1. Introduction

The title of this chapter is a bit misleading because the most basic material on restricted power series is in the chapter on formal algebraic spaces. For example, Formal Spaces, Section 21 defines the restricted power series ring $A\{x_1, \ldots, x_n\}$ given a linearly topologized ring $A$. In Formal Spaces, Section 22 we discuss the relationship between these restricted power series rings and morphisms of finite type between locally countably indexed formal algebraic spaces.

Let $A$ be a Noetherian ring and let $I \subset A$ be an ideal. In the first part of this chapter (Sections 2 – 8) we discuss the category of $I$-adically complete algebras $B$ topologically of finite type over a Noetherian ring $A$. It is shown that $B = A\{x_1, \ldots, x_n\}/J$ for some (closed) ideal $J$ in the restricted power series ring (where $A$ is endowed with the $I$-adic topology). We show there is a good notion of a naive cotangent complex $NL_{B/A}$. If some power of $I$ annihilates $NL_{B/A}$, then we think of $\text{Spf}(B)$ as a rig-étale formal algebraic space over $A$. This leads to a definition of rig-étale morphisms of Noetherian formal algebraic spaces. After a certain amount of work we are able to prove the main result of the first part: if $\text{Spf}(B)$ is rig-étale over $A$ as above, then there exists a finite type $A$-algebra $C$ such that $B$ is isomorphic to the $I$-adic completion of $C$, see Lemma 7.4. One thing to note here is that we prove this without assuming the ring $A$ is excellent or even a G-ring. In the last section of the first part we show that under the assumption that $A$ is a G-ring there is a straightforward proof of the lemma based on Popescu’s theorem.
Many of the results discussed in the first part can be found in the paper [Elk73]. Other general references for this part are [DG67], [Abb10], and [FK].

In the second part of this chapter we use the main result of the first part to prove Artin’s result on dilatations from [Art70]. The result on contractions will be the subject of a later chapter (insert future reference here). The main existence theorem is the equivalence of categories in Theorem 10.9. It is more general than Artin’s result in that it shows that any rig-étale morphism \( f : W \to \text{Spec}(A)/V(I) \) is the completion of a morphism \( Y \to \text{Spec}(A) \) of algebraic spaces \( X \) which is locally of finite type and isomorphism away from \( V(I) \). In Artin’s work the morphism \( f \) is assumed proper and rig-surjective. A special case of this is the main lemma mentioned above and the general case follows from this by a straightforward (somewhat lengthy) glueing procedure. There are several lemmas modifying the main theorem the final one of which is (almost) exactly the statement in Artin’s paper. In the last section we apply the results to modifications of \( \text{Spec}(A) \) before and after completion.

2. Two categories

Let \( A \) be a ring and let \( I \subset A \) be an ideal. In this section \( {}^\wedge \) will mean \( I \)-adic completion. Set \( A_n = A/I^n \) so that the \( I \)-adic completion of \( A \) is \( A = \lim_{n} A_n \).

\[
\begin{align*}
C &= \left\{ \text{systems } (B_n, B_{n+1} \to B_n)_{n \in \mathbb{N}} \text{ where } \\
&\quad B_n \text{ is a finite type } A_n\text{-algebra, } \\
&\quad B_{n+1} \to B_n \text{ is an } A_{n+1}\text{-algebra map} \\
&\quad \text{which induces } B_{n+1}/I^nB_{n+1} \cong B_n \right\}
\end{align*}
\]

Morphisms in \( C \) are given by systems of homomorphisms. Let \( C' \) be the category

\[
C' = \left\{ \text{ } A\text{-algebras } B \text{ which are } I\text{-adically complete} \right\}
\]

such that \( B/IB \) is of finite type over \( A/I \).

There is a functor

\[
C' \to C, \quad B \mapsto (B/I^nB)
\]

Indeed, since \( B/IB \) is of finite type over \( A/I \) the ring maps \( A_n = A/I^n \to B/I^nB \) are of finite type (apply Algebra, Lemma 19.1 to a ring map \( A/I^n[x_1, \ldots, x_r] \to B/I^nB \) such that the images of \( x_1, \ldots, x_r \) generate \( B/IB \) over \( A/I \)).

**Lemma 2.1.** Let \( A \) be a ring and let \( I \subset A \) be a finitely generated ideal. The functor

\[
C \to C', \quad (B_n) \mapsto B = \lim B_n
\]

is a quasi-inverse to \( (2.0.3) \). The completions \( A[x_1, \ldots, x_r]^{\wedge} \) are in \( C' \) and any object of \( C' \) is of the form

\[
B = A[x_1, \ldots, x_r]^{\wedge}/J
\]

for some ideal \( J \subset A[x_1, \ldots, x_r]^{\wedge} \).

**Proof.** Let \((B_n)\) be an object of \( C \). By Algebra, Lemma 97.1 we see that \( B = \lim B_n \) is \( I \)-adically complete and \( B/I^nB = B_n \). Hence we see that \( B \) is an object of \( C' \) and that we can recover the object \((B_n)\) by taking the quotients. Conversely, if \( B \) is an object of \( C' \), then \( B = \lim B/I^nB \) by assumption. Thus \( B \mapsto (B/I^nB) \) is a quasi-inverse to the functor of the lemma.
Let \( A \) be ideals such that (2.3.1) There is a base change functor which tells us that every finite Noetherian ring. Part (2) follows from Algebra, Lemma 96.1. Polynomial ring. This case follows from Algebra, Lemma 96.6 as our map \( A[x_1, \ldots, x_r] \to B \) is surjective modulo \( I \) and as \( B = B^h \).

We warn the reader that, in case \( A \) is not Noetherian, the quotient of an object of \( C' \) may not be an object of \( C' \). See Examples, Lemma 7.1. Next we show this does not happen when \( A \) is Noetherian.

**Lemma 2.2.** Let \( A \) be a Noetherian ring and let \( I \subset A \) be an ideal. Then

1. every object of the category \( C' \), in particular the completion \( A[x_1, \ldots, x_r]^\wedge \), is Noetherian,
2. if \( B \) is an object of \( C' \) and \( J \subset B \) is an ideal, then \( B/J \) is an object of \( C' \).

**Proof.** To see (1) by Lemma 2.1 we reduce to the case of the completion of the polynomial ring. This case follows from Algebra, Lemma 96.6 as \( A[x_1, \ldots, x_r] \) is Noetherian (Algebra, Lemma 30.1). Part (2) follows from Algebra, Lemma 96.1 which tells us that every finite \( B \)-module is \( IB \)-adically complete.

**Remark 2.3** (Base change). Let \( \varphi : A_1 \to A_2 \) be a ring map and let \( I_i \subset A_i \) be ideals such that \( \varphi(I_i) \subset I_2 \) for some \( c \geq 1 \). This induces ring maps \( A_1, cn = A_1/I_1^n \to A_2/I_2^n = A_2, n \) for all \( n \geq 1 \). Let \( C_i \) be the category \( (2.0.1) \) for \( (A_i, I_i) \). There is a base change functor

\[
C_1 \to C_2, \quad (B_n) \mapsto (B_{cn} \otimes_{A_1, cn} A_2, n)
\]

Let \( C'_i \) be the category \( (2.0.2) \) for \( (A_i, I_i) \). If \( I_2 \) is finitely generated, then there is a base change functor

\[
C'_1 \to C'_2, \quad B \mapsto (B \otimes_{A_1} A_2)^\wedge
\]

because in this case the completion is complete (Algebra, Lemma 95.3). If both \( I_1 \) and \( I_2 \) are finitely generated, then the two base change functors agree via the functors \( (2.0.3) \) which are equivalences by Lemma 2.1.

**Remark 2.4** (Base change by closed immersion). Let \( A \) be a Noetherian ring and \( I \subset A \) an ideal. Let \( a \subset A \) be an ideal. Denote \( \bar{A} = A/a \). Let \( \bar{I} \subset \bar{A} \) be an ideal such that \( \bar{I}^c \bar{A} \subset \bar{I} \) and \( \bar{I}^d \subset \bar{I} \bar{A} \) for some \( c, d \geq 1 \). In this case the base change functor \( (2.3.2) \) for \( (A, I) \) to \( (\bar{A}, \bar{I}) \) is given by \( B \mapsto \bar{B} = B/aB \). Namely, we have

\[
\bar{B} = (B \otimes_A \bar{A})^\wedge = (B/aB)^\wedge = B/aB
\]

the last equality because any finite \( B \)-module is \( I \)-adically complete by Algebra, Lemma 96.1 and if annihilated by \( a \) also \( I \)-adically complete by Algebra, Lemma 95.9.

**3. A naive cotangent complex**

Let \( A \) be a Noetherian ring and let \( I \subset A \) be an ideal. Let \( B \) be an \( A \)-algebra which is \( I \)-adically complete such that \( A/I \to B/IB \) is of finite type, i.e., an object of \( (2.0.2) \). By Lemma 2.2 we can write

\[
B = A[x_1, \ldots, x_r]^\wedge / J
\]
Let two term complex (termwise limit) and the transition maps in the system are termwise surjective. The morphism of (2.0.2). Then there is an exact sequence

\[ NL_{B/A}^\wedge = (J/J^2 \longrightarrow \bigoplus Bdx_i) \]

with terms sitting in degrees \(-1\) and \(0\) where the map sends the residue class of \(g \in J\) to the differential \(dg = \sum (\partial g/\partial x_i) dx_i\). Here the partial derivative is taken by thinking of \(g\) as a power series. The following lemma shows that \(NL_{B/A}^\wedge\) is well defined in \(D(B)\), i.e., independent of the chosen presentation, although this could be shown directly by comparing presentations as in Algebra, Section 132.

**Lemma 3.1.** Let \(A\) be a Noetherian ring and let \(I \subset A\) be an ideal. Let \(B\) be an object of \((2.0.2)\). Then \(NL_{B/A}^\wedge = \text{R\ lim NL}_{B_n/A_n}\) in \(D(B)\).

**Proof.** In fact, the presentation \(B = A[x_1, \ldots, x_r]^\wedge/J\) defines presentations

\[ B_n = B/I^nB = A_n[x_1, \ldots, x_r]/J_n \]

where

\[ J_n =JA_n[x_1, \ldots, x_r] = J/(J \cap I^n A[x_1, \ldots, x_r]^\wedge) \]

By Artin-Rees (Algebra, Lemma 50.2) in the Noetherian ring \(A[x_1, \ldots, x_r]^\wedge\) (Lemma 2.2) we see that we have canonical surjections

\[ J/I^nJ \to J_n \to J/I^{n-c}J, \quad n \geq c \]

for some \(c \geq 0\). It follows that \(\text{lim} J_n/J_n^2 = J/J^2\) as any finite \(A[x_1, \ldots, x_r]^\wedge\)-module is \(I\)-adically complete (Algebra, Lemma 96.1). Thus

\[ NL_{B/A}^\wedge = \text{lim}(J_n/J_n^2 \longrightarrow \bigoplus B_n dx_i) \]

(termwise limit) and the transition maps in the system are termwise surjective. The two term complex \(J_n/J_n^2 \longrightarrow \bigoplus B_n dx_i\) represents \(NL_{B_n/A_n}\) by Algebra, Section 132. It follows that \(NL_{B/A}^\wedge\) represents \(\text{R\ lim NL}_{B_n/A_n}\) in the derived category by More on Algebra, Lemma 78.1. □

**Lemma 3.2.** Let \(A\) be a Noetherian ring and let \(I \subset A\) be an ideal. Let \(B \to C\) be a morphism of \((2.0.2)\). Then there is an exact sequence

\[ C \otimes_B H^0(NL_{B/A}^\wedge) \longrightarrow H^0(NL_{C/A}^\wedge) \longrightarrow H^0(NL_{C/B}^\wedge) \longrightarrow 0 \]

\[ H^{-1}(NL_{B/A}^\wedge \otimes_B C) \longrightarrow H^{-1}(NL_{C/A}^\wedge) \longrightarrow H^{-1}(NL_{C/B}^\wedge) \]

**Proof.** Choose a presentation \(B = A[x_1, \ldots, x_r]^\wedge/J\). Note that \((B, IB)\) is a pair consisting of a Noetherian ring and an ideal, and \(C\) is in the corresponding category \((2.0.2)\) for this pair. Hence we can choose a presentation \(C = B[y_1, \ldots, y_s]^\wedge/J'\). Combing these presentations gives a presentation

\[ C = A[x_1, \ldots, x_r, y_1, \ldots, y_s]^\wedge/K \]

Then the reader verifies that we obtain a commutative diagram

\[ 0 \longrightarrow \bigoplus Cdx_i \longrightarrow \bigoplus Cdx_i \oplus \bigoplus Cdy_j \longrightarrow \bigoplus Cdy_j \longrightarrow 0 \]

\[ J/J^2 \otimes_B C \longrightarrow K/K^2 \longrightarrow J'/(J')^2 \longrightarrow 0 \]
with exact rows. Note that the vertical arrow on the left hand side is the tensor product of the arrow defining \( NL_{B/A} \) with \( \text{id}_C \). The lemma follows by applying the snake lemma (Algebra, Lemma 4.1). □

**Lemma 3.3.** With assumptions as in Lemma 3.2 assume that \( B/I^nB \rightarrow C/I^nC \) is a local complete intersection homomorphism for all \( n \). Then \( H^{-1}(NL_{B/A}^\wedge \otimes BC) \rightarrow H^{-1}(NL_{C/A}) \) is injective.

**Proof.** By More on Algebra, Lemma 32.6 we see that this holds for the map between naive cotangent complexes of the situation modulo \( I^n \) for all \( n \). In other words, we obtain a distinguished triangle in \( D(C/I^nC) \) for every \( n \). Using Lemma 3.1 this implies the lemma; details omitted. □

Maps in the derived category out of a complex such as (3.0.1) are easy to understand by the result of the following lemma.

**Lemma 3.4.** Let \( R \) be a ring. Let \( M^\bullet \) be a complex of modules over \( R \) with \( M^i = 0 \) for \( i > 0 \) and \( M^0 \) a projective \( R \)-module. Let \( K^\bullet \) be a second complex.

1. If \( K^i = 0 \) for \( i \leq -2 \), then \( \text{Hom}_{D(R)}(M^\bullet, K^\bullet) = \text{Hom}_{K(R)}(M^\bullet, K^\bullet) \).
2. If \( K^i = 0 \) for \( i \leq -3 \) and \( \alpha \in \text{Hom}_{D(R)}(M^\bullet, K^\bullet) \) composed with \( K^\bullet \rightarrow K^{-2}[2] \) comes from an \( R \)-module map \( a : M^{-2} \rightarrow K^{-2} \) with \( a \circ d_M^{-2} = 0 \), then \( \alpha \) can be represented by a map of complexes \( a^\bullet : M^\bullet \rightarrow K^\bullet \) with \( a^{-2} = a \).
3. In (2) for any second map of complexes \( (a')^\bullet : M^\bullet \rightarrow K^\bullet \) representing \( \alpha \) with \( a = (a')^{-2} \) there exist \( h' : M^0 \rightarrow K^{-1} \) and \( h : M^{-1} \rightarrow K^{-2} \) such that \( h \circ d_M^2 = 0 \), \( (a')^{-1} = a^{-1} + d_K^{-2} \circ h + h' \circ d_M^{-1} \), \( (a')^0 = a^0 + d_K^{-1} \circ h' \).

**Proof.** Set \( F^0 = M^0 \). Choose a free \( R \)-module \( F^{-1} \) and a surjection \( F^{-1} \rightarrow M^{-1} \). Choose a free \( R \)-module \( F^{-2} \) and a surjection \( F^{-2} \rightarrow M^{-2} \times_{M^{-1}} F^{-1} \). Continuing in this way we obtain a quasi-isomorphism \( p^\bullet : F^\bullet \rightarrow M^\bullet \) which is termwise surjective and with \( F^i \) free for all \( i \).

Proof of (1). By Derived Categories, Lemma 19.8 we have
\[
\text{Hom}_{D(R)}(M^\bullet, K^\bullet) = \text{Hom}_{K(R)}(F^\bullet, K^\bullet)
\]
If \( K^i = 0 \) for \( i \leq -2 \), then any morphism of complexes \( F^\bullet \rightarrow K^\bullet \) factors through \( p^\bullet \). Similarly, any homotopy \( \{ h^i : F^i \rightarrow K^{i-1} \} \) factors through \( p^\bullet \). Thus (1) holds.

