

## DIFFERENTIAL GRADED SHEAVES

0FQS

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## 1. Introduction

0FQT This chapter is a continuation of the discussion started in Differential Graded Algebra, Section 1. A survey paper is [Kel06].

## 2. Conventions

0FQU In this chapter we hold on to the convention that *ring* means commutative ring with 1. If  $R$  is a ring, then an  $R$ -algebra  $A$  will be an  $R$ -module  $A$  endowed with an  $R$ -bilinear map  $A \times A \rightarrow A$  (multiplication) such that multiplication is associative and has an identity. In other words, these are unital associative  $R$ -algebras such that the structure map  $R \rightarrow A$  maps into the center of  $A$ .

## 3. Sheaves of graded algebras

0FQV Please skip this section.

0FQW **Definition 3.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. A *sheaf of graded  $\mathcal{O}$ -algebras* or a *sheaf of graded algebras* on  $(\mathcal{C}, \mathcal{O})$  is given by a family  $\mathcal{A}^n$  indexed by  $n \in \mathbf{Z}$  of  $\mathcal{O}$ -modules endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \rightarrow \mathcal{A}^{n+m}, \quad (a, b) \mapsto ab$$

called the multiplication maps with the following properties

- (1) multiplication is associative, and
- (2) there is a global section 1 of  $\mathcal{A}^0$  which is a two-sided identity for multiplication.

We often denote such a structure  $\mathcal{A}$ . A *homomorphism of graded  $\mathcal{O}$ -algebras*  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a family of maps  $f^n : \mathcal{A}^n \rightarrow \mathcal{B}^n$  of  $\mathcal{O}$ -modules compatible with the multiplication maps.

Given a graded  $\mathcal{O}$ -algebra  $\mathcal{A}$  and an object  $U \in \text{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a graded  $\mathcal{O}(U)$ -algebra.

0FQX **Remark 3.2.** Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. We have

- (1) Let  $\mathcal{A}$  be a graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. The multiplication maps of  $\mathcal{A}$  induce multiplication maps  $f_*\mathcal{A}^n \times f_*\mathcal{A}^m \rightarrow f_*\mathcal{A}^{n+m}$  and via  $f^\sharp$  we may view these as  $\mathcal{O}_{\mathcal{D}}$ -bilinear maps. We will denote  $f_*\mathcal{A}$  the graded  $\mathcal{O}_{\mathcal{D}}$ -algebra we so obtain.
- (2) Let  $\mathcal{B}$  be a graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. The multiplication maps of  $\mathcal{B}$  induce multiplication maps  $f^*\mathcal{B}^n \times f^*\mathcal{B}^m \rightarrow f^*\mathcal{B}^{n+m}$  and using  $f^\sharp$  we may view these as  $\mathcal{O}_{\mathcal{C}}$ -bilinear maps. We will denote  $f^*\mathcal{B}$  the graded  $\mathcal{O}_{\mathcal{C}}$ -algebra we so obtain.
- (3) The set of homomorphisms  $f^*\mathcal{B} \rightarrow \mathcal{A}$  of graded  $\mathcal{O}_{\mathcal{C}}$ -algebras is in 1-to-1 correspondence with the set of homomorphisms  $\mathcal{B} \rightarrow f_*\mathcal{A}$  of graded  $\mathcal{O}_{\mathcal{D}}$ -algebras.

Part (3) follows immediately from the usual adjunction between  $f^*$  and  $f_*$  on sheaves of modules.

#### 4. Sheaves of graded modules

0FQY Please skip this section.

0FQZ **Definition 4.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A (right) *graded  $\mathcal{A}$ -module* or (right) *graded module* over  $\mathcal{A}$  is given by a family  $\mathcal{M}^n$  indexed by  $n \in \mathbf{Z}$  of  $\mathcal{O}$ -modules endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \rightarrow \mathcal{M}^{n+m}, \quad (x, a) \mapsto xa$$

called the multiplication maps with the following properties

- (1) multiplication satisfies  $(xa)a' = x(aa')$ ,
- (2) the identity section 1 of  $\mathcal{A}^0$  acts as the identity on  $\mathcal{M}^n$  for all  $n$ .

We often say “let  $\mathcal{M}$  be a graded  $\mathcal{A}$ -module” to indicate this situation. A *homomorphism of graded  $\mathcal{A}$ -modules*  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a family of maps  $f^n : \mathcal{M}^n \rightarrow \mathcal{N}^n$  of  $\mathcal{O}$ -modules compatible with the multiplication maps. The category of (right) graded  $\mathcal{A}$ -modules is denoted  $Mod(\mathcal{A})$ .

We can define *left graded modules* in exactly the same manner but our default in the chapter will be right modules.

Given a graded  $\mathcal{A}$ -module  $\mathcal{M}$  and an object  $U \in \text{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) graded  $\mathcal{A}(U)$ -module.

