact that with $S$ as in Lemma 13.5 the limit $\lim_{n \in S} \mu_n(\pi)$ is equal to $\lim_{n \text{ prime to } p} \mu_n(\kappa^{\text{sep}})$. To see this it suffices to show that every integer $n$ prime to $p$ divides an element of $S$. Let $\pi \in A$ be a uniformizer and consider the splitting field $L$ of the polynomial $X^n - \pi$. Since the polynomial is separable we see that $L$ is a finite Galois extension of $K$. Choose an embedding $L \to K^{\text{sep}}$. Observe that if $B$ is the integral closure of $A$ in $L$, then the ramification index of $A \to B_{\text{int}}(B)$ is divisible by $n$ (because $\pi$ has an $n$th root in $B$; in fact the ramification index equals $n$ but we do not need this). Then it follows from the construction of the $S$ in the proof of Lemma 13.5 that $n$ divides an element of $S$. □

14. Geometric and arithmetic fundamental groups

0BTU In this section we work out what happens when comparing the fundamental group of a scheme $X$ over a field $k$ with the fundamental group of $X_{\overline{k}}$ where $\overline{k}$ is the algebraic closure of $k$.

0BTV Lemma 14.1. Let $I$ be a directed set. Let $X_i$ be an inverse system of quasi-compact and quasi-separated schemes over $I$ with affine transition morphisms. Let $X = \lim X_i$ as in Limits, Section 2. Then there is an equivalence of categories $\text{colim} \text{FÉt}_{X_i} = \text{FÉt}_{X}$

If $X_i$ is connected for all sufficiently large $i$ and $x$ is a geometric point of $X$, then $\pi_1(X, x) = \lim \pi_1(X_i, x)$

Proof. The equivalence of categories follows from Limits, Lemmas 10.1, 8.3, and 8.10. The second statement is formal given the statement on categories. □

0BTW Lemma 14.2. Let $k$ be a field with perfection $k^{\text{perf}}$. Let $X$ be a connected scheme over $k$. Then $X_{k^{\text{perf}}}$ is connected and $\pi_1(X_{k^{\text{perf}}}) \to \pi_1(X)$ is an isomorphism.

Proof. Special case of topological invariance of the fundamental group. See Proposition 8.4. To see that $\text{Spec}(k^{\text{perf}}) \to \text{Spec}(k)$ is a universal homeomorphism you can use Algebra, Lemma 45.10. □

0BTX Lemma 14.3. Let $k$ be a field with algebraic closure $\overline{k}$. Let $X$ be a quasi-compact and quasi-separated scheme over $k$. If the base change $X_{\overline{k}}$ is connected, then there is a short exact sequence $1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(\text{Spec}(k)) \to 1$ of profinite topological groups.

Proof. Connected objects of $\text{FÉt}_{\text{Spec}(k)}$ are of the form $\text{Spec}(k') \to \text{Spec}(k)$ with $k'/k$ a finite separable extension. Then $X_{\text{Spec} k'}$ is connected, as the morphism $X_{\overline{k}} \to X_{\text{Spec}(k')}$ is surjective and $X_{\overline{k}}$ is connected by assumption. Thus $\pi_1(X) \to \pi_1(\text{Spec}(k))$ is surjective by Lemma 4.1.

Before we go on, note that we may assume that $k$ is a perfect field. Namely, we have $\pi_1(X_{k^{\text{perf}}}) = \pi_1(X)$ and $\pi_1(\text{Spec}(k^{\text{perf}})) = \pi_1(\text{Spec}(k))$ by Lemma 14.2.
It is clear that the composition of the functors $\text{FÉt}_{\text{Spec}(k)} \to \text{FÉt}_X \to \text{FÉt}_{X_T}$ sends objects to disjoint unions of copies of $X_{\text{Spec}(\overline{k})}$. Therefore the composition $\pi_1(X_T) \to \pi_1(X) \to \pi_1(\text{Spec}(k))$ is the trivial homomorphism by Lemma 4.2. Let $U \to X$ be a finite étale morphism with $U$ connected. Observe that $U \times_X X_T = U_T$. Suppose that $U_T \to X_T$ has a section $s : X_T \to U_T$. Then $s(X_T)$ is an open connected component of $U_T$. For $\sigma \in \text{Gal}(\overline{k}/k)$ denote $s^\sigma$ the base change of $s$ by $\text{Spec}(\sigma)$. Since $U_T \to X_T$ is finite étale it has only a finite number of sections. Thus

$$T = \bigcup s^\sigma(X_T)$$

is a finite union and we see that $T$ is a $\text{Gal}(\overline{k}/k)$-stable open and closed subset. By Varieties, Lemma 7.10 we see that $T$ is the inverse image of a closed subset $T \subset U$. Since $U_T \to U$ is open (Morphisms, Lemma 22.4) we conclude that $T$ is open as well. As $U$ is connected we see that $T = U$. Hence $U_T$ is a (finite) disjoint union of copies of $X_T$. By Lemma 4.5 we conclude that the image of $\pi_1(X_T) \to \pi_1(X)$ is normal.

Let $V \to X_T$ be a finite étale cover. Recall that $\overline{k}$ is the union of finite separable extensions of $k$. By Lemma 14.1 we find a finite separable extension $k'/k$ and a finite étale morphism $U \to X_{k'}$ such that $V = X_{\overline{k}} \times_{X_{k'}} U = U \times_{\text{Spec}(k')} \text{Spec}(\overline{k})$. Then the composition $U \to X_{k'} \to X$ is finite étale and $U \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$ contains $V = U \times_{\text{Spec}(k')} \text{Spec}(\overline{k})$ as an open and closed subscheme. (Because $\text{Spec}(\overline{k})$ is an open and closed subscheme of $\text{Spec}(k') \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$ via the multiplication map $k' \otimes_k \overline{k} \to \overline{k}$.) By Lemma 4.4 we conclude that $\pi_1(X_T) \to \pi_1(X)$ is injective.

Finally, we have to show that for any finite étale morphism $U \to X$ such that $U_T$ is a disjoint union of copies of $X_T$ there is a finite étale morphism $V \to \text{Spec}(k)$ and a surjection $V \times_{\text{Spec}(k)} X \to U$. See Lemma 14.1 Arguing as above using Lemma 14.1 we find a finite separable extension $k'/k$ such that there is an isomorphism $U_{k'} \cong \bigsqcup_{i=1,\ldots,n} X_{k_i}$. Thus setting $V = \bigsqcup_{i=1,\ldots,n} \text{Spec}(k'_i)$ we conclude. \hfill \Box

15. Homotopy exact sequence

0BUM In this section we discuss the following result. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume $S$ is connected and let $\overline{s}$ be a geometric point of $S$. Then there is an exact sequence

$$\pi_1(X_s) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups. See Proposition 15.2

0BUN Lemma 15.1. Let $f : X \to S$ be a proper morphism of schemes. Let $X \to S' \to S$ be the Stein factorization of $f$, see Morphisms, Theorem 68.5. If $f$ is of finite presentation, flat, with geometrically reduced fibres, then $S' \to S$ is finite étale.

Proof. Let $s \in S$. Set $n$ be the number of connected components of the geometric fibre $X_s$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at $s$ is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. By More on Morphisms, Lemma 18.2 there are finitely many points $s'_1, \ldots, s'_m \in S'$ lying over $s$ and for each $i$ the extension $\kappa(s'_i)/\kappa(s)$ is finite. More on Morphisms, Lemma 37.1 tells us that after replacing $S$ by an étale neighbourhood of $s$ we
Lemma 51.4). Thus we see that the functors normalization in More on Morphisms, Lemma 48.1 and factorization by Morphisms, a morphism factors uniquely through the Stein factorization, i.e., corresponds to a adjoints. Note that the Stein factorization of $Y \to X$ is connected (as the image of a connected scheme) and by the surjectivity above $\kappa(s_i) = \kappa(s_i')$ purely inseparable. In this case the schemes $X_i$ are geometrically connected over $\kappa(s)$, hence $m = n$. The schemes $X_i = (f')^{-1}(V_i), i = 1, \ldots, n$ are proper, flat, of finite presentation, with geometrically reduced fibres over $S$. It suffices to prove the lemma for each of the morphisms $X_i \to S$. This reduces us to the case where $X_\pi$ is connected.

Assume that $X_\pi$ is connected. By More on Morphisms, Lemma 48.8 we see that $X \to S$ has geometrically connected fibres in a neighbourhood of $s$. Thus we may assume the fibres of $X \to S$ are geometrically connected. Then $f_* O_X = O_S$ by Derived Categories of Schemes, Lemma 28.6 which finishes the proof.

Proposition 15.2. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume $S$ is connected and let $\pi$ be a geometric point of $S$. Then there is an exact sequence

$$\pi_1(X_\pi) \to \pi_1(X) \to \pi_1(S) \to 1$$

of fundamental groups.

Proof. Let $Y \to X$ be a finite étale morphism. Consider the Stein factorization

$$\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}$$

of $Y \to S$. By Lemma 15.1 the morphism $T \to S$ is finite étale. In this way we obtain a functor $F\text{Et}_X \to F\text{Et}_S$. For any finite étale morphism $U \to S$ a morphism $Y \to U \times_S X$ over $X$ is the same thing as a morphism $Y \to U$ over $S$ and such a morphism factors uniquely through the Stein factorization, i.e., corresponds to a unique morphism $T \to U$ (by the construction of the Stein factorization as a relative normalization in More on Morphisms, Lemma 48.1 and factorization by Morphisms, Lemma 51.4). Thus we see that the functors $F\text{Et}_X \to F\text{Et}_S$ and $F\text{Et}_S \to F\text{Et}_X$ are adjoints. Note that the Stein factorization of $U \times_S X \to S$ is $U$, because the fibres of $U \times_S X \to U$ are geometrically connected.

By the discussion above and Categories, Lemma 24.3 we conclude that $F\text{Et}_S \to F\text{Et}_X$ is fully faithful, i.e., $\pi_1(X) \to \pi_1(S)$ is surjective (Lemma 4.1).

It is immediate that the composition $F\text{Et}_S \to F\text{Et}_X \to F\text{Et}_X$ sends any $U$ to a disjoint union of copies of $X_\pi$. Hence $\pi_1(X_\pi) \to \pi_1(X) \to \pi_1(S)$ is trivial by Lemma 4.1.

Let $Y \to X$ be a finite étale morphism with $Y$ connected such that $Y \times_X X_\pi$ contains a connected component $Z$ isomorphic to $X_\pi$. Consider the Stein factorization $T$ as above. Let $\tilde{t} \in T_\pi$ be the point corresponding to the fibre $Z$. Observe that $T$ is connected (as the image of a connected scheme) and by the surjectivity above $T \times_S X$ is connected. Now consider the factorization

$$\pi : Y \longrightarrow T \times_S X$$

Let $\pi \in X_\pi$ be any closed point. Note that $\kappa(\tilde{t}) = \kappa(\pi) = \kappa(\pi)$ is an algebraically closed field. Then the fibre of $\pi$ over $(\tilde{t}, \pi)$ consists of a unique point, namely the unique point $\pi \in Z$ corresponding to $\pi \in X_\pi$ via the isomorphism $Z \to X_\pi$. We conclude that the finite étale morphism $\pi$ has degree 1 in a neighbourhood of $(\tilde{t}, \pi)$. 


Since $T \times_S X$ is connected it has degree 1 everywhere and we find that $Y \cong T \times_S X$. Thus $Y \times_X X_\tau$ splits completely. Combining all of the above we see that Lemmas \ref{lem:4.3} and \ref{lem:4.5} both apply and the proof is complete. \hfill \qed

16. Specialization maps

In this section we construct specialization maps. Let $f : X \to S$ be a proper morphism of schemes with geometrically connected fibres. Let $s' \leadsto s$ be a specialization of points in $S$. Let $\tau$ and $\tau'$ be geometric points lying over $s$ and $s'$. Then there is a specialization map

$$sp : \pi_1(X_{\tau'}) \longrightarrow \pi_1(X_\tau)$$

The construction of this map is as follows. Let $A$ be the strict henselization of $\mathcal{O}_{S,s}$ with respect to $\kappa(s) \subset \kappa(s)_{sep} \subset \kappa(\tau)$, see Algebra, Definition \ref{def:150.3}. Since $s' \leadsto s$ the point $s'$ corresponds to a point of $\text{Spec}(\mathcal{O}_{S,s})$ and hence there is at least one point (and potentially many points) of $\text{Spec}(A)$ over $s'$ whose residue field is a separable algebraic extension of $\kappa(s')$. Since $\kappa(\tau')$ is algebraically closed we can choose a morphism $\varphi : \tau' \to \text{Spec}(A)$ giving rise to a commutative diagram

$$\begin{array}{ccc}
\tau' & \xrightarrow{\varphi} & \text{Spec}(A) \\
\downarrow & & \downarrow \\
S & & \tau
\end{array}$$

The specialization map is the composition

$$\pi_1(X_{\tau'}) \longrightarrow \pi_1(X_A) = \pi_1(X_{\kappa(s)_{sep}}) = \pi_1(X_\tau)$$

where the first equality is Lemma \ref{lem:9.1} and the second follows from Lemmas \ref{lem:14.2} and \ref{lem:9.2} By construction the specialization map fits into a commutative diagram

$$\begin{array}{ccc}
\pi_1(X_{\tau'}) & \xrightarrow{sp} & \pi_1(X_\tau) \\
\downarrow & & \downarrow \\
\pi_1(X) & & \pi_1(X)
\end{array}$$

provided that $X$ is connected. The specialization map depends on the choice of $\varphi : \tau' \to \text{Spec}(A)$ above and we will write $sp_\varphi$ if we want to indicate this.

\textbf{Lemma 16.1.} Consider a commutative diagram

$$\begin{array}{ccc}
Y & \longrightarrow & X \\
g & \downarrow & f \\
T & \longrightarrow & S
\end{array}$$

of schemes where $f$ and $g$ are proper with geometrically connected fibres. Let $t' \leadsto t$ be a specialization of points in $T$ and consider a specialization map $sp : \pi_1(Y_{\tau'}) \to \pi_1(Y_\tau)$ as above. Then there is a commutative diagram

$$\begin{array}{ccc}
\pi_1(Y_{\tau'}) & \xrightarrow{sp} & \pi_1(Y_\tau) \\
\downarrow & & \downarrow \\
\pi_1(X_{\tau'}) & \xrightarrow{sp} & \pi_1(X_\tau)
\end{array}$$
of specialization maps where \( \overline{s} \) and \( \overline{s}' \) are the images of \( t \) and \( t' \).

**Proof.** Let \( B \) be the strict henselization of \( \mathcal{O}_{T,t} \) with respect to \( \kappa(t) \subset \kappa(t)_{\text{sep}} \subset \kappa(\overline{t}) \). Pick \( \psi : \overline{t} \rightarrow \text{Spec}(B) \) lifting \( t' \rightarrow T \) as in the construction of the specialization map. Let \( s' \) and \( s'' \) denote the images of \( t \) and \( t' \) in \( S \). Let \( A \) be the strict henselization of \( \mathcal{O}_{S,s} \) with respect to \( \kappa(s) \subset \kappa(s)_{\text{sep}} \subset \kappa(\overline{s}) \). Since \( \kappa(\overline{s}) = \kappa(\overline{t}) \), by the functoriality of strict henselization (Algebra, Lemma \[150.12\]) we obtain a ring map \( A \rightarrow B \) fitting into the commutative diagram

\[
\begin{array}{ccc}
\overline{t} & \xrightarrow{\psi} & \text{Spec}(B) \\
\downarrow & & \downarrow \\
\overline{s}' & \xrightarrow{\varphi} & \text{Spec}(A) \\
\end{array}
\]

Here the morphism \( \varphi : \overline{s}' \rightarrow \text{Spec}(A) \) is simply taken to be the composition \( \overline{t} \rightarrow \text{Spec}(B) \rightarrow \text{Spec}(A) \). Applying base change we obtain a commutative diagram

\[
\begin{array}{ccc}
Y_{\overline{t}} & \longrightarrow & Y_B \\
\downarrow & & \downarrow \\
X_{\overline{s}'} & \longrightarrow & X_A \\
\end{array}
\]

and from the construction of the specialization map the commutativity of this diagram implies the commutativity of the diagram of the lemma. \( \square \)

**Lemma 16.2.** Let \( f : X \rightarrow S \) be a proper morphism with geometrically connected fibres. Let \( s'' \sim s' \sim s \) be specializations of points of \( S \). A composition of specialization maps \( \pi_1(X_{s''}) \rightarrow \pi_1(X_{s'}) \rightarrow \pi_1(X_s) \) is a specialization map \( \pi_1(X_{s''}) \rightarrow \pi_1(X_s) \).

**Proof.** Let \( \mathcal{O}_{S,s} \rightarrow A \) be the strict henselization constructed using \( \kappa(s) \rightarrow \kappa(\overline{s}) \). Let \( A' \rightarrow \kappa(\overline{s}') \) be the map used to construct the first specialization map. Let \( \mathcal{O}_{S,s'} \rightarrow A' \) be the strict henselization constructed using \( \kappa(s') \subset \kappa(\overline{s}') \). By functoriality of strict henselization, there is a map \( A \rightarrow A' \) such that the composition with \( A' \rightarrow \kappa(\overline{s}') \) is the given map (Algebra, Lemma \[149.3\]). Next, let \( A' \rightarrow \kappa(\overline{s}'') \) be the map used to construct the second specialization map. Then it is clear that the composition of the first and second specialization maps is the specialization map \( \pi_1(X_{s''}) \rightarrow \pi_1(X_{s'}) \rightarrow \pi_1(X_s) \) constructed using \( A \rightarrow A' \rightarrow \kappa(\overline{s}'') \). \( \square \)

Let \( X \rightarrow S \) be a proper morphism with geometrically connected fibres. Let \( R \) be a strictly henselian valuation ring with algebraically closed fraction field and let \( \text{Spec}(R) \rightarrow S \) be a morphism. Let \( \eta, s \in \text{Spec}(R) \) be the generic and closed point. Then we can consider the specialization map

\[ sp_R : \pi_1(X_\eta) \rightarrow \pi_1(X_s) \]

for the base change \( X_R / \text{Spec}(R) \). Note that this makes sense as both \( \eta \) and \( s \) have algebraically closed residue fields.