Proof of (2). Choose \( b^\bullet : F^\bullet \rightarrow K^\bullet \) representing \( \alpha \). The composition of \( \alpha \) with \( K^\bullet \rightarrow K^{-2}[2] \) is represented by \( b^{-2} : F^{-2} \rightarrow K^{-2} \). As this is homotopic to \( a \circ p^{-2} : F^{-2} \rightarrow M^{-2} \rightarrow K^{-2} \), there is a map \( h : F^{-1} \rightarrow K^{-2} \) such that \( b^{-2} = a \circ p^{-2} + h \circ d_F^{-2} \). Adjusting \( b^\bullet \) by \( h \) viewed as a homotopy from \( F^\bullet \) to \( K^\bullet \), we find that \( b^{-2} = a \circ p^{-2} \). Hence \( b^{-2} \) factors through \( p^{-2} \). Since \( F^0 = M^0 \) the kernel of \( p^{-2} \) surjects onto the kernel of \( p^{-1} \) (for example because the kernel of \( p^\bullet \) is an acyclic complex or by a diagram chase). Hence \( b^{-1} \) necessarily factors through \( p^{-1} \) as well and we see that (2) holds for these factorizations and \( a^0 = b^0 \).

Proof of (3) is omitted. Hint: There is a homotopy between \( a^\bullet \circ p^\bullet \) and \( (a')^\bullet \circ p^\bullet \) and we argue as before that this homotopy factors through \( p^\bullet \). □

**Lemma 3.5.** Let \( R \) be a ring. Let \( M^\bullet \) be a two term complex \( M^{-1} \rightarrow M^0 \) over \( R \). If \( \varphi, \psi \in \text{End}_{D(R)}(M^\bullet) \) are zero on \( H^i(M^\bullet) \), then \( \varphi \circ \psi = 0 \).
Proof. Apply Derived Categories, Lemma \[12.5\] to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2} M^* = 0$. \[\square\]

4. Rig-étale homomorphisms

0ALP In this and some of the later sections we will study ring maps as in Lemma \[1.1\] Condition \[4\] is one of the conditions used in [Art70] to define modifications. Ring maps like this are sometimes called rig-étale or rigid-étale ring maps in the literature. These and the analogously defined rig-smooth ring maps were studied in [Elk73]. A detailed exposition can also be found in [Abb10]. Our main goal will be to show that rig-étale ring maps are completions of finite type algebras, a result very similar to results found in Elkik’s paper [Elk73].

0AJU \textbf{Lemma 4.1.} Let $A$ be a Noetherian ring and let $I \subset A$ be an ideal. Let $B$ be an object of \[2.0.2\]. The following are equivalent

0AJV  (1) there exists an $a \geq 0$ such that multiplication by $a$ on $NL_{B/A}^\wedge$ is zero in $D(B)$ for all $a \in I^c$.

0AJW  (2) there exists an $a \geq 0$ such that $H^i(NL_{B/A}^\wedge)$, $i = -1, 0$ is annihilated by $I^c$.

0AXJ  (3) there exists an $a \geq 0$ such that $H^i(NL_{B_n/A_n})$, $i = -1, 0$ is annihilated by $I^c$ for all $n \geq 1$.

0AY  (4) $B = A[x_1, \ldots, x_r]^\wedge / J$ and for every $a \in I$ there exists an $a \geq 0$ such that

(a) $a^c$ annihilates $H^0(NL_{B/A}^\wedge)$, and
(b) there exist $f_1, \ldots, f_r \in J$ such that $a^c J \subset (f_1, \ldots, f_r) + J^2$.

Proof. The equivalence of (1) and (2) follows from Lemma \[3.5\] The equivalence of (1) and (2) follows from Lemma \[3.1\] Some details omitted.

Assume the equivalent conditions (1), (2), (3) holds and let $B = A[x_1, \ldots, x_r]^\wedge / J$ be a presentation (see Lemma \[2.1\]). Let $a \in I$. Let $c$ be such that multiplication by $a$ is zero on $NL_{B/A}^\wedge$ which exists by (1). By Lemma \[3.4\] there exists a map

\[
\alpha : \bigoplus B dx_i \to J / J^2
\]

such that $d \circ \alpha$ and $\alpha \circ d$ are both multiplication by $a^c$. Let $f_e \in J$ be an element whose class modulo $J^2$ is equal to $\alpha(dx_i)$. Then we see that \[4\](a), (b) hold.

Assume \[4\] holds. Say $I = (a_1, \ldots, a_l)$. Let $c_i \geq 0$ be the integer such that \[4\](a), (b) hold for $a_i^{c_i}$. Then we see that $I \sum c_i$ annihilates $H^0(NL_{B/A}^\wedge)$. Let $f_{i,1}, \ldots, f_{i,r} \in J$ be as in \[4\](b) for $a_i$. Consider the composition

\[
B^{\oplus r} \to J / J^2 \to \bigoplus B dx_i
\]

where the $j$th basis vector is mapped to the class of $f_{i,j}$ in $J / J^2$. By \[4\](a) and (b) the cokernel of the composition is annihilated by $a_i^{2c_i}$. Thus this map is surjective after inverting $a_i^{c_i}$, and hence an isomorphism (Algebra, Lemma \[15.2\]). Thus the kernel of $B^{\oplus r} \to \bigoplus B dx_i$ is $a_i$-power torsion, and hence $H^{-1}(NL_{B/A}^\wedge) = \text{Ker}(J / J^2 \to \bigoplus B dx_i)$ is $a_i$-power torsion. Since $B$ is Noetherian (Lemma \[2.2\], all modules including $H^{-1}(NL_{B/A}^\wedge)$ are finite. Thus $a_i^{d_i}$ annihilates $H^{-1}(NL_{B/A}^\wedge)$ for some $d_i \geq 0$. It follows that $I \sum d_i$ annihilates $H^{-1}(NL_{B/A}^\wedge)$ and we see that (2) holds. \[\square\]

0ALQ \textbf{Lemma 4.2.} Let $A$ be a Noetherian ring and let $I$ be an ideal. Let $B$ be a finite type $A$-algebra.

\begin{enumerate}
\item If $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$, then $B^\wedge$ satisfies the equivalent conditions of Lemma \[4.1\].
\end{enumerate}
(2) If $B^\wedge$ satisfies the equivalent conditions of Lemma 4.1 then there exists $g \in 1 + IB$ such that $\text{Spec}(B_g)$ is étale over $\text{Spec}(A) \setminus V(I)$.

**Proof.** Assume $B^\wedge$ satisfies the equivalent conditions of Lemma 4.1. The naive cotangent complex $\text{NL}_{B/A}$ is a complex of finite type $B$-modules and hence $H^{-1}$ and $H^0$ are finite $B$-modules. Completion is an exact functor on finite $B$-modules (Algebra, Lemma 96.2) and $\text{NL}_{B/A}^\wedge$ is the completion of the complex $\text{NL}_{B/A}$ (this is easy to see by choosing presentations). Hence the assumption implies there exists a $c \geq 0$ such that $H^{-1}/I^nH^{-1}$ and $H^0/I^nH^0$ are annihilated by $I^c$ for all $n$. By Nakayama’s lemma (Algebra, Lemma 19.1) this means that $I^nH^{-1}$ and $I^nH^0$ are annihilated by an element of the form $g = 1 + x$ with $x \in IB$. After inverting $g$ (which does not change the quotients $B/I^nB$) we see that $\text{NL}_{B/A}$ has cohomology annihilated by $I^c$. Thus $A \to B$ is étale at any prime of $B$ not lying over $V(I)$ by the definition of étale ring maps, see Algebra, Definition 141.1.

Conversely, assume that $\text{Spec}(B) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists a $c \geq 0$ such that multiplication by $a^c$ is zero $\text{NL}_{B/A}$. Since $\text{NL}_{B/A}^\wedge$ is the derived completion of $\text{NL}_{B/A}$ (see Lemma 3.1) it follows that $B^\wedge$ satisfies the equivalent conditions of Lemma 4.1. □

**Lemma 4.3.** Assume the map $(A_1, I_1) \to (A_2, I_2)$ is as in Remark 2.3 with $A_1$ and $A_2$ Noetherian. Let $B_1$ be in (2.0.3) for $(A_1, I_1)$. Let $B_2$ be the base change of $B_1$. If multiplication by $f_1 \in B_1$ on $\text{NL}_{B_1/A_1}$ is zero in $D(B_1)$, then multiplication by the image $f_2 \in B_2$ on $\text{NL}_{B_2/A_2}$ is zero in $D(B_2)$.

**Proof.** Choose a presentation $B_1 = A_1[x_1, \ldots, x_r]/J_1$. Since $A_2/I^n_2[x_1, \ldots, x_r] = A_1/I^n_1[x_1, \ldots, x_r] \otimes_{A_1/I^n_1} A_2/I^n_2$ we have

$$A_2[x_1, \ldots, x_r]^\wedge = (A_1[x_1, \ldots, x_r]^\wedge \otimes_{A_1} A_2)^\wedge$$

where we use $I_2$-adic completion on both sides (but of course $I_1$-adic completion for $A_1[x_1, \ldots, x_r]^\wedge$). Set $J_2 = J_1A_2[x_1, \ldots, x_r]^\wedge$. Arguing similarly we get the presentation

$$B_2 = (B_1 \otimes_{A_1} A_2)^\wedge = \lim \frac{A_1/I^n_1[x_1, \ldots, x_r]}{J_1(A_1/I^n_1[x_1, \ldots, x_r])} \otimes_{A_1/I^n_1} A_2/I^n_2 = \lim \frac{A_2/I^n_2[x_1, \ldots, x_r]}{J_2(A_2/I^n_2[x_1, \ldots, x_r])} = A_2[x_1, \ldots, x_r]^\wedge/J_2$$

for $B_2$ over $A_2$. Consider the commutative diagram

$$\begin{array}{ccc}
\text{NL}_{B_1/A_1} & \xrightarrow{d} & B_1 \text{d}x_i \\
\downarrow & & \downarrow \\
\text{NL}_{B_2/A_2} & \xrightarrow{\text{d}} & B_2 \text{d}x_i
\end{array}$$

The induced arrow $J_1/J_1^2 \otimes_{B_1} B_2 \to J_2/J_2^2$ is surjective because $J_2$ is generated by the image of $J_1$. By Lemma 3.4 there is a map $\alpha_1 : \bigoplus B \text{d}x_i \to J_1/J_1^2$ such that $f_1 \text{id}_{B_1 \text{d}x_i} = d \circ \alpha_1$ and $f_1 \text{id}_{J_1/J_1^2} = \alpha_1 \circ d$. We define $\alpha_2 : \bigoplus B_1 \text{d}x_i \to J_2/J_2^2$ by mapping $\text{d}x_i$ to the image of $\alpha_1(\text{d}x_i)$ in $J_2/J_2^2$. Because the image of the vertical
arrows contains generators of the modules \( J_2/J_2^2 \) and \( \bigoplus B_2 \mathrm{d} x_i \) it follows that \( \alpha_2 \) also defines a homotopy between multiplication by \( f_2 \) and the zero map. \( \square \)

**Lemma 4.4.** Let \( A \) be a Noetherian ring. Let \( I \subset A \) be an ideal. Let \( B \) be a finite type \( A \)-algebra such that \( \operatorname{Spec}(B) \to \operatorname{Spec}(A) \) is étale over \( \operatorname{Spec}(A) \setminus V(I) \). Let \( C \) be a Noetherian \( A \)-algebra. Then any \( A \)-algebra map \( B^\wedge \to C^\wedge \) of \( I \)-adic completions comes from a unique \( A \)-algebra map

\[
B \to C^h
\]

where \( C^h \) is the henselization of the pair \( (C, IC) \) as in More on Algebra, Lemma 12.4. Moreover, any \( A \)-algebra homomorphism \( B \to C^h \) factors through some étale \( C \)-algebra \( C' \) such that \( C/IC \to C'/IC' \) is an isomorphism.

**Proof.** Uniqueness follows from the fact that \( C^h \) is a subring of \( C^\wedge \), see for example More on Algebra, Lemma 12.4. The final assertion follows from the fact that \( C^h \) is the filtered colimit of these \( C \)-algebras \( C' \), see proof of More on Algebra, Lemma 12.1. Having said this we now turn to the proof of existence.

Let \( \varphi : B^\wedge \to C^\wedge \) be the given map. This defines a section

\[
\sigma : (B \otimes_A C)^\wedge \to C^\wedge
\]

of the completion of the map \( C \to B \otimes_A C \). We may replace \( (A, I, B, C, \varphi) \) by \( (C, IC, B \otimes_A C, C, \sigma) \). In this way we see that we may assume that \( A = C \).

Proof of existence in the case \( A = C \). In this case the map \( \varphi : B^\wedge \to A^\wedge \) is necessarily surjective. By Lemmas 1.2 and 3.2 we see that the cohomology groups of \( NL_{A^\wedge/B^\wedge}^A \) are annihilated by a power of \( I \). Since \( \varphi \) is surjective, this implies that \( \operatorname{Ker}(\varphi)/\operatorname{Ker}(\varphi)^2 \) is annihilated by a power of \( I \). Hence \( \varphi : B^\wedge \to A^\wedge \) is the completion of a finite type \( B \)-algebra \( B \to D \), see More on Algebra, Lemma 94.4. Hence \( A \to D \) is a finite type algebra map which induces an isomorphism \( A^\wedge \to D^\wedge \). By Lemma 4.2 we may replace \( D \) by a localization and assume that \( A \to D \) is étale away from \( V(I) \). Since \( A^\wedge \to D^\wedge \) is an isomorphism, we see that \( \operatorname{Spec}(D) \to \operatorname{Spec}(A) \) is also étale in a neighbourhood of \( V(ID) \) (for example by More on Morphisms, Lemma 12.3). Thus \( \operatorname{Spec}(D) \to \operatorname{Spec}(A) \) is étale. Therefore \( D \) maps to \( A^h \) and the lemma is proved. \( \square \)

### 5. Rig-étale morphisms

We can use the notion introduced in the previous section to define a new type of morphism of locally Noetherian formal algebraic spaces. Before we do so, we have to check it is a local property.

**Lemma 5.1.** For morphisms \( A \to B \) of the category \( \text{WAdm}^{\text{Noeth}} \) (Formal Spaces, Section 16) consider the condition \( P = \{ \text{for some ideal of definition } I \text{ of } A \text{ the topology on } B \text{ is the } I \text{-adic topology, the ring map } A/I \to B/IB \text{ is of finite type and } A \to B \text{ satisfies the equivalent conditions of Lemma 4.4} \} \). Then \( P \) is a local property, see Formal Spaces, Remark 16.2.

**Proof.** We have to show that Formal Spaces, Axioms (1), (2), and (3) hold for maps between Noetherian adic rings. For a Noetherian adic ring \( A \) with ideal of definition \( I \) we have \( A(x_1, \ldots, x_r) = A[x_1, \ldots, x_r]^\wedge \) as topological \( A \)-algebras (see Formal Spaces, Remark 21.2). We will use without further mention that we know
the axioms hold for the property “$B$ is a quotient of $A[x_1,\ldots,x_r]^\wedge$”, see Formal Spaces, Lemma 22.6.

Let a diagram as in Formal Spaces, Diagram 16.2.1 be given with $A$ and $B$ in the category $\textit{WAdm}^{\text{Noeth}}$. Pick an ideal of definition $I \subset A$. By the remarks above the topology on each ring in the diagram is the $I$-adic topology. Since $A \to A'$ and $B \to B'$ are étale we see that $\text{NL}_{(A')}^\wedge/I_A$ and $\text{NL}_{(B')}^\wedge/I_B$ are zero. By Lemmas 3.2 and 3.3 we get

$$H^i(\text{NL}_{(B')}^\wedge/(A')^\wedge) \cong H^i(\text{NL}_{B/A}^\wedge \otimes_B (B')^\wedge) \cong H^i(\text{NL}_{(B')}^\wedge/I_A)$$

for $i = -1, 0$. Since $B$ is Noetherian the ring map $B \to B' \to (B')^\wedge$ is flat (Algebra, Lemma 96.2) hence the tensor product comes out. Moreover, as $B$ is $I$-adically complete, then if $B \to B'$ is faithfully flat, so is $B \to (B')^\wedge$. From these observations Formal Spaces, Axioms (1) and (2) follow immediately.

We omit the proof of Formal Spaces, Axiom (3). \[\square\]

**Definition 5.2.** Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces over $S$. We say $f$ is rig-étale if $f$ satisfies the equivalent conditions of Formal Spaces, Lemma 16.3 (in the setting of locally Noetherian formal algebraic spaces, see Formal Spaces, Remark 16.4) for the property $P$ of Lemma 5.1.