0FR0 **Lemma 4.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a graded  $\mathcal{O}$ -algebra. The category  $Mod(\mathcal{A})$  is an abelian category with the following properties*

- (1)  $Mod(\mathcal{A})$  has arbitrary direct sums,
- (2)  $Mod(\mathcal{A})$  has arbitrary colimits,
- (3) filtered colimit in  $Mod(\mathcal{A})$  are exact,
- (4)  $Mod(\mathcal{A})$  has arbitrary products,
- (5)  $Mod(\mathcal{A})$  has arbitrary limits.

The functor

$$Mod(\mathcal{A}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{M} \mapsto \mathcal{M}^n$$

sending a graded  $\mathcal{A}$ -module to its  $n$ th term commutes with all limits and colimits.

The lemma says that we may take limits and colimits termwise. It also says (or implies if you like) that the forgetful functor

$$Mod(\mathcal{A}) \longrightarrow \text{graded } \mathcal{O}\text{-modules}$$

commutes with all limits and colimits.

**Proof.** Let us denote  $\text{gr}^n : Mod(\mathcal{A}) \rightarrow Mod(\mathcal{O})$  the functor in the statement of the lemma. Consider a homomorphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  of graded  $\mathcal{A}$ -modules. The kernel and cokernel of  $f$  as maps of graded  $\mathcal{O}$ -modules are additionally endowed with multiplication maps as in Definition 4.1. Hence these are also the kernel and cokernel in  $Mod(\mathcal{A})$ . Thus  $Mod(\mathcal{A})$  is an abelian category and taking kernels and cokernels commutes with  $\text{gr}^n$ .

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 14.11 and 14.12. The same

lemmas show that proving the commutation of limits and colimits with  $\text{gr}^n$  follows if  $\text{gr}^n$  commutes with direct sums and products.

Let  $\mathcal{M}_t$ ,  $t \in T$  be a set of graded  $\mathcal{A}$ -modules. Then we can consider the graded  $\mathcal{A}$ -module whose degree  $n$  term is  $\bigoplus_{t \in T} \mathcal{M}_t^n$  (with obvious multiplication maps). The reader easily verifies that this is a direct sum in  $\text{Mod}(\mathcal{A})$ . Similarly for products.

Observe that  $\text{gr}^n$  is an exact functor for all  $n$  and that a complex  $\mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3$  of  $\text{Mod}(\mathcal{A})$  is exact if and only if  $\text{gr}^n \mathcal{M}_1 \rightarrow \text{gr}^n \mathcal{M}_2 \rightarrow \text{gr}^n \mathcal{M}_3$  is exact in  $\text{Mod}(\mathcal{O})$  for all  $n$ . Hence we conclude that (3) holds as filtered colimits are exact in  $\text{Mod}(\mathcal{O})$ ; it is a Grothendieck abelian category, see Cohomology on Sites, Section 19.  $\square$

### 5. The graded category of sheaves of graded modules

0FR1 Please skip this section. This section is the analogue of Differential Graded Algebra, Example 25.6. For our conventions on graded categories, please see Differential Graded Algebra, Section 25.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . We will construct a graded category  $\text{Mod}^{\text{gr}}(\mathcal{A})$  over  $R = \Gamma(\mathcal{C}, \mathcal{O})$  whose associated category  $(\text{Mod}^{\text{gr}}(\mathcal{A}))^0$  is the category of graded  $\mathcal{A}$ -modules. As objects of  $\text{Mod}^{\text{gr}}(\mathcal{A})$  we take right graded  $\mathcal{A}$ -modules (see Section 4). Given graded  $\mathcal{A}$ -modules  $\mathcal{L}$  and  $\mathcal{M}$  we set

$$\text{Hom}_{\text{Mod}^{\text{gr}}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \text{Hom}^n(\mathcal{L}, \mathcal{M})$$

where  $\text{Hom}^n(\mathcal{L}, \mathcal{M})$  is the set of right  $\mathcal{A}$ -module maps  $f : \mathcal{L} \rightarrow \mathcal{M}$  which are homogeneous of degree  $n$ . More precisely,  $f$  is given by a family of maps  $f : \mathcal{L}^i \rightarrow \mathcal{M}^{i+n}$  for  $i \in \mathbf{Z}$  compatible with the multiplication maps. In terms of components, we have that

$$\text{Hom}^n(\mathcal{L}, \mathcal{M}) \subset \prod_{p+q=n} \text{Hom}_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

(observe reversal of indices) is the subset consisting of those  $f = (f_{p,q})$  such that

$$f_{p,q}(ma) = f_{p-i, q+i}(m)a$$

for local sections  $a$  of  $\mathcal{A}^i$  and  $m$  of  $\mathcal{L}^{-q-i}$ . For graded  $\mathcal{A}$ -modules  $\mathcal{K}$ ,  $\mathcal{L}$ ,  $\mathcal{M}$  we define composition in  $\text{Mod}^{\text{gr}}(\mathcal{A})$  via the maps

$$\text{Hom}^m(\mathcal{L}, \mathcal{M}) \times \text{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

by simple composition of right  $\mathcal{A}$ -module maps:  $(g, f) \mapsto g \circ f$ .