**Lemma 16.3.** Let \( f : X \rightarrow S \) be a proper morphism with geometrically connected fibres. Let \( s' \sim s \) be a specialization of points of \( S \) and let \( sp : \pi_1(X_{s''}) \rightarrow \pi_1(X_s) \) be a specialization map. Then there exists a strictly henselian valuation ring \( R \) over \( S \) with algebraically closed fraction field such that \( sp \) is isomorphic to \( sp_R \) defined above.
Proof. Let \( \mathcal{O}_{S,s} \rightarrow A \) be the strict henselization constructed using \( \kappa(s) \rightarrow \kappa(\overline{s}) \). Let \( A \rightarrow \kappa(\overline{s}') \) be the map used to construct \( \text{sp} \). Let \( R \subset \kappa(\overline{s}') \) be a valuation ring with fraction field \( \kappa(\overline{s}') \) dominating the image of \( A \). See Algebra, Lemma 49.2. Observe that \( R \) is strictly henselian for example by Lemma 12.2 and Algebra, Lemma 49.10. Then the lemma is clear. \( \square \)

Let \( X \rightarrow S \) be a proper morphism with geometrically connected fibres. Let \( R \) be a strictly henselian discrete valuation ring and let \( \text{Spec}(R) \rightarrow S \) be a morphism. Let \( \eta, s \in \text{Spec}(R) \) be the generic and closed point. Then we can consider the specialization map

\[
\text{sp}_R : \pi_1(X_\eta) \rightarrow \pi_1(X_s)
\]

for the base change \( X_R/\text{Spec}(R) \). Note that this makes sense as \( s \) has algebraically closed residue field.

0C0N Lemma 16.4. Let \( f : X \rightarrow S \) be a proper morphism with geometrically connected fibres. Let \( s' \rightarrow s \) be a specialization of points of \( S \) and let \( \text{sp} : \pi_1(X_{\overline{s}'}) \rightarrow \pi_1(X_{\overline{s}}) \) be a specialization map. If \( S \) is Noetherian, then there exists a strictly henselian discrete valuation ring \( R \) over \( S \) such that \( \text{sp} \) is isomorphic to \( \text{sp}_R \) defined above.

Proof. Let \( \mathcal{O}_{S,s} \rightarrow A \) be the strict henselization constructed using \( \kappa(s) \rightarrow \kappa(\overline{s}) \). Let \( A \rightarrow \kappa(\overline{s}') \) be the map used to construct \( \text{sp} \). Let \( R \subset \kappa(\overline{s}') \) be a discrete valuation ring dominating the image of \( A \), see Algebra, Lemma 118.13. Choose a diagram of fields

\[
\begin{array}{c}
\kappa(\overline{s}) \downarrow \downarrow \kappa \\
A/m_A \xrightarrow{\text{inclusion}} R/m_R \xrightarrow{\text{inclusion}} R \\
\end{array}
\]

with \( k \) algebraically closed. Let \( R^{sh} \) be the strict henselization of \( R \) constructed using \( R \rightarrow k \). Then \( R^{sh} \) is a discrete valuation ring by More on Algebra, Lemma 44.11. Denote \( \eta, o \) the generic and closed point of \( \text{Spec}(R^{sh}) \). Since the diagram of schemes

\[
\begin{array}{c}
\overline{s}' \xrightarrow{} \text{Spec}(R^{sh}) \rightarrow \text{Spec}(k) \\
\downarrow \downarrow \downarrow \\
\overline{s} \rightarrow \text{Spec}(A) \leftarrow \overline{s}
\end{array}
\]

commutes, we obtain a commutative diagram

\[
\begin{array}{c}
\pi_1(X_{\overline{s}'}) \xrightarrow{\text{sp}_{R^{sh}}} \pi_1(X_{\overline{s}}) \\
\downarrow \downarrow \downarrow \\
\pi_1(X_{\overline{s}}) \xrightarrow{\text{sp}} X_{\overline{s}}
\end{array}
\]

of specialization maps by the construction of these maps. Since the vertical arrows are isomorphisms (Lemma 9.2), this proves the lemma. \( \square \)
17. Restriction to a closed subscheme

In this section we prove some results about the restriction functor

$$F\text{Ét}_X \to F\text{Ét}_Y, \quad U \mapsto V = U \times_X Y$$

where $X$ is a scheme and $Y$ is a closed subscheme. Using the topological invariance of the fundamental group, we can relate the study of this functor to the completion functor on finite locally free modules.

In the following lemmas we use the concept of coherent formal modules defined in Cohomology of Schemes, Section 23. Given a Noetherian scheme and a quasi-coherent sheaf of ideals $I \subset \mathcal{O}_X$ we will say an object $(F_n)$ of $\text{Coh}(X, I)$ is finite locally free if each $F_n$ is a finite locally free $\mathcal{O}_X/I^n$-module.

**Lemma 17.1.** Let $X$ be a Noetherian scheme and let $Y \subset X$ be a closed subscheme with ideal sheaf $I \subset \mathcal{O}_X$. Assume the completion functor

$$\text{Coh}(%3% \mathcal{O}_X) \to \text{Coh}(X, I), \quad \mathcal{F} \mapsto \mathcal{F}^\wedge$$

is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor $F\text{Ét}_X \to F\text{Ét}_Y$ is fully faithful.

**Proof.** Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given $U$ finite étale over $X$ and a morphism $t : Y \to U$ over $X$ there exists a unique section $s : X \to U$ such that $t = s|_Y$. Picture

$$\begin{array}{ccc}
U & \downarrow f & Y \\
\downarrow t & & \downarrow f \\
X & & X
\end{array}$$

Finding the dotted arrow $s$ is the same thing as finding an $\mathcal{O}_X$-algebra map

$$s^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_X$$

which reduces modulo the ideal sheaf of $Y$ to the given algebra map $t^\sharp : f_*\mathcal{O}_U \to \mathcal{O}_Y$. By Lemma 8.3 we can lift $t$ uniquely to a compatible system of maps $t_n : Y_n \to U$ and hence a map

$$\lim t_n^\sharp : f_*\mathcal{O}_U \to \lim \mathcal{O}_{Y_n}$$

of sheaves of algebras on $X$. Since $f_*\mathcal{O}_U$ is a finite locally free $\mathcal{O}_X$-module, we conclude that we get a unique $\mathcal{O}_X$-module map $\sigma : f_*\mathcal{O}_U \to \mathcal{O}_X$ whose completion is $\lim t_n^\sharp$. To see that $\sigma$ is an algebra homomorphism, we need to check that the diagram

$$\begin{array}{ccc}
f_*\mathcal{O}_U \otimes_{\mathcal{O}_X} f_*\mathcal{O}_U & \to & f_*\mathcal{O}_U \\
\sigma \otimes \sigma & \downarrow & \sigma \\
\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X & \to & \mathcal{O}_X
\end{array}$$

commutes. For every $n$ we know this diagram commutes after restricting to $Y_n$, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we conclude. \qed
Lemma 17.2. Let \( X \) be a Noetherian scheme and let \( Y \subset X \) be a closed subscheme with ideal sheaf \( \mathcal{I} \subset \mathcal{O}_X \). Assume the completion functor
\[
\text{Coh}(\mathcal{O}_X) \to \text{Coh}(X, \mathcal{I}), \quad \mathcal{F} \mapsto \mathcal{F}^\wedge
\]
is an equivalence on full subcategories of finite locally free objects (see above). Then the restriction functor \( \mathcal{F} \mathcal{E}t_X \to \mathcal{F} \mathcal{E}t_Y \) is an equivalence.

Proof. The restriction functor is fully faithful by Lemma 17.1. Let \( U_1 \to Y \) be a finite étale morphism. To finish the proof we will show that \( U_1 \) is in the essential image of the restriction functor.

For \( n \geq 1 \) let \( Y_n \) be the \( n \)th infinitesimal neighbourhood of \( Y \). By Lemma 8.3 there is a unique finite étale morphism \( \pi_n : U_n \to Y_n \) whose base change to \( Y = Y_1 \) recovers \( U_1 \to Y_1 \). Consider the sheaves \( \mathcal{F}_n = \pi_n^* \mathcal{O}_{U_n} \). We may and do view \( \mathcal{F}_n \) as an \( \mathcal{O}_X \)-module on \( X \) which is locally isomorphic to \( (\mathcal{O}_X/f_n^{n+1}\mathcal{O}_X)^{\oplus r} \). This \( (\mathcal{F}_n) \) is a finite locally free object of \( \text{Coh}(X, \mathcal{I}) \). By assumption there exists a finite locally free \( \mathcal{O}_X \)-module \( \mathcal{F} \) and a compatible system of isomorphisms \( \mathcal{F}/\mathcal{I}^n \mathcal{F} \to \mathcal{F}_n \) of \( \mathcal{O}_X \)-modules.

To construct an algebra structure on \( \mathcal{F} \) consider the multiplication maps \( \mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{F}_n \to \mathcal{F}_n \) coming from the fact that \( \mathcal{F}_n = \pi_n^* \mathcal{O}_{U_n} \) are sheaves of algebras. These define a map
\[
(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F})^\wedge \to \mathcal{F}^\wedge
\]
in the category \( \text{Coh}(X, \mathcal{I}) \). Hence by assumption we may assume there is a map \( \mu : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \) whose restriction to \( Y_n \) gives the multiplication maps above. After possibly shrinking further we may assume \( \mu \) defines a commutative \( \mathcal{O}_X \)-algebra structure on \( \mathcal{F} \) compatible with the given algebra structures on \( \mathcal{F}_n \). Setting \( U = \text{Spec}_X((\mathcal{F}, \mu)) \)
we obtain a finite locally free scheme \( \pi : U \to X \) whose restriction to \( Y \) is isomorphic to \( U_1 \). The discriminant of \( \pi \) is the zero set of the section
\[
\text{det}(\mathcal{Q}_\pi) : \mathcal{O}_X \to \wedge^\top(\pi^* \mathcal{O}_U)^{\oplus -2}
\]
constructed in Discriminants, Section 3. Since the restriction of this to \( Y_n \) is an isomorphism for all \( n \) by Discriminants, Lemma 3.1 we conclude that it is an isomorphism. Thus \( \pi \) is étale by Discriminants, Lemma 3.1. \( \square \)

Lemma 17.3. Let \( X \) be a Noetherian scheme and let \( Y \subset X \) be a closed subscheme with ideal sheaf \( \mathcal{I} \subset \mathcal{O}_X \). Let \( \mathcal{V} \) be the set of open subschemes \( V \subset X \) containing \( Y \) ordered by reverse inclusion. Assume the completion functor
\[
\text{colim}_V \text{Coh}(\mathcal{O}_V) \to \text{Coh}(X, \mathcal{I}), \quad \mathcal{F} \mapsto \mathcal{F}^\wedge
\]
defines is fully faithful on the full subcategory of finite locally free objects (see above). Then the restriction functor \( \text{colim}_V \mathcal{F} \mathcal{E}t_V \to \mathcal{F} \mathcal{E}t_Y \) is fully faithful.

Proof. Observe that \( \mathcal{V} \) is a directed set, so the colimits are as in Categories, Section 19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 17.1 we urge the reader to skip it.

Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given \( U \) finite étale over \( V \in \mathcal{V} \) and a morphism
t : Y → U over V there exists a V' ≥ V and a morphism s : V' → U over V such that t = s|Y. Picture

Finding the dotted arrow s is the same thing as finding an O_{V'}-algebra map

s^\#: f_*O_U|_{V'} → O_{V'}

which reduces modulo the ideal sheaf of Y to the given algebra map t^\#: f_*O_U → O_Y. By Lemma 8.3 we can lift t uniquely to a compatible system of maps t_n : Y_n → U and hence a map

\lim t_n : f_*O_U → \lim O_{Y_n}

of sheaves of algebras on V. Observe that f_*O_U is a finite locally free O_V-module. Hence we get a V' ≥ V a map σ : f_*O_U|_{V'} → O_{V'} whose completion is \lim t_n^\#. To see that σ is an algebra homomorphism, we need to check that the diagram

\begin{align*}
(f_*O_U \otimes_{O_V} f_*O_U)|_{V'} & \longrightarrow f_*O_U|_{V'} \\
\sigma \otimes \sigma & \quad \sigma \\
O_{V'} \otimes_{O_V} O_{V'} & \quad O_{V'}
\end{align*}

commutes. For every n we know this diagram commutes after restricting to Y_n, i.e., the diagram commutes after applying the completion functor. Hence by faithfulness of the completion functor we deduce that there exists a V'' ≥ V' such that σ|_{V''} is an algebra homomorphism as desired.

□

Lemma 17.4. Let X be a Noetherian scheme and let Y ⊂ X be a closed subscheme with ideal sheaf I ⊂ O_X. Let V be the set of open subschemes V ⊂ X containing Y ordered by reverse inclusion. Assume the completion functor

\[ \text{colim}_V \text{Coh}(O_V) \longrightarrow \text{Coh}(X, I), \quad F \longrightarrow F^{\wedge} \]

defines an equivalence of the full subcategories of finite locally free objects (see explanation above). Then the restriction functor

\[ \text{colim}_V F^{\text{Ét}}_V \rightarrow F^{\text{Ét}}_Y \]

is an equivalence.

Proof. Observe that V is a directed set, so the colimits are as in Categories, Section 19. The rest of the argument is almost exactly the same as the argument in the proof of Lemma 17.3. We urge the reader to skip it.

The restriction functor is fully faithful by Lemma 17.3.

Let U_1 → Y be a finite étale morphism. To finish the proof we will show that U_1 is in the essential image of the restriction functor.

For n ≥ 1 let Y_n be the n-th infinitesimal neighbourhood of Y. By Lemma 8.3 there is a unique finite étale morphism π_n : U_n → Y_n whose base change to Y = Y_1 recovers U_1 → Y_1. Consider the sheaves F_n = π_n_* O_{U_n}. We may and do view F_n as an O_X-module on X which is locally isomorphic to (O_X/f^{n+1}O_X)^{\otimes r}. This (F_n)
is a finite locally free object of $\text{Coh}(X, \mathcal{I})$. By assumption there exists a $V \in \mathcal{V}$ and a finite locally free $\mathcal{O}_V$-module $\mathcal{F}$ and a compatible system of isomorphisms

$$\mathcal{F}/\mathcal{I}^n \mathcal{F} \to \mathcal{F}_n$$

of $\mathcal{O}_V$-modules.

To construct an algebra structure on $\mathcal{F}$ consider the multiplication maps $\mathcal{F}_n \otimes \mathcal{O}_V \mathcal{F}_n \to \mathcal{F}_n$ coming from the fact that $\mathcal{F}_n = \pi_{n,*}\mathcal{O}_{U_n}$ are sheaves of algebras. These define a map

$$(\mathcal{F} \otimes \mathcal{O}_V \mathcal{F})^\wedge \to \mathcal{F}^\wedge$$

in the category $\text{Coh}(X, \mathcal{I})$. Hence by assumption after shrinking $V$ we may assume there is a map $\mu : \mathcal{F} \otimes \mathcal{O}_V \mathcal{F} \to \mathcal{F}$ whose restriction to $Y_n$ gives the multiplication maps above. After possibly shrinking further we may assume $\mu$ defines a commutative $\mathcal{O}_V$-algebra structure on $\mathcal{F}$ compatible with the given algebra structures on $\mathcal{F}_n$. Setting $U = \text{Spec}_V((\mathcal{F}, \mu))$ we obtain a finite locally free scheme over $V$ whose restriction to $Y$ is isomorphic to $U_1$. It follows that $U \to V$ is étale at all points lying over $Y$, see More on Morphisms, Lemma 12.3. Thus after shrinking $V$ once more we may assume $U \to V$ is finite étale. This finishes the proof.

**Lemma 17.5.** Let $X$ be a scheme and let $Y \subset X$ be a closed subscheme. If every connected component of $X$ meets $Y$, then the restriction functor $\mathcal{F}\text{Ét}_X \to \mathcal{F}\text{Ét}_Y$ is faithful.

**Proof.** Let $a, b : U \to U'$ be two morphisms of schemes finite étale over $X$ whose restriction to $Y$ are the same. The image of a connected component of $U$ is an connected component of $X$; this follows from Topology, Lemma 7.6 applied to the restriction of $U \to X$ to a connected component of $X$. Hence the image of every connected component of $U$ meets $Y$ by assumption. We conclude that $a = b$ after restriction to each connected component of $U$ by Étale Morphisms, Proposition 6.3. Since the equalizer of $a$ and $b$ is an open subscheme of $U$ (as the diagonal of $U'$ over $X$ is open) we conclude. □

**Lemma 17.6.** Let $X$ be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the $n$th infinitesimal neighbourhood of $Y$ in $X$. Assume one of the following holds

1. $X$ is quasi-affine and $\Gamma(X, \mathcal{O}_X) \to \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$ is an isomorphism, or
2. $X$ has an ample invertible module $\mathcal{L}$ and $\Gamma(X, \mathcal{L}^\otimes m) \to \lim \Gamma(Y_n, \mathcal{L}^\otimes m|_{Y_n})$ is an isomorphism for all $m \gg 0$, or
3. for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ the map $\Gamma(X, \mathcal{E}) \to \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})$ is an isomorphism.

Then the restriction functor $\mathcal{F}\text{Ét}_X \to \mathcal{F}\text{Ét}_Y$ is fully faithful.

**Proof.** This lemma follows formally from Lemma 17.1 and Algebraic and Formal Geometry, Lemma 15.1. □

**Lemma 17.7.** Let $X$ be a Noetherian scheme and let $Y \subset X$ be a closed subscheme. Let $Y_n \subset X$ be the $n$th infinitesimal neighbourhood of $Y$ in $X$. Let $\mathcal{V}$ be the set of open subschemes $V \subset X$ containing $Y$ ordered by reverse inclusion. Assume one of the following holds
(1) $X$ is quasi-affine and
$$
\colim V \Gamma(V, \mathcal{O}_V) \longrightarrow \lim \Gamma(Y_n, \mathcal{O}_{Y_n})
$$
is an isomorphism, or

(2) $X$ has an ample invertible module $\mathcal{L}$ and
$$
\colim V \Gamma(V, \mathcal{L}^\otimes m) \longrightarrow \lim \Gamma(Y_n, \mathcal{L}^\otimes m|_{Y_n})
$$
is an isomorphism for all $m \gg 0$, or

(3) for every $V \in \mathcal{V}$ and every finite locally free $\mathcal{O}_V$-module $\mathcal{E}$ the map
$$
\colim V' \Gamma(V', \mathcal{E}|_{V'}) \longrightarrow \lim \Gamma(Y_n, \mathcal{E}|_{Y_n})
$$
is an isomorphism.

Then the functor
$$
\colim V \text{FÉt}_V \rightarrow \text{FÉt}_Y
$$
is fully faithful.