To be sure, a rig-étale morphism is locally of finite type.

**Lemma 5.3.** A rig-étale morphism of locally Noetherian formal algebraic spaces is locally of finite type.

**Proof.** The property $P$ in Lemma 5.1 implies the equivalent conditions (a), (b), (c), and (d) in Formal Spaces, Lemma 22.6. Hence this follows from Formal Spaces, Lemma 22.9. \[\square\]

### 6. Glueing rings along a principal ideal

In this situation we prove some results about the categories $C$ and $C'$ of Section 2 in case $A$ is a Noetherian ring and $I = (a)$ is a principal ideal.

**Remark 6.1 (Linear approximation).** Let $A$ be a ring and $I \subset A$ be a finitely generated ideal. Let $C$ be an $I$-adically complete $A$-algebra. Let $\psi : A[x_1,\ldots,x_r]^\wedge \to C$ be a continuous $A$-algebra map. Suppose given $\delta_i \in C$, $i = 1,\ldots,r$. Then we can consider

$$\psi' : A[x_1,\ldots,x_r]^\wedge \to C, \quad x_i \mapsto \psi(x_i) + \delta_i$$

see Formal Spaces, Remark 21.1. Then we have

$$\psi'(g) = \psi(g) + \sum \psi(\partial g/\partial x_i) \delta_i + \xi$$

with error term $\xi \in (\delta_i \delta_j)$. This follows by writing $g$ as a power series and working term by term. Convergence is automatic as the coefficients of $g$ tend to zero. Details omitted.

**Lemma 6.2.** Let $A$ be a Noetherian ring and $I = (a)$ a principal ideal. Let $B$ be an objects of 2.0.2. Assume given an integer $c \geq 0$ such that multiplication by $a^c$ on $\text{NL}_{B/A}^\wedge$ is zero in $D(B)$. Let $C$ be an $I$-adically complete $A$-algebra such that $a$ is a nonzerodivisor on $C$. Let $n > 2c$. For any $A_n$-algebra map $\psi_n : B/a^nB \to C/a^nC$ there exists an $A$-algebra map $\phi : B \to C$ such that $\psi_n \mod a^{n-c} = \phi \mod a^{n-c}$.
Proof. Choose a presentation $B = A[x_1, \ldots, x_r] \wedge/J$. Choose a lift

$$\psi : A[x_1, \ldots, x_r] \wedge \rightarrow C$$

of $\psi_n$. Then $\psi(J) \subset a^nC$ and $\psi(J^2) \subset a^{2n}C$ which determines a linear map

$$J/J^2 \rightarrow a^nC/a^{2n}C, \quad g \mapsto \psi(g)$$

By assumption and Lemma 3.4 there is a $B$-module map $\bigoplus Bdx_i \rightarrow a^nC/a^{2n}C$, $dx_i \mapsto \delta_i$ such that $a^c\psi(g) = \sum (\partial g/\partial x_i)\delta_i$ for all $g \in J$. Write $\delta_i = -a^c\delta'_i$ for some $\delta'_i \in a^{n-c}C$. Since $a$ is a nonzerodivisor on $C$ we see that $\psi(g) = -\sum (\partial g/\partial x_i)\delta'_i$ in $C/a^{2n-c}C$. Then we look at the map

$$\psi' : A[x_1, \ldots, x_r] \rightarrow C, \quad x_i \mapsto \psi(x_i) + \delta'_i$$

A computation with power series (see Remark 6.1) shows that $\psi'(J) \subset a^{2n-2c}C$. Since $n > 2c$ we see that $n' = 2n - 2c = n + (n - 2c) > n$. Thus we obtain a morphism $\psi_n' : B/a^nB \rightarrow C/a^nC$ agreeing with $\psi_n$ modulo $a^{n-c}$. Continuing in this fashion and taking the limit into $C = \lim C/a^nC$ we obtain the lemma. □

Lemma 6.3. Let $A$ be a Noetherian ring and $I = (a)$ a principal ideal. Let $B$ be an object of (2.0.2). Assume given an integer $c \geq 0$ such that multiplication by $a^c$ on $N_{(I)}^{\wedge}B/A$ is zero in $D(B)$. Let $C$ be an $I$-adically complete $A$-algebra. Assume given an integer $d \geq 0$ such that $C[a^{\infty}] \cap a^dC = 0$. Let $n > \max(2c, c + d)$. For any $A_n$-algebra map $\psi_n : B/a^nB \rightarrow C/a^nC$ there exists an $A$-algebra map $\varphi : B \rightarrow C$ such that $\psi_n \bmod a^{n-c} = \varphi \bmod a^{n-c}$.

If $C$ is Noetherian we have $C[a^{\infty}] = C[a^c]$ for some $c \geq 0$. By Artin-Rees (Algebra, Lemma 50.2) there exists an integer $f$ such that $a^nC \cap C[a^{\infty}] \subset a^{n-f}C[a^{\infty}]$ for all $n \geq f$. Then $d = e + f$ is an integer as in the lemma. This argument works in particular if $C$ is an object of (2.0.2) by Lemma 2.2.

Proof. Let $C \rightarrow C'$ be the quotient of $C$ by $C[a^{\infty}]$. The $A$-algebra $C'$ is $I$-adically complete by Algebra, Lemma 95.10 and the fact that $\bigcap C[a^{\infty}] + a^nC = C[a^{\infty}]$ because for $n \geq d$ the sum $C[a^{\infty}] + a^nC$ is direct. For $m \geq d$ the diagram

$$\begin{array}{ccc}
0 & \rightarrow & C[a^{\infty}] & \rightarrow & C & \rightarrow & C' & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C[a^m] & \rightarrow & C/a^mC & \rightarrow & C'/a^mC' & \rightarrow & 0
\end{array}$$

has exact rows. Thus $C$ is the fibre product of $C'$ and $C/a^mC$ over $C'/a^mC'$. Thus the lemma now follows formally from the lifting result of Lemma 6.2. □

Lemma 6.4. Let $A$ be a Noetherian ring and $I = (a)$ a principal ideal. Let $B$ be an object of (2.0.2). Assume given an integer $c \geq 0$ such that multiplication by $a^c$ on $N_{(I)}^{\wedge}B/A$ is zero in $D(B)$. Then there exists a finite type $A$-algebra $C$ and an isomorphism $B \cong C\wedge$.

Proof. Choose a presentation $B = A[x_1, \ldots, x_r] \wedge/J$. By Lemma 3.4 we can find a map $\alpha : \bigoplus Bdx_i \rightarrow J/J^2$ such that $d \circ \alpha$ and $\alpha \circ d$ are both multiplication by $a^c$. Pick an element $f_i \in J$ whose class modulo $J^2$ is equal to $\alpha(dx_i)$. Then we see that $df_i = a^cdx_i$ in $\bigoplus dx_i$. In particular we have a ring map

$$A[x_1, \ldots, x_r] \wedge/(f_1, \ldots, f_r, \Delta(f_1, \ldots, f_r) - a^c) \rightarrow B$$

The rig-étale case of [Elk73] III Theorem 7 which handles the rig-smooth case.
where $\Delta(f_1, \ldots, f_r) \in A[x_1, \ldots, x_r]^\wedge$ is the determinant of the matrix of partial derivatives of the $f_i$.

Pick a large integer $N$. Pick $F_1, \ldots, F_r \in A[x_1, \ldots, x_r]$ such that $F_i - f_i \in I^N A[x_1, \ldots, x_r]^\wedge$. Set

$$C = A[x_1, \ldots, x_r, z]/(F_1, \ldots, F_r, z\Delta(F_1, \ldots, F_r) - a^{rc})$$

We claim that multiplication by $a^{2rc}$ is zero on $NL_{C/A}$ in $D(C)$. Namely, the determinant of the matrix of the partial derivatives of the $r + 1$ generators of the ideal of $C$ with respect to the variables $x_1, \ldots, x_{r+1}, z$ is $\Delta(F_1, \ldots, F_r)^2$. Since $\Delta(F_1, \ldots, F_r)$ divides $a^{rc}$ we in $C$ the claim follows from example from Algebra, Lemma \[14.5.\] Let $C^\wedge$ be the $I$-adic completion of $C$. Since $NL_{C^\wedge/A}$ is the $I$-adic completion of $NL_{C/A}$ we conclude that multiplication by $a^{2rc}$ is zero on $NL_{C^\wedge/A}$ as well.

By construction there is a (surjective) map $\psi_N : C/I^N C \to B/I^N B$ sending $x_i$ to $x_i$ and $z$ to 1. By Lemma \[6.3\] (with the roles of $B$ and $C$ reversed) for $N$ large enough we get a map $\varphi : C^\wedge \to B$ which agrees with $\psi_N$ modulo $I^{N-2rc}$.

Since $\varphi : C^\wedge \to B$ is surjective modulo $I$ we see that it is surjective (for example use Algebra, Lemma \[95.1\]). By construction and assumption the naive cotangent complexes $NL_{C^\wedge/A}$ and $NL_{B/A}$ have cohomology annihilated by a fixed power of $a$. Thus the same thing is true for $NL_{B/C^\wedge}$ by Lemma \[3.2\]. Since $\varphi$ is surjective we conclude that $\operatorname{Ker}(\varphi)/\operatorname{Ker}(\varphi)^2$ is annihilated by a power of $a$. The result of the lemma now follows from More on Algebra, Lemma \[94.4\]. \hfill \qed

7. Glueing rings along an ideal

0AK8 Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. In this section we study $I$-adically complete $A$-algebras which are, in some vague sense, étale over the complement of $V(I)$ in $\text{Spec}(A)$.

0AK9 \textbf{Lemma 7.1.} Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. Let $t$ be the minimal number of generators for $I$. Let $C$ be a Noetherian $I$-adically complete $A$-algebra. There exists an integer $d \geq 0$ depending only on $I \subset A \to C$ with the following property: given

1. $c \geq 0$ and $B$ in $I^c$ such that for $a \in I^c$ multiplication by $a$ on $NL_{B/A}$ is zero in $D(B)$,
2. an integer $n > 2t \max(c, d)$,
3. an $A/I^n$-algebra map $\psi_n : B/I^n B \to C/I^n C$,

there exists a map $\varphi : B \to C$ of $A$-algebras such that $\psi_n \bmod I^{m-c} = \varphi \bmod I^{m-c}$ with $m = [\frac{n}{t}]$. \hfill \medskip

\textbf{Proof.} We prove this lemma by induction on the number of generators of $I$. Say $I = (a_1, \ldots, a_t)$. If $t = 0$, then $I = 0$ and there is nothing to prove. If $t = 1$, then the lemma follows from Lemma \[6.3\] because $2 \max(c, d) \geq \max(2c, c + d)$. Assume $t > 1$.

Set $m = [\frac{n}{t}]$ as in the lemma. Set $\tilde{A} = A/(a_1^m)$. Consider the ideal $\tilde{I} = (a_1, \ldots, a_{t-1})$ in $\tilde{A}$. Set $\tilde{C} = C/(a_1^m)$. Note that $\tilde{C}$ is a $\tilde{I}$-adically complete Noetherian $\tilde{A}$-algebra (use Algebra, Lemmas \[96.1\] and \[95.9\]). Let $\tilde{d}$ be the integer for $\tilde{I} \subset \tilde{A} \to \tilde{C}$ which exists by induction hypothesis.
Let \( d_1 \geq 0 \) be an integer such that \( C[a_1^\infty] \cap a_1^{d_1} C = 0 \) as in Lemma 6.3 (see discussion following the lemma and before the proof).

We claim the lemma holds with \( d = \max(\bar{d}, d_1) \). To see this, let \( c, B, n, \psi_n \) be as in the lemma.

Note that \( \bar{I} \subset I \bar{A} \). Hence by Lemma 4.3 multiplication by an element of \( \bar{I}^c \) on the cotangent complex of \( \bar{B} = B/(a_1^m) \) is zero in \( D(\bar{B}) \). Also, we have

\[
\bar{I}^{n-m+1} \supset \bar{I}_n \bar{A}
\]

Thus \( \psi_n \) gives rise to a map

\[
\bar{\psi}_{n-m+1} : \bar{B}/\bar{I}^{n-m+1} \bar{B} \longrightarrow \bar{C}/\bar{I}^{n-m+1} \bar{C}
\]

Since \( n > 2t \max(c, d) \) and \( d \geq \bar{d} \) we see that

\[
n - m + 1 \geq (t-1)n/t > 2(t-1) \max(c, d) \geq 2(t-1) \max(c, \bar{d})
\]

Hence we can find a morphism \( \varphi_m : \bar{B} \to \bar{C} \) agreeing with \( \bar{\psi}_{n-m+1} \) modulo the ideal \( \bar{I}^{m'-c} \) where \( m' = \floor{\frac{n-m+1}{t-1}} \).

Since \( m \geq n/t > 2 \max(c, d) \geq 2 \max(c, d_1) \geq \max(2c, c+d_1) \), we can apply Lemma 6.3 for the ring map \( A \to B \) and the ideal \( (a_t) \) to find a morphism \( \varphi : B \to C \) agreeing modulo \( a_t^{m-c} \) with \( \varphi_m \).

All in all we find \( \varphi : B \to C \) which agrees with \( \psi_n \) modulo

\[
(a_t^{m-c}) + (a_1, \ldots, a_{t-1})^{m'-c} \subset \bar{I}^{\min(m-c, m'-c)}
\]

We leave it to the reader to see that \( \min(m-c, m'-c) = m-c \). This concludes the proof. \( \square \)

**Lemma 7.2.** Let \( A \) be a Noetherian ring and \( I \subset A \) an ideal. Let \( J \subset A \) be a nilpotent ideal. Consider a diagram

\[
\begin{array}{ccc}
C & \longrightarrow & C/JC \\
\uparrow & & \uparrow \\
B_0 & \longrightarrow & C/JC \\
A & \longrightarrow & A/J
\end{array}
\]

whose vertical arrows are of finite type such that

1. \( \text{Spec}(C) \to \text{Spec}(A) \) is étale over \( \text{Spec}(A) \setminus V(I) \),
2. \( \text{Spec}(B_0) \to \text{Spec}(A/J) \) is étale over \( \text{Spec}(A/J) \setminus V((I + J)/J) \), and
3. \( B_0 \to C/JC \) is étale and induces an isomorphism \( B_0/IB_0 = C/(I + J)C \).
Then we can fill in the diagram

\[
\begin{array}{c}
C \\
\downarrow \downarrow \\
B \\
\downarrow \\
A
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\rightarrow \rightarrow \\
\rightarrow \\
\rightarrow \\
A/J
\end{array}
\quad
\begin{array}{c}
C/JC \\
B_0 \\
B_0 \\
B_0 \\
B
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
A/J
\end{array}
\]

with \(A \to B\) of finite type, \(B/\mathcal{J}B = B_0\), \(B \to C\) étale, and \(\text{Spec}(B) \to \text{Spec}(A)\) étale over \(\text{Spec}(A) \setminus V(I)\).

**First proof.** This proof uses algebraic spaces to construct \(B\). Set \(X = \text{Spec}(A)\), \(X_0 = \text{Spec}(A/J)\), \(Y_0 = \text{Spec}(B_0)\), \(Z = \text{Spec}(C)\), \(Z_0 = \text{Spec}(C/JC)\). Furthermore, denote \(U \subset X\), \(U_0 \subset X_0\), \(V_0 \subset Y_0\), \(W \subset Z\), \(W_0 \subset Z_0\) the complement of the vanishing set of \(I\). The conditions in the lemma guarantee that

\[
\begin{array}{c}
W_0 \\\nV_0
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
Z_0 \\
Y_0
\end{array}
\]

is an elementary distinguished square. In addition we know that \(W_0 \to U_0\) and \(V_0 \to U_0\) are étale. The morphism \(X_0 \subset X\) is a finite order thickening. By the topological invariance of the étale site we can find a unique étale morphism \(V \to X\) with \(V_0 = V \times_X X_0\) and we can lift the given morphism \(W_0 \to V_0\) to a unique morphism \(W \to V\). See More on Morphisms of Spaces, Theorem 8.1. By Pushouts of Spaces, Lemma 4.2 we can construct an elementary distinguished square

\[
\begin{array}{c}
W \\
V
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\quad
\begin{array}{c}
Z \\
Y
\end{array}
\]

in the category of algebraic spaces over \(X\). Since the base change of an elementary distinguished square is an elementary distinguished square (Derived Categories of Spaces, Lemma 9.2) and since elementary distinguished squares are pushouts (Pushouts of Spaces, Lemma 4.1) we see that the base change of this diagram by \(X_0 \to X\) gives the previous diagram. It follows that \(Y\) is affine by Limits of Spaces, Proposition 15.2. Write \(Y = \text{Spec}(B)\). Then \(B\) fits into the desired diagram and satisfies all the properties required of it. □

**Second proof.** This proof uses a little bit of deformation theory to construct \(B\). By induction on the smallest \(n\) such that \(J^n = 0\) we reduce to the case \(J^2 = 0\). Denote by a subscript zero the base change of objects to \(A_0 = A/J\). Since \(J^2 = 0\) we see that \(JC\) is a \(C_0\)-module.