### 6. Tensor product for sheaves of graded modules

0FR2 Please skip this section. This section is the analogue of part of Differential Graded Algebra, Section 12.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right graded  $\mathcal{A}$ -module and let  $\mathcal{N}$  be a left graded  $\mathcal{A}$ -module. Then we define the *tensor product*  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  to be the graded  $\mathcal{O}$ -module whose degree  $n$  term is

$$(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^n = \text{Coker} \left( \bigoplus_{r+s+t=n} \mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t \longrightarrow \bigoplus_{p+q=n} \mathcal{M}^p \otimes_{\mathcal{O}} \mathcal{N}^q \right)$$

where the map sends the local section  $x \otimes a \otimes y$  of  $\mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t$  to  $xa \otimes y - x \otimes ay$ . With this definition we have that  $(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^n$  is the sheafification of the presheaf  $U \mapsto (\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))^n$  where the tensor product of graded modules is as defined in Differential Graded Algebra, Section 12.

If we fix the left graded  $\mathcal{A}$ -module  $\mathcal{N}$  we obtain a functor

$$- \otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}(\mathcal{A}) \longrightarrow \text{Gr}(\text{Mod}(\mathcal{O})) = \text{graded } \mathcal{O}\text{-modules}$$

For the notation  $\text{Gr}(-)$  please see Homology, Definition 16.1. The graded category of graded  $\mathcal{O}$ -modules is denoted  $\text{Gr}^{gr}(\text{Mod}(\mathcal{O}))$ , see Differential Graded Algebra, Example 25.5. The functor above can be upgraded to a functor of graded categories

$$- \otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Gr}^{gr}(\text{Mod}(\mathcal{O}))$$

by sending homomorphisms of degree  $n$  from  $\mathcal{M} \rightarrow \mathcal{M}'$  to the induced map of degree  $n$  from  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  to  $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$ .

### 7. Internal hom for sheaves of graded modules

0FR3 We urge the reader to skip this section.

We are going to need the sheafified version of the construction in Section 5. Let  $(\mathcal{C}, \mathcal{O})$ ,  $\mathcal{A}$ ,  $\mathcal{M}$ ,  $\mathcal{L}$  be as in Section 5. Then we define

$$\mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{M}, \mathcal{L})$$

as the graded  $\mathcal{O}$ -module whose degree  $n$  term

$$\mathcal{H}om_{\mathcal{A}}^n(\mathcal{M}, \mathcal{L}) \subset \prod_{p+q=n} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

is the subsheaf consisting of those local sections  $f = (f_{p,q})$  such that

$$f_{p,q}(ma) = f_{p-i, q+i}(m)a$$

for local sections  $a$  of  $\mathcal{A}^i$  and  $m$  of  $\mathcal{L}^{-q-i}$ . As in Section 5 there is a composition map

$$\mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of graded  $\mathcal{O}$ -modules defined in Section 6. This map is given by the composition map

$$\mathcal{H}om_{\mathcal{A}}^m(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{A}}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om_{\mathcal{A}}^{n+m}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally).

With these definitions we have

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{L}, \mathcal{M}))$$

as graded  $R$ -modules compatible with composition.

### 8. Sheaves of graded bimodules and tensor-hom adjunction

0FR4 Please skip this section.

0FR5 **Definition 8.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A *graded  $(\mathcal{A}, \mathcal{B})$ -bimodule* is given by a family  $\mathcal{M}^n$  indexed by  $n \in \mathbf{Z}$  of  $\mathcal{O}$ -modules endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \rightarrow \mathcal{M}^{n+m}, \quad (x, b) \longmapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \rightarrow \mathcal{M}^{n+m}, \quad (a, x) \longmapsto ax$$

called the multiplication maps with the following properties

- (1) multiplication satisfies  $a(a'x) = (aa')x$  and  $(xb)b' = x(bb')$ ,
- (2)  $(ax)b = a(xb)$ ,

- (3) the identity section 1 of  $\mathcal{A}^0$  acts as the identity by multiplication, and
- (4) the identity section 1 of  $\mathcal{B}^0$  acts as the identity by multiplication.

We often denote such a structure  $\mathcal{M}$ . A *homomorphism of graded  $(\mathcal{A}, \mathcal{B})$ -bimodules*  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a family of maps  $f^n : \mathcal{M}^n \rightarrow \mathcal{N}^n$  of  $\mathcal{O}$ -modules compatible with the multiplication maps.