**Proof.** This lemma follows formally from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.2. □

18. Pushouts and fundamental groups

**Lemma 18.1.** In More on Morphisms, Situation 59.1, for example if $Z \rightarrow Y$ and $Z \rightarrow X$ are closed immersions of schemes, there is an equivalence of categories
$$
\text{FÉt}_{Y \amalg Z} \rightarrow \text{FÉt}_Y \times_{\text{FÉt}_Z} \text{FÉt}_X
$$

**Proof.** The pushout exists by More on Morphisms, Proposition 59.3. The functor is given by sending a scheme $U$ finite étale over the pushout to the base changes $Y' = U \times_{Y \amalg Z} Y$ and $X' = U \times_{Y \amalg Z} X$ and the natural isomorphism $Y' \times_Y Z \rightarrow X' \times_X Z$ over $Z$. To prove this functor is an equivalence we use More on Morphisms, Lemma 59.7 to construct a quasi-inverse functor. The only thing left to prove is to show that given a morphism $U \rightarrow Y \amalg Z$ which is separated, quasi-finite and étale such that $X' \rightarrow X$ and $Y' \rightarrow Y$ are finite, then $U \rightarrow Y \amalg Z$ is finite. This can either be deduced from the corresponding algebra fact (More on Algebra, Lemma 6.7) or it can be seen because

$$
X' \amalg Y' \rightarrow U
$$
is surjective and $X'$ and $Y'$ are proper over $Y \amalg Z$ $X$ (this uses the description of the pushout in More on Morphisms, Proposition 59.3) and then we can apply Morphisms, Lemma 39.10 to conclude that $U$ is proper over $Y \amalg Z$ $X$. Since a quasi-finite and proper morphism is finite (More on Morphisms, Lemma 39.1) we win. □

19. Finite étale covers of punctured spectra, I

**Situation 19.1.** Let $(A, m)$ be a Noetherian local ring and $f \in m$. We set $X = \text{Spec}(A)$ and $X_0 = \text{Spec}(A/fA)$ and we let $U = X \setminus \{m\}$ and $U_0 = X_0 \setminus \{m\}$ be the punctured spectrum of $A$ and $A/fA$. We first prove some results à la Lefschetz.
Recall that for a scheme $X$ the category of schemes finite étale over $X$ is denoted $\text{FÉt}_X$, see Section 5. In Situation 19.1 we will study the base change functors

$$
\begin{align*}
\text{FÉt}_X & \longrightarrow \text{FÉt}_U \\
\downarrow & \\
\text{FÉt}_{X_0} & \longrightarrow \text{FÉt}_{U_0}
\end{align*}
$$

In many case the right vertical arrow is faithful.

**Lemma 19.2.** In Situation 19.1 Assume one of the following holds

1. $\dim(A/p) \geq 2$ for every minimal prime $p \subset A$ with $f \notin p$, or
2. every connected component of $U$ meets $U_0$.

Then

$$
\text{FÉt}_U \longrightarrow \text{FÉt}_{U_0}, \quad V \mapsto V = V \times_U U_0
$$

is a faithful functor.

**Proof.** Case (2) is immediate from Lemma 17.5. Assumption (1) implies every irreducible component of $U$ meets $U_0$, see Algebra, Lemma 59.12. Hence (1) follows from (2). □

Before we prove something more interesting, we need a couple of lemmas.

**Lemma 19.3.** In Situation 19.1 Let $V \to U$ be a finite morphism. Let $A^\wedge$ be the $m$-adic completion of $A$, let $X = \text{Spec}(A^\wedge)$ and let $U'$ and $V'$ be the base changes of $U$ and $V$ to $X'$. If $Y' \to X'$ is a finite morphism such that $V' = Y' \times_{X'} U'$, then there exists a finite morphism $Y \to X$ such that $V = Y \times_X U$ and $Y' = Y \times_X X'$.

**Proof.** This is a straightforward application of More on Algebra, Proposition 80.15. Namely, choose generators $f_1, \ldots, f_n$ of $m$. For each $i$ write $V \times U D(f_i) = \text{Spec}(B_i)$. For $1 \leq i, j \leq n$ we obtain an isomorphism $\alpha_{ij} : (B_i)_{f_i} \to (B_j)_{f_i}$ of $A_{f_i}f_j$-algebras because the spectrum of both represent $V \times_U D(f_if_j)$. Write $Y' = \text{Spec}(B')$. Since $V \times_U U' = Y \times_{X'} U'$ we get isomorphisms $\alpha_i : B'_{f_i} \to B_i \otimes_A A^\wedge$. A straightforward argument shows that $(B', B_1, \alpha_1, \alpha_2)$ is an object of $\text{Glue}(A \to A^\wedge, f_1, \ldots, f_n)$, see More on Algebra, Remark 80.10. Applying the proposition cited above (and using More on Algebra, Remark 80.19 to obtain the algebra structure) we find an $A$-algebra $B$ such that $\text{Can}(B)$ is isomorphic to $(B', B_1, \alpha_1, \alpha_2)$. Setting $Y = \text{Spec}(B)$ we see that $Y \to X$ is a morphism which comes equipped with compatible isomorphisms $V \cong Y \times_X U$ and $Y' = Y \times_X X'$ as desired. □

**Lemma 19.4.** In Situation 19.1 assume $A$ is henselian or more generally that $(A, f)$ is a henselian pair. Let $A^\wedge$ be the $m$-adic completion of $A$, let $X = \text{Spec}(A^\wedge)$ and let $U'$ and $U'_0$ be the base changes of $U$ and $U_0$ to $X'$. If $\text{FÉt}_{U'} \to \text{FÉt}_{U'_0}$ is fully faithful, then $\text{FÉt}_U \to \text{FÉt}_{U_0}$ is fully faithful.

**Proof.** Assume $\text{FÉt}_{U'} \to \text{FÉt}_{U'_0}$ is a fully faithful. Since $X' \to X$ is faithfully flat, it is immediate that the functor $V \to V_0 = V \times_U U_0$ is faithful. Since the category of finite étale coverings has an internal hom (Lemma 5.4) it suffices to prove the following: Given $V$ finite étale over $U$ we have

$$
\text{Mor}_U(U, V) = \text{Mor}_{U_0}(U_0, V_0)
$$
The we assume we have a morphism $s_0 : U_0 \rightarrow V_0$ over $U_0$ and we will produce a morphism $s : U \rightarrow V$ over $U$.

By our assumption there does exist a morphism $s' : U' \rightarrow V'$ whose restriction to $V_0'$ is the base change $s_0'$ of $s_0$. Since $V' \rightarrow U'$ is finite étale this means that $V' = s'(U') \amalg W'$ for some $W' \rightarrow U'$ finite and étale. Choose a finite morphism $Z' \rightarrow X'$ such that $W' = Z' \times_{X'} U'$. This is possible by Zariski’s main theorem in the form stated in More on Morphisms, Lemma \[148.3\] (small detail omitted). Then

$$V' = s'(U') \amalg W' \rightarrow X' \amalg Z' = Y'$$

is an open immersion such that $V' = Y' \times_{X'} U'$. By Lemma \[19.3\] we can find $Y \rightarrow X$ finite such that $V = Y \times_X U$ and $Y' = Y \times_{X'} X'$. Write $Y = \text{Spec}(B)$ so that $Y' = \text{Spec}(B \otimes_A A^\wedge)$. Then $B \otimes_A A^\wedge$ has an idempotent $e'$ corresponding to the open and closed subscheme $X'$ of $Y' = X' \amalg Z'$.

The case where $A$ is henselian (slightly easier). The image $\sigma$ of $e'$ in $B \otimes_A \kappa(m) = B/fB$ lifts to an idempotent $e$ of $B$ as $A$ is henselian (because $B$ is a product of local rings by Algebra, Lemma \[148.3\]). Then we see that $e$ maps to $e'$ by uniqueness of lifts of idempotents (using that $B \otimes_A A^\wedge$ is a product of local rings). Let $Y_1 \subset Y$ be the open and closed subscheme corresponding to $e$. Then $Y_1 \times_X X' = s'(X')$ which implies that $Y_1 \rightarrow X$ is an isomorphism (by faithfully flat descent) and gives the desired section.

The case where $(A, (f))$ is a henselian pair. Here we use that $s'$ is a lift of $s_0'$. Namely, let $Y_{0,1} \subset Y_0 = Y \times_X X_0$ be the closure of $s_0(U_0) \subset V_0 = Y_0 \times_{X_0} U_0$. As $X' \rightarrow X$ is flat, the base change $Y_{0,1}' \subset Y_0'$ is the closure of $s_0'(U_0')$ which is equal to $X_0' \subset Y_0'$ (see Morphisms, Lemma \[24.15\]). Since $Y_0' \rightarrow Y_0$ is submersive (Morphisms, Lemma \[24.11\]) we conclude that $Y_{0,1}$ is open and closed in $Y_0$. Let $e_0 \in B/fB$ be the corresponding idempotent. By More on Algebra, Lemma \[11.6\] we can lift $e_0$ to an idempotent $e \in B$. Then we conclude as before.

In Situation \[19.1\] fully faithfulness of the restriction functor $F\text{Ét}_U \rightarrow F\text{Ét}_{U_0}$ holds under fairly mild assumptions. In particular, the assumptions often do not imply $U$ is a connected scheme, but the conclusion guarantees that $U$ and $U_0$ have the same number of connected components.

**Lemma 19.5.** In Situation \[19.1\] Assume

(a) $A$ has a dualizing complex,

(b) the pair $(A, (f))$ is henselian,

(c) one of the following is true

(i) $A_f$ is $(S_2)$ and every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 3$, or

(ii) for every prime $p \subset A$, $f \not\in p$ we have $\text{depth}(A_p) + \text{dim}(A/p) > 2$.

Then the restriction functor $F\text{Ét}_U \rightarrow F\text{Ét}_{U_0}$ is fully faithful.

**Proof.** Let $A'$ be the $m$-adic completion of $A$. We will show that the hypotheses remain true for $A'$. This is clear for conditions (a) and (b). Condition (c)(ii) is preserved by Local Cohomology, Lemma \[10.3\] Next, assume (c)(i) holds. Since $A$ is universally catenary (Dualizing Complexes, Lemma \[17.4\]) we see that every irreducible component of $\text{Spec}(A')$ not contained in $V(f)$ has dimension $\geq 3$, see More on Algebra, Proposition \[95.5\]. Since $A \rightarrow A'$ is flat with Gorenstein fibres, the condition that $A_f$ is $(S_2)$ implies that $A'_f$ is $(S_2)$. References used: Dualizing
Thus by Lemma 19.4 we may assume that $A$ is a Noetherian complete local ring.

Assume $A$ is a complete local ring in addition to the other assumptions. By Lemma 17.1 the result follows from Algebraic and Formal Geometry, Lemmas 15.5 and 15.7.

Lemma 19.6. In Situation 19.1 Assume

1. $H^1_m(A)$ and $H^2_m(A)$ are annihilated by a power of $f$, and
2. $A$ is henselian or more generally $(A,(f))$ is a henselian pair.

Then the restriction functor $F\text{Ét}_U \rightarrow F\text{Ét}_{U_0}$ is fully faithful.

Proof. By Lemma 19.4 we may assume that $A$ is a Noetherian complete local ring. (The assumptions carry over; use Dualizing Complexes, Lemma 9.3.) By Lemma 17.1 the result follows from Algebraic and Formal Geometry, Lemma 15.6.

Lemma 19.7. In Situation 19.1 assume $A$ has depth $\geq 3$ and $A$ is henselian or more generally $(A,(f))$ is a henselian pair. Then the restriction functor $F\text{Ét}_U \rightarrow F\text{Ét}_{U_0}$ is fully faithful.

Proof. The assumption of depth forces $H^1_m(A) = H^2_m(A) = 0$, see Dualizing Complexes, Lemma 11.1. Hence Lemma 19.6 applies.

20. Purity in local case, I

Let $(A,m)$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{m\}$ be the punctured spectrum. We say purity holds for $(A,m)$ if the restriction functor $F\text{Ét}_X \rightarrow F\text{Ét}_U$ is essentially surjective. In this section we try to understand how the question changes when one passes from $X$ to a hypersurface $X_0$ in $X$, in other words, we study a kind of local Lefschetz property for the fundamental groups of punctured spectra. These results will be useful to proceed by induction on dimension in the proofs of our main results on local purity, namely, Lemma 21.3, Proposition 25.3, and Proposition 26.4.

Lemma 20.1. Let $(A,m)$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{m\}$. Let $\pi : Y \rightarrow X$ be a finite morphism such that depth$(\mathcal{O}_{Y,y}) \geq 2$ for all closed points $y \in Y$. Then $Y$ is the spectrum of $B = \mathcal{O}_Y(\pi^{-1}(U))$.

Proof. Set $V = \pi^{-1}(U)$ and denote $\pi' : V \rightarrow U$ the restriction of $\pi$. Consider the $\mathcal{O}_X$-module map

$$\pi_*\mathcal{O}_Y \rightarrow j_*\pi'_*\mathcal{O}_V$$

where $j : U \rightarrow X$ is the inclusion morphism. We claim Divisors, Lemma 5.11 applies to this map. If so, then $B = \Gamma(Y,\mathcal{O}_Y)$ and we see that the lemma holds. Let $x \in X$ be the closed point. It suffices to show that depth$((\pi_*\mathcal{O}_Y)_x) \geq 2$. Let $y_1, \ldots, y_n \in Y$ be the points mapping to $x$. By Algebra, Lemma 7.10 it suffices to show that depth$(\mathcal{O}_{Y,y_i}) \geq 2$ for $i = 1, \ldots, n$. Since this is the assumption of the lemma the proof is complete.

Lemma 20.2. Let $(A,m)$ be a Noetherian local ring. Set $X = \text{Spec}(A)$ and let $U = X \setminus \{m\}$. Let $V$ be finite étale over $U$. Assume $A$ has depth $\geq 2$. The following are equivalent

0BLK
(1) \( V = Y \times_X U \) for some \( Y \to X \) finite étale,
(2) \( B = \Gamma(V, \mathcal{O}_V) \) is finite étale over \( A \).

**Proof.** Denote \( \pi : V \to U \) the given finite étale morphism. Assume \( Y \) as in (1) exists. Let \( x \in X \) be the point corresponding to \( \mathfrak{m} \). Let \( y \in Y \) be a point mapping to \( x \). We claim that \( \text{depth}(\mathcal{O}_{Y,y}) \geq 2 \). This is true because \( Y \to X \) is étale and hence \( A = \mathcal{O}_{X,x} \) and \( \mathcal{O}_{Y,y} \) have the same depth (Algebra, Lemma \([157.2]\)). Hence Lemma \([20.1]\) applies and \( \mathcal{O}_Y = \text{Spec}(B) \).

The implication (2) \( \Rightarrow \) (1) is easier and the details are omitted. \( \square \)

**Lemma 20.3.** Let \( (A, \mathfrak{m}) \) be a Noetherian local ring. Set \( X = \text{Spec}(A) \) and let \( U = X \setminus \{ \mathfrak{m} \} \). Assume \( A \) is normal of dimension \( \geq 2 \). The functor

\[ \text{FÉt}_U \longrightarrow \left\{ \begin{array}{ll}
\text{finite normal } A\text{-algebras } B \text{ such that } \text{Spec}(B) \to X \text{ is étale over } U
\end{array} \right\} ,
\]  
\[ V \longmapsto \Gamma(V, \mathcal{O}_V) \]

is an equivalence. Moreover, \( V = Y \times_X U \) for some \( Y \to X \) finite étale if and only if \( B = \Gamma(V, \mathcal{O}_V) \) is finite étale over \( A \).

**Proof.** Observe that \( \text{depth}(A) \geq 2 \) because \( A \) is normal (Serre’s criterion for normality, Algebra, Lemma \([151.4]\)). Thus the final statement follows from Lemma \([20.2]\).

Given \( \pi : V \to U \) finite étale, set \( B = \Gamma(V, \mathcal{O}_V) \). If we can show that \( B \) is normal and finite over \( A \), then we obtain the displayed functor. Since there is an obvious quasi-inverse functor, this is also all that we have to show.

Since \( A \) is normal, the scheme \( V \) is normal (Descent, Lemma \([15.2]\)). Hence \( V \) is a finite disjoint union of integral schemes (Properties, Lemma \([7.6]\)). Thus we may assume \( V \) is integral. In this case the function field \( L \) of \( V \) (Morphisms, Section \([47]\)) is a finite separable extension of the function field of \( A \) (because we get it by looking at the generic fibre of \( V \to U \) and using Morphisms, Lemma \([34.7]\)). By Algebra, Lemma \([155.8]\) the integral closure \( B' \subset L \) of \( A \) in \( L \) is finite over \( A \). By More on Algebra, Lemma \([23.20]\) we see that \( B' \) is a reflexive \( A \)-module, which in turn implies that \( \text{depth}(A(B')) \geq 2 \) by More on Algebra, Lemma \([23.18]\).

Let \( f \in \mathfrak{m} \). Then \( B'_f = \Gamma(V \times_U D(f), \mathcal{O}_V) \) (Properties, Lemma \([17.1]\)). Hence \( B'_f = B_f \) because \( B_f \) is normal (see above), finite over \( A_f \) with fraction field \( L \). It follows that \( V = \text{Spec}(B'_f) \times_X U \). Then we conclude that \( B = B' \) from Lemma \([20.1]\) applied to \( \text{Spec}(B') \to X \). This lemma applies because the localizations \( B'_{\mathfrak{m}} \) of \( B' \) at maximal ideals \( \mathfrak{m}' \subset B' \) lying over \( \mathfrak{m} \) have depth \( \geq 2 \) by Algebra, Lemma \([71.10]\) and the remark on depth in the preceding paragraph. \( \square \)

**Lemma 20.4.** Let \( (A, \mathfrak{m}) \) be a Noetherian local ring. Set \( X = \text{Spec}(A) \) and let \( U = X \setminus \{ \mathfrak{m} \} \). Let \( V \) be finite étale over \( U \). Let \( A' \) be the \( \mathfrak{m} \)-adic completion of \( A \), let \( X' = \text{Spec}(A') \) and let \( U' = U' \text{ and } V' = \text{Spec}(V) \) be the base changes of \( U \) and \( V \) to \( X' \). The following are equivalent

(1) \( V = Y \times_X U \) for some \( Y \to X \) finite étale,
(2) \( V' = Y' \times_{X'} U' \) for some \( Y' \to X' \) finite étale.

**Proof.** The implication (1) \( \Rightarrow \) (2) follows from taking the base change of a solution \( Y \to X \). Let \( Y' \to X' \) be as in (2). By Lemma \([19.3]\) we can find \( Y \to X \) finite such that \( V = Y \times_X U \) and \( Y' = Y \times_X X' \). By descent we see that \( Y \to X \) is finite étale (Algebra, Lemmas \([82.2]\) and \([141.3]\)). This finishes the proof. \( \square \)
The point of the following two lemmas is that the assumptions do not force $A$ to have depth $\geq 3$. For example if $A$ is a complete normal local domain of dimension $\geq 3$ and $f \in m$ is nonzero, then the assumptions are satisfied.

**Lemma 20.5.** In Situation 19.1. Let $V$ be finite étale over $U$. Assume

(a) $A$ has a dualizing complex,
(b) the pair $(A, (f))$ is henselian,
(c) one of the following is true
   (i) $A_f$ is $(S_2)$ and every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 3$, or
   (ii) for every prime $p \subset A$, $f \not\in p$ we have $\text{depth}(A_p) + \dim(A/p) > 2$.
(d) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

**Proof.** We reduce to the complete case using Lemma 20.4. (The assumptions carry over; see proof of Lemma 19.5.)

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 13.2; observe that $(A, fA)$ is a henselian pair by More on Algebra, Lemma 11.4. Then we can use Lemma 19.5 to see that $V$ is isomorphic to $Y \times_X U$ and the proof is complete.