Consider the canonical map

\[\gamma : J \otimes_{A_0} C_0 \to JC\]

Since \(\text{Spec}(C) \to \text{Spec}(A)\) is étale over the complement of \(V(I)\) (and hence flat) we see that \(\gamma\) is an isomorphism away from \(V(IC_0)\), see More on Morphisms, Lemma
In particular, the kernel and cokernel of $\gamma$ are annihilated by a power of $I$ (use that $C_0$ is Noetherian and that the modules in question are finite). Observe that $J \otimes_{A_0} C_0 = (J \otimes_{A_0} B_0) \otimes_{B_0} C_0$. Hence by More on Algebra, Lemma 80.16 there exists a unique $B_0$-module homomorphism

$$c : J \otimes_{A_0} B_0 \to N$$

with $c \otimes \text{id}_{C_0} = \gamma$ and $\text{Ker}(\gamma) = \text{Ker}(c)$ and $\text{Coker}(\gamma) = \text{Coker}(c)$. Moreover, $N$ is a finite $B_0$-module, see More on Algebra, Remark 80.19.

Choose a presentation $B_0 = A[x_1, \ldots, x_r]/I$. To construct $B$ we try to find the dotted arrow $m$ fitting into the following pushout diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & N & \rightarrow & B & \rightarrow & B_0 & \rightarrow & 0 \\
& & \downarrow{m} & & & & \downarrow & & \\
0 & \rightarrow & K/K^2 & \rightarrow & A[x_1, \ldots, x_r]/K^2 & \rightarrow & A[x_1, \ldots, x_r]/K & \rightarrow & 0 \\
J \otimes_{A_0} B_0 & \rightarrow & & & & & & \\
\end{array}
$$

where the curved arrow is the map $c$ constructed above and the map $J \otimes_{A_0} B_0 \to K/K^2$ is the obvious one.

As $B_0 \to C_0$ is étale we can write $C_0 = B_0[y_1, \ldots, y_s]/(g_{0,1}, \ldots, g_{0,s})$ such that the determinant of the partial derivatives of the $g_{0,j}$ is invertible in $C_0$, see Algebra, Lemma 141.2. We combine this with the chosen presentation of $B_0$ to get a presentation $C_0 = A[x_1, \ldots, x_r, y_1, \ldots, y_s]/L$. Choose a lift $\psi : A[x_i, y_j] \to C$ of the map to $C_0$. Then it is the case that $C$ fits into the diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & JC & \rightarrow & C & \rightarrow & C_0 & \rightarrow & 0 \\
& & \downarrow{\mu} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L/L^2 & \rightarrow & A[x_i, y_j]/L^2 & \rightarrow & A[x_i, y_j]/L & \rightarrow & 0 \\
J \otimes_{A_0} C_0 & \rightarrow & & & & & & \\
\end{array}
$$

where the curved arrow is the map $\gamma$ constructed above and the map $J \otimes_{A_0} C_0 \to L/L^2$ is the obvious one. By our choice of presentations and the fact that $C_0$ is a complete intersection over $B_0$ we have

$$L/L^2 = K/K^2 \otimes_{B_0} C_0 \oplus \bigoplus g_j$$

where $g_j \in L$ is any lift of $g_{0,j}$, see More on Algebra, Lemma 32.6.

Consider the three term complex

$$K^\bullet : J \otimes_{A_0} B_0 \to K/K^2 \to \bigoplus B_0 dx_i$$

where the second arrow is the differential in the naive cotangent complex of $B_0$ over $A$ for the given presentation and the last term is placed in degree 0. Since $\text{Spec}(B_0) \rightarrow \text{Spec}(A_0)$ is étale away from $V(I)$ the cohomology modules of this complex are supported on $V(IB_0)$. Namely, for $a \in I$ after inverting $a$ we can apply More on Algebra, Lemma 32.6 for the ring maps $A_a \to A_{0,a} \to B_{0,a}$ and
use that $NL_{A_0/A} = J_\alpha$ and $NL_{B_0/A_0} = 0$ (some details omitted). Hence these cohomology groups are annihilated by a power of $I$.

Similarly, consider the three term complex

$$L^\bullet : J \otimes A_0 C_0 \to L/L^2 \to \bigoplus C_0 dx_i \oplus \bigoplus C_0 dy_j$$

By our direct sum decomposition of $L/L^2$ above and the fact that the determinant of the partial derivatives of the $g_{0,j}$ is invertible in $C_0$ we see that the natural map $K^\bullet \to L^\bullet$ induces a quasi-isomorphism

$$K^\bullet \otimes_{B_0} C_0 \to L^\bullet$$

Applying Dualizing Complexes, Lemma 9.8 we find that

$$\text{Hom}_{D(B_0)}(K^\bullet, E) = \text{Hom}_{D(C_0)}(L^\bullet, E \otimes_{B_0} C_0)$$

for any object $E \in D(B_0)$.

The maps id$_{J \otimes A_0 C_0}$ and $\mu$ define an element in

$$\text{Hom}_{D(C_0)}(L^\bullet, (J \otimes A_0 C_0 \to JC))$$

(the target two term complex is placed in degree $-2$ and $-1$) such that the composition with the map to $J \otimes A_0 C_0[2]$ is the element in $\text{Hom}_{D(C_0)}(L^\bullet, J \otimes A_0 C_0[2])$ corresponding to id$_{J \otimes A_0 C_0}$. Picture

$$\begin{array}{cccc}
J \otimes A_0 C_0 & \to & L/L^2 & \to \bigoplus C_0 dx_i \oplus \bigoplus C_0 dy_j \\
\text{id}_{J \otimes A_0 C_0} & \downarrow & \mu & \\
J \otimes A_0 C_0 & \to & JC
\end{array}$$

Applying (7.2.1) we obtain a unique element

$$\xi \in \text{Hom}_{D(B_0)}(K^\bullet, (J \otimes A_0 B_0 \to N))$$

Its composition with the map to $J \otimes A_0 B_0[2]$ is the element in $\text{Hom}_{D(C_0)}(K^\bullet, J \otimes A_0 B_0[2])$ corresponding to id$_{J \otimes A_0 B_0}$. By Lemma 3.4 we can find a map of complexes $K^\bullet \to (J \otimes A_0 B_0 \to N)$ representing $\xi$ and equal to id$_{J \otimes A_0 B_0}$ in degree $-2$. Denote $m : K^\bullet \to N$ the degree $-1$ part of this map. Picture

$$\begin{array}{cccc}
J \otimes A_0 B_0 & \to & K/K^2 & \to \bigoplus B_0 dx_i \\
\text{id}_{J \otimes A_0 B_0} & \downarrow & \overline{m} & \\
J \otimes A_0 B_0 & \to & N
\end{array}$$

Thus we can use $m$ to create an algebra $B$ by push out as explained above. However, we may still have to change $m$ a bit to make sure that $B$ maps to $C$ in the correct manner.

Denote $m \otimes \text{id}_{C_0} \otimes 0 : L/L^2 \to JC$ the map coming from the direct sum decomposition of $L/L^2$ (see above), using that $N \otimes_{B_0} C_0 = JC$, and using 0 on the second factor. By our choice of $m$ above the maps of complexes $(\text{id}_{J \otimes A_0 C_0}, \mu, 0)$ and $(\text{id}_{J \otimes A_0 C_0}, m \otimes \text{id}_{C_0} \otimes 0, 0)$ define the same element of $\text{Hom}_{D(C_0)}(L^\bullet, (J \otimes A_0 C_0 \to JC))$. By Lemma 3.4 there exist maps $h : L^{-1} \to J \otimes A_0 C_0$ and $h' : L^0 \to JC$
which define a homotopy between \((\text{id}_{J \otimes A_0} C_0, \mu, 0)\) and \((\text{id}_{J \otimes A_0} C_0, m \otimes \text{id}_{C_0} \otimes 0, 0)\). Picture

\[
\begin{array}{c}
J \otimes A_0 C_0 & \xrightarrow{h} & K/K^2 \otimes B_0 C_0 \oplus C_0 d_y \\
\downarrow \text{id}_{J \otimes A_0} C_0 & & \downarrow \mu \\
J \otimes A_0 C_0 & \xrightarrow{\gamma} & JC
\end{array}
\]

Since \(h\) precomposed with \(d_L^2\) is zero it defines an element in \(\text{Hom}_{D(C_0)}(L^\bullet, J \otimes A_0 C_0[1])\) which comes from a unique element \(\chi\) of \(\text{Hom}_{D(B_0)}(K^\bullet, J \otimes A_0 B_0[1])\) by \((7.2.1)\). Applying Lemma \(3.4\) again we represent \(\chi\) by a map \(g : K/K^2 \to J \otimes A_0 B_0\). Then the base change \(g \otimes \text{id}_{C_0}\) and \(h\) differ by a homotopy \(h'' : L^0 \to J \otimes A_0 C_0\). Hence if we modify \(m\) into \(m + c \circ g\), then we find that \(m \otimes \text{id}_{C_0} \otimes 0\) and \(\mu\) just differ by a map \(h' : L^0 \to JC\).

Changing our choice of the map \(\psi : A[x_i, y_j] \to C\) by sending \(x_i\) to \(\psi(x_i) + h'(dx_i)\) and sending \(y_j\) to \(\psi(y_j) + h'(dy_j)\), we find a commutative diagram

\[
\begin{array}{c}
N & \xrightarrow{\gamma} & JC \\
\downarrow m & & \downarrow \mu \\
K/K^2 & \xrightarrow{\epsilon} & L/L^2 \\
\downarrow \gamma & & \downarrow \gamma \\
J \otimes A_0 B_0 & \xrightarrow{\gamma} & J \otimes A_0 C_0
\end{array}
\]

At this point we can define \(B\) as the pushout in the first commutative diagram of the proof. The commutativity of the diagram just displayed, shows that there is an \(A\)-algebra map \(B \to C\) compatible with the given map \(N = JB \to JC\). As \(N \otimes_{B_0} C_0 = JC\) it follows from More on Morphisms, Lemma \((10.1)\) that \(B \to C\) is flat. From this it easily follows that it is étale. We omit the proof of the other properties as they are mostly self evident at this point. □

Lemma 7.3. Let \(A\) be a Noetherian ring. Let \(I \subset A\) be an ideal. Let \(B\) be an object of \((2.0.3)\). Assume there is an integer \(c \geq 0\) such that for \(a \in I^c\) multiplication by \(a\) on \(NL^\wedge_{B/A}\) is zero in \(D(B)\). Then there exists a finite type \(A\)-algebra \(C\) and an isomorphism \(B \cong C^\wedge\).

In Section 8 we will give a simpler proof of this result in case \(A\) is a G-ring.

Proof. We prove this lemma by induction on the number of generators of \(I\). Say \(I = (a_1, \ldots, a_t)\). If \(t = 0\), then \(I = 0\) and there is nothing to prove. If \(t = 1\), then the lemma follows from Lemma \((6.4)\) Assume \(t > 1\).

For any \(m \geq 1\) set \(\tilde{A}_m = A/I(a_i^m)\). Consider the ideal \(\tilde{I}_m = (\tilde{a}_1, \ldots, \tilde{a}_{t-1})\) in \(\tilde{A}_m\). Let \(B_m = B/(a_t^m)\) be the base change of \(B\) for the map \((A, I) \to (\tilde{A}_m, \tilde{I}_m)\), see \((2.4.1)\). By Lemma \((4.3)\) the assumption of the lemma holds for \(\tilde{I}_m \subset \tilde{A}_m \to B_m\).

By induction hypothesis (on \(t\)) we can find a finite type \(\tilde{A}_m\)-algebra \(C_m\) and a map \(C_m \to B_m\) which induces an isomorphism \(C_m^\wedge \cong B_m\) where the completion is with respect to \(\tilde{I}_m\). By Lemma \((4.2)\) we may assume that \(\text{Spec}(C_m) \to \text{Spec}(\tilde{A}_m)\) is étale over \(\text{Spec}(\tilde{A}_m) \setminus V(\tilde{I}_m)\).
We claim that we may choose $A_m \to C_m \to B_m$ as in the previous paragraph such that moreover there are isomorphisms $C_m/(a_i^{m-1}) \to C_{m-1}$ compatible with the given $A$-algebra structure and the maps to $B_{m-1} = B_m/(a_i^{m-1})$. Namely, first fix a choice of $A_1 \to C_1 \to B_1$. Suppose we have found $C_{m-1} \to C_{m-2} \to \ldots \to C_1$ with the desired properties. Note that $C_m/(a_i^{m-1})$ is étale over $\text{Spec}(\hat{A}_{m-1}) \setminus V(I_{m-1})$. Hence by Lemma 4.3 there exists an étale extension $C_{m-1} \to C'_m$ which induces an isomorphism modulo $I_{m-1}$ and an $A_{m-1}$-algebra map $C_m/(a_i^{m-1}) \to C'_m$ inducing the isomorphism $B_m/(a_i^{m-1}) \to B_{m-1}$ on completions. Note that $C_m/(a_i^{m-1}) \to C'_{m-1}$ is étale over the complement of $V(I_{m-1})$ by Morphisms, Lemma 34.18 and over $V(I_{m-1})$ induces an isomorphism on completions hence is étale there too (for example by More on Morphisms, Lemma 12.3). Thus $C_m/(a_i^{m-1}) \to C'_m$ is étale. By the topological invariance of étale morphisms (Étale Morphisms, Theorem 15.2) there exists an étale ring map $C_m \to C'_m$ such that $C_m/(a_i^{m-1}) \to C'_m$ is isomorphic to $C_m/(a_i^{m-1}) \to C'_m/(a_i^{m-1})$. Observe that the $I_m$-adic completion of $C'_m$ is equal to the $I_{m-1}$-adic completion of $C_m$, i.e., to $B_m$ (details omitted). We apply Lemma 7.2 to the diagram

\[
\begin{array}{ccc}
C''_m & \to & C'_m/(a_i^{m-1}) \\
\downarrow & & \downarrow \\
C'_m & \to & C_{m-1} \\
\downarrow & & \downarrow \\
A_m & \to & \hat{A}_{m-1}
\end{array}
\]

to see that there exists a “lift” of $C''_m$ of $C_{m-1}$ to an algebra over $\hat{A}_m$ with all the desired properties.

By construction ($C_m$) is an object of the category $\mathcal{C}$ for the principal ideal $(a_i)$. Thus the inverse limit $B' = \varprojlim C_m$ is an $(a_i)$-adically complete $A$-algebra such that $B'/(a_i)$ is of finite type over $A/(a_i)$, see Lemma 2.1. By construction the $I$-adic completion of $B'$ is isomorphic to $B$ (details omitted). Consider the complex $\mathcal{N}_B'^A$ constructed using the $(a_i)$-adic topology. Choosing a presentation for $B'$ (which induces a similar presentation for $B$) the reader immediately sees that $\mathcal{N}_B'^A \otimes_B B = \mathcal{N}_B'^A A$. Since $a_i \in I$ and since the cohomology modules of $\mathcal{N}_B'^A$ are finite $B'$-modules (hence complete for the $a_i$-adic topology), we conclude that $a_i^\infty$ acts as zero on these cohomologies as the same thing is true by assumption for $\mathcal{N}_B'^A A$. Thus multiplication by $a_i^\infty$ is zero on $\mathcal{N}_B'^A A$ by Lemma 3.5. Hence finally, we may apply Lemma 6.4 to $(a_i) \subset A \to B'$ to finish the proof.

0AKG Lemma 7.4. Let $A$ be a Noetherian ring. Let $I \subset A$ be an ideal. Let $B$ be an $I$-adically complete $A$-algebra with $A/I \to B/I$ of finite type. The equivalent conditions of Lemma 7.4 are also equivalent to

0AKH (5) there exists a finite type $A$-algebra $C$ with $\text{Spec}(C) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$ such that $B \cong C^\wedge$.