Given a graded  $(\mathcal{A}, \mathcal{B})$ -bimodule  $\mathcal{M}$  and an object  $U \in \text{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a graded  $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodule.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right graded  $\mathcal{A}$ -module and let  $\mathcal{N}$  be a graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. In this case the graded tensor product defined in Section 6

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

is a right graded  $\mathcal{B}$ -module with obvious multiplication maps. This construction defines a functor and a functor of graded categories

$$\otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{B})$$

by sending homomorphisms of degree  $n$  from  $\mathcal{M} \rightarrow \mathcal{M}'$  to the induced map of degree  $n$  from  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  to  $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$ .

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{N}$  be a graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Let  $\mathcal{L}$  be a right graded  $\mathcal{B}$ -module. In this case the graded internal hom defined in Section 7

$$\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, \mathcal{L})$$

is a right graded  $\mathcal{A}$ -module with multiplication maps<sup>1</sup>

$$\mathcal{H}om_{\mathcal{B}}^n(\mathcal{N}, \mathcal{L}) \times \mathcal{A}^m \longrightarrow \mathcal{H}om_{\mathcal{B}}^{n+m}(\mathcal{N}, \mathcal{L})$$

sending a section  $f = (f_{p,q})$  of  $\mathcal{H}om_{\mathcal{B}}^n(\mathcal{N}, \mathcal{L})$  over  $U$  and a section  $a$  of  $\mathcal{A}^m$  over  $U$  to the section  $fa$  if  $\mathcal{H}om_{\mathcal{B}}^{n+m}(\mathcal{N}, \mathcal{L})$  over  $U$  defined as the family of maps

$$\mathcal{N}^{-q-m}|_U \xrightarrow{a \cdot} \mathcal{N}^{-q}|_U \xrightarrow{f_{p,q}} \mathcal{M}^p|_U$$

We omit the verification that this is well defined. This construction defines a functor and a functor of graded categories

$$\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, -) : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, -) : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

by sending homomorphisms of degree  $n$  from  $\mathcal{L} \rightarrow \mathcal{L}'$  to the induced map of degree  $n$  from  $\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, \mathcal{L})$  to  $\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, \mathcal{L}')$ .

0FR6 **Lemma 8.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right graded  $\mathcal{A}$ -module. Let  $\mathcal{N}$  be a graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Let  $\mathcal{L}$  be a right graded  $\mathcal{B}$ -module. With conventions as above we have*

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{M}, \mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{M}, \mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{N}, \mathcal{L}))$$

functorially in  $\mathcal{M}, \mathcal{N}, \mathcal{L}$ .

<sup>1</sup>Our conventions are here that this does not involve any signs.

**Proof.** Omitted. Hint: This follows by interpreting both sides as  $\mathcal{A}$ -bilinear graded maps  $\psi : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{L}$  which are  $\mathcal{B}$ -linear on the right.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be sheaves of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . As a special case of the above, suppose we are given a homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  of graded  $\mathcal{O}$ -algebras. Then we obtain a functor and a functor of graded categories

$$\otimes_{\mathcal{A}, \varphi} \mathcal{B} : \text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A}, \varphi} \mathcal{B} : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{B})$$

On the other hand, we have the restriction functors

$$\text{res}_\varphi : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A}) \quad \text{and} \quad \text{res}_\varphi : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write  ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$  for  $\mathcal{B}$  viewed as a graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Then for any right graded  $\mathcal{B}$ -module  $\mathcal{L}$  we have

$$\mathcal{H}om_{\mathcal{B}}^{gr}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = \text{res}_\varphi(\mathcal{L})$$

as right graded  $\mathcal{A}$ -modules. Thus Lemma 8.2 tells us that we have a functorial isomorphism

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A}, \varphi} \mathcal{B}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{M}, \text{res}_\varphi(\mathcal{L}))$$

We usually drop the dependence on  $\varphi$  in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om_{\mathcal{B}}^{gr}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om_{\mathcal{A}}^{gr}(\mathcal{M}, \mathcal{L})$$

of graded  $\mathcal{O}$ -modules.

## 9. Pull and push for sheaves of graded modules

0FR7 We advise the reader to skip this section.

Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \rightarrow \mathcal{A}$$

of graded  $f^{-1}\mathcal{O}_{\mathcal{D}}$ -algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map  $\varphi : f^*\mathcal{B} \rightarrow \mathcal{A}$  of graded  $\mathcal{O}_{\mathcal{C}}$ -algebras or equivalently  $\varphi$  can be viewed as a map

$$\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$$

of graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. See Remark 3.2.