**Lemma 20.6.** In Situation 19.1. Let $V$ be finite étale over $U$. Assume

(1) $H^1_m(A)$ and $H^2_m(A)$ are annihilated by a power of $f$,
(2) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

**Proof.** We reduce to the complete case using Lemma 20.4 (The assumptions carry over; use Dualizing Complexes, Lemma 9.3).

In the complete case we can lift $Y_0 \to X_0$ to a finite étale morphism $Y \to X$ by More on Algebra, Lemma 13.2; observe that $(A, fA)$ is a henselian pair by More on Algebra, Lemma 11.4. Then we can use Lemma 19.6 to see that $V$ is isomorphic to $Y \times_X U$ and the proof is complete.

**Lemma 20.7.** In Situation 19.1. Let $V$ be finite étale over $U$. Assume

(1) $A$ has depth $\geq 3$,
(2) $V_0 = V \times_U U_0$ is equal to $Y_0 \times_{X_0} U_0$ for some $Y_0 \to X_0$ finite étale.

Then $V = Y \times_X U$ for some $Y \to X$ finite étale.

**Proof.** The assumption of depth forces $H^1_m(A) = H^2_m(A) = 0$, see Dualizing Complexes, Lemma 11.1. Hence Lemma 20.6 applies.

## 21. Purity of branch locus

We will use the discriminant of a finite locally free morphism. See Discriminants, Section 3.

**Lemma 21.1.** Let $(A, m)$ be a Noetherian local ring with $\dim(A) \geq 1$. Let $f \in m$. Then there exist a $p \in V(f)$ with $\dim(A_p) = 1$. 
Proof. By induction on \( \dim(A) \). If \( \dim(A) = 1 \), then \( p = m \) works. If \( \dim(A) > 1 \), then let \( Z \subset \Spec(A) \) be an irreducible component of dimension \( > 1 \). Then \( V(f) \cap Z \) has dimension \( > 0 \) (Algebra, Lemma \[29.12\]). Pick a prime \( q \in V(f) \cap Z \), \( q \not\in m \) corresponding to a closed point of the punctured spectrum of \( A \); this is possible by Properties, Lemma \[6.4\]. Then \( q \) is not the generic point of \( Z \). Hence \( 0 < \dim(A_q) < \dim(A) \) and \( f \in qA_q \). By induction on the dimension we can find \( f \in p \subset A_q \) with \( \dim((A_q)_p) = 1 \). Then \( p \cap A \) works. \( \square \)

**Lemma 21.2.** Let \( f : X \to Y \) be a morphism of locally Noetherian schemes. Let \( x \in X \). Assume

1. \( f \) is flat,
2. \( f \) is quasi-finite at \( x \),
3. \( x \) is not a generic point of an irreducible component of \( X \),
4. for specializations \( x' \leadsto x \) with \( \dim(O_{X,x}) = 1 \) our \( f \) is unramified at \( x' \).

Then \( f \) is étale at \( x \).

**Proof.** Observe that the set of points where \( f \) is unramified is the same as the set of points where \( f \) is étale and that this set is open. See Morphisms, Definitions \[33.1\] and \[34.1\] and Lemma \[34.16\]. To check \( f \) is étale at \( x \) we may work étale locally on the base and on the target (Descent, Lemmas \[20.29\] and \[28.1\]). Thus we can apply More on Morphisms, Lemma \[36.1\] and assume that \( f : X \to Y \) is finite and that \( x \) is the unique point of \( X \) lying over \( y = f(x) \). Then it follows that \( f \) is finite locally free (Morphisms, Lemma \[46.2\]).

Assume \( f \) is finite locally free and that \( x \) is the unique point of \( X \) lying over \( y = f(x) \). By Discriminants, Lemma \[3.1\] we find a locally principal closed subscheme \( D_\pi \subset Y \) such that \( y' \in D_\pi \) if and only if there exists an \( x' \in X \) with \( f(x') = y' \) and \( f \) ramified at \( x' \). Thus we have to prove that \( y \not\in D_\pi \). Assume \( y \in D_\pi \) to get a contradiction.

By condition (3) we have \( \dim(O_{X,x}) \geq 1 \). We have \( \dim(O_{X,x}) = \dim(O_{Y,y}) \) by Algebra, Lemma \[111.1\]. By Lemma \[21.1\] we can find \( y' \in D_\pi \) specializing to \( y \) with \( \dim(O_{Y,y'}) = 1 \). Choose \( x' \in X \) with \( f(x') = y' \) where \( f \) is ramified. Since \( f \) is finite it is closed, and hence \( x' \leadsto x \). We have \( \dim(O_{X,x'}) = \dim(O_{Y,y'}) = 1 \) as before. This contradicts property (4). \( \square \)

**Lemma 21.3.** Let \( (A, m) \) be a regular local ring of dimension \( d \geq 2 \). Set \( X = \Spec(A) \) and \( U = X \setminus \{m\} \). Then

1. the functor \( F\Et_X \to F\Et_U \) is essentially surjective, i.e., purity holds for \( A \),
2. any finite \( A \to B \) with \( B \) normal which induces a finite étale morphism on punctured spectra is étale.

**Proof.** Recall that a regular local ring is normal by Algebra, Lemma \[151.5\]. Hence (1) and (2) are equivalent by Lemma \[20.3\]. We prove the lemma by induction on \( d \).

The case \( d = 2 \). In this case \( A \to B \) is flat. Namely, we have going down for \( A \to B \) by Algebra, Proposition \[37.7\]. Then \( \dim(B_{m'}) = 2 \) for all maximal ideals \( m' \subset B \) by Algebra, Lemma \[111.7\]. Then \( B_{m'} \) is Cohen-Macaulay by Algebra, Lemma \[151.4\]. Hence and this is the important step Algebra, Lemma \[127.1\] applies to show \( A \to B_{m'} \) is flat. Then Algebra, Lemma \[38.19\] shows \( A \to B \) is flat. Thus we can apply Lemma \[21.2\] (or you can directly argue using the easier Discriminants, Lemma \[3.1\]) to see that \( A \to B \) is étale.
The case \( d \geq 3 \). Let \( V \to U \) be finite étale. Let \( f \in \mathfrak{m}_A, f \not\in \mathfrak{m}_A^2 \). Then \( A/fA \) is a regular local ring of dimension \( d - 1 \geq 2 \), see Algebra, Lemma \[105.3\]. Let \( U_0 \) be the punctured spectrum of \( A/fA \) and let \( V_0 = V \times_U U_0 \). By Lemma \[20.7\] it suffices to show that \( V_0 \) is in the essential image of \( F\tilde{\text{Et}}_{\text{Spec}(A/fA)} \to F\tilde{\text{Et}}_{U_0} \). This follows from the induction hypothesis. □

**Lemma 21.4** (Purity of branch locus). Let \( f : X \to Y \) be a morphism of locally Noetherian schemes. Let \( x \in X \) and set \( y = f(x) \). Assume

1. \( O_{X,x} \) is normal,
2. \( O_{Y,y} \) is regular,
3. \( f \) is quasi-finite at \( x \),
4. \( \dim(O_{X,x}) = \dim(O_{Y,y}) \geq 1 \),
5. for specializations \( x' \leadsto x \) with \( \dim(O_{X,x'}) = 1 \) our \( f \) is unramified at \( x' \).

Then \( f \) is étale at \( x \).

**Proof.** We will prove the lemma by induction on \( d = \dim(O_{X,x}) = \dim(O_{Y,y}) \).

An uninteresting case is when \( d = 1 \). In that case we are assuming that \( f \) is unramified at \( x \) and that \( O_{Y,y} \) is a discrete valuation ring (Algebra, Lemma \[118.7\]). Then \( O_{X,x} \) is flat over \( O_{Y,y} \) (otherwise the map would not be quasi-finite at \( x \)) and we see that \( f \) is flat at \( x \). Since flat + unramified is étale we conclude (some details omitted).

The case \( d \geq 2 \). We will use induction on \( d \) to reduce to the case discussed in Lemma \[21.3\]. To check \( f \) is étale at \( x \) we may work étale locally on the base and on the target (Descent, Lemmas \[20.29\] and \[28.1\]). Thus we can apply More on Morphisms, Lemma \[36.1\] and assume that \( f : X \to Y \) is finite and that \( x \) is the unique point of \( X \) lying over \( y \). Here we use that étale extensions of local rings do not change dimension, normality, and regularity, see More on Algebra, Section \[143\] and Étale Morphisms, Section \[19\].

Next, we can base change by \( \text{Spec}(O_{Y,y}) \) and assume that \( Y \) is the spectrum of a regular local ring. It follows that \( X = \text{Spec}(O_{X,x}) \) as every point of \( X \) necessarily specializes to \( x \).

The ring map \( O_{Y,y} \to O_{X,x} \) is finite and necessarily injective (by equality of dimensions). We conclude we have going down for \( O_{Y,y} \to O_{X,x} \) by Algebra, Proposition \[37.7\] (and the fact that a regular ring is a normal ring by Algebra, Lemma \[151.5\]). Pick \( x' \in X, x' \neq x \) with image \( y' = f(x') \). Then \( O_{X,x'} \) is normal as a localization of a normal domain. Similarly, \( O_{Y,y'} \) is regular (see Algebra, Lemma \[109.6\]). We have \( \dim(O_{X,x'}) = \dim(O_{Y,y'}) \) by Algebra, Lemma \[111.7\] (we checked going down above). Of course these dimensions are strictly less than \( d \) as \( x' \neq x \) and by induction on \( d \) we conclude that \( f \) is étale at \( x' \).

Thus we arrive at the following situation: We have a finite local homomorphism \( A \to B \) of Noetherian local rings of dimension \( d \geq 2 \), with \( A \) regular, \( B \) normal, which induces a finite étale morphism \( V \to U \) on punctured spectra. Our goal is to show that \( A \to B \) is étale. This follows from Lemma \[21.3\] and the proof is complete. □

The following lemma is sometimes useful to find the maximal open subset over which a finite étale morphism extends.
Lemma 21.5. Let $j : U \to X$ be an open immersion of locally Noetherian schemes such that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \not\in U$. Let $\pi : V \to U$ be finite étale. Then

1. $\mathcal{B} = j_\ast \pi_\ast \mathcal{O}_V$ is a reflexive coherent $\mathcal{O}_X$-algebra, set $Y = \text{Spec}_X(\mathcal{B})$.
2. $Y \to X$ is the unique finite morphism such that $V = Y \times_X \overline{U}$ and $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$.
3. $Y \to X$ is étale at $y$ if and only if $Y \to X$ is flat at $y$, and
4. $Y \to X$ is étale if and only if $\mathcal{B}$ is finite locally free as an $\mathcal{O}_X$-module.

Moreover, (a) the construction of $\mathcal{B}$ and $Y \to X$ commutes with base change by flat morphisms $X' \to X$ of locally Noetherian schemes, and (b) if $V' \to U'$ is a finite étale morphism with $U \subset U' \subset X$ open which restricts to $V \to U$ over $U$, then there is a unique isomorphism $Y' \times_X U' = V' \to U'$.

Proof. Observe that $\pi_\ast \mathcal{O}_V$ is a finite locally free $\mathcal{O}_U$-module, in particular reflexive. By Divisors, Lemma 12.12 the module $j_\ast \pi_\ast \mathcal{O}_V$ is the unique reflexive coherent module on $X$ restricting to $\pi_\ast \mathcal{O}_V$ over $U$. This proves (1).

By construction $Y \times_X \overline{U} = V$. Since $\mathcal{B}$ is coherent, we see that $Y \to X$ is finite. We have $\text{depth}(\mathcal{B}_x) \geq 2$ for $x \in X \setminus \overline{U}$ by Divisors, Lemma 12.11. Hence $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$ for $y \in Y \setminus V$ by Algebra, Lemma 71.10. Conversely, suppose that $\pi' : Y' \to X$ is a finite morphism such that $V = Y' \times_X \overline{U}$ and $\text{depth}(\mathcal{O}_{Y',y'}) \geq 2$ for $y' \in Y' \setminus V$. Then $\pi'_* \mathcal{O}_{Y'}$ restricts to $\pi_* \mathcal{O}_V$ over $U$ and satisfies $\text{depth}(\pi'_* \mathcal{O}_{Y'y'}) \geq 2$ for $x \in X \setminus U$ by Algebra, Lemma 71.10. Then $\pi'_* \mathcal{O}_{Y'}$ is canonically isomorphic to $j_\ast \pi_* \mathcal{O}_V$ for example by Divisors, Lemma 5.11. This proves (2).

If $Y \to X$ is étale at $y$, then $Y \to X$ is flat at $y$. Conversely, suppose that $Y \to X$ is flat at $y$. If $y \in V$, then $Y \to X$ is étale at $y$. If $y \not\in V$, then we check (1), (2), (3), and (4) of Lemma 21.2 hold to see that $Y \to X$ is étale at $y$. Parts (1) and (2) are clear and so is (3) since $\text{depth}(\mathcal{O}_{Y,y}) \geq 2$. If $y' \sim y$ is a specialization and $\text{dim}(\mathcal{O}_{Y,y'}) = 1$, then $y' \in V$ since otherwise the depth of this local ring would be 2 a contradiction by Divisors, Lemma 71.3. Hence $Y \to X$ is étale at $y'$ and we conclude (4) of Lemma 21.2 holds too. This finishes the proof of (3).

Part (4) follows from (3) and the fact that $(Y \to X)_* \mathcal{O}_Y$ is a flat $\mathcal{O}_{X,x}$-module if and only if $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-module for all $y \in Y$ mapping to $x$, see Algebra, Lemma 38.19. Here we also use that a finite flat module over a Noetherian ring is finite locally free, see Algebra, Lemma 77.2 (and Algebra, Lemma 30.4).

As to the final assertions of the lemma, part (a) follows from flat base change, see Cohomology of Schemes, Lemma 5.2, and part (b) follows from the uniqueness in (2) applied to the restriction $Y \times_X U'$.

Lemma 21.6. Let $j : U \to X$ be an open immersion of Noetherian schemes such that purity holds for $\mathcal{O}_{X,x}$ for all $x \not\in U$. Then

$$F\acute{e}t_X \to F\acute{e}t_U$$

is essentially surjective.

Proof. Let $V \to U$ be a finite étale morphism. By Noetherian induction it suffices to extend $V \to U$ to a finite étale morphism to a strictly larger open subset of $X$. Let $x \in X \setminus U$ be the generic point of an irreducible component of $X \setminus U$. Then the inverse image $U_x$ of $U$ in $\text{Spec}(\mathcal{O}_{X,x})$ is the punctured spectrum of $\mathcal{O}_{X,x}$. By assumption $V_x = V \times_U U_x$ is the restriction of a finite étale morphism $Y_x \to \text{Spec}(\mathcal{O}_{X,x})$ to
By Limits, Lemma 18.3 we find an open subscheme $U \subset U' \subset X$ containing $x$ and a morphism $V' \to U'$ of finite presentation whose restriction to $U$ recovers $V \to U$ and whose restriction to $\text{Spec}(\mathcal{O}_{X,x})$ recovering $Y_x$. Finally, the morphism $V' \to U'$ is finite étale after possible shrinking $U'$ to a smaller open by Limits, Lemma 18.4. □

22. Finite étale covers of punctured spectra, II

In this section we prove some variants of the material discussed in Section 19. Suppose we have a Noetherian local ring $(A, \mathfrak{m})$ and $f \in \mathfrak{m}$. We set $X = \text{Spec}(A)$ and $X_0 = \text{Spec}(A/fA)$ and we let $U = X \setminus \{\mathfrak{m}\}$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$ be the punctured spectrum of $A$ and $A/fA$. All of this is exactly as in Situation 19.1. The difference is that we will consider the restriction functor $\colim_{U_0 \subset U' \subset U} F\acute{E}t_{U'} \to F\acute{E}t_{U_0}$ in other words, we will not try to lift finite étale coverings of $U_0$ to all of $U$, but just to some open neighbourhood $U'$ of $U_0$ in $U$.

Lemma 22.1. In Situation 19.1. Let $U' \subset U$ be open and contain $U_0$. Assume for $p \subset A$ minimal with $p \subseteq U'$, $p \notin U_0$ we have $\dim(A/p) \geq 2$. Then $F\acute{E}t_{U'}, V' \mapsto V_0 = V' \times_{U'} U_0$ is a faithful functor. Moreover, there exists a $U'$ satisfying the assumption and any smaller open $U'' \subset U'$ containing $U_0$ also satisfies this assumption. In particular, the restriction functor $\colim_{U_0 \subset U' \subset U} F\acute{E}t_{U'} \to F\acute{E}t_{U_0}$ is faithful.

Proof. By Algebra, Lemma 59.12 we see that $V(p)$ meets $U_0$ for every prime $p$ of $A$ with $\dim(A/p) \geq 2$. Thus the displayed functor is faithful for a $U$ as in the statement by Lemma 17.5. To see the existence of such a $U'$ note that for $p \subset A$ with $p \notin U_0$ with $\dim(A/p) = 1$ then $p$ corresponds to a closed point of $U$ and hence $V(p) \cap U_0 = \emptyset$. Thus we can take $U'$ to be the complement of the irreducible components of $X$ which do not meet $U_0$ and have dimension 1. □

Lemma 22.2. In Situation 19.1 assume

1. $A$ has a dualizing complex and is $f$-adically complete,
2. every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 3$.

Then the restriction functor $\colim_{U_0 \subset U' \subset U} F\acute{E}t_{U'} \to F\acute{E}t_{U_0}$ is fully faithful.

Proof. To prove this we may replace $A$ by its reduction by the topological invariance of the fundamental group, see Lemma 8.3. Then the result follows from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.8. □

Lemma 22.3. In Situation 19.1 assume

1. $A$ is $f$-adically complete,
2. $f$ is a nonzerodivisor,
3. $H^1_{\mathfrak{m}}(A/fA)$ is a finite $A$-module.
Then the restriction functor
$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \to \text{FÉt}_{U_0}$$
is fully faithful.

**Proof.** Follows from Lemma 17.3 and Algebraic and Formal Geometry, Lemma 15.9. □

### 23. Finite étale covers of punctured spectra, III

In this section we study when in Situation 19.1. the restriction functor
$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \to \text{FÉt}_{U_0}$$
is an equivalence of categories.

**Lemma 23.1.** In Situation 19.1 assume
1. $A$ has a dualizing complex and is $f$-adically complete,
2. one of the following is true
   a. $A_f$ is $(S_2)$ and every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 4$, or
   b. if $p \notin V(f)$ and $V(p) \cap V(f) \neq \{m\}$, then $\text{depth}(A_p) + \dim(A/p) > 3$.