Proof. First, assume conditions (1) – (4) hold. Then there exists a finite type $A$-algebra $C$ with such that $B \cong C^\wedge$ by Lemma 7.3. In other words, $B_n = C/I^n C$. The naive cotangent complex $\mathcal{N}_C^A$ is a complex of finite type $C$-modules and
If the base ring $A$ is a Noetherian G-ring, then some of the material above simplifies somewhat and we obtain some additional results.

**Proof of Lemma 7.3 in case $A$ is a G-ring.** This proof is easier in that it does not depend on the somewhat delicate deformation theory argument given in the proof of Lemma 7.2, but of course it requires a very strong assumption on the Noetherian ring $A$.

Choose a presentation $B = A[x_1, \ldots, x_r]/J$. Choose generators $g_1, \ldots, g_m \in J$. Choose generators $k_1, \ldots, k_l$ of the module of relations between $g_1, \ldots, g_m$, i.e., such that

$$(A[x_1, \ldots, x_r]^\wedge)^{\oplus l} \xrightarrow{k_1, \ldots, k_l} (A[x_1, \ldots, x_r]^\wedge)^{\oplus m} g_1, \ldots, g_m, A[x_1, \ldots, x_r]^\wedge$$

is exact in the middle. Write $k_i = (k_{i1}, \ldots, k_{il})$ so that we have

$$\sum k_{ij} g_j = 0 \quad (8.0.1)$$

for $i = 1, \ldots, t$. Let $I^c = (a_1, \ldots, a_s)$. For each $l \in \{1, \ldots, s\}$ we know that multiplication by $a_l$ on $\text{NL}^\wedge_{B/A}$ is zero in $\text{D}(B)$. By Lemma 7.4 we can find a map $\alpha_l : \bigoplus Bdz_i \to J/J^2$ such that $d \circ \alpha_l$ and $\alpha_l \circ d$ are both multiplication by $a_l$. Pick an element $f_{l,i} \in J$ whose class modulo $J^2$ is equal to $\alpha_l(dx_i)$. Then we have for all $l = 1, \ldots, s$ and $i = 1, \ldots, r$

$$\sum j' \partial f_{l,i}/\partial x_{j'} dx_{j'} = a_l dx_i + \sum h_{l,i}^{j',j''} g_j dx_{j'} \quad (8.0.2)$$

for some $h_{l,i}^{j',j''} \in A[x_1, \ldots, x_r]^\wedge$. We also have for $j = 1, \ldots, m$ and $l = 1, \ldots, s$ that

$$a_l g_j = \sum h_{l,j}^{j',j''} f_{l,i} + \sum h_{l,j}^{j',j''} g_j g_{j'} \quad (8.0.3)$$

for some $h_{l,j}^{j',j''} \in A[x_1, \ldots, x_r]^\wedge$. Of course, since $f_{l,i} \in J$ we can write for $l = 1, \ldots, s$ and $i = 1, \ldots, r$

$$f_{l,i} = \sum h_{l,i}^j g_j \quad (8.0.4)$$

for some $h_{l,i}^j$ in $A[x_1, \ldots, x_r]^\wedge$.

Let $A[x_1, \ldots, x_r]^h$ be the henselization of the pair $(A[x_1, \ldots, x_r], IA[x_1, \ldots, x_r])$, see More on Algebra, Lemma 12.1. Since $A$ is a Noetherian G-ring, so is $A[x_1, \ldots, x_r]$,

$$H^{-1} \text{ and } H^0 \text{ are finite } C\text{-modules. By assumption there exists a } c \geq 0 \text{ such that }$$

$$H^{-1}/H^0H^{-1} \text{ and } H^0/H^0H^0 \text{ are annihilated by } I^c \text{ for some } n. \text{ By Nakayama's lemma this means that }$$

$$H^{-1}/H^0 \text{ and } H^0 \text{ are annihilated by an element of the form } f = 1 + x \text{ with } x \in IC. \text{ After inverting } f \text{ (which does not change the quotients } B_n = C/I^nC) \text{ we see that }$$

$$\text{NL}_{C/A} \text{ has cohomology annihilated by } I^c. \text{ Thus } A \to C \text{ is étale at any prime of } C \text{ not lying over } V(I) \text{ by the definition of étale ring maps, see Algebra, Definition 141.1}.$$  

Conversely, assume that $A \to C$ of finite type is given such that $\text{Spec}(C) \to \text{Spec}(A)$ is étale over $\text{Spec}(A) \setminus V(I)$. Then for every $a \in I$ there exists an $c \geq 0$ such that multiplication by $a^c$ is zero $\text{NL}_{C/A}$. Since $\text{NL}^\wedge_{C/A}$ is the derived completion of $\text{NL}_{C/A}$ (see Lemma 3.1) it follows that $B = C^\wedge$ satisfies the equivalent conditions of Lemma 141.1. 

\[\square\]
see More on Algebra, Proposition \[19.10\] Hence we have approximation for the map \(A[x_1, \ldots, x_r]^h \to A[x_1, \ldots, x_r]^h\) with respect to the ideal generated by \(I\), see Smoothing Ring Maps, Lemma \[14.1\]. Choose a large integer \(M\). Choose
\[
G_j, K_{ij}, F_{l,i}, H_{l,j}^{i,j''}, H_{l,i}^j \in A[x_1, \ldots, x_r]^h
\]
such that analogues of equations \((8.0.1)\), \((8.0.3)\), and \((8.0.4)\) hold for these elements in \(A[x_1, \ldots, x_r]^h\), i.e.,
\[
\sum K_{ij}G_j = 0, \quad a_lG_j = \sum H_{l,j}^{i,j''}G_j + \sum H_{l,i}^jG_j, \quad F_{l,i} = \sum H_{l,i}^jG_j
\]
and such that we have
\[
G_j - g_j, K_{ij} - k_{ij}, F_{l,i} - f_{l,i}, H_{l,j}^{i,j''} - h_{l,j}^{i,j''}, H_{l,i}^j - h_{l,i}^j \in I^M A[x_1, \ldots, x_r]^h
\]
where we take liberty of thinking of \(A[x_1, \ldots, x_r]^h\) as a subring of \(A[x_1, \ldots, x_r]^\Lambda\). Note that we cannot guarantee that the analogue of \((8.0.2)\) holds in \(A[x_1, \ldots, x_r]^h\), because it is not a polynomial equation. But since taking partial derivatives is \(A\)-linear, we do get the analogue modulo \(I^M\). More precisely, we see that
\[
0_{\text{AKF}} \quad (8.0.5) \quad \sum \frac{\partial F_{l,i}}{\partial x_i} dx_i - a_l dx_i - \sum h_{l,j}^{i,j''}G_j dx_i \in I^M A[x_1, \ldots, x_r]^\Lambda
\]
for \(l = 1, \ldots, s\) and \(i = 1, \ldots, r\).

With these choices, consider the ring
\[
C^h = A[x_1, \ldots, x_r]^h/(G_1, \ldots, G_r)
\]
and denote \(C^\Lambda\) its \(I\)-adic completion, namely
\[
C^\Lambda = A[x_1, \ldots, x_r]^\Lambda/J', \quad J' = (G_1, \ldots, G_r)A[x_1, \ldots, x_r]^\Lambda
\]
In the following paragraphs we establish the fact that \(C^\Lambda\) is isomorphic to \(B\). Then in the final paragraph we deal with show that \(C^h\) comes from a finite type algebra over \(A\) as in the statement of the lemma.

First consider the cokernel
\[
\Omega = \text{Coker}(J'/J')^2 \longrightarrow \bigoplus C^\Lambda dx_i
\]
This \(C^\Lambda\) module is generated by the images of the elements \(dx_i\). Since \(F_{l,i} \in J'\) by the analogue of \((8.0.4)\) we see from \((8.0.5)\) we see that \(a_l dx_i \in I^M \Omega\). As \(I^c = (a_l)\) we see that \(I^c \Omega \subset I^M \Omega\). Since \(M > c\) we conclude that \(I^c \Omega = 0\) by Algebra, Lemma \[19.3\].

Next, consider the kernel
\[
H_1 = \text{Ker}(J'/J')^2 \longrightarrow \bigoplus C^\Lambda dx_i
\]
By the analogue of \((8.0.3)\) we see that \(a_l J' \subset (F_{l,i}) + (J')^2\). On the other hand, the determinant \(\Delta_l\) of the matrix \((\partial F_{l,i}/\partial x_i)\) satisfies \(\Delta_l = a_l^2 \mod I^MC^\Lambda\) by \((8.0.5)\). It follows that \(a_l^{-1}H_1 \subset I^M H_1\) (some details omitted; use Algebra, Lemma \[14.5\]). Now \(a_1^{-1}, \ldots, a_s^{-1}\) \(\supset \) \(I^{(sr+1)c}\). Hence \(I^{(sr+1)c}H_1 \subset I^M H_1\) and since \(M > (sr+1)c\) we conclude that \(I^{(sr+1)c}H_1 = 0\).

By Lemma \[3.5\] we conclude that multiplication by an element of \(I^{2(s+1)c}\) on \(NL_{C^\Lambda/A}^h\) is zero (note that the bound does not depend on \(M\) or the choice of
the approximation, as long as $M$ is large enough). Since $G_j - g_j$ is in the ideal generated by $I^M$ we see that there is an isomorphism
\[
\psi_M : C^\wedge / I^M C^\wedge \to B/I^M B
\]
As $M$ is large enough we can use Lemma 1 with $d = d(I \subset A \to B)$, with $C^\wedge$ playing the role of $B$, with $2(rs + 1)c$ instead of $c$, to find a morphism
\[
\psi : C^\wedge \longrightarrow B
\]
which agrees with $\psi_M$ modulo $I^{q-2(rs+1)c}$ where $q$ is the quotient of $M$ by the number of generators of $I$. We claim $\psi$ is an isomorphism. Since $C^\wedge$ and $B$ are $I$-adically complete the map $\psi$ is surjective because it is surjective modulo $I$ (see Algebra, Lemma 95.1). On the other hand, as $M$ is large enough we see that
\[
\text{Gr}_I(C^\wedge) \cong \text{Gr}_I(B)
\]
as graded $\text{Gr}_I(A[x_1, \ldots, x_r]^\wedge)$-modules by More on Algebra, Lemma 12.2. Since $\psi$ is compatible with this isomorphism as it agrees with $\psi_M$ modulo $I$, this means that $\text{Gr}_I(\psi)$ is an isomorphism. As $C^\wedge$ and $B$ are $I$-adically complete, it follows that $\psi$ is an isomorphism.

This paragraph serves to deal with the issue that $C^h$ is not of finite type over $A$. Namely, the ring $A[x_1, \ldots, x_r]^h$ is a filtered colimit of étale $A[x_1, \ldots, x_r]$ algebras $A'$ such that $A/I[x_1, \ldots, x_r] \to A'/IA'$ is an isomorphism (see proof of More on Algebra, Lemma 12.2). Pick an $A'$ such that $G_1, \ldots, G_m$ are the images of $G_1', \ldots, G_m' \in A'$. Setting $C = A'/(G_1', \ldots, G_m')$ we get the finite type algebra we were looking for.

The following lemma isn’t true in general if $A$ is not a G-ring but just Noetherian. Namely, if $(A, \mathfrak{m})$ is local and $I = \mathfrak{m}$, then the lemma is equivalent to Artin approximation for $A^h$ (as in Smoothing Ring Maps, Theorem 13.1) which does not hold for every Noetherian local ring.

**Lemma 8.1.** Let $A$ be a Noetherian G-ring. Let $I \subset A$ be an ideal. Let $B, C$ be finite type $A$-algebras. For any $A$-algebra map $\varphi : B^\wedge \to C^\wedge$ of $I$-adic completions and any $N \geq 1$ there exist

1. an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$,
2. an $A$-algebra map $\varphi : B \to C'$

such that $\varphi$ and $\psi$ agree modulo $I^N$ into $C^\wedge = (C')^\wedge$.

**Proof.** The statement of the lemma makes sense as $C \to C'$ is flat (Algebra, Lemma 141.3) hence induces an isomorphism $C/I^nC \to C'/I^nC'$ for all $n$ (More on Algebra, Lemma 80.2) and hence an isomorphism on completions. Let $C^h$ be the henselization of the pair $(C, IC)$, see More on Algebra, Lemma 12.1. Then $C^h$ is the filtered colimit of the algebras $C'$ and the maps $C \to C' \to C^h$ induce isomorphism on completions (More on Algebra, Lemma 12.4). Thus it suffices to prove there exists an $A$-algebra map $B \to C^h$ which is congruent to $\psi$ modulo $I^N$. Write $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. The ring map $\psi$ corresponds to elements $\hat{c}_1, \ldots, \hat{c}_n \in C^\wedge$ with $f_j(\hat{c}_1, \ldots, \hat{c}_n) = 0$ for $j = 1, \ldots, m$. Namely, as $A$ is a Noetherian G-ring, so is $C$, see More on Algebra, Proposition 49.10. Thus Smoothing Ring Maps, Lemma 14.1 applies to give elements $c_1, \ldots, c_n \in C^h$ such that $f_j(c_1, \ldots, c_n) = 0$ for $j = 1, \ldots, m$ and such that $\hat{c}_i - c_i \in IC^h$. This determines the map $B \to C^h$ as desired. \[\square\]
9. Rig-surjective morphisms

For morphisms locally of finite type between locally Noetherian formal algebraic spaces a definition borrowed from [Art70] can be used. See Remark 9.10 for a discussion of what to do in more general cases.

Definition 9.1. Let $S$ be a scheme. Let $f : X \rightarrow Y$ be a morphism of formal algebraic spaces over $S$. Assume that $X$ and $Y$ are locally Noetherian and that $f$ is locally of finite type. We say $f$ is rig-surjective if for every solid diagram

$$
\begin{array}{ccc}
\text{Spf}(R) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spf}(R) & \longrightarrow & Y
\end{array}
$$

where $R$ is a complete discrete valuation ring and where $p$ is an adic morphism there exists an extension of complete discrete valuation rings $R \subset R'$ and a morphism $\text{Spf}(R') \rightarrow X$ making the displayed diagram commute.

We prove a few lemmas to explain what this means.

Lemma 9.2. Let $S$ be a scheme. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of formal algebraic spaces over $S$. Assume $X$, $Y$, $Z$ are locally Noetherian and $f$ and $g$ locally of finite type. Then if $f$ and $g$ are rig-surjective, so is $g \circ f$.

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemma [18.3]).

Lemma 9.3. Let $S$ be a scheme. Let $f : X \rightarrow Y$ and $Z \rightarrow Y$ be morphisms of formal algebraic spaces over $S$. Assume $X$, $Y$, $Z$ are locally Noetherian and $f$ and $g$ locally of finite type. If $f$ is rig-surjective, then the base change $Z \times_Y X \rightarrow Z$ is too.

Proof. Follows in a straightforward manner from the definitions (and Formal Spaces, Lemmas [18.9] and [18.4]).

Lemma 9.4. Let $S$ be a scheme. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of formal algebraic spaces over $S$. Assume $X$, $Y$, $Z$ locally Noetherian and $f$ and $g$ locally of finite type. If $g \circ f : X \rightarrow Z$ is rig-surjective, so is $g : Y \rightarrow Z$.

Proof. Immediate from the definition.

Lemma 9.5. Let $S$ be a scheme. Let $f : X \rightarrow Y$ be a morphism of formal algebraic spaces which is representable by algebraic spaces, étale, and surjective. Assume $X$ and $Y$ locally Noetherian. Then $f$ is rig-surjective.

Proof. Let $p : \text{Spf}(R) \rightarrow Y$ be an adic morphism where $R$ is a complete discrete valuation ring. Let $Z = \text{Spf}(R) \times_Y X$. Then $Z \rightarrow \text{Spf}(R)$ is representable by algebraic spaces, étale, and surjective. Hence $Z$ is nonempty. Pick a nonempty affine formal algebraic space $V$ and an étale morphism $V \rightarrow Z$ (possible by our definitions). Then $V \rightarrow \text{Spf}(R)$ corresponds to $R \rightarrow A^\wedge$ where $R \rightarrow A$ is an étale ring map, see Formal Spaces, Lemma [14.13] Since $A^\wedge \neq 0$ (as $V \neq \emptyset$) we can find a maximal ideal $m$ of $A$ lying over $m_R$. Then $A_m$ is a discrete valuation ring (More on Algebra, Lemma [43.4]). Then $R' = A^\wedge_m$ is a complete discrete valuation ring (More on Algebra, Lemma [12.5]). Applying Formal Spaces, Lemma [5.10] we find the desired morphism $\text{Spf}(R') \rightarrow V \rightarrow Z \rightarrow X$. 