Let us define a functor

$$f_* : \text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{B})$$

Given a graded  $\mathcal{A}$ -module  $\mathcal{M}$  we define  $f_*\mathcal{M}$  to be the graded  $\mathcal{B}$ -module whose degree  $n$  term is  $f_*\mathcal{M}^n$ . As multiplication we use

$$f_*\mathcal{M}^n \times \mathcal{B}^m \xrightarrow{(\text{id}, \varphi^m)} f_*\mathcal{M}^n \times f_*\mathcal{A}^m \xrightarrow{f_*\mu_{n,m}} f_*\mathcal{M}^{n+m}$$

where  $\mu_{n,m} : \mathcal{M}^n \times \mathcal{A}^m \rightarrow \mathcal{M}^{n+m}$  is the multiplication map for  $\mathcal{M}$  over  $\mathcal{A}$ . This uses that  $f_*$  commutes with products. The construction is clearly functorial in  $\mathcal{M}$  and we obtain our functor.

Let us define a functor

$$f^* : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A})$$

We will define this functor as a composite of functors

$$\text{Mod}(\mathcal{B}) \xrightarrow{f^{-1}} \text{Mod}(f^{-1}\mathcal{B}) \xrightarrow{-\otimes_{f^{-1}\mathcal{B}}\mathcal{A}} \text{Mod}(\mathcal{A})$$

First, given a graded  $\mathcal{B}$ -module  $\mathcal{N}$  we define  $f^{-1}\mathcal{N}$  to be the graded  $f^{-1}\mathcal{B}$ -module whose degree  $n$  term is  $f^{-1}\mathcal{N}^n$ . As multiplication we use

$$f^{-1}\nu_{n,m} : f^{-1}\mathcal{N}^n \times f^{-1}\mathcal{B}^m \longrightarrow f^{-1}\mathcal{N}^{n+m}$$

where  $\nu_{n,m} : \mathcal{N}^n \times \mathcal{B}^m \rightarrow \mathcal{N}^{n+m}$  is the multiplication map for  $\mathcal{N}$  over  $\mathcal{B}$ . This uses that  $f^{-1}$  commutes with products. The construction is clearly functorial in  $\mathcal{N}$  and we obtain our functor  $f^{-1}$ . Having said this, we can use the tensor product discussion in Section 8 to define the functor

$$-\otimes_{f^{-1}\mathcal{B}}\mathcal{A} : \text{Mod}(f^{-1}\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A})$$

Finally, we set

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B},\varphi}\mathcal{A}$$

as already foretold above.

The functors  $f_*$  and  $f^*$  are readily enhanced to give functors of graded categories

$$f_* : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{B}) \quad \text{and} \quad f^* : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogeneous morphisms of degree  $n$ .

0FR8 **Lemma 9.1.** *In the situation above we have*

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{N}, f_*\mathcal{M}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(f^*\mathcal{N}, \mathcal{M})$$

**Proof.** Omitted. Hints: First prove that  $f^{-1}$  and  $f_*$  are adjoint as functors between  $\text{Mod}(\mathcal{B})$  and  $\text{Mod}(f^{-1}\mathcal{B})$  using the adjunction between  $f^{-1}$  and  $f_*$  on sheaves of abelian groups. Next, use the adjunction between base change and restriction given in Section 8.  $\square$

## 10. Localization and sheaves of graded modules

0FR9 We advise the reader to skip this section.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$  and denote

$$j : (\text{Sh}(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (\text{Sh}(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 19). Below we will use the following fact: for  $\mathcal{O}_U$ -modules  $\mathcal{M}_i$ ,  $i = 1, 2$  and a  $\mathcal{O}$ -module  $\mathcal{A}$  there is a canonical map

$$j_! : \text{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \text{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have  $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$  by Modules on Sites, Lemma 27.9.

Let  $\mathcal{A}$  be a graded  $\mathcal{O}$ -algebra. We will denote  $\mathcal{A}_U$  the restriction of  $\mathcal{A}$  to  $\mathcal{C}/U$ , in other words, we have  $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$ . In Section 9 we have constructed adjoint functors

$$j_* : \text{Mod}^{gr}(\mathcal{A}_U) \longrightarrow \text{Mod}^{gr}(\mathcal{A}) \quad \text{and} \quad j^* : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{A}_U)$$

with  $j^*$  left adjoint to  $j_*$ . We claim there is in addition an exact functor

$$j_! : \text{Mod}^{gr}(\mathcal{A}_U) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$



left adjoint to  $j^*$ . Namely, given a graded  $\mathcal{A}_U$ -module  $\mathcal{M}$  we define  $j_!\mathcal{M}$  to be the graded  $\mathcal{A}$ -module whose degree  $n$  term is  $j_!\mathcal{M}^n$ . As multiplication map we use

$$j_!\mu_{n,m} : j_!\mathcal{M}^n \times \mathcal{A}^m \rightarrow j_!\mathcal{M}^{n+m}$$

where  $\mu_{m,n} : \mathcal{M}^n \times \mathcal{A}^m \rightarrow \mathcal{M}^{n+m}$  is the given multiplication map. Given a homogeneous map  $f : \mathcal{M} \rightarrow \mathcal{M}'$  of degree  $n$  of graded  $\mathcal{A}_U$ -modules, we obtain a homogeneous map  $j_!f : j_!\mathcal{M} \rightarrow j_!\mathcal{M}'$  of degree  $n$ . Thus we obtain our functor.