Then the restriction functor
$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \to \text{FÉt}_{U_0}$$
is an equivalence.

**Proof.** This follows from Lemma 17.4 and Algebraic and Formal Geometry, Lemma 24.1. □

**Lemma 23.2.** In Situation 19.1 assume
1. $A$ is $f$-adically complete,
2. $f$ is a nonzerodivisor,
3. $H^1_m(A/fA)$ and $H^2_m(A/fA)$ are finite $A$-modules.

Then the restriction functor
$$\text{colim}_{U_0 \subset U' \subset U \text{ open}} \text{FÉt}_{U'} \to \text{FÉt}_{U_0}$$
is an equivalence.

**Proof.** This follows from Lemma 17.4 and Algebraic and Formal Geometry, Lemma 24.2. □

**Remark 23.3.** Let $(A, m)$ be a complete local Noetherian ring and $f \in m$ nonzero. Suppose that $A_f$ is $(S_2)$ and every irreducible component of $\text{Spec}(A)$ has dimension $\geq 4$. Then Lemma 23.1 tells us that the category
$$\text{colim}_{U' \subset U \text{ open}, U_0 \subset U} \text{ category of schemes finite étale over } U'$$is equivalent to the category of schemes finite étale over $U_0$. For example this holds if $A$ is a normal domain of dimension $\geq 4$!
24. Finite étale covers of punctured spectra, IV

Let $X, X_0, U, U_0$ be as in Situation 19.1. In this section we ask when the restriction functor

$$FÉt_U \rightarrow FÉt_{U_0}$$

is essentially surjective. We will do this by taking results from Section 23 and then filling in the gaps using purity. Recall that we say purity holds for a Noetherian local ring $(A, m)$ if the restriction functor $FÉt_X \rightarrow FÉt_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{m\}$.

0EK9 Lemma 24.1. In Situation 19.1 assume

1. $A$ has a dualizing complex and is $f$-adically complete,
2. one of the following is true
   a. $A_f$ is $(S_2)$ and every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 4$, or
   b. if $p \notin V(f)$ and $V(p) \cap V(f) \neq \{m\}$, then $\text{depth}(A_p) + \dim(A/p) > 3$.
3. for every maximal ideal $p \subset A_f$ purity holds for $(A_f)_p$.

Then the restriction functor $FÉt_U \rightarrow FÉt_{U_0}$ is essentially surjective.

Proof. Let $V_0 \rightarrow U_0$ be a finite étale morphism. By Lemma 23.1 there exists an open $U' \subset U$ containing $U_0$ and a finite étale morphism $V' \rightarrow U$ whose base change to $U_0$ is isomorphic to $V_0 \rightarrow U_0$. Since $U' \supset U_0$ we see that $U \setminus U'$ consists of points corresponding to prime ideals $p_1, \ldots, p_n$ as in (3). By assumption we can find finite étale morphisms $V' \rightarrow \text{Spec}(A_{p_i})$ agreeing with $V' \rightarrow U'$ over $U' \times_U \text{Spec}(A_{p_i})$. By Limits, Lemma 18.1 applied $n$ times we see that $V' \rightarrow U'$ extends to a finite étale morphism $V \rightarrow U$.

0EKA Lemma 24.2. Let $(A, m)$ be a Noetherian local ring. Let $f \in m$. Assume

1. $A$ is $f$-adically complete,
2. $f$ is a nonzerodivisor,
3. $H^2_m(A/fA)$ and $H^2_m(A/A_f)$ are finite $A$-modules,
4. for every maximal ideal $p \subset A_f$ purity holds for $(A_f)_p$.

Then the restriction functor $FÉt_U \rightarrow FÉt_{U_0}$ is essentially surjective.

Proof. The proof is identical to the proof of Lemma 24.1 using Lemma 23.2 in stead of Lemma 23.1.

25. Purity in local case, II

This section is the continuation of Section 20. Recall that we say purity holds for a Noetherian local ring $(A, m)$ if the restriction functor $FÉt_X \rightarrow FÉt_U$ is essentially surjective where $X = \text{Spec}(A)$ and $U = X \setminus \{m\}$.

0DXZ Lemma 25.1. Let $(A, m)$ be a Noetherian local ring. Let $f \in m$. Assume

1. $A$ has a dualizing complex and is $f$-adically complete,
2. one of the following is true
   a. $A_f$ is $(S_2)$ and every irreducible component of $X$ not contained in $X_0$ has dimension $\geq 4$, or
   b. if $p \notin V(f)$ and $V(p) \cap V(f) \neq \{m\}$, then $\text{depth}(A_p) + \dim(A/p) > 3$.
3. for every maximal ideal $p \subset A_f$ purity holds for $(A_f)_p$, and
4. purity holds for $A$. 
Then purity holds for $A/ fA$.

**Proof.** Denote $X = \text{Spec}(A)$ and $U = X \setminus \{m\}$ the punctured spectrum. Similarly we have $X_0 = \text{Spec}(A/ fA)$ and $U_0 = X_0 \setminus \{m\}$. Let $V_0 \rightarrow U_0$ be a finite étale morphism. By Lemma 24.1 we find a finite étale morphism $V \rightarrow U$ whose base change to $U_0$ is isomorphic to $V_0 \rightarrow U_0$. By assumption (5) we find that $V \rightarrow U$ extends to a finite étale morphism $Y \rightarrow X$. Then the restriction of $Y$ to $X_0$ is the desired extension of $V_0 \rightarrow U_0$. \hfill \Box

**Lemma 25.2.** Let $(A, m)$ be a Noetherian local ring. Let $f \in m$. Assume

1. $A$ is $f$-adically complete,
2. $f$ is a nonzerodivisor,
3. $H^1_m(A/ fA)$ and $H^2_m(A/ fA)$ are finite $A$-modules,
4. for every maximal ideal $p \subset A_f$ purity holds for $(A_f)_p$,
5. purity holds for $A/ fA$.

Then purity holds for $A/ fA$.

**Proof.** The proof is identical to the proof of Lemma 25.1 using Lemma 24.2 in stead of Lemma 24.1 \hfill \Box

Now we can bootstrap the earlier results to prove that purity holds for complete intersections of dimension $\geq 3$. Recall that a Noetherian local ring is called a complete intersection if its completion is the quotient of a regular local ring by the ideal generated by a regular sequence. See the discussion in Divided Power Algebra, Section 8.

**Proposition 25.3.** Let $(A, m)$ be a Noetherian local ring. If $A$ is a complete intersection of dimension $\geq 3$, then purity holds for $A$ in the sense that any finite étale cover of the punctured spectrum extends.

**Proof.** By Lemma 20.4 we may assume that $A$ is a complete local ring. By assumption we can write $A = B/(f_1, \ldots, f_r)$ where $B$ is a complete regular local ring and $f_1, \ldots, f_r$ is a regular sequence. We will finish the proof by induction on $r$. The base case is $r = 0$ which follows from Lemma 21.3 which applies to regular rings of dimension $\geq 2$.

Assume that $A = B/(f_1, \ldots, f_r)$ and that the proposition holds for $r - 1$. Set $A' = B/(f_1, \ldots, f_{r-1})$ and apply Lemma 25.2 to $f_r \in A'$. This is permissible: condition (1) holds as $f_1, \ldots, f_r$ is a regular sequence, condition (2) holds as $B$ and hence $A'$ is complete, condition (3) holds as $A = A'/ f_r A'$ is Cohen-Macaulay of dimension $\text{dim}(A) \geq 3$, see Dualizing Complexes, Lemma 11.1, condition (4) holds by induction hypothesis as $\text{dim}((A'_{f_r})_p) \geq 3$ for a maximal prime $p$ of $A'_{f_r}$ and as $(A'_{f_r})_p = B_q/(f_1, \ldots, f_{r-1})$ for some $q \subset B$, condition (5) holds by induction hypothesis. \hfill \Box

**26. Purity in local case, III**

In this section is a continuation of the discussion in Sections 20 and 25.
is fully faithful.

**Proof.** Set $I = (x_1, \ldots, x_d)$. Set $X = \text{Spec}(A)$. If we use the map $Y \to X$ to identify $Y_0$ with $X$, then $V_0$ is identified with the punctured spectrum $U$ of $A$. Pushing forward modules by this affine morphism we get

$$\lim_n \Gamma(V_0, \mathcal{O}_V / I^n \mathcal{O}_V) = \lim_n \Gamma(V_0, \mathcal{O}_Y / I^n \mathcal{O}_Y) = \lim_n \Gamma(U, \mathcal{O}_U[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n) = \lim_n A[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n = B$$

Namely, as the depth of $A$ is at least 2 we have $\Gamma(U, \mathcal{O}_U) = A$, see Local Cohomology, Lemma 7.2. Thus for any $V \subset Y$ open as in the lemma we get

$$B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(V, \mathcal{O}_V) \to \lim_n \Gamma(V_0, \mathcal{O}_Y/I^n \mathcal{O}_Y) = B$$

which implies both arrows are isomorphisms (small detail omitted). By Algebraic and Formal Geometry, Lemma 15.1 we conclude that $\text{Coh}(\mathcal{O}_V) \to \text{Coh}(V, I \mathcal{O}_V)$ is fully faithful on the full subcategory of finite locally free objects. Thus we conclude by Lemma 7.1. □

**Lemma 26.2.** Let $(A, m)$ be a Noetherian local ring of depth $\geq 2$. Let $B = A[[x_1, \ldots, x_d]]$ with $d \geq 1$. For any open $V \subset Y = \text{Spec}(B)$ which contains

1. any prime $q \subset B$ such that $q \cap A \neq m$,
2. the prime $mB$

the functor $\text{FÉt}_V \to \text{FÉt}_V$ is an equivalence. In particular purity holds for $B$.

**Proof.** A prime $q \subset B$ which is not contained in $V$ lies over $m$. In this case $A \to B_q$ is a flat local homomorphism and hence depth$(B_q) \geq 2$ (Algebra, Lemma 157.2). Thus the functor is fully faithful by Lemma 10.4 combined with Lemma 10.3.

Let $W \to V$ be a finite étale morphism. Let $B \to C$ be the unique finite ring map such that $\text{Spec}(C) \to Y$ is the finite morphism extending $W \to V$ constructed in Lemma 21.5. Observe that $C = \Gamma(W, \mathcal{O}_W)$.

Set $Y_0 = V(x_1, \ldots, x_d)$ and $V_0 = V \cap Y_0$. Set $X = \text{Spec}(A)$. If we use the map $Y \to X$ to identify $Y_0$ with $X$, then $V_0$ is identified with the punctured spectrum $U$ of $A$. Thus we may view $W_0 = W \times_Y Y_0$ as a finite étale scheme over $U$. Then

$W_0 \times_U (U \times_Y Y) \text{ and } W \times_V (U \times_Y Y)$

are schemes finite étale over $U \times_Y Y$ which restrict to isomorphic finite étale schemes over $V_0$. By Lemma 26.1 applied to the open $U \times_Y Y$ we obtain an isomorphism

$W_0 \times_U (U \times_Y Y) \to W \times_V (U \times_Y Y)$

over $U \times_Y Y$.

Observe that $C_0 = \Gamma(W_0, \mathcal{O}_{W_0})$ is a finite $A$-algebra by Lemma 21.5 applied to $W_0 \to U \subset X$ (exactly as we did for $B \to C$ above). Since the construction in Lemma 21.5 is compatible with flat base change and with change of opens, the isomorphism above induces an isomorphism

$$\Psi : C \to C_0 \otimes_A B$$

of finite $B$-algebras. However, we know that $\text{Spec}(C) \to Y$ is étale at all points above at least one point of $Y$ lying over $m \in X$. Since $\Psi$ is an isomorphism, we
conclude that $\text{Spec}(C_0) \to X$ is étale above $m$ (small detail omitted). Of course this means that $A \to C_0$ is finite étale and hence $B \to C$ is finite étale. □

**Lemma 26.3.** Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ be an open subscheme. Assume

1. $f$ is smooth,
2. $S$ is Noetherian,
3. for $s \in S$ with $\text{depth}(\mathcal{O}_{S,s}) \leq 1$ we have $X_s = U_s$,
4. $U_s \subset X_s$ is dense for all $s \in S$.

Then $\text{FÉt}_X \to \text{FÉt}_U$ is an equivalence.

**Proof.** The functor is fully faithful by Lemma 10.4 combined with Lemma 10.3 (plus an application of Algebra, Lemma 157.2 to check the depth condition).

Let $\pi : V \to U$ be a finite étale morphism. Let $Y \to X$ be the finite morphism constructed in Lemma 21.5. We have to show that $Y \to X$ is finite étale. To show that this is true for all points $x \in X$ mapping to a given point $s \in S$ we may perform a base change by a flat morphism $S' \to S$ of Noetherian schemes such that $s$ is in the image. This follows from the compatibility of the construction in Lemma 21.5 with flat base change.

After enlarging $U$ we may assume $U \subset X$ is the maximal open over which $Y \to X$ is finite étale. Let $Z \subset X$ be the complement of $U$. To get a contradiction, assume $Z \neq \emptyset$. Let $s \in S$ be a point in the image of $Z \to S$ such that no strict generalization of $s$ is in the image. Then after base change to $\text{Spec}(\mathcal{O}_{S,s})$ we see that $S = \text{Spec}(A)$ with $(A, m, \kappa)$ a local Noetherian ring of depth $\geq 2$ and $Z$ contained in the closed fibre $X_s$ and nowhere dense in $X_s$. Choose a closed point $z \in Z$. Then $\kappa(z)/\kappa$ is finite (by the Hilbert Nullstellensatz, see Algebra, Theorem 33.1). Choose a finite flat morphism $(S', s') \to (S, s)$ of local schemes realizing the residue field extension $\kappa(z)/\kappa$, see Algebra, Lemma 153.3. After doing a base change by $S' \to S$ we reduce to the case where $\kappa(z) = \kappa$.

By More on Morphisms, Lemma 34.5 there exists a locally closed subscheme $S' \subset X$ passing through $z$ such that $S' \to S$ is étale at $z$. After performing the base change by $S' \to S$, we may assume there is a section $\sigma : S \to X$ such that $\sigma(s) = z$. Choose an affine neighbourhood $\text{Spec}(B) \subset X$ of $s$. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \text{Ker}(\sigma)$ and denote $B^\wedge$ the $I$-adic completion of $B$. Then $B^\wedge \cong A[[x_1, \ldots, x_d]]$ for some $d \geq 0$, see Algebra, Lemma 137.4. Observe that $d > 0$ since otherwise we see that $X \to S$ is étale at $z$ which would imply that $z$ is a generic point of $X_s$ and hence $z \in U$ by assumption (4). Similarly, if $d > 0$, then $mB^\wedge$ maps into $U$ via the morphism $\text{Spec}(B^\wedge) \to X$. It suffices prove $Y \to X$ is finite étale after base change to $\text{Spec}(B^\wedge)$. Since $B \to B^\wedge$ is flat (Algebra, Lemma 36.2) this follows from Lemma 26.2 and the uniqueness in the construction of $Y \to X$. □

**Proposition 26.4.** Let $A \to B$ be a local homomorphism of local Noetherian rings. Assume $A$ has depth $\geq 2$, $A \to B$ is formally smooth for the $m_B$-adic topology, and $\dim(B) > \dim(A)$. For any open $V \subset Y = \text{Spec}(B)$ which contains

1. any prime $q \subset B$ such that $q \cap A \neq m_A$,
2. the prime $m_AB$

the functor $\text{FÉt}_V \to \text{FÉt}_U$ is an equivalence. In particular purity holds for $B$. 
Proof. A prime \( q \subset B \) which is not contained in \( V \) lies over \( m_A \). In this case \( A \to B_q \) is a flat local homomorphism and hence \( \text{depth}(B_q) \geq 2 \) (Algebra, Lemma 157.2). Thus the functor is fully faithful by Lemma 10.4 combined with Lemma 10.3.

Denote \( A^\wedge \) and \( B^\wedge \) the completions of \( A \) and \( B \) with respect to their maximal ideals. Observe that the assumptions of the proposition hold for \( A^\wedge \to B^\wedge \), see More on Algebra, Lemmas 42.1, 42.2, and 36.4. By the uniqueness and compatibility with flat base change of the construction of Lemma 21.5 it suffices to prove the essential surjectivity for \( A^\wedge \to B^\wedge \) and the inverse image of \( V \) (details omitted; compare with Lemma 20.4 for the case where \( V \) is the punctured spectrum). By More on Algebra, Proposition 48.2 this means we may assume \( A \to B \) is regular.

Let \( W \to V \) be a finite étale morphism. By Popescu’s theorem (Smoothing Ring Maps, Theorem 12.1) we can write \( B = \text{colim} B_i \) as a filtered colimit of smooth \( A \)-algebras. We can pick an \( i \) and an open \( V_i \subset \text{Spec}(B_i) \) whose inverse image is \( V \) (Limits, Lemma 4.11). After increasing \( i \) we may assume there is a finite étale morphism \( W_i \to V_i \) whose base change to \( V \) is \( W \to V \), see Limits, Lemmas 10.1, 8.3, and 8.10. We may assume the complement of \( V_i \) is contained in the closed fibre of \( \text{Spec}(B_i) \to \text{Spec}(A) \) (details omitted; use that a scheme smooth over a field is a disjoint union of irreducible schemes). At this point we may apply Lemma 26.3 to see that \( W_i \to V_i \) extends to a finite étale morphism \( \text{Spec}(C_i) \to \text{Spec}(B_i) \) and pulling back to \( \text{Spec}(B) \) we conclude that \( W \) is in the essential image of the functor \( \mathcal{F} \text{Et}_Y \to \mathcal{F} \text{Et}_V \) as desired. \( \square \)

27. Lefschetz for the fundamental group

Of course we have already proven a bunch of results of this type in the local case. In this section we discuss the projective case.

Proposition 27.1. Let \( k \) be a field. Let \( X \) be a proper scheme over \( k \). Let \( L \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( s \in \Gamma(X, L) \). Let \( Y = Z(s) \) be the zero scheme of \( s \). Assume that for all \( x \in X \setminus Y \) we have

\[
\text{depth} \left( \mathcal{O}_{X,x} \right) + \text{dim} \left( \{ x \} \right) > 1
\]

Then the restriction functor \( \mathcal{F} \text{Et}_X \to \mathcal{F} \text{Et}_Y \) is fully faithful. In fact, for any open subscheme \( V \subset X \) containing \( Y \) the restriction functor \( \mathcal{F} \text{Et}_V \to \mathcal{F} \text{Et}_Y \) is fully faithful.