□
Remark 9.6. Let $S$ be a scheme. Let $f : X \to Y$ be a morphism of locally Noetherian formal algebraic spaces which is locally of finite type. The upshot of the lemmas above is that we may check whether $f : X \to Y$ is rig-surjective, étale locally on $Y$. For example, suppose that $\{ Y_i \to Y \}$ is a covering as in Formal Spaces, Definition 7.1. Then $f$ is rig-surjective if and only if $f_i : X \times_Y Y_i \to Y_i$ is rig-surjective. Namely, if $f$ is rig-surjective, so is any base change (Lemma 9.3). Conversely, if all $f_i$ are rig-surjective, so is $\coprod f_i : \coprod X \times_Y Y_i \to \coprod Y_i$. By Lemma 9.5 the morphism $\coprod Y_i \to Y$ is rig-surjective. Hence $\coprod X \times_Y Y_i \to Y$ is rig-surjective (Lemma 9.2). Since this morphism factors through $X \to Y$ we see that $X \to Y$ is rig-surjective by Lemma 9.4.

Lemma 9.7. Let $S$ be a scheme. Let $f : X \to Y$ be a proper surjective morphism of locally Noetherian algebraic spaces over $S$. Let $T \subset |Y|$ be a closed subset and let $T' = |f|^{-1}(T) \subset |X|$. Then $X_{/T'} \to Y_{/T}$ is rig-surjective.

Proof. The statement makes sense by Formal Spaces, Lemmas 15.6 and 18.10. Let $Y_j \to Y$ be a jointly surjective family of étale morphisms from schemes $Y_j$ is an affine scheme for each $j$. Denote $T_j \subset Y_j$ the inverse image of $T$. Then $\{ (Y_j)_{/T_j} \to Y_{/T} \}$ is a covering as in Formal Spaces, Definition 7.1. Moreover, setting $X_j = Y_j \times_Y X$ and $T'_j \subset |X_j|$ the inverse image of $T$, we have

$$(X_j)_{/T'_j} = (Y_j)_{/T_j} \times_{(Y_{/T})} X_{/T'}.$$ 

By the discussion in Remark 9.6 we reduce to the case where $Y$ is an affine Noetherian scheme treated in the next paragraph.

Assume $Y = \text{Spec}(A)$ where $A$ is a Noetherian ring. This implies that $Y_{/T} = \text{Spf}(A^\wedge)$ where $A^\wedge$ is the $I$-adic completion of $A$ for some ideal $I \subset A$. Let $p : \text{Spf}(R) \to \text{Spf}(A^\wedge)$ be an adic morphism where $R$ is a complete discrete valuation ring. Let $K$ be the field of fractions of $R$. Consider the composition $A \to A^\wedge R$. Since $X \to Y$ is surjective, the fibre $X_K = \text{Spec}(K) \times_Y X$ is nonempty. Thus we may choose an affine scheme $U$ and an étale morphism $U \to X$ such that $U_K$ is nonempty. Let $u \in U_K$ be a closed point (possible as $U_K$ is affine). By Morphisms, Lemma 19.3 the residue field $L = \kappa(u)$ is a finite extension of $K$. Let $R' \subset L$ be the integral closure of $R$ in $L$. By More on Algebra, Remark 97.6 we see that $R'$ is a discrete valuation ring. Because $X \to Y$ is proper we see that the given morphism $\text{Spec}(L) = u \to U_K \to X_K \to X$ extends to a morphism $\text{Spec}(R') \to X$ over the given morphism $\text{Spec}(R) \to Y$ (Morphisms of Spaces, Lemma 44.1). By commutativity of the diagram the induced morphisms $\text{Spec}(R'/m_{R'}^n) \to X$ are points of $X_{/T'}$ and we find

$$\text{Spf}(R'^\wedge) = \text{colim} \text{Spec}(R'/m_{R'}^n) \to X_{/T'},$$

as desired (note that $(R')^\wedge$ is a complete discrete valuation ring by More on Algebra, Lemma 42.5 in fact in the current situation $R' = (R')^\wedge$ but we do not need this).

Lemma 9.8. Let $A$ be a Noetherian ring complete with respect to an ideal $I$. Let $B$ be an $I$-adically complete $A$-algebra. If $A/I^n \to B/I^n B$ is of finite type and flat for all $n$ and faithfully flat for $n = 1$, then $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective.

Proof. We will use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10. Let $\varphi : A \to R$ be a continuous map into a complete
discrete valuation ring $A$. This implies that $\varphi(I) \subset \mathfrak{m}_R$. On the other hand, since we only need to produce the lift $\varphi' : B' \to R'$ in the case that $\varphi$ corresponds to an adic morphism, we may assume that $\varphi(I) \neq 0$. Thus we may consider the base change $C = B \widehat{\otimes}_A R$, see Remark 2.3 for example. Then $C$ is an $\mathfrak{m}_R$-adically complete $R$-algebra such that $C/\mathfrak{m}_R^n C$ is of finite type and flat over $R/\mathfrak{m}_R^n$ and such that $C/\mathfrak{m}_R C$ is nonzero. Pick any maximal ideal $\mathfrak{m} \subset C$ lying over $\mathfrak{m}_R$. By flatness (which implies going down) we see that $\text{Spec}(C) \setminus V(\mathfrak{m}_R C)$ is a nonempty open. Hence, we can pick a prime $\mathfrak{q} \subset \mathfrak{m}$ such that $\mathfrak{q}$ defines a closed point of $\text{Spec}(C) \setminus \{\mathfrak{m}\}$ and such that $\mathfrak{q} \not\subset V(I C)$, see Properties, Lemma 6.4. Then $C/\mathfrak{q}$ is a dimension 1-local domain and we can find $C/\mathfrak{q} \subset R'$ with $R'$ a discrete valuation ring (Algebra, Lemma 118.13). By construction $\mathfrak{m}_R R' \subset \mathfrak{m}_R'$ and we see that $C \to R'$ extends to a continuous map $C \to (R')^\wedge$ (in fact we can pick $R'$ such that $R' = (R')^\wedge$ in our current situation but we do not need this). Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption gives a commutative diagram of rings

\[
\begin{array}{ccc}
(R')^\wedge & \leftarrow & C \\
\uparrow & & \downarrow \\
R & \leftarrow & B \\
\uparrow & & \uparrow \\
A & \leftarrow & \end{array}
\]

which gives the desired lift. □

0AQY Lemma 9.9. Let $A$ be a Noetherian ring complete with respect to an ideal $I$. Let $B$ be an $I$-adically complete $A$-algebra. Assume that

1. the $I$-torsion in $A$ is 0,
2. $A/I^n \to B/I^n B$ is flat and of finite type for all $n$.

Then $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective if and only if $A/I \to B/I B$ is faithfully flat.

Proof. Faithful flatness implies rig-surjectivity by Lemma 9.8. To prove the converse we will use without further mention that the vanishing of $I$-torsion is equivalent to the vanishing of $I$-power torsion (More on Algebra, Lemma 79.3). We will also use without further mention that morphisms between formal spectra are given by continuous maps between the corresponding topological rings, see Formal Spaces, Lemma 5.10.

Assume $\text{Spf}(B) \to \text{Spf}(A)$ is rig-surjective. Choose a maximal ideal $I \subset \mathfrak{m} \subset A$. The open $U = \text{Spec}(A_\mathfrak{m}) \setminus V(I_\mathfrak{m})$ of $\text{Spec}(A_\mathfrak{m})$ is nonempty as the $I_\mathfrak{m}$-torsion of $A_\mathfrak{m}$ is zero (use Algebra, Lemma 118.14). Thus we can find a prime $\mathfrak{q} \subset A_\mathfrak{m}$ which defines a point of $U$ (i.e., $I_\mathfrak{m} \not\subset \mathfrak{q}$) and which corresponds to a closed point of $\text{Spec}(A_\mathfrak{m}) \setminus \{\mathfrak{m}\}$, see Properties, Lemma 6.4. Then $A_\mathfrak{m}/\mathfrak{q}$ is a dimension 1 local domain. Thus we can find an injective local homomorphism of local rings $A_\mathfrak{m}/\mathfrak{q} \subset R$ where $R$ is a discrete valuation ring (Algebra, Lemma 118.13). By construction $IR \subset \mathfrak{m}_R$ and we see that $A \to R$ extends to a continuous map $A \to R^\wedge$. Since the completion of a discrete valuation ring is a discrete valuation ring, we see that the assumption
gives a commutative diagram of rings

\[
\begin{array}{ccc}
R' & \leftarrow & B \\
\uparrow & & \uparrow \\
R \land & \leftarrow & A
\end{array}
\]

Thus we find a prime ideal of \(B\) lying over \(m\). It follows that \(\text{Spec}(B/IB) \to \text{Spec}(A/I)\) is surjective, whence \(A/I \to B/IB\) is faithfully flat (Algebra, Lemma 38.16).

\[\square\]

\textbf{Remark 9.10.} The condition as formulated in Definition 9.1 is not right for morphisms of locally adic* formal algebraic spaces. For example, if \(A = \left( \bigcup_{n \geq 1} k[t^{1/n}] \right)^{\land}\) where the completion is the \(t\)-adic completion, then there are no adic morphisms \(\text{Spf}(R) \to \text{Spf}(A)\) where \(R\) is a complete discrete valuation ring. Thus any morphism \(X \to \text{Spf}(A)\) would be rig-surjective, but since \(A\) is a domain and \(t \in A\) is not zero, we want to think of \(A\) as having at least one “rig-point”, and we do not want to allow \(X = \emptyset\). To cover this particular case, one can consider adic morphisms \(\text{Spf}(R) \to Y\) where \(R\) is a valuation ring complete with respect to a principal ideal \(J\) whose radical is \(m_R = \sqrt{J}\). In this case the value group of \(R\) can be embedded into \((R, +)\) and one obtains the point of view used by Berkovich in defining an analytic space associated to \(Y\), see [Ber90]. Another approach is championed by Huber. In his theory, one drops the hypothesis that \(\text{Spec}(R/J)\) is a singleton, see [Hub93].

\textbf{Lemma 9.11.} Let \(S\) be a scheme. Let \(f : X \to Y\) be a morphism of formal algebraic spaces. Assume \(X\) and \(Y\) are locally Noetherian, \(f\) locally of finite type, and \(f\) a monomorphism. Then \(f\) is rig surjective if and only if every adic morphism \(\text{Spf}(R) \to Y\) where \(R\) is a complete discrete valuation ring factors through \(X\).

\textbf{Proof.} One direction is trivial. For the other, suppose that \(\text{Spf}(R) \to Y\) is an adic morphism such that there exists an extension of complete discrete valuation rings \(R \subset R'\) with \(\text{Spf}(R') \to \text{Spf}(R) \to X\) factoring through \(Y\). Then \(\text{Spec}(R'/m_R^{n}R') \to \text{Spec}(R/m_R^{n})\) is surjective and flat, hence the morphisms \(\text{Spec}(R/m_R^{n}) \to X\) factor through \(X\) as \(X\) satisfies the sheaf condition for fpqc coverings, see Formal Spaces, Lemma 25.1. In other words, \(\text{Spf}(R) \to Y\) factors through \(X\). \(\square\)

\textbf{10. Algebraization}

In this section we prove a generalization of the result on dilatations from the paper of Artin [Art70]. We first reformulate the algebra results proved above into the language of formal algebraic spaces.

Let \(S\) be a scheme. Let \(V\) be a locally Noetherian formal algebraic space over \(S\). We denote \(C_V\) the category of formal algebraic spaces \(W\) over \(V\) such that the structure morphism \(W \to V\) is rig-étale.

Let \(S\) be a scheme. Let \(X\) be an algebraic space over \(S\). Let \(T \subset |X|\) be a closed subset. Recall that \(X_T\) denotes the formal completion of \(X\) along \(T\), see Formal Spaces, Section 9. More generally, for any algebraic space \(Y\) over \(X\) we denote \(Y_T\) the completion of \(Y\) along the inverse image of \(T\) in \(|Y|\), so that \(Y_T\) is a formal algebraic space over \(X_T\).
Lemma 10.1. Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. If $Y \to X$ is morphism of algebraic spaces which is locally of finite type and étale over $X \setminus T$, then $Y_{/T} \to X_{/T}$ is rig-étale, i.e., $Y_{/T}$ is an object of $\mathcal{C}_{X_{/T}}$ defined above.

Proof. Choose a surjective étale morphism $U \to X$ with $U = \bigsqcup U_i$ a disjoint union of affine schemes, see Properties of Spaces, Lemma 6.1. For each $i$ choose a surjective étale morphism $V_i \to Y \times_X U_i$ where $V_i = \bigsqcup V_{ij}$ is a disjoint union of affines. Write $U_i = \text{Spec}(A_i)$ and $V_{ij} = \text{Spec}(B_{ij})$. Let $I_i \subset A_i$ be an ideal cutting out the inverse image of $U_i$. Then we may apply Lemma 4.2 to see that the map of $I_i$-adic completions $\hat{A}^i \to B^{i}_{ij}$ has the property $P$ of Lemma 5.1. Since $\{\text{Spf}(A^i) \to X_{/T}\}$ and $\{\text{Spf}(B_{ij}) \to Y_{/T}\}$ are coverings as in Formal Spaces, Definition 7.1 we see that $Y_{/T} \to X_{/T}$ is rig-étale by definition. □

Lemma 10.2. Let $X$ be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let $U$ be an affine scheme and let $U \to X$ a finite type morphism étale over $X \setminus T$. Let $V$ be a Noetherian affine scheme over $X$. For any morphism $c : V_{/T} \to U_{/T}$ over $X_{/T}$ there exists an étale morphism $b : V' \to V$ of affine schemes which induces an isomorphism $b_{/T} : V'_{/T} \to V_{/T}$ and a morphism $a : V' \to U$ such that $c = a_{/T} \circ b^{-1}_{/T}$.

Proof. This is a reformulation of Lemma 4.4. □

Lemma 10.3. Let $X$ be a Noetherian affine scheme. Let $T \subset X$ be a closed subset. Let $W$ be a rig-étale morphism of formal algebraic spaces with $W$ an affine formal algebraic space. Then there exists an affine scheme $U$, a finite type morphism $U \to X$ étale over $X \setminus T$ such that $W \cong U_{/T}$. Moreover, if $W \to X_{/T}$ is étale, then $U \to X$ is étale.

Proof. The existence of $U$ is a restatement of Lemma 7.4. The final statement follows from More on Morphisms, Lemma 12.3. □

Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$ and let $T \subset |X|$ be a closed subset. Let us denote $\mathcal{C}_{X,T}$ the category of algebraic spaces $Y$ over $X$ such that the structure morphism $f : Y \to X$ is locally of finite type and an isomorphism over the complement of $T$. Formal completion defines a functor

\[
F_{X,T} : \mathcal{C}_{X,T} \longrightarrow \mathcal{C}_{X_{/T}}, \quad (f : Y \to X) \longmapsto (f_{/T} : Y_{/T} \to X_{/T})
\]

see Lemma 10.1.

Lemma 10.4. Let $S$ be a scheme. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic spaces. Let $T \subset |X|$ be closed. Assume that

1. $X$ is locally Noetherian,
2. $g$ is a monomorphism and locally of finite type,
3. $f|_{X \setminus T} : X \setminus T \to Y$ factors through $g$, and
4. $f_{/T} : X_{/T} \to Y$ factors through $g$,

then $f$ factors through $g$.

Proof. Consider the fibre product $E = X \times_Y Z \to X$. By assumption the open immersion $X \setminus T \to X$ factors through $E$ and any morphism $\varphi : X' \to X$ with $|\varphi|(|X'|) \subset T$ factors through $E$ as well, see Formal Spaces, Section 9. By More on Morphisms of Spaces, Lemma 20.3 this implies that $E \to X$ is étale at every point.
of $E$ mapping to a point of $T$. Hence $E \to X$ is an étale monomorphism, hence an open immersion (Morphisms of Spaces, Lemma \[51.2\]). Then it follows that $E = X$ since our assumptions imply that $|X| = |E|$.

\[\square\]

**Lemma 10.5.** Let $S$ be a scheme. Let $X, Y$ be locally Noetherian algebraic spaces over $S$. Let $T \subset |X|$ and $T' \subset |Y|$ be closed subsets. Let $a, b : X \to Y$ be morphisms of algebraic spaces over $S$ such that $a|_{X\setminus T} = b|_{X\setminus T}$, such that $|a|(T) \subset T'$ and $|b|(T) \subset T'$, and such that $a_{/T} = b_{/T}$ as morphisms $X_{/T} \to Y_{/T'}$. Then $a = b$.