0FRA **Lemma 10.1.** *In the situation above we have*

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A}_U)}(\mathcal{M}, j^*\mathcal{N})$$

**Proof.** By the discussion in Modules on Sites, Section 19 the functors  $j_!$  and  $j^*$  on  $\mathcal{O}$ -modules are adjoint. Thus if we only look at the  $\mathcal{O}$ -module structures we know that

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}))}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}_U))}(\mathcal{M}, j^*\mathcal{N})$$

(Recall that  $\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}))$  denotes the graded category of graded  $\mathcal{O}$ -modules.) Then one has to check that these identifications map the  $\mathcal{A}$ -module maps on the left hand side to the  $\mathcal{A}_U$ -module maps on the right hand side. To check this, given  $\mathcal{O}_U$ -linear maps  $f^n : \mathcal{M}^n \rightarrow j^*\mathcal{N}^{n+d}$  corresponding to  $\mathcal{O}$ -linear maps  $g^n : j_!\mathcal{M}^n \rightarrow \mathcal{N}^{n+d}$  it suffices to show that

$$\begin{array}{ccc} \mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m & \xrightarrow{f^n \otimes 1} & j^*\mathcal{N}^{n+d} \otimes_{\mathcal{O}_U} \mathcal{A}_U^m \\ \downarrow & & \downarrow \\ \mathcal{M}^{n+m} & \xrightarrow{f^{n+m}} & j^*\mathcal{N}^{n+m+d} \end{array}$$

commutes if and only if

$$\begin{array}{ccc} j_!\mathcal{M}^n \otimes_{\mathcal{O}} \mathcal{A}^m & \xrightarrow{g^n \otimes 1} & \mathcal{N}^{n+d} \otimes_{\mathcal{O}} \mathcal{A}_U^m \\ \downarrow & & \downarrow \\ j_!\mathcal{M}^{n+m} & \xrightarrow{g^{n+m}} & \mathcal{N}^{n+m+d} \end{array}$$

commutes. However, we know that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m, j^*\mathcal{N}^{n+d+m}) &= \mathrm{Hom}_{\mathcal{O}}(j_!(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m), \mathcal{N}^{n+d+m}) \\ &= \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}^n \otimes_{\mathcal{O}} \mathcal{A}^m, \mathcal{N}^{n+d+m}) \end{aligned}$$

by the already used Modules on Sites, Lemma 27.9. We omit the verification that shows that the obstruction to the commutativity of the first diagram in the first group maps to the obstruction to the commutativity of the second diagram in the last group.  $\square$

0FRB **Lemma 10.2.** *In the situation above, let  $\mathcal{M}$  be a right graded  $\mathcal{A}_U$ -module and let  $\mathcal{N}$  be a left graded  $\mathcal{A}$ -module. Then*

$$j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

as graded  $\mathcal{O}$ -modules functorially in  $\mathcal{M}$  and  $\mathcal{N}$ .

**Proof.** Recall that the degree  $n$  component of  $j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  is the cokernel of the canonical map

$$\bigoplus_{r+s+t=n} j_!\mathcal{M}^r \otimes_{\mathcal{O}} \mathcal{A}^s \otimes_{\mathcal{O}} \mathcal{N}^t \longrightarrow \bigoplus_{p+q=n} j_!\mathcal{M}^p \otimes_{\mathcal{O}} \mathcal{N}^q$$

See Section 6. By Modules on Sites, Lemma 27.9 this is the same thing as the cokernel of

$$\bigoplus_{r+s+t=n} j_!(\mathcal{M}^r \otimes_{\mathcal{O}_U} \mathcal{A}^s|_U \otimes_{\mathcal{O}_U} \mathcal{N}^t|_U) \longrightarrow \bigoplus_{p+q=n} j_!(\mathcal{M}^p \otimes_{\mathcal{O}_U} \mathcal{N}^q|_U)$$

and we win. An alternative proof would be to redo the Yoneda argument given in the proof of the lemma cited above.  $\square$