Proof. The first statement is a formal consequence of Lemma 17.6 and Algebraic and Formal Geometry, Proposition 28.1. The second statement follows from Lemma 17.6 and Algebraic and Formal Geometry, Lemma 28.2. \( \square \)

Proposition 27.2. Let \( k \) be a field. Let \( X \) be a proper scheme over \( k \). Let \( L \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( s \in \Gamma(X, L) \). Let \( Y = Z(s) \) be the zero scheme
of $s$. Let $V$ be the set of open subschemes of $X$ containing $Y$ ordered by reverse inclusion. Assume that for all $x \in X \setminus Y$ we have
\[ \text{depth}(\mathcal{O}_{X,x}) + \dim(\{x\}) > 2 \]
Then the restriction functor
\[ \text{colim}_V F^\text{ét}_V \to F^\text{ét}_Y \]
is an equivalence.

**Proof.** This is a formal consequence of Lemma 17.4 and Algebraic and Formal Geometry, Proposition 28.7. □

**Proposition 27.3.** Let $k$ be a field. Let $X$ be a proper scheme over $k$. Let $L$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, L)$. Let $Y = Z(s)$ be the zero scheme of $s$. Assume that for all $x \in X \setminus Y$ we have
\[ \text{depth}(\mathcal{O}_{X,x}) + \dim(\{x\}) > 2 \]
and that for $x \in X \setminus Y$ closed purity holds for $\mathcal{O}_{X,x}$. Then the restriction functor $F^\text{ét}_X \to F^\text{ét}_Y$ is an equivalence. If $X$ or equivalently $Y$ is connected, then $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{y})$ is an isomorphism for any geometric point $\overline{y}$ of $Y$.

**Proof.** Fully faithfulness holds by Proposition 27.1. By Proposition 27.2 any object of $F^\text{ét}_Y$ is isomorphic to the fibre product $U \times V_Y$ for some finite étale morphism $U \to V$ where $V \subset X$ is an open subscheme containing $Y$. The complement $T = X \setminus V$ is a finite set of closed points of $X \setminus Y$. Say $T = \{x_1, \ldots, x_n\}$. By assumption we can find finite étale morphisms $V'_i \to \text{Spec}(\mathcal{O}_{X, x_i})$ agreeing with $U \to V$ over $V \times_X \text{Spec}(\mathcal{O}_{X, x_i})$. By Limits, Lemma [8.1] applied $n$ times we see that $U \to V$ extends to a finite étale morphism $U' \to X$ as desired. See Lemma [8.1] for the final statement. □

28. Purity of ramification locus

0EA1 In this section we discuss the analogue of purity of branch locus for generically finite morphisms. Apparently, this result is due to Gabber. A special case is van der Waerden’s purity theorem for the locus where a birational morphism from a normal variety to a smooth variety is not an isomorphism.

0EA2 **Lemma 28.1.** Let $A$ be a Noetherian normal local domain of dimension 2. Assume $A$ is Nagata, has a dualizing module $\omega_A$, and has a resolution of singularities $f : X \to \text{Spec}(A)$. Let $\omega_X$ be as in Resolution of Surfaces, Remark 7.7. If $\omega_X \cong \mathcal{O}_X(E)$ for some effective Cartier divisor $E \subset X$ supported on the exceptional fibre, then $A$ defines a rational singularity. If $f$ is a minimal resolution, then $E = 0$.

**Proof.** There is a trace map $Rf_* \omega_X \to \omega_A$, see Duality for Schemes, Section 7. By Grauert-Riemenschneider (Resolution of Surfaces, Proposition 7.8) we have $R^1f_* \omega_X = 0$. Thus the trace map is a map $f_* \omega_X \to \omega_A$. Then we can consider
\[ \mathcal{O}_{\text{Spec}(A)} = f_* \mathcal{O}_X \to f_* \omega_X \to \omega_A \]

3Namely, $T$ is proper over $k$ (being closed in $X$) and affine (being closed in the affine scheme $X \setminus Y$, see Morphisms, Lemma 41.18) and hence finite over $k$ (Morphisms, Lemma 42.11). Thus $T$ is a finite set of closed points.
where the first map comes from the map $\mathcal{O}_X \to \mathcal{O}_X(E) = \omega_X$ which is assumed to exist in the statement of the lemma. The composition is an isomorphism by Divisors, Lemma 21.11 as it is an isomorphism over the punctured spectrum of $A$ (by the assumption in the lemma and the fact that $f$ is an isomorphism over the punctured spectrum) and $A$ and $\omega_A$ are $A$-modules of depth 2 (by Algebra, Lemma 151.4 and Dualizing Complexes, Lemma 17.5). Hence $f_*\omega_X \to \omega_A$ is surjective whence an isomorphism. Thus $Rf_*\omega_X = \omega_A$ which by duality implies $Rf_*\mathcal{O}_X = \mathcal{O}_{\text{Spec}(A)}$. Whence $H^1(X, \mathcal{O}_X) = 0$ which implies that $A$ defines a rational singularity (see discussion in Resolution of Surfaces, Section 8 in particular Lemmas 8.7 and 8.1). If $f$ is minimal, then $E = 0$ because the map $f^*\omega_A \to \omega_X$ is surjective by a repeated application of Resolution of Surfaces, Lemma 9.7 and in particular Lemmas 8.7 and 8.1.

0EA3 Lemma 28.2. Let $f : X \to \text{Spec}(A)$ be a finite type morphism. Let $x \in X$ be a point. Assume

1. $A$ is an excellent local regular ring,
2. $\mathcal{O}_{X,x}$ is normal of dimension 2,
3. $f$ is étale outside of $\{x\}$.

Then $f$ is étale at $x$.

Proof. We first replace $X$ by an affine open neighbourhood of $x$. Observe that $\mathcal{O}_{X,x}$ is an excellent local ring (More on Algebra, Lemma 51.2). Thus we can choose a minimal resolution of singularities $W \to \text{Spec}(\mathcal{O}_{X,x})$, see Resolution of Surfaces, Theorem 14.3. After possibly replacing $X$ by an affine open neighbourhood of $x$ we can find a proper morphism $b : X' \to X$ such that $X' \times_X \text{Spec}(\mathcal{O}_{X,x}) = W$, see Limits, Lemma 15.1. After shrinking $X$ further, we may assume $X'$ is regular. Namely, we know $W$ is regular and $X'$ is excellent and the regular locus of the spectrum of an excellent ring is open. Since $W \to \text{Spec}(\mathcal{O}_{X,x})$ is projective (as a sequence of normalized blowing ups), we may assume after shrinking $X$ that $b$ is projective (details omitted). Let $U = X \setminus \{x\}$. Since $W \to \text{Spec}(\mathcal{O}_{X,x})$ is an isomorphism over the punctured spectrum, we may assume $b : X' \to X$ is an isomorphism over $U$. Thus we may and will think of $U$ as an open subscheme of $X'$ as well. Set $f' = f \circ b : X' \to \text{Spec}(A)$.

Since $A$ is regular we see that $\mathcal{O}_Y$ is a dualizing complex for $Y$. Hence $f'^*\mathcal{O}_Y$ is a dualizing complex on $X$ (Duality for Schemes, Lemma 17.6). The Cohen-Macaulay locus of $X$ is open by Duality for Schemes, Lemma 23.1 (this can also be proven using excellency). Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay, after shrinking $X$ we may assume $X$ is Cohen-Macaulay. Observe that an étale morphism is a local complete intersection. Thus Duality for Schemes, Lemma 28.3 applies with $r = 0$ and we get a map

$$\mathcal{O}_X \longrightarrow \omega_{X/Y} = H^0(f'^*\mathcal{O}_Y)$$

which is an isomorphism over $X \setminus \{x\}$. Since $\omega_{X/Y}$ is $(S_2)$ by Duality for Schemes, Lemma 21.5 we find this map is an isomorphism by Divisors, Lemma 21.11. This already shows that $X$ and in particular $\mathcal{O}_{X,x}$ is Gorenstein.

Set $\omega_{X'/Y} = H^0((f')^*\mathcal{O}_Y)$. Arguing in exactly the same manner as above we find that $(f')^*\mathcal{O}_Y = \omega_{X'/Y}[0]$ is a dualizing complex for $X'$. Since $X'$ is regular the morphism $X' \to Y$ is a local complete intersection morphism, see More on
Morphisms, Lemma \ref{duality-lem}. By Duality for Schemes, Lemma \ref{dual-schemes-lem} there exists a map
\[ \mathcal{O}_{X'} \to \omega_{X'/Y} \]
which is an isomorphism over \( U \). We conclude \( \omega_{X'/Y} = \mathcal{O}_{X'}(E) \) for some effective Cartier divisor \( E \subset X' \) disjoint from \( U \).

Since \( \omega_{X'/Y} = \mathcal{O}_Y \) we see that \( \omega_{X'/Y} = b'f'\mathcal{O}_Y = b'\mathcal{O}_X \). Returning to \( W \to \text{Spec}(\mathcal{O}_{X,x}) \) we see that \( \omega_W = \mathcal{O}_W(E|_W) \). By Lemma \ref{dual-schemes-lem} we find \( E|_W = 0 \). This means that \( f' : X' \to Y \) is étale by (the already used) Duality for Schemes, Lemma \ref{dual-schemes-lem}. This immediately finishes the proof, as étalement of \( f' \) forces \( b \) to be an isomorphism.

\[ \square \]

\begin{lemma}[Purity of ramification locus]
Let \( f : X \to Y \) be a morphism of locally Noetherian schemes. Let \( x \in X \) and set \( y = f(x) \). Assume
\begin{enumerate}
\item \( \mathcal{O}_{X,x} \) is normal of dimension \( \geq 1 \),
\item \( \mathcal{O}_{Y,y} \) is regular,
\item \( f \) is locally of finite type, and
\item for specializations \( x' \to x \) with \( \dim(\mathcal{O}_{X,x'}) = 1 \) our \( f \) is étale at \( x' \).
\end{enumerate}
Then \( f \) is étale at \( x \).
\end{lemma}

\textbf{Proof.} We will prove the lemma by induction on \( d = \dim(\mathcal{O}_{X,x}) \).

An uninteresting case is \( d = 1 \) since in that case the morphism \( f \) is étale at \( x \) by assumption. Assume \( d \geq 2 \).

We can base change by \( \text{Spec}(\mathcal{O}_{Y,y}) \to Y \) without affecting the conclusion of the lemma, see Morphisms, Lemma \ref{spec-lem}. Thus we may assume \( Y = \text{Spec}(A) \) where \( A \) is a regular local ring and \( y \) corresponds to the maximal ideal \( m \) of \( A \).

Let \( x' \to x \) be a specialization with \( x' \neq x \). Then \( \mathcal{O}_{X,x'} \) is normal as a localization of \( \mathcal{O}_{X,x} \). If \( x' \) is not a generic point of \( X \), then \( 1 \leq \dim(\mathcal{O}_{X,x'}) < d \) and we conclude that \( f \) is étale at \( x' \) by induction hypothesis. Thus we may assume that \( f \) is étale at all points specializing to \( x \). Since the set of points where \( f \) is étale is open in \( X \) (by definition) we may after replacing \( X \) by an open neighbourhood of \( x \) assume that \( f \) is étale away from \( \overline{\{x\}} \). In particular, we see that \( f \) is étale except at points lying over the closed point \( y \in Y = \text{Spec}(A) \).

Let \( X' = X \times_{\text{Spec}(A)} \text{Spec}(A^\wedge) \). Let \( x' \in X' \) be the unique point lying over \( x \). By the above we see that \( X' \) is étale over \( \text{Spec}(A^\wedge) \) away from the closed fibre and hence \( X' \) is normal away from the closed fibre. Since \( X \) is normal we conclude that \( X' \) is normal by Resolution of Surfaces, Lemma \ref{res-surf-lem}. Then if we can show \( X' \to \text{Spec}(A^\wedge) \) is étale at \( x' \), then \( f \) is étale at \( x \) (by the aforementioned Morphisms, Lemma \ref{spec-lem}). Thus we may and do assume \( A \) is a regular complete local ring.

The case \( d = 2 \) now follows from Lemma \ref{dual-schemes-lem}.

Assume \( d > 2 \). Let \( t \in m, t \notin m^2 \). Set \( Y_0 = \text{Spec}(A/tA) \) and \( X_0 = X \times_Y Y_0 \). Then \( X_0 \to Y_0 \) is étale away from the fibre over the closed point. Since \( d > 2 \) we have \( \dim(\mathcal{O}_{X_0,x}) = d - 1 \) is \( \geq 2 \). The normalization \( X'_0 \to X_0 \) is surjective and finite (as we’re working over a complete local ring and such rings are Nagata). Let \( x' \in X'_0 \) be a point mapping to \( x \). By induction hypothesis the morphism \( X'_0 \to Y \) is étale at \( x' \). From the inclusions \( \kappa(y) \subset \kappa(x) \subset \kappa(x') \) we conclude that \( \kappa(x) \) is finite over \( \kappa(y) \). Hence \( x \) is a closed point of the fibre of \( X \to Y \) over \( y \). But since \( x \) is also a
FUNDAMENTAL GROUPS OF SCHEMES

generic point of this fibre, we conclude that \( f \) is quasi-finite at \( x \) and we reduce to the case of purity of branch locus, see Lemma 21.4

29. Affineness of complement of ramification locus

Let \( f : X \to Y \) be a finite type morphism of Noetherian schemes with \( X \) normal and \( Y \) regular. Let \( V \subset X \) be the maximal open subscheme where \( f \) is étale. The discussion in [DG67, Chapter IV, Section 21.12] suggests that \( V \to X \) might be an affine morphism. Observe that if \( V \to X \) is affine, then we deduce purity of ramification locus (Lemma 28.3) by using Divisors, Lemma 16.4. Thus affineness of \( V \to X \) is a “strong” form of purity for the ramification locus. In this section we prove \( V \to X \) is affine when \( X \) and \( Y \) are equicharacteristic and excellent, see Theorem 29.3. It seems reasonable to guess the result remains true for \( X \) and \( Y \) of mixed characteristic (but still excellent).

**Lemma 29.1.** Let \( (A, \mathfrak{m}) \) be a regular local ring which contains a field. Let \( f : V \to \text{Spec}(A) \) be étale and quasi-compact. Assume that \( \mathfrak{m} \notin f(V) \) and assume that \( g : V \to \text{Spec}(A) \setminus \{ \mathfrak{m} \} \) is affine. Then \( H^i(V, \mathcal{O}_V) \), \( i > 0 \) is isomorphic to a direct sum of copies of the injective hull of the residue field of \( A \).

**Proof.** Denote \( U = \text{Spec}(A) \setminus \{ \mathfrak{m} \} \) the punctured spectrum. Thus \( g : V \to U \) is affine. We have \( H^i(V, \mathcal{O}_V) = H^i(U, g_*\mathcal{O}_V) \) by Cohomology of Schemes, Lemma 2.4. The \( \mathcal{O}_U \)-module \( g_*\mathcal{O}_V \) is quasi-coherent by Schemes, Lemma 21.1. For any quasi-coherent \( \mathcal{O}_U \)-module \( F \) the cohomology \( H^i(U, F) \), \( i > 0 \) is \( \mathfrak{m} \)-power torsion, see for example Local Cohomology, Lemma 17.2. In particular, the \( A \)-modules \( H^i(V, \mathcal{O}_V) \), \( i > 0 \) are \( \mathfrak{m} \)-power torsion. For any flat ring map \( A \to A' \) we have \( R\Gamma(V, \mathcal{O}_V) \otimes_A A' = H^i(V', \mathcal{O}_{V'}) \) where \( V' = V \times_{\text{Spec}(A)} \text{Spec}(A') \) by flat base change Cohomology of Schemes, Lemma 5.2. If we take \( A' \) to be the completion of \( A \) (flat by More on Algebra, Section 42), then we see that

\[
H^i(V, \mathcal{O}_V) = H^i(V', \mathcal{O}_{V'}) \otimes_A A' = H^i(V', \mathcal{O}_{V'}), \quad \text{for } i > 0
\]

The first equality by the torsion property we just proved and More on Algebra, Lemma 80.3. Moreover, the injective hull of the residue field \( k \) is the same for \( A \) and \( A' \), see Dualizing Complexes, Lemma 7.4. In this way we reduce to the case \( A = k[[x_1, \ldots, x_d]] \), see Algebra, Section 154.

Assume the characteristic of \( k \) is \( p > 0 \). Since \( F : A \to A', a \mapsto a^p \) is flat (Local Cohomology, Lemma 16.6) we find \( H^i(V, \mathcal{O}_V) \otimes_{A, F} A \cong H^i(V', \mathcal{O}_{V'}) \). Thus we get the result by Local Cohomology, Lemma 17.2.

Assume the characteristic of \( k \) is 0. By Local Cohomology, Lemma 18.3 there are additive operators \( D_j \), \( j = 1, \ldots, d \) on \( H^i(V, \mathcal{O}_V) \) satisfying the Leibniz rule with respect to \( \partial_j = \partial/\partial x_j \). Thus we get the result by Local Cohomology, Lemma 17.1.

**Lemma 29.2.** In the situation of Lemma 29.1 assume that \( H^i(V, \mathcal{O}_V) = 0 \) for \( i \geq \dim(A) - 1 \). Then \( V \) is affine.

**Proof.** Let \( k = A/\mathfrak{m} \). Since \( V \times_{\text{Spec}(A)} \text{Spec}(k) = \emptyset \), by cohomology and base change we have

\[
R\Gamma(V, \mathcal{O}_V) \otimes_A^L k = 0
\]
In this section we discuss the following result. Let $f : X \to Y$ be a finite type morphism of schemes with $Y$ normal. Let $V \subset X$ be the maximal open sub scheme where $f$ is "étale. Then the inclusion morphism $V \to X$ is affine.

**Proof.** Let $x \in X$ with image $y \in Y$. It suffices to prove that $V \cap W$ is affine for some affine open neighbourhood $W$ of $x$. Since $\text{Spec}(O_{X,x})$ is the limit of the schemes $W$, this holds if and only if

$$V_x = V \times_X \text{Spec}(O_{X,x})$$

is affine (Limits, Lemma 4.13). Thus, if the theorem holds for the morphism $X \times_Y \text{Spec}(O_{Y,y}) \to \text{Spec}(O_{Y,y})$, then the theorem holds. In particular, we may assume $Y$ is regular of finite dimension, which allows us to do induction on the dimension $d = \dim(Y)$. Combining this with the same argument again, we may assume that $Y$ is local with closed point $y$ and that $V \cap (X \setminus f^{-1}([y])) \to X \setminus f^{-1}([y])$ is affine.

Let $x \in X$ be a point lying over $y$. If $x \in V$, then there is nothing to prove. Observe that $f^{-1}([y]) \cap V$ is a finite set of closed points (the fibres of an "étale morphism are discrete). Thus after replacing $X$ by an affine open neighbourhood of $x$ we may assume $y \notin f(V)$. We have to prove that $V$ is affine.