**Proof.** Let $E$ be the equalizer of $a$ and $b$. Then $E$ is an algebraic space and $E \to X$ is locally of finite type and a monomorphism, see Morphisms of Spaces, Lemma \[4.1\]. Our assumptions imply we can apply Lemma \[10.4\] to the two morphisms $f = \text{id} : X \to X$ and $g : E \to X$ and the closed subset $T$ of $|X|$.

\[\square\]

**Lemma 10.6.** Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. Let $s, t : R \to U$ be two morphisms of algebraic spaces over $X$. Assume

1. $R, U$ are locally of finite type over $X$,
2. the base change of $s$ and $t$ to $X \setminus T$ is an étale equivalence relation, and
3. the formal completion $(t_{/T}, s_{/T}) : R_{/T} \to U_{/T} \times_{X_{/T}} U_{/T}$ is an equivalence relation too.

Then $(t, s) : R \to U \times_X U$ is an étale equivalence relation.

**Proof.** The morphisms $s, t : R \to U$ are étale over $X \setminus T$ by assumption. Since the formal completions of the maps $s, t : R \to U$ are étale, we see that $s$ and $t$ are étale for example by More on Morphisms, Lemma \[12.3\]. Applying Lemma \[10.4\] to the morphisms $\text{id} : R \times_{U \times X} U \to R \times_{U \times X} U$ and $\Delta : R \to R \times_{U \times X} U$ we conclude that $(t, s)$ is a monomorphism. Applying it again to $(t \circ \text{pr}_0, s \circ \text{pr}_1) : R \times_{s, U, t} R \to U \times_X U$ and $(t, s) : R \to U \times_X U$ we find that “transitivity” holds. We omit the proof of the other two axioms of an equivalence relation.

\[\square\]

**Remark 10.7.** Let $S$, $X$, and $T \subset |X|$ be as in (10.3.1). Let $U \to X$ be an algebraic space over $X$ such that $U \to X$ is locally of finite type and étale outside of $T$. We will construct a factorization

$$U \longrightarrow Y \longrightarrow X$$

with $Y$ in $\mathcal{C}_{X, T}$ such that $Y_{/T} \to Y_{/T}$ is an isomorphism. We may assume the image of $U \to X$ contains $X \setminus T$, otherwise we replace $U$ by $U \amalg (X \setminus T)$. For an algebraic space $Z$ over $X$, let us denote $Z^\circ$ the open subspace which is the inverse image of $X \setminus T$. Let

$$R = U \amalg U^\circ (U \times_X U)^\circ$$

be the pushout of $U^\circ \to U$ and the diagonal morphism $U^\circ \to U^\circ \times_X U^\circ = (U \times_X U)^\circ$. Since $U^\circ \to X$ is étale, the diagonal is an open immersion and we see that $R$ is an algebraic space (this follows for example from Spaces, Lemma \[8.5\]). The two projections $(U \times_X U)^\circ \to U$ extend to $R$ and we obtain two étale morphisms $s, t : R \to U$. Checking on each piece separately we find that $R$ is an étale equivalence relation on $U$. Set $Y = U/R$ which is an algebraic space by Bootstrap, Theorem \[10.1\]. Since $U^\circ \to X \setminus T$ is a surjective étale morphism and since $R^\circ = U^\circ \times_X U^\circ$ we see that $Y^\circ \to X \setminus T$ is an isomorphism. In other words, $Y \to X$ is an object of $\mathcal{C}_{X, T}$. On the other hand, the morphism $U \to Y$ induces an isomorphism $U_{/T} \to Y_{/T}$. Namely, the formal completion of $R$ along the inverse image of $T$ is
equal to the formal completion of $U$ along the inverse image of $T$ by our choice of $R$. By our construction of the formal completion in Formal Spaces, Section 9 we conclude that $U/T = Y/T$.

**Lemma 10.8.** Let $S$ be a scheme. Let $X$ be a Noetherian affine algebraic space over $S$. Let $T \subset |X|$ be a closed subset. Then the functor $F_{X,T}$ is an equivalence.

Before we prove this lemma let us discuss an example. Suppose that $S = \text{Spec}(k)$, $X = \mathbb{A}^1_k$, and $T = \{0\}$. Then $X/T = \text{Spf}(k[[x]])$. Let $W = \text{Spf}(k[[x]] \times k[[x]])$. Then the corresponding $Y$ is the affine line with zero doubled (Schemes, Example 14.3). Moreover, this is the output of the construction in Remark 10.7 starting with $U = X \amalg X$.

**Proof.** For any scheme or algebraic space $Z$ over $X$, let us denote $Z_0 \subset Z$ the inverse image of $T$ with the induced reduced closed subscheme or subspace structure. Note that $Z_0 = (Z/T)_{red}$ is the reduction of the formal completion.

The functor $F_{X,T}$ is faithful by Lemma 10.3.

Let $Y, Y'$ be objects of $\mathcal{C}_{X,T}$ and let $a' : Y/T \to Y'/T$ be a morphism in $\mathcal{C}_{X,T}$. To prove $F_{X,T}$ is fully faithful, we will construct a morphism $a : Y \to Y'$ in $\mathcal{C}_{X,T}$ such that $a' = a/T$.

Let $U$ be an affine scheme and let $U \to Y$ be an étale morphism. Because $U$ is affine, $U_0$ is affine and the image of $U_0 \to Y_0 \to Y'_0$ is a quasi-compact subspace of $|Y'_0|$. Thus we can choose an affine scheme $V$ and an étale morphism $V \to Y'$ such that the image of $|V_0| \to |Y'_0|$ contains this quasi-compact subset. Consider the formal algebraic space

$$W = U/T \times_{Y'/T} V/T$$

By our choice of $V$ the above, the map $W \to U/T$ is surjective. Thus there exists an affine formal algebraic space $W'$ and an étale morphism $W' \to W$ such that $W' \to W \to U/T$ is surjective. Then $W' \to U/T$ is étale. By Lemma 10.3 $W' = U'/T$ for $U' \to U$ étale and $U'$ affine. Write $V = \text{Spec}(C)$. By Lemma 10.2 there exists an étale morphism $U'' \to U'$ of affines which is an isomorphism on completions and a morphism $U'' \to V$ whose completion is the composition $U''/T \to U'/T \to W \to V/T$. Thus we get

$$Y \leftarrow U'' \rightarrow Y'$$

over $X$ agreeing with the given map on formal completions such that the image of $U''_0 \to Y_0$ is the same as the image of $U_0 \to Y_0$.

Taking a disjoint union of $U''$ as constructed in the previous paragraph, we find a scheme $U$, an étale morphism $U \to Y$, and a morphism $b : U \to Y'$ over $X$, such that the diagram

$$\begin{array}{ccc}
U/T & \xrightarrow{b/T} & Y'/T \\
\downarrow & & \downarrow a'
\end{array}$$

is commutative and such that $U_0 \to Y_0$ is surjective. Taking a disjoint union with the open $X \setminus T$ (which is also open in $Y$ and $Y'$), we find that we may even assume that $U \to Y$ is a surjective étale morphism. Let $R = U \times_Y U$. Then the two compositions $R \to U \to Y'$ agree both over $X \setminus T$ and after formal completion
along $T$, whence are equal by Lemma \[10.5\] This means exactly that $b$ factors as $U \to Y \to Y'$ to give us our desired morphism $a : Y \to Y'$.

Essential surjectivity. Let $W$ be an object of $\mathcal{C}_{X/T}$. We prove $W$ is in the essential image in a number of steps.

Step 1: $W$ is an affine formal algebraic space. Then we can find $U \to X$ of finite type and étale over $X \setminus T$ such that $U/T$ is isomorphic to $W$, see Lemma \[10.3\]. Thus we see that $W$ is in the essential image by the construction in Remark \[10.7\].

Step 2: $W$ is separated. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition \[7.1\]. By Step 1 the formal algebraic spaces $W_i$ and $W_i \times_W W_j$ are in the essential image. Say $W_i = (Y_i)/T$ and $W_i \times_W W_j = (Y_{ij})/T$. By fully faithfulness we obtain morphisms $t_{ij} : Y_{ij} \to Y_i$ and $s_{ij} : Y_{ij} \to Y_j$ matching the projections $W_i \times_W W_j \to W_i$ and $W_i \times_W W_j \to W_j$. Set $R = \coprod Y_{ij}$ and $U = \coprod Y_i$ and denote $s = \coprod s_{ij} : R \to U$ and $t = \coprod t_{ij} : R \to U$. Applying Lemma \[10.6\] we find that $(t,s) : R \to U \times_X U$ is an étale equivalence relation. Thus we can take the quotient $Y = U/R$ and it is an algebraic space, see Bootstrap, Theorem \[10.1\]. Since completion commutes with fibre products and taking quotient sheaves, we find that $Y/T \cong W$ in $\mathcal{C}_{X/T}$.

Step 3: $W$ is general. Choose $\{W_i \to W\}$ as in Formal Spaces, Definition \[7.1\]. The formal algebraic spaces $W_i$ and $W_i \times_W W_j$ are separated. Hence by Step 2 the formal algebraic spaces $W_i$ and $W_i \times_W W_j$ are in the essential image. Then we argue exactly as in the previous paragraph to see that $W$ is in the essential image as well. This concludes the proof.

**Theorem 10.9.** Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. The functor $F_{X,T}$ \[10.3.1\]

\[
\begin{cases}
\text{algebraic spaces } Y \text{ locally of finite type over } X \text{ such that } Y \to X \\
\text{is an isomorphism over } X \setminus T
\end{cases}
\to
\begin{cases}
\text{formal algebraic spaces } W \text{ endowed with a rig-étale morphism } W \to X/T
\end{cases}
\]

given by formal completion is an equivalence.

**Proof.** The theorem is essentially a formal consequence of Lemma \[10.8\]. We give the details but we encourage the reader to think it through for themselves. Let $g : U \to X$ be a surjective étale morphism with $U = \coprod U_i$ and each $U_i$ affine. Denote $F_{U,T}$ the functor for $U$ and the inverse image of $T$ in $|U|$.

Since $U = \coprod U_i$ both the category $\mathcal{C}_{U,T}$ and the category $\mathcal{C}_{U/T}$ decompose as a product of categories, one for each $i$. Since the functors $F_{U_i,T}$ are equivalences for all $i$ by the lemma we find that the same is true for $F_{U,T}$.

Since $F_{U,T}$ is faithful, it follows that $F_{X,T}$ is faithful too. Namely, if $a,b : Y \to Y'$ are morphisms in $\mathcal{C}_{X,T}$ such that $a/T = b/T$, then we find on pulling back that the base changes $a_U, b_U : U \times_X Y \to U \times_X Y'$ are equal. Since $U \times_X Y \to Y$ is surjective étale, this implies that $a = b$.

At this point we know that $F_{X,T}$ is faithful for every situation as in the theorem. Let $R = U \times_X U$ where $U$ is as above. Let $t,s : R \to U$ be the projections. Since $X$ is Noetherian, so is $R$. Thus the functor $F_{R,T}$ (defined in the obvious manner) is faithful. Let $Y \to X$ and $Y' \to X$ be objects of $\mathcal{C}_{X,T}$. Let $a' : Y/T \to Y'/T$, be a morphism in the category $\mathcal{C}_{X,T}$. Taking the base change to $U$ we obtain a morphism $a'_U : (U \times_X Y)/T \to (U \times_X Y')/T$ in the category $\mathcal{C}_{U,T}$. Since the functor $F_{U,T}$ is
fully faithful we obtain a morphism \( a_U : U \times_X Y \to U \times_X Y' \) with \( F_{U,T}(a_U) = a'_U \).

Since \( s^*(a'_U) = t^*(a'_U) \) and since \( F_{R,T} \) is faithful, we find that \( s^*(a_U) = t^*(a_U) \).

Since

\[
\begin{array}{c}
R \times_X Y \\
\downarrow \quad \downarrow \\
U \times_X Y \\
\downarrow \\
Y
\end{array}
\]

is an equalizer diagram of sheaves, we find that \( a_U \) descends to a morphism \( a : Y \to Y' \). We omit the proof that \( F_{X,T}(a) = a' \).

At this point we know that \( F_{X,T} \) is faithful for every situation as in the theorem. To finish the proof we show that \( F_{X,T} \) is essentially surjective. Let \( W \to X/T \) be an object of \( \mathcal{C}_{X/T} \). Then \( U \times_X W \) is an object of \( \mathcal{C}_{U,T} \). By the affine case we find an object \( V \to U \) of \( \mathcal{C}_{U,T} \) and an isomorphism \( \alpha : F_{U,T}(V) \to U \times_X W \) in \( \mathcal{C}_{U,T} \).

By fully faithfulness of \( F_{R,T} \) we find a unique morphism \( h : s^*V \to t^*V \) in the category \( \mathcal{C}_{R,T} \) such that \( F_{R,T}(h) \) corresponds, via the isomorphism \( \alpha \), to the canonical descent datum on \( U \times_X W \) in the category \( \mathcal{C}_{R,T} \). Using faithfulness of our functor on \( R \times_{s,T} R \) we see that \( h \) satisfies the cocycle condition. We conclude, for example by the much more general Bootstrap, Lemma \( \ref{bootstrap-lemma} \) that there exists an object \( Y \to X \) of \( \mathcal{C}_{X,T} \) and an isomorphism \( \beta : U \times_X Y \to V \) such that the descent datum \( h \) corresponds, via \( \beta \), to the canonical descent datum on \( U \times_X Y \). We omit the verification that \( F_{X,T}(Y) \) is isomorphic to \( W \); hint: in the category of formal algebraic spaces there is descent for morphisms along étale coverings. \( \square \)

We are often interested as to whether the output of the construction of Theorem \( \ref{0aru} \) is a separated algebraic space. In the next few lemmas we match properties of \( Y \to X \) and the corresponding completion \( Y/T \to X/T \).

0ARU \textbf{Lemma} \( \ref{0aru} \). Let \( S \) be a scheme. Let \( X \) be a locally Noetherian algebraic space over \( S \). Let \( T \subseteq |X| \) be a closed subset. Let \( W \to X/T \) be an object of the category \( \mathcal{C}_{X/T} \) and let \( Y \to X \) be the object corresponding to \( W \) via Theorem \( \ref{0aru} \). Then \( Y \to X \) is quasi-compact if and only if \( W \to X/T \) is so.

\textbf{Proof.} These conditions may be checked after base change to an affine scheme étale over \( X \), resp. a formal affine algebraic space étale over \( X/T \), see Morphisms of Spaces, Lemma \( \ref{morphism-of-spaces-lemma} \) as well as Formal Spaces, Lemma \( \ref{formal-spaces-lemma} \). If \( U \to X \) ranges over étale morphisms with \( U \) affine, then the formal completions \( U/T \to X/T \) give a family of formal affine coverings as in Formal Spaces, Definition \( \ref{formal-spaces-definition} \). Thus we may and do assume \( X \) is affine.

Let \( V \to Y \) be a surjective étale morphism where \( V = \coprod_{j \in J} V_j \) is a disjoint union of affines. Then \( V/T \to Y/T = W \) is a surjective étale morphism. Thus if \( Y \) is quasi-compact, we can choose \( J \) is finite, and we conclude that \( W \) is quasi-compact. Conversely, if \( W \) is quasi-compact, then we can find a finite subset \( J' \subseteq J \) such that \( \coprod_{j \in J'} (V_j)/T \to W \) is surjective. Then it follows that

\[
(X \setminus T) \amalg \coprod_{j \in J'} V_j \longrightarrow Y
\]

is surjective. This either follows from the construction of \( Y \) in the proof of Lemma \( \ref{0aru} \) or it follows since we have

\[
|Y| = |X \setminus T| \amalg |W_{red}|
\]

as \( Y/T = W \). \( \square \)
Lemma 10.11. Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to $W$ via Theorem 10.9. Then $Y \to X$ is quasi-separated if and only if $W \to X_{/T}$ is so.

Proof. These conditions may be checked after base change to an affine scheme étale over $X$, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 23.5. If $U \to X$ ranges over étale morphisms with $U$ affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume $X$ is affine.