### 11. Shift functors on sheaves of graded modules

0FRC We urge the reader to skip this section. It turns out that sheaves of graded modules over a graded algebra are an example of the phenomenon discussed in Differential Graded Algebra, Remark 25.7.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a graded  $\mathcal{A}$ -module. Let  $k \in \mathbf{Z}$ . We define the  $k$ th shift of  $\mathcal{M}$ , denoted  $\mathcal{M}[k]$ , to be the graded  $\mathcal{A}$ -module whose  $n$ th part is given by

$$(\mathcal{M}[k])^n = \mathcal{M}^{n+k}$$

is the  $(n+k)$ th part of  $\mathcal{M}$ . As multiplication maps

$$(\mathcal{M}[k])^n \times \mathcal{A}^m \longrightarrow (\mathcal{M}[k])^{n+m}$$

we simply use the multiplication maps

$$\mathcal{M}^{n+k} \times \mathcal{A}^m \longrightarrow \mathcal{M}^{n+m+k}$$

of  $\mathcal{M}$ . It is clear that we have defined a functor  $[k]$ , that we have  $[k+l] = [k] \circ [l]$ , and that we have

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M})[k]$$

(without the intervention of signs) functorially in  $\mathcal{M}$  and  $\mathcal{L}$ . Thus we see indeed that the graded category of graded  $\mathcal{A}$ -modules can be recovered from the ordinary category of graded  $\mathcal{A}$ -modules and the shift functors as discussed in Differential Graded Algebra, Remark 25.7.

0FRD **Lemma 11.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a graded  $\mathcal{O}$ -algebra. The category  $\mathrm{Mod}(\mathcal{A})$  is a Grothendieck abelian category.*

**Proof.** By Lemma 4.2 and the definition of a Grothendieck abelian category (Injectives, Definition 10.1) it suffices to show that  $\mathrm{Mod}(\mathcal{A})$  has a generator. We claim that

$$\mathcal{G} = \bigoplus_{k,U} j_{U!} \mathcal{A}_U[k]$$

is a generator where the sum is over all objects  $U$  of  $\mathcal{C}$  and  $k \in \mathbf{Z}$ . Indeed, given a graded  $\mathcal{A}$ -module  $\mathcal{M}$  if there are no nonzero maps from  $\mathcal{G}$  to  $\mathcal{M}$ , then we see that for all  $k$  and  $U$  we have

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}(j_{U!} \mathcal{A}_U[k], \mathcal{M}) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A}_U)}(\mathcal{A}_U[k], \mathcal{M}|_U) = \Gamma(U, \mathcal{M}^{-k})$$

is equal to zero. Hence  $\mathcal{M}$  is zero.  $\square$

## 12. Sheaves of differential graded algebras

0FRE This section is the analogue of Differential Graded Algebra, Section 3.

0FRF **Definition 12.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. A *sheaf of differential graded  $\mathcal{O}$ -algebras* or a *sheaf of differential graded algebras* on  $(\mathcal{C}, \mathcal{O})$  is a cochain complex  $\mathcal{A}^\bullet$  of  $\mathcal{O}$ -modules endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \rightarrow \mathcal{A}^{n+m}, \quad (a, b) \longmapsto ab$$

called the multiplication maps with the following properties

- (1) multiplication is associative,
- (2) there is a global section 1 of  $\mathcal{A}^0$  which is a two-sided identity for multiplication,
- (3) for  $U \in \text{Ob}(\mathcal{C})$ ,  $a \in \mathcal{A}^n(U)$ , and  $b \in \mathcal{A}^m(U)$  we have

$$d^{n+m}(ab) = d^n(a)b + (-1)^n ad^m(b)$$

We often denote such a structure  $(\mathcal{A}, d)$ . A *homomorphism of differential graded  $\mathcal{O}$ -algebras* from  $(\mathcal{A}, d)$  to  $(\mathcal{B}, d)$  is a map  $f : \mathcal{A}^\bullet \rightarrow \mathcal{B}^\bullet$  of complexes of  $\mathcal{O}$ -modules compatible with the multiplication maps.

Given a differential graded  $\mathcal{O}$ -algebra  $(\mathcal{A}, d)$  and an object  $U \in \text{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a differential graded  $\mathcal{O}(U)$ -algebra.

As much as possible, we will think of a differential graded  $\mathcal{O}$ -algebra  $(\mathcal{A}, d)$  as a graded  $\mathcal{O}$ -algebra  $\mathcal{A}$  endowed with the operator  $d : \mathcal{A} \rightarrow \mathcal{A}$  of degree 1 (where  $\mathcal{A}$  is viewed as a graded  $\mathcal{O}$ -module) satisfying the Leibniz rule given in the definition.

0FRG **Remark 12.2.** Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi.