Let $e(V)$ be the maximum $i$ with $H^i(V, O_Y) \neq 0$. As $X$ is affine the integer $e(V)$ is the maximum of the numbers $e(V_x)$ where $x \in X \setminus V$, see Local Cohomology, Lemma 3.6 and the characterization of cohomological dimension in Local Cohomology, Lemma 3.7. If $\dim(O_{X,x}) \geq 2$ then purity of ramification locus (Lemma 28.3) shows that $V_x$ is strictly smaller than the punctured spectrum of $O_{X,x}$. Since $O_{X,x}$ is normal and excellent, this implies $e(V_x) \leq \dim(O_{X,x}) - 2$ by Hartshorne-Lichtenbaum vanishing (Local Cohomology, Lemma 15.7). On the other hand, since $X \to Y$ is of finite type and $V \subset X$ is dense (after possibly replacing $X$ by the closure of $V$), we see that $\dim(O_{X,x}) \leq d$ by the dimension formula (Morphisms, Lemma 50.1). Whence $e(V) \leq \max(0, d - 2)$. Thus $V$ is affine by Lemma 29.2 if $d \geq 2$. If $d = 1$ or $d = 0$, then the punctured spectrum of $O_{Y,y}$ is affine and hence $V$ is affine.

---

**30. Specialization maps in the smooth proper case**

In this section we discuss the following result. Let $f : X \to S$ be a proper smooth morphism of schemes. Let $s \leadsto s'$ be a specialization of points in $S$. Then the specialization map

$$sp : \pi_1(X_s) \to \pi_1(X_{s'})$$

See Derived Categories of Schemes, Lemma 21.3. Thus there is a spectral sequence (More on Algebra, Example 60.4)

$$E_2^{p,q} = \text{Tor}_p(k, H^q(V, O_Y)),$$

and $d_2 : E_2^{p,q} \to E_2^{p+2, q-1}$ converging to zero. By Lemma Dualizing Complexes, Lemma 21.9 and our assumption $H^i(V, O_Y) = 0$ for $i \geq \dim(A) - 1$ we conclude that there is no nonzero differential entering or leaving the $(p,q) = (0,0)$ spot. Thus $H^0(V, O_Y) \otimes_A k = 0$. This means that if $m = (x_1, \ldots, x_d)$ then we have an open covering $V = \bigcup V \times_{\text{Spec}(A)} \text{Spec}(A_{x_i})$ by affine open subschemes $V \times_{\text{Spec}(A)} \text{Spec}(A_{x_i})$ (because $V$ is affine over the punctured spectrum of $A$) such that $x_1, \ldots, x_d$ generate the unit ideal in $\Gamma(V, O_Y)$. This implies $V$ is affine by Properties, Lemma 27.3. □
of Section [16] is surjective and

(1) if the characteristic of $\kappa(s')$ is zero, then it is an isomorphism, or
(2) if the characteristic of $\kappa(s')$ is $p > 0$, then it induces an isomorphism on maximal prime-to-$p$ quotients.

Lemma 30.1. Let $f : X \to S$ be a flat proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If $X_s$ is geometrically reduced, then the specialization map $sp : \pi_1(X_{s'}) \to \pi_1(X_s)$ is surjective.

Proof. Since $X_s$ is geometrically reduced, we may assume all fibres are geometrically reduced after possibly shrinking $S$, see More on Morphisms, Lemma 24.7. Let $O_{S,s} \to A \to \kappa(s')$ be as in the construction of the specialization map, see Section 16. Thus it suffices to show that $\pi_1(X_{s'}) \to \pi_1(X_A)$ is surjective. This follows from Proposition 15.2 and $\pi_1(\text{Spec}(A)) = \{1\}$. □

Proposition 30.2. Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres. Let $s' \rightsquigarrow s$ be a specialization. If the characteristic to $\kappa(s)$ is zero, then the specialization map $sp : \pi_1(X_{s'}) \to \pi_1(X_s)$ is an isomorphism.

Proof. The map is surjective by Lemma 30.1. Thus we have to show it is injective. We may assume $S$ is affine. Then $S$ is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to S$ is the base change of $X_0 \to S_0$ where $S_0$ is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to S_0$ is smooth and proper. See Limits, Lemma 10.1, 8.9, and 13.1. By Lemma 16.1 we reduce to the case where the base is Noetherian. Applying Lemma 16.4 we reduce to the case where the base $S$ is the spectrum of a strictly henselian discrete valuation ring $A$ and we are looking at the specialization map over $A$. Let $K$ be the fraction field of $A$. Choose an algebraic closure $\overline{K}$ which corresponds to a geometric generic point $\overline{\eta}$ of $\text{Spec}(A)$. For $\overline{K}/L/K$ finite separable, let $B \subset L$ be the integral closure of $A$ in $L$. This is a discrete valuation ring by More on Algebra, Remark 97.6.

Let $X \to \text{Spec}(A)$ be as in the previous paragraph. To show injectivity of the specialization map it suffices to prove that every finite étale cover $V$ of $X_\overline{\eta}$ is the base change of a finite étale cover $Y \to X$. Namely, then $\pi_1(X_\overline{\eta}) \to \pi_1(X) = \pi_1(X_s)$ is injective by Lemma 4.4.

Given $V$ we can first descend $V$ to $V' \to X_{K^{sep}}$ by Lemma 14.2 and then to $V'' \to X_L$ by Lemma 14.1. Let $Z \to X_B$ be the normalization of $X_B$ in $V''$. Observe that $Z$ is normal and that $Z_L = V''$ as schemes over $X_L$. Hence $Z \to X_B$ is finite étale over the generic fibre. The problem is that we do not know that $Z \to X_B$ is everywhere étale. Since $X \to \text{Spec}(A)$ has geometrically connected smooth fibres, we see that the special fibre $X_s$ is geometrically irreducible. Hence the special fibre of $X_B \to \text{Spec}(B)$ is irreducible; let $\xi_B$ be its generic point. Let
Let \( \xi_1, \ldots, \xi_r \) be the points of \( Z \) mapping to \( \xi_B \). Our first (and it will turn out only) problem is now that the extensions

\[
\mathcal{O}_{X_B, \xi_B} \subset \mathcal{O}_{Z, \xi_i}
\]

discrete valuation rings may be ramified. Let \( e_i \) be the ramification index of this extension. Note that since the characteristic of \( \kappa(s) \) is zero, the ramification is tame.

To get rid of the ramification we are going to choose a further finite separable extension \( K^{sep}/L'/L/K \) such that the ramification index \( e \) of the induced extensions \( B'/B \) is divisible by \( e_i \).

Consider the normalized base change \( Z' \) of \( Z \) with respect to \( \text{Spec}(B') \to \text{Spec}(B) \), see discussion in More on Morphisms, Section \( \underline{57} \). Let \( \xi_{i,j} \) be the points of \( Z' \) mapping to \( \xi_{B'} \) and to \( \xi_i \) in \( Z \). Then the local rings

\[
\mathcal{O}_{Z', \xi_{i,j}}
\]

are localizations of the integral closure of \( \mathcal{O}_{Z, \xi_i} \) in \( L' \otimes_L F_i \) where \( F_i \) is the fraction field of \( \mathcal{O}_{Z, \xi_i} \); details omitted. Hence Abhyankar’s lemma (More on Algebra, Lemma \( \underline{100.4} \)) tells us that

\[
\mathcal{O}_{X_{B'}, \xi_{B'}} \subset \mathcal{O}_{Z', \xi_{i,j}}
\]

is unramified. We conclude that the morphism \( Z' \to X_{B'} \) is étale away from codimension 1. Hence by purity of branch locus (Lemma \( \underline{21.4} \)) we see that \( Z' \to X_{B'} \) is finite étale!

However, since the residue field extension induced by \( A \to B' \) is trivial (as the residue field of \( A \) is algebraically closed being separably closed of characteristic zero) we conclude that \( Z' \) is the base change of a finite étale cover \( Y \to X \) by applying Lemma \( \underline{9.1} \) twice (first to get \( Y \) over \( A \), then to prove that the pullback to \( B \) is isomorphic to \( Z' \)). This finishes the proof. \( \Box \)

Let \( G \) be a profinite group. Let \( p \) be a prime number. The **maximal prime-to-\( p \) quotient** is by definition

\[
G' = \lim_{\longrightarrow} U \subset G \text{ open, normal, index prime to } p \text{ } G/U
\]

If \( X \) is a connected scheme and \( p \) is given, then the maximal prime-to-\( p \) quotient of \( \pi_1(X) \) is denoted \( \pi_1'(X) \).

**Theorem 30.3.** Let \( f : X \to S \) be a smooth proper morphism with geometrically connected fibres. Let \( s' \rightsquigarrow s \) be a specialization. If the characteristic of \( \kappa(s) \) is \( p \), then the specialization map

\[
sp : \pi_1(X_{\overline{s'}}) \to \pi_1(X_{\overline{s}})
\]

is surjective and induces an isomorphism

\[
\pi_1'(X_{\overline{s'}}) \cong \pi_1'(X_{\overline{s}})
\]

of the maximal prime-to-\( p \) quotients

**Proof.** This is proved in exactly the same manner as Proposition \( \underline{30.2} \) with the following differences

1. Given \( X/A \) we no longer show that the functor \( \text{FÉt}_X \to \text{FÉt}_{X_{\overline{s'}}} \) is essentially surjective. We show only that Galois objects whose Galois group has order prime to \( p \) are in the essential image. This will be enough to conclude the injectivity of \( \pi_1'(X_{\overline{s'}}) \to \pi_1'(X_{\overline{s}}) \) by exactly the same argument.
(2) The extensions \( O_{X_B, \xi_B} \subset O_{Z, \xi} \) are tamely ramified as the associated extension of fraction fields is Galois with group of order prime to \( p \). See More on Algebra, Lemma \[88.2\].

(3) The extension \( \kappa_A \subset \kappa_B \) is no longer necessarily trivial, but it is purely inseparable. Hence the morphism \( X_{\kappa_B} \to X_{\kappa_A} \) is a universal homeomorphism and induces an isomorphism of fundamental groups by Proposition \[3.4\]. □

31. Tame ramification

Let \( X \to Y \) be a finite étale morphism of schemes of finite type over \( \mathbb{Z} \). There are many ways to define what it means for \( f \) to be tamely ramified at \( \infty \). The article \[KS10\] discusses to what extent these notions agree.

In this section we discuss a different more elementary question which precedes the notion of tameness at infinity. Please compare with the (slightly different) discussion in \[GM71\]. Assume we are given

1. a locally Noetherian scheme \( X \),
2. a dense open \( U \subset X \),
3. a finite étale morphism \( f : Y \to U \)

such that for every every prime divisor \( Z \subset X \) with \( Z \cap U = \emptyset \) the local ring \( O_{X, \xi} \) of \( X \) at the generic point \( \xi \) of \( Z \) is a discrete valuation ring. Setting \( K_\xi \) equal to the fraction field of \( O_{X, \xi} \) we obtain a cartesian square

\[
\begin{array}{ccc}
\text{Spec}(K_\xi) & \longrightarrow & U \\
\downarrow & & \downarrow \\
\text{Spec}(O_{X, \xi}) & \longrightarrow & X
\end{array}
\]

of schemes. In particular, we see that \( Y \times_U \text{Spec}(K_\xi) \) is the spectrum of a finite separable algebra \( L_\xi/K \). Then we say \( Y \) is unramified over \( X \) in codimension 1, resp. \( Y \) is tamely ramified over \( X \) in codimension 1 if \( L_\xi/K \) is unramified, resp. tamely ramified with respect to \( O_{X, \xi} \) for every \( (Z, \xi) \) as above, see More on Algebra, Definition \[107.7\]. More precisely, we decompose \( L_\xi \) into a product of finite separable field extensions of \( K_\xi \) and we require each of these to be unramified, resp. tamely ramified with respect to \( O_{X, \xi} \).

Lemma 31.1. Let \( X' \to X \) be a morphism of locally Noetherian schemes. Let \( U \subset X' \) be a dense open. Assume

1. \( U' = f^{-1}(U) \) is dense open in \( X' \),
2. for every prime divisor \( Z \subset X \) with \( Z \cap U = \emptyset \) the local ring \( O_{X, \xi} \) of \( X \) at the generic point \( \xi \) of \( Z \) is a discrete valuation ring,
3. for every prime divisor \( Z' \subset X' \) with \( Z' \cap U' = \emptyset \) the local ring \( O_{X', \xi'} \) of \( X' \) at the generic point \( \xi' \) of \( Z' \) is a discrete valuation ring,
4. if \( \xi' \in X' \) is as in (3), then \( \xi = f(\xi') \) is as in (2).

Then if \( f : Y \to U \) is finite étale and \( Y \) is unramified, resp. tamely ramified over \( X \) in codimension 1, then \( Y' = Y \times_X X' \to U' \) is finite étale and \( Y' \) is unramified, resp. tamely ramified over \( X' \) in codimension 1.
Proof. The only interesting fact in this lemma is the commutative algebra result given in More on Algebra, Lemma \[100.9\]

Using the terminology introduced above, we can reformulate our purity results obtained earlier in the following pleasing manner.

**Lemma 31.2.** Let \( X \) be a locally Noetherian scheme. Let \( D \subset X \) be an effective Cartier divisor such that \( D \) is a regular scheme. Let \( Y \to X \setminus D \) be a finite étale morphism. If \( Y \) is unramified over \( X \) in codimension 1, then there exists a finite étale morphism \( Y' \to X \) whose restriction to \( X \setminus D \) is \( Y \).

**Proof.** Before we start we note that \( \mathcal{O}_{X,x} \) is a regular local ring for all \( x \in D \). This follows from Algebra, Lemma \[105.7\] and our assumption that \( \mathcal{O}_{D,x} \) is regular. Let \( \xi \in D \) be a generic point of an irreducible component of \( D \). By the above \( \mathcal{O}_{X,\xi} \) is a discrete valuation ring. Hence the statement of the lemma makes sense. As in the discussion above, write \( Y \times_U \text{Spec}(K_\xi) = \text{Spec}(L_\xi) \). Denote \( B_\xi \) the integral closure of \( \mathcal{O}_{X,\xi} \) in \( L_\xi \). Our assumption that \( Y \) is unramified over \( X \) in codimension 1 signifies that \( \mathcal{O}_{X,\xi} \to B_\xi \) is finite étale. Thus we get \( Y_\xi \to \text{Spec}(\mathcal{O}_{X,\xi}) \) finite étale and an isomorphism

\[
\text{Spec}(K_\xi) \cong Y_\xi \times_{\text{Spec}(\mathcal{O}_{X,\xi})} \text{Spec}(K_\xi)
\]

over \( \text{Spec}(K_\xi) \). By Limits, Lemma \[18.3\] we find an open subscheme \( X \setminus D \subset U' \subset X \) containing \( \xi \) and a morphism \( Y' \to U' \) of finite presentation whose restriction to \( X \setminus D \) recovers \( Y \) and whose restriction to \( \text{Spec}(\mathcal{O}_{X,\xi}) \) recovers \( Y_\xi \). Finally, the morphism \( Y' \to U' \) is finite étale after possible shrinking \( U' \) to a smaller open by Limits, Lemma \[18.4\]. Repeating the argument with the other generic points of \( D \) we may assume that we have a finite étale morphism \( Y' \to U' \) extending \( Y \to X \setminus D \) to an open subscheme containing \( U' \subset X \) containing \( D \) and all generic points of \( D \). We finish by applying Lemma \[21.6\] to \( Y' \to U' \). Namely, all local rings \( \mathcal{O}_{X,x} \) for \( x \in D \) are regular (see above) and if \( x \notin U' \) we have \( \dim(\mathcal{O}_{X,x}) \geq 2 \). Hence we have purity for \( \mathcal{O}_{X,x} \) by Lemma \[21.3\].

**Example 31.3** (Standard tamely ramified morphism). Let \( A \) be a Noetherian ring. Let \( f \in A \) be a nonzerodivisor such that \( A/fA \) is reduced. This implies that \( A_p \) is a discrete valuation ring with uniformizer \( f \) for any minimal prime \( p \) over \( f \).

Let \( e \geq 1 \) be an integer which is invertible in \( A \). Set

\[
C = A[x]/(x^e - f)
\]

Then \( \text{Spec}(C) \to \text{Spec}(A) \) is a finite locally free morphism which is étale over the spectrum of \( A_f \). The finite étale morphism

\[
\text{Spec}(C_f) \to \text{Spec}(A_f)
\]

is tamely ramified over \( \text{Spec}(A) \) in codimension 1. The tameness follows immediately from the characterization of tamely ramified extensions in More on Algebra, Lemma \[100.7\].

Here is a version of Abhyankar’s lemma for regular divisors.

**Lemma 31.4** (Abhyankar’s lemma for regular divisor). Let \( X \) be a locally Noetherian scheme. Let \( D \subset X \) be an effective Cartier divisor such that \( D \) is a regular scheme. Let \( Y \to X \setminus D \) be a finite étale morphism. If \( Y \) is tamely ramified over \( X \) in codimension 1, then étale locally on \( X \) the morphism \( Y \to X \) is as given
as a finite disjoint union of standard tamely ramified morphisms as described in Example 31.3.

**Proof.** Before we start we note that $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in D$. This follows from Algebra, Lemma 105.7 and our assumption that $\mathcal{O}_{D,x}$ is regular. Below we will also use that regular rings are normal, see Algebra, Lemma 151.5.

To prove the lemma we may work locally on $X$. Thus we may assume $X = \text{Spec}(A)$ and $D \subset X$ is given by a nonzerodivisor $f \in A$. Then $Y = \text{Spec}(B)$ as a finite étale scheme over $A_f$. Let $p_1, \ldots, p_r$ be the minimal primes of $A$ over $f$. Then $A_i = A_{p_i}$ is a discrete valuation ring; denote its fraction field $K_i$. By assumption

$$K_i \otimes_{A_f} B = \prod L_{ij}$$

is a finite product of fields each tamely ramified with respect to $A_i$. Choose $e \geq 1$ sufficiently divisible (namely, divisible by all ramification indices for $L_{ij}$ over $A_i$ as in More on Algebra, Remark 97.6). Warning: at this point we do not know that $e$ is invertible on $A$.

Consider the finite free $A$-algebra

$$A' = A[x]/(x^e - f)$$

Observe that $f' = x$ is a nonzerodivisor in $A'$ and that $A'/f A' \cong A/f A$ is a regular ring. Set $B' = B \otimes_A A' = B \otimes_{A_f} A'$. By Abhyankar’s lemma (More on Algebra, Lemma 100.4) we see that $\text{Spec}(B')$ is unramified over $\text{Spec}(A')$ in codimension 1. Namely, by Lemma 31.1 we see that $\text{Spec}(B')$ is still at least tamely ramified over $\text{Spec}(A')$ in codimension 1. But Abhyankar’s lemma tells us that the ramification indices have all become equal to 1. By Lemma 31.2 we conclude that $\text{Spec}(B') \to \text{Spec}(A')$ extends to a finite étale morphism $\text{Spec}(C) \to \text{Spec}(A')$.