Let $V \to Y$ be a surjective étale morphism where $V = \coprod_{j \in J} V_j$ is a disjoint union of affines. Then $Y$ is quasi-separated if and only if $V_j \times_Y V_{j'}$ is quasi-compact for all $j, j' \in J$. Similarly, $W$ is quasi-separated if and only if $(V_j \times_Y V_{j'})_{/T} = (V_j)_{/T} \times_{Y_{/T}} (V_{j'})_{/T}$ is quasi-compact for all $j, j' \in J$. Since $X$ is Noetherian affine, we see that

$$(V_j \times_Y V_{j'}) \times_X (X \setminus T)$$

is quasi-compact. Hence we conclude the equivalence holds by the equality

$$|V_j \times_Y V_{j'}| = |(V_j \times_Y V_{j'}) \times_X (X \setminus T)| \cup |(V_j \times_Y V_{j'})_{/T}|$$

and the fact that the second summand is closed in the left hand side. □

Lemma 10.12. Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to $W$ via Theorem 10.9. Then $Y \to X$ is separated if and only if $W \to X_{/T}$ is separated and $\Delta : W \to W \times_{X_{/T}} W$ is rig-surjective.

Proof. These conditions may be checked after base change to an affine scheme étale over $X$, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms of Spaces, Lemma 4.12 as well as Formal Spaces, Lemma 23.5. If $U \to X$ ranges over étale morphisms with $U$ affine, then the formal completions $U_{/T} \to X_{/T}$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume $X$ is affine. In the proof of both directions we may assume that $Y \to X$ and $W \to X_{/T}$ are quasi-separated by Lemma 10.11.

Proof of easy direction. Assume $Y \to X$ is separated. Then $Y \to Y \times_X Y$ is a closed immersion and it follows that $W \to W \times_{X_{/T}} W$ is a closed immersion too, i.e., we see that $W \to X_{/T}$ is separated. Let

$$p : \text{Spf}(R) \to W \times_{X_{/T}} W = (Y \times_X Y)/T$$

be an adic morphism where $R$ is a complete discrete valuation ring with fraction field $K$. The composition into $Y \times_X Y$ corresponds to a morphism $g : \text{Spec}(R) \to Y \times_X Y$, see Formal Spaces, Lemma 26.3. Since $p$ is an adic morphism, so is the composition $\text{Spf}(R) \to X$. Thus we see that $g(\text{Spec}(K))$ is a point of

$$(Y \times_X Y) \times_X (X \setminus T) \cong X \setminus T \cong Y \times_X (X \setminus T)$$
(small detail omitted). Hence this lifts to a $K$-point of $Y$ and we obtain a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & Y \times_X Y
\end{array}
$$

Since $Y \to X$ was assumed separated we find the dotted arrow exists (Cohomology of Spaces, Lemma \[19.1\]). Applying the functor completion along $T$ we find that $p$ can be lifted to a morphism into $W$, i.e., $W \to W \times_{X/T} W$ is rig-surjective.

Proof of hard direction. Assume $W \to X_{/T}$ separated and $W \to W \times_{X_{/T}} W$ rig-surjective. By Cohomology of Spaces, Lemma \[19.1\] and Remark \[19.3\] it suffices to show that given any commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \rightarrow & Y \times_X Y
\end{array}
$$

where $R$ is a complete discrete valuation ring with fraction field $K$, there is at most one dotted arrow making the diagram commute. Let $h : \text{Spec}(R) \to X$ be the composition of $g$ with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\text{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: $h$ maps $\text{Spec}(R)$ into $T$. This case follows from our assumption that $W \to X_{/T}$ is separated. Namely, if $T$ denotes the reduced induced closed subspace structure on $T$, then $h$ factors through $T$ and

$$
W \times_{X_{/T}} T = Y \times_X T \longrightarrow T
$$

is separated by assumption (and for example Formal Spaces, Lemma \[23.5\] which implies we get the lifting property by Cohomology of Spaces, Lemma \[19.1\] applied to the displayed arrow. Case III: $h(\text{Spec}(K))$ is not in $T$ but $h$ maps the closed point of $\text{Spec}(R)$ into $T$. In this case the corresponding morphism

$$
g_{/T} : \text{Spf}(R) \longrightarrow (Y \times_X Y)/T = W \times_{X_{/T}} W
$$

is an adic morphism (detail omitted). Hence our assumption that $W \to W \times_{X_{/T}} W$ be rig-surjective implies we can lift $g_{/T}$ to a morphism $e : \text{Spf}(R) \to W = Y_{/T}$ (see Lemma \[9.11\] for why we do not need to extend $R$). Algebraizing the composition $\text{Spf}(R) \to Y$ using Formal Spaces, Lemma \[26.3\] we find a morphism $\text{Spec}(R) \to Y$ lifting $g$ as desired.

0ARX

**Lemma 10.13.** Let $S$ be a scheme. Let $X$ be a locally Noetherian algebraic space over $S$. Let $T \subset |X|$ be a closed subset. Let $W \to X_{/T}$ be an object of the category $\mathcal{C}_{X_{/T}}$ and let $Y \to X$ be the object corresponding to $W$ via Theorem \[10.9\]. Then $Y \to X$ is proper if and only if the following conditions hold

1. $W \to X_{/T}$ is proper,
2. $W \to X_{/T}$ is rig-surjective, and
3. $A : W \to W \times_{X_{/T}} W$ is rig-surjective.

**Proof.** These conditions may be checked after base change to an affine scheme étale over $X$, resp. a formal affine algebraic space étale over $X_{/T}$, see Morphisms
of Spaces, Lemma 40.2 as well as Formal Spaces, Lemma 24.2. If $U \to X$ ranges over étale morphisms with $U$ affine, then the formal completions $U/T \to X/T$ give a family of formal affine coverings as in Formal Spaces, Definition 7.1. Thus we may and do assume $X$ is affine. In the proof of both directions we may assume that $Y \to X$ and $W \to X/T$ are separated and quasi-compact and that $W \to W \times_{X/T} W$ is rig-surjective by Lemmas 10.10 and 10.12.

Proof of the easy direction. Assume $Y \to X$ is proper. Then $Y/T = Y \times_X X/T \to X/T$ is proper too. Let
\[ p : \text{Spf}(R) \to X/T \]
be an adic morphism where $R$ is a complete discrete valuation ring with fraction field $K$. Then $p$ corresponds to a morphism $g : \text{Spec}(R) \to X$, see Formal Spaces, Lemma 26.3. Since $p$ is an adic morphism, we have $p(\text{Spec}(K)) \notin T$. Since $Y \to X$ is an isomorphism over $X \setminus T$ we can lift to $X$ and obtain a commutative diagram
\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & X
\end{array}
\]
Since $Y \to X$ was assumed proper we find the dotted arrow exists. (Cohomology of Spaces, Lemma 19.2). Applying the functor completion along $T$ we find that $p$ can be lifted to a morphism into $W$, i.e., $W \to X/T$ is rig-surjective.

Proof of hard direction. Assume $W \to X/T$ proper, $W \to W \times_{X/T} W$ rig-surjective, and $W \to X/T$ rig-surjective. By Cohomology of Spaces, Lemma 19.2 and Remark 19.3 it suffices to show that given any commutative diagram
\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \stackrel{g}{\longrightarrow} & X
\end{array}
\]
where $R$ is a complete discrete valuation ring with fraction field $K$, there is a dotted arrow making the diagram commute. Let $h : \text{Spec}(R) \to X$ be the composition of $g$ with the morphism $Y \times_X Y \to X$. There are three cases: Case I: $h(\text{Spec}(R)) \subset (X \setminus T)$. This case is trivial because $Y \times_X (X \setminus T) = X \setminus T$. Case II: $h$ maps $\text{Spec}(R)$ into $T$. This case follows from our assumption that $W \to X/T$ is proper. Namely, if $T$ denotes the reduced induced closed subspace structure on $T$, then $h$ factors through $T$ and
\[
W \times_{X/T} T = Y \times_X T \longrightarrow T
\]
is proper by assumption which implies we get the lifting property by Cohomology of Spaces, Lemma 19.2 applied to the displayed arrow. Case III: $h(\text{Spec}(K))$ is not in $T$ but $h$ maps the closed point of $\text{Spec}(R)$ into $T$. In this case the corresponding morphism
\[
g/T : \text{Spf}(R) \to Y/T = W
\]
is an adic morphism (detail omitted). Hence our assumption that $W \to X/T$ be rig-surjective implies we can lift $g/T$ to a morphism $e : \text{Spf}(R') \to W = Y/T$ for some extension of complete discrete valuation rings $R \subset R'$. Algebraizing the composition $\text{Spf}(R') \to Y$ using Formal Spaces, Lemma 26.3 we find a morphism.
Spec($R' \to Y$) lifting $g$. By the discussion in Cohomology of Spaces, Remark 19.3, this is sufficient to conclude that $Y \to X$ is proper.

11. Application to modifications

Let $A$ be a Noetherian ring and let $I \subset A$ be an ideal. We set $S = \text{Spec}(A)$ and $U = S \setminus V(I)$. In this section we will consider the category

$$\left\{ f : X \to S \mid \begin{array}{l} X \text{ is an algebraic space} \\ f \text{ is locally of finite type} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{array} \right\}$$

A morphism from $X/S$ to $X'/S$ will be a morphism of algebraic spaces $X \to X'$ compatible with the structure morphisms over $S$.

Let $A \to B$ be a homomorphism of Noetherian rings and let $J \subset B$ be an ideal such that $J = \sqrt{IB}$. Then base change along the morphism $\text{Spec}(B) \to \text{Spec}(A)$ gives a functor from the category (11.0.1) for $A$ to the category (11.0.1) for $B$.

Lemma 11.1. Let $(A, I)$ be a pair consisting of a Noetherian ring and an ideal $I$. Let $A^\wedge$ be the $I$-adic completion of $A$. Then base change defines an equivalence of categories between the category (11.0.1) for $A$ with the completion $A^\wedge$.

Proof. Set $S = \text{Spec}(A)$ as in (11.0.1) and $T = V(I)$. Similarly, write $S' = \text{Spec}(A^\wedge)$ and $T' = V(IA^\wedge)$. The morphism $S' \to S$ defines an isomorphism $S'/T' \to S/T$ of formal completions. Let $\mathcal{C}_{S,T}$, $\mathcal{C}_{S'/T'}$, and $\mathcal{C}_{S'/T'}$ be the corresponding categories as used in (10.3.1). By Theorem 10.9 (in fact we only need the affine case treated in Lemma 10.8) we see that $\mathcal{C}_{S,T} = \mathcal{C}_{S'/T'}$. Since $\mathcal{C}_{S,T}$ is the category (11.0.1) for $A$ and $\mathcal{C}_{S'/T'}$ the category (11.0.1) for $A^\wedge$, this proves the lemma.

Lemma 11.2. Notation and assumptions as in Lemma 11.1. Let $f : X \to \text{Spec}(A)$ correspond to $g : Y \to \text{Spec}(A^\wedge)$ via the equivalence. Then $f$ is quasi-compact, quasi-separated, separated, proper, finite, and add more here if and only if $g$ is so.

Proof. You can deduce this for the statements quasi-compact, quasi-separated, separated, and proper by using Lemmas 10.10, 10.11, 10.12, and 10.13 to translate the corresponding property into a property of the formal completion and using the argument of the proof of Lemma 11.1. However, there is a direct argument using fpqc descent as follows. First, note that $\{U \to \text{Spec}(A), \text{Spec}(A) \to \text{Spec}(A)\}$ is an fpqc covering with $U = \text{Spec}(A) \setminus V(I)$ as before. The base change of $f$ by $U \to \text{Spec}(A)$ is $\text{id}_U$ by definition of our category (11.0.1). Let $P$ be a property of morphisms of algebraic spaces which is fpqc local on the base (Descent on Spaces, Definition 9.1) such that $P$ holds for identity morphisms. Then we see that $P$ holds for $f$ if and only if $P$ holds for $g$. This applies to $P$ equal to quasi-compact, quasi-separated, separated, proper, and finite by Descent on Spaces, Lemmas 10.1, 10.2, 10.18, 10.19, and 10.23.

Lemma 11.3. Let $A \to B$ be a local map of local Noetherian rings such that

1. $A \to B$ is flat,
(2) $m_B = m_A B$, and
(3) $\kappa(m_A) = \kappa(m_B)$
(equivalently, $A \to B$ induces an isomorphism on completions, see More on Algebra, Lemma \cite{11.0.1}. Then the base change functor from the category \cite{11.0.1} for $(A, m_A)$ to the category \cite{11.0.1} for $(B, m_B)$ is an equivalence.

**Proof.** This follows immediately from Lemma \cite{11.1}.

**Lemma 11.4.** Let $(A, m, \kappa)$ be a Noetherian local ring. Let $f : X \to S$ be an object of \cite{11.0.1}. Then there exists a $U$-admissible blowup $S' \to S$ which dominates $X$.

**Proof.** Special case of More on Morphisms of Spaces, Lemma \cite{39.4}.

### 12. Other chapters

**Preliminaries**
1. Introduction
2. Conventions
3. Set Theory
4. Categories
5. Topology
6. Sheaves on Spaces
7. Sites and Sheaves
8. Stacks
9. Fields
10. Commutative Algebra
11. Brauer Groups
12. Homological Algebra
13. Derived Categories
14. Simplicial Methods
15. More on Algebra
16. Smoothing Ring Maps
17. Sheaves of Modules
18. Modules on Sites
19. Injectives
20. Cohomology of Sheaves
21. Cohomology on Sites
22. Differential Graded Algebra
23. Divided Power Algebra
24. Hypercoverings

**Schemes**
25. Schemes
26. Constructions of Schemes
27. Properties of Schemes
28. Morphisms of Schemes
29. Cohomology of Schemes
30. Divisors
31. Limits of Schemes
32. Varieties

**Topics in Scheme Theory**
33. Topologies on Schemes
34. Descent
35. Derived Categories of Schemes
36. More on Morhphisms
37. More on Flatness
38. Groupoid Schemes
39. More on Groupoid Schemes
40. Étale Morphisms of Schemes
41. Chow Homology
42. Intersection Theory
43. Picard Schemes of Curves
44. Adequate Modules
45. Dualizing Complexes
46. Duality for Schemes
47. Discriminants and Differents
48. Local Cohomology
49. Algebraic and Formal Geometry
50. Algebraic Curves
51. Resolution of Surfaces
52. Semistable Reduction
53. Fundamental Groups of Schemes
54. Étale Cohomology
55. Crystalline Cohomology
56. Pro-étale Cohomology

**Algebraic Spaces**
57. Algebraic Spaces
58. Properties of Algebraic Spaces
59. Morphisms of Algebraic Spaces
60. Decent Algebraic Spaces
61. Cohomology of Algebraic Spaces
62. Limits of Algebraic Spaces
63. Divisors on Algebraic Spaces
64. Algebraic Spaces over Fields
RESTRICTED POWER SERIES 35

(65) Topologies on Algebraic Spaces
(66) Descent and Algebraic Spaces
(67) Derived Categories of Spaces
(68) More on Morphisms of Spaces
(69) Flatness on Algebraic Spaces
(70) Groupoids in Algebraic Spaces
(71) More on Groupoids in Spaces
(72) Bootstrap
(73) Pushouts of Algebraic Spaces

Topics in Geometry
(74) Chow Groups of Spaces
(75) Quotients of Groupoids
(76) More on Cohomology of Spaces
(77) Simplicial Spaces
(78) Duality for Spaces
(79) Formal Algebraic Spaces
(80) Restricted Power Series
(81) Resolution of Surfaces Revisited

Deformation Theory
(82) Formal Deformation Theory
(83) Deformation Theory
(84) The Cotangent Complex
(85) Deformation Problems

Algebraic Stacks
(86) Algebraic Stacks

(87) Examples of Stacks
(88) Sheaves on Algebraic Stacks
(89) Criteria for Representability
(90) Artin’s Axioms
(91) Quot and Hilbert Spaces
(92) Properties of Algebraic Stacks
(93) Morphisms of Algebraic Stacks
(94) Limits of Algebraic Stacks
(95) Cohomology of Algebraic Stacks

Topics in Moduli Theory
(96) Derived Categories of Stacks
(97) Introducing Algebraic Stacks
(98) More on Morphisms of Stacks
(99) The Geometry of Stacks
(100) Moduli Stacks
(101) Moduli of Curves

Miscellany
(102) Examples
(103) Exercises
(104) Guide to Literature
(105) Desirables
(106) Coding Style
(107) Obsolete
(108) GNU Free Documentation License
(109) Auto Generated Index

References