For a point $x \in D$ corresponding to $p \in V(f)$ denote $A^{sh}$ a strict henselization of $A_p = \mathcal{O}_{X,x}$. Observe that $A^{sh}$ and $A^{sh}/f A^{sh} = (A/f A)^{sh}$ (Algebra, Lemma 150.10) are regular local rings, see More on Algebra, Lemma 44.10. Observe that $A'$ has a unique prime $p'$ lying over $p$ with identical residue field. Thus

$$(A')^{sh} = A^{sh} \otimes_A A' = A^{sh}[x]/(x^e - f)$$

is a strictly henselian local ring finite over $A^{sh}$ (Algebra, Lemma 150.15). Since $f'$ is a nonzerodivisor in $(A')^{sh}$ and since $(A')^{sh}/f'(A')^{sh} = A^{sh}/f A^{sh}$ is regular, we conclude that $(A')^{sh}$ is a regular local ring (see above). Observe that the induced extension

$$Q(A^{sh}) \subset Q((A')^{sh}) = Q(A^{sh}[x]/(x^e - f))$$

of fraction fields has degree $e$ (and not less). Since $A' \to C$ is finite étale we see that $A^{sh} \otimes_A C$ is a finite product of copies of $(A')^{sh}$ (Algebra, Lemma 148.6). We have the inclusions

$$A^{sh}_f \subset A^{sh} \otimes_A B \subset A^{sh} \otimes_A B' = A^{sh} \otimes_A C_{p'}$$

and each of these rings is Noetherian and normal; this follows from Algebra, Lemma 157.9 for the ring in the middle. Taking total quotient rings, using the product decomposition of $A^{sh} \otimes_A C$ and using Fields, Lemma 24.3 we conclude that there is an isomorphism

$$Q(A^{sh}) \otimes_A B \cong \prod_{i \in I} F_i, \quad F_i \cong Q(A^{sh})[x]/(x^{e_i} - f)$$
of $Q(A^h)$-algebras for some finite set $I$ and integers $e_i|e$. Since $A^h \otimes_A B$ is a normal ring, it must be the integral closure of $A^h$ in its total quotient ring. We conclude that we have an isomorphism

$$A^h \otimes_A B \cong \prod_{i \in I} A^{sh}_{f_i}[x]/(x^{e_i} - f)$$

over $A^{sh}_f$ because the algebras $A^{sh}_f[x]/(x^{e_i} - f)$ are regular and hence normal. The discriminant of $A^{sh}_f[x]/(x^{e_i} - f)$ over $A^h$ is $e_i^e f_i^{e-1}$ (up to sign; calculation omitted). Since $A_f \to B$ is finite étale we see that $e_i$ must be invertible in $A^{sh}_f$. On the other hand, since $A_f \to B$ is tamely ramified over $\Spec(A)$ in codimension 1, by Lemma 31.1 the ring map $A^{sh}_f \to A^h \otimes_A B$ is tamely ramified over $\Spec(A^h)$ in codimension 1. This implies $e_i$ is nonzero in $A^h/fA^h$ (as it must map to an invertible element of the fraction field of this domain by definition of tamely ramified extensions). We conclude that $V(e_i) \subset \Spec(A^h)$ has codimension $\geq 2$ which is absurd unless it is empty. In other words, $e_i$ is an invertible element of $A^h$. We conclude that the pullback of $Y$ to $\Spec(A^h)$ is indeed a finite disjoint union of standard tamely ramified morphisms.

To finish the proof, we write $A^h = \colim A_\lambda$ as a filtered colimit of étale $A$-algebras $A_\lambda$. The isomorphism

$$A^h \otimes_A B \cong \prod_{i \in I} A^{sh}_{f_i}[x]/(x^{e_i} - f)$$

descends to an isomorphism

$$A_\lambda \otimes_A B \cong \prod_{i \in I} (A_\lambda)_{f_i}[x]/(x^{e_i} - f)$$

for suitably large $\lambda$. After increasing $\lambda$ a bit more we may assume $e_i$ is invertible in $A_\lambda$. Then $\Spec(A_\lambda) \to \Spec(A)$ is the desired étale neighbourhood of $x$ and the proof is complete. \qed

**Lemma 31.5.** In the situation of Lemma 31.4 the normalization of $X$ in $Y$ is a finite locally free morphism $\pi : Y' \to X$ such that

1. the restriction of $Y'$ to $X \setminus D$ is isomorphic to $Y$,
2. $D' = \pi^{-1}(D)_{\text{red}}$ is an effective Cartier divisor on $Y'$, and
3. $D'$ is a regular scheme.

Moreover, étale locally on $X$ the morphism $Y' \to X$ is a finite disjoint union of morphisms

$$\Spec(A[x]/(x^e - f)) \to \Spec(A)$$

where $A$ is a Noetherian ring, $f \in A$ is a nonzerodivisor with $A/fA$ regular, and $e \geq 1$ is invertible in $A$.

**Proof.** This is just an addendum to Lemma 31.4 and in fact the truth of this lemma follows almost immediately if you’ve read the proof of that lemma. But we can also deduce the lemma from the result of Lemma 31.4. Namely, taking the normalization of $X$ in $Y$ commutes with étale base change, see More on Morphisms, Lemma 17.2. Hence we see that we may prove the statements on the local structure of $Y' \to X$ étale locally on $X$. Thus, by Lemma 31.4 we may assume that $X = \Spec(A)$ where $A$ is a Noetherian ring, that we have a nonzerodivisor $f \in A$ such that $A/fA$ is regular, and that $Y$ is a finite disjoint union of spectra of rings $A_f[x]/(x^e - f)$ where $e$ is invertible in $A$. We omit the verification that the integral closure of
A in \(A[x]/(x^e - f)\) is equal to \(A' = A[x]/(x^e - f)\). (To see this argue that the localizations of \(A'\) at primes lying over \((f)\) are regular.) We omit the details. \(\square\)

**Lemma 31.6.** In the situation of Lemma 31.4 let \(Y' \to X\) be as in Lemma 31.5. Let \(R\) be a discrete valuation ring with fraction field \(K\). Let \(t : \text{Spec}(R) \to X\) be a morphism such that the scheme theoretic inverse image \(t^{-1}D\) is the reduced closed point of \(\text{Spec}(R)\).

(1) If \(t|_{\text{Spec}(K)}\) lifts to a point of \(Y\), then we get a lift \(t' : \text{Spec}(R) \to Y'\) such that \(Y' \to X\) is étale along \(t'|_{\text{Spec}(R)}\).

(2) If \(\text{Spec}(K) \times_X Y\) is isomorphic to a disjoint union of copies of \(\text{Spec}(K)\), then \(Y' \to X\) is finite étale over an open neighbourhood of \(t(\text{Spec}(R))\).

**Proof.** By the valuative criterion of properness applied to the finite morphism \(Y' \to X\) we see that \(\text{Spec}(K)\)-valued points of \(Y\) matching \(t|_{\text{Spec}(K)}\) as maps into \(X\) lift uniquely to morphisms \(t' : \text{Spec}(R) \to Y'\). Thus statement (1) make sense.

Choose an étale neighbourhood \((U, u) \to (X, t(\mathfrak{m}_R))\) such that \(U = \text{Spec}(A)\) and such that \(Y' \times_X U \to U\) has a description as in Lemma 31.5 for some \(f \in A\). Then \(\text{Spec}(R) \times_X U \to \text{Spec}(R)\) is étale and surjective. If \(R'\) denotes the local ring of \(\text{Spec}(R) \times_X U\) lying over the closed point of \(\text{Spec}(R)\), then \(R'\) is a discrete valuation ring and \(R \subset R'\) is an unramified extension of discrete valuation rings (More on Algebra, Lemma 43.4). The assumption on \(t\) signifies that the map \(A \to R'\) corresponding to \(\text{Spec}(R') \to \text{Spec}(R) \times_X U \to U\) maps \(f\) to a uniformizer \(\pi \in R'\). Now suppose that \(Y' \times_X U = \coprod_{i \in I} \text{Spec}(A[x]/(x^{e_i} - f))\) for some \(e_i \geq 1\). Then we see that \(\text{Spec}(R') \times_U (Y' \times_X U) = \coprod_{i \in I} \text{Spec}(R'[x]/(x^{e_i} - \pi))\).

The rings \(R'[x]/(x^{e_i} - f)\) are discrete valuation rings (More on Algebra, Lemma 100.2) and hence have no map into the fraction field of \(R'\) unless \(e_i = 1\).

Proof of (1). In this case the map \(t' : \text{Spec}(R) \to Y'\) base changes to determine a corresponding map \(t'' : \text{Spec}(R') \to Y' \times_X U\) which must map into a summand corresponding to \(i \in I\) with \(e_i = 1\) by the discussion above. Thus clearly we see that \(Y' \times_X U \to U\) is étale along the image of \(t''\). Since being étale is a property one can check after étale base change, this proves (1).

Proof of (2). In this case the assumption implies that \(e_i = 1\) for all \(i \in I\). Thus \(Y' \times_X U \to U\) is finite étale and we conclude as before. \(\square\)

**Lemma 31.7.** Let \(S\) be an integral normal Noetherian scheme with generic point \(\eta\). Let \(f : X \to S\) be a smooth morphism with geometrically connected fibres. Let \(\sigma : S \to X\) be a section of \(f\). Let \(Z \to X_\eta\) be a finite étale Galois cover (Section 7) with group \(G\) of order invertible on \(S\) such that \(Z\) has a \(\kappa(\eta)\)-rational point mapping to \(\sigma(\eta)\). Then there exists a finite étale Galois cover \(Y \to X\) with group \(G\) whose restriction to \(X_\eta\) is \(Z\).
Proof. First assume $S = \text{Spec}(R)$ is the spectrum of a discrete valuation ring $R$ with closed point $s \in S$. Then $X_s$ is an effective Cartier divisor in $X$ and $X_s$ is regular as a scheme smooth over a field. Moreover the generic fibre $X_g$ is the open subscheme $X \setminus X_s$. It follows from More on Algebra, Lemma 18.2 and the assumption on $G$ that $Z$ is tamely ramified over $X$ in codimension 1. Let $Z' \to X$ be as in Lemma 31.5. Observe that the action of $G$ on $Z$ extends to an action of $G$ on $Z'$. By Lemma 31.6 we see that $Z' \to X$ is finite étale over an open neighbourhood of $\sigma(y)$. Since $X_s$ is irreducible, this implies $Z \to X_g$ is unramified over $X$ in codimension 1. Then we get a finite étale morphism $Y \to X$ whose restriction to $X_g$ is $Z$ by Lemma 31.2. Of course $Y \cong Z'$ (details omitted; hint: compute étale locally) and hence $Y$ is a Galois cover with group $G$.

General case. Let $U \subset S$ be a maximal open subscheme such that there exists a finite étale Galois cover $Y \to X \times S U$ with group $G$ whose restriction to $X_g$ is isomorphic to $Z$. Assume $U \neq S$ to get a contradiction. Let $s \in S \setminus U$ be a generic point of an irreducible component of $S \setminus U$. Then the inverse image $U_s$ of $U$ in $\text{Spec}(\mathcal{O}_{S,s})$ is the punctured spectrum of $\mathcal{O}_{S,s}$. We claim $Y \times_S U_s \to X \times S U_s$ is the restriction of a finite étale Galois cover $Y'_s \to X \times S \text{Spec}(\mathcal{O}_{S,s})$ with group $G$.

Let us first prove the claim produces the desired contradiction. By Limits, Lemma 18.3 we find an open subscheme $U \subset U' \subset S$ containing $s$ and a morphism $Y'' \to U'$ of finite presentation whose restriction to $U$ recovers $Y' \to U$ and whose restriction to $\text{Spec}(\mathcal{O}_{S,s})$ recovers $Y'_s$. Moreover, by the equivalence of categories given in the lemma, we may assume after shrinking $U'$ there is a morphism $Y'' \to U' \times_S X$ and there is an action of $G$ on $Y''$ over $U' \times_S X$ compatible with the given morphisms and actions after base change to $U$ and $\text{Spec}(\mathcal{O}_{S,s})$. After shrinking $U'$ further if necessary, we may assume $Y'' \to U \times_S X$ is finite étale, see Limits, Lemma 18.4. This means we have found a strictly larger open of $S$ over which $Y$ extends to a finite étale Galois cover with group $G$ which gives the contradiction we were looking for.

Proof of the claim. We may and do replace $S$ by $\text{Spec}(\mathcal{O}_{S,s})$. Then $S = \text{Spec}(A)$ where $(A, m)$ is a local normal domain. Also $U \subset S$ is the punctured spectrum and we have a finite étale Galois cover $Y \to X \times_S U$ with group $G$. If $\dim(A) = 1$, then we can construct the extension of $Y$ to a Galois covering of $X$ by the first paragraph of the proof. Thus we may assume $\dim(A) \geq 2$ and hence $\text{depth}(A) \geq 2$ as $S$ is normal, see Algebra, Lemma 151.4. Since $X \to S$ is flat, we conclude that $\text{depth}(\mathcal{O}_{X,s}) \geq 2$ for every point $x \in X$ mapping to $s$, see Algebra, Lemma 157.2.

Let

$$Y' \to X$$

be the finite morphism constructed in Lemma 21.5 using $Y \to X \times_S U$. Observe that we obtain a canonical $G$-action on $Y$. Thus all that remains is to show that $Y'$ is étale over $X$. In fact, by Lemma 26.3 (for example) it even suffices to show that $Y' \to X$ is étale over the (unique) generic point of the fibre $X_s$. This we do by a local calculation in a (formal) neighbourhood of $\sigma(s)$.

Choose an affine open $\text{Spec}(B) \subset X$ containing $\sigma(s)$. Then $A \to B$ is a smooth ring map which has a section $\sigma : B \to A$. Denote $I = \text{Ker}(\sigma)$ and denote $B^\wedge$ the $I$-adic completion of $B$. Then $B^\wedge \cong A[[x_1,\ldots,x_d]]$ for some $d \geq 0$, see Algebra, Lemma 137.4. Of course $B \to B^\wedge$ is flat (Algebra, Lemma 96.2) and the image of
Spec($B^\wedge$) $\to$ $X$ contains the generic point of $X_s$. Let $V \subset \text{Spec}(B^\wedge)$ be the inverse image of $U$. Consider the finite étale morphism

$$W = Y \times_{(X \times SU)} V \to V$$

By the compatibility of the construction of $Y'$ with flat base change in Lemma 21.5 we find that the base change $Y' \times_X \text{Spec}(B^\wedge) \to \text{Spec}(B^\wedge)$ is constructed from $W \to V$ over $\text{Spec}(B^\wedge)$ by the procedure in Lemma 21.5. Set $V_0 = V \cap V(x_1, \ldots, x_d) \subset V$ and $W_0 = W \times V V_0$. This is a normal integral scheme which maps into $\sigma(S)$ by the morphism $\text{Spec}(B^\wedge) \to X$ and in fact is identified with $\sigma(U)$. Hence we know that $W_0 \to V_0 = U$ completely decomposes as this is true for its generic fibre by our assumption on $Z \to X_\eta$ having a $\kappa(\eta)$-rational point lying over $\sigma(\eta)$ (and of course the $G$-action then implies the whole fibre $Z_{\kappa(\eta)}$ is a disjoint union of copies of the scheme $\eta = \text{Spec}(\kappa(\eta))$). Finally, by Lemma 26.1 we have

$$W_0 \times_U V \cong W$$

This shows that $W$ is a disjoint union of copies of $V$ and hence $Y' \times_X \text{Spec}(B^\wedge)$ is a disjoint union of copies of $\text{Spec}(B^\wedge)$ and the proof is complete. \hfill \square

**Lemma 31.8.** Let $S$ be a quasi-compact and quasi-separated integral normal scheme with generic point $\eta$. Let $f : X \to S$ be a quasi-compact and quasi-separated smooth morphism with geometrically connected fibres. Let $\sigma : S \to X$ be a section of $f$. Let $Z \to X_\eta$ be a finite étale Galois cover (Section 7) with group $G$ of order invertible on $S$ such that $Z$ has a $\kappa(\eta)$-rational point mapping to $\sigma(\eta)$. Then there exists a finite étale Galois cover $Y \to X$ with group $G$ whose restriction to $X_\eta$ is $Z$.

**Proof.** If $S$ is Noetherian, then this is the result of Lemma 31.7. The general case follows from this by a standard limit argument. We strongly urge the reader to skip the proof.

We can write $S = \text{lim } S_i$ as a directed limit of a system of schemes with affine transition morphisms and with $S_i$ of finite type over $\mathbb{Z}$, see Limits, Proposition 5.4. For each $i$ let $S \to S_i' \to S_i$ be the normalization of $S_i$ in $S$, see Morphisms, Section 31. Combining Algebra, Proposition 156.10 Morphisms, Lemmas 51.13 and 51.15 we conclude that $S_i'$ is of finite type over $\mathbb{Z}$, finite over $S_i$, and that $S_i'$ is an integral normal scheme such that $S \to S_i'$ is dominant. By Morphisms, Lemma 51.15 we obtain transition morphisms $S_i' \to S_i'$ compatible with the transition morphisms $S_i \to S_i$ and with the morphisms with source $S$. We claim that $S = \text{lim } S_i'$. Proof of claim omitted (hint: look on affine opens over a chosen affine open in $S_i$ for some $i$ to translate this into a straightforward algebra problem). We conclude that we may write $S = \text{lim } S_i$ as a directed limit of a system of normal integral schemes $S_i$ with affine transition morphisms and with $S_i$ of finite type over $\mathbb{Z}$.

For some $i$ we can find a smooth morphism $X_i \to S_i$ of finite presentation whose base change to $S$ is $X \to S$. See Limits, Lemmas 10.1 and 8.3. After increasing $i$ we may assume the section $\sigma$ lifts to a section $\sigma_i : S_i \to X_i$ (by the equivalence of categories in Limits, Lemma 10.1). We may replace $X_i$ by the open subscheme $X_i^0$ of it studied in More on Morphisms, Section 27.2. Since the image of $X \to X_i$ clearly maps into it (openness by More on Morphisms, Lemma 27.6). Thus we may assume the fibres of $X_i \to S_i$ are geometrically connected. After increasing $i$ we may assume $|G|$ is invertible on $S_i$. Let $\eta_i \in S_i$ be the generic point. Since $X_{\eta_i}$ is the
limit of the schemes $X_{i,\eta}$ we can use the exact same arguments to descent $Z \to X_{\eta}$ to some finite étale Galois cover $Z_i \to X_{i,\eta}$ after possibly increasing $i$. See Lemma 14.1. After possibly increasing $i$ once more we may assume $Z_i$ has a $\kappa(\eta_i)$-rational point mapping to $\sigma_i(\eta_i)$. Then we apply the lemma in the Noetherian case and we pullback to $X$ to conclude. □