- (1) Let  $(\mathcal{A}, d)$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. The pushforward will be the differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra  $(f_*\mathcal{A}, d)$  where  $f_*\mathcal{A}$  is as in Remark 3.2 and  $d = f_*d$  as maps  $f_*\mathcal{A}^n \rightarrow f_*\mathcal{A}^{n+1}$ . We omit the verification that the Leibniz rule is satisfied.
- (2) Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. The pullback will be the differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra  $(f^*\mathcal{B}, d)$  where  $f^*\mathcal{B}$  is as in Remark 3.2 and  $d = f^*d$  as maps  $f^*\mathcal{B}^n \rightarrow f^*\mathcal{B}^{n+1}$ . We omit the verification that the Leibniz rule is satisfied.
- (3) The set of homomorphisms  $f^*\mathcal{B} \rightarrow \mathcal{A}$  of differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebras is in 1-to-1 correspondence with the set of homomorphisms  $\mathcal{B} \rightarrow f_*\mathcal{A}$  of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras.

Part (3) follows immediately from the usual adjunction between  $f^*$  and  $f_*$  on sheaves of modules.

## 13. Sheaves of differential graded modules

0FRH This section is the analogue of Differential Graded Algebra, Section 4.

0FRI **Definition 13.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A (right) *differential graded  $\mathcal{A}$ -module* or (right) *differential graded module* over  $\mathcal{A}$  is a cochain complex  $\mathcal{M}^\bullet$  endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \rightarrow \mathcal{M}^{n+m}, \quad (x, a) \mapsto xa$$

called the multiplication maps with the following properties

- (1) multiplication satisfies  $(xa)a' = x(aa')$ ,
- (2) the identity section 1 of  $\mathcal{A}^0$  acts as the identity on  $\mathcal{M}^n$  for all  $n$ ,
- (3) for  $U \in \text{Ob}(\mathcal{C})$ ,  $x \in \mathcal{M}^n(U)$ , and  $a \in \mathcal{A}^m(U)$  we have

$$d^{n+m}(xa) = d^n(x)a + (-1)^n x d^m(a)$$

We often say “let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module” to indicate this situation. A *homomorphism of differential graded  $\mathcal{A}$ -modules* from  $\mathcal{M}$  to  $\mathcal{N}$  is a map  $f : \mathcal{M}^\bullet \rightarrow \mathcal{N}^\bullet$  of complexes of  $\mathcal{O}$ -modules compatible with the multiplication maps. The category of (right) differential graded  $\mathcal{A}$ -modules is denoted  $\text{Mod}(\mathcal{A}, d)$ .

We can define *left differential graded modules* in exactly the same manner but our default in the chapter will be right modules.

Given a differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  and an object  $U \in \text{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) differential graded  $\mathcal{A}(U)$ -module.

0FRJ **Lemma 13.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a differential graded  $\mathcal{O}$ -algebra. The category  $\text{Mod}(\mathcal{A}, d)$  is an abelian category with the following properties*

- (1)  $\text{Mod}(\mathcal{A}, d)$  has arbitrary direct sums,
- (2)  $\text{Mod}(\mathcal{A}, d)$  has arbitrary colimits,
- (3) filtered colimit in  $\text{Mod}(\mathcal{A}, d)$  are exact,
- (4)  $\text{Mod}(\mathcal{A}, d)$  has arbitrary products,
- (5)  $\text{Mod}(\mathcal{A}, d)$  has arbitrary limits.

The forgetful functor

$$\text{Mod}(\mathcal{A}, d) \longrightarrow \text{Mod}(\mathcal{A})$$

sending a differential graded  $\mathcal{A}$ -module to its underlying graded module commutes with all limits and colimits.

**Proof.** Let us denote  $F : \text{Mod}(\mathcal{A}, d) \rightarrow \text{Mod}(\mathcal{A})$  the functor in the statement of the lemma. Observe that the category  $\text{Mod}(\mathcal{A})$  has properties (1) – (5), see Lemma 4.2.

Consider a homomorphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  of graded  $\mathcal{A}$ -modules. The kernel and cokernel of  $f$  as maps of graded  $\mathcal{A}$ -modules are additionally endowed with differentials as in Definition 13.1. Hence these are also the kernel and cokernel in  $\text{Mod}(\mathcal{A}, d)$ . Thus  $\text{Mod}(\mathcal{A}, d)$  is an abelian category and taking kernels and cokernels commutes with  $F$ .

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 14.11 and 14.12. The same lemmas show that proving the commutation of limits and colimits with  $F$  follows if  $F$  commutes with direct sums and products.

Let  $\mathcal{M}_t, t \in T$  be a set of differential graded  $\mathcal{A}$ -modules. Then we can consider the direct sum  $\bigoplus \mathcal{M}_t$  as a graded  $\mathcal{A}$ -module. Since the direct sum of graded modules is done termwise, it is clear that  $\bigoplus \mathcal{M}_t$  comes endowed with a differential. The reader easily verifies that this is a direct sum in  $Mod(\mathcal{A}, d)$ . Similarly for products.

Observe that  $F$  is an exact functor and that a complex  $\mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3$  of  $Mod(\mathcal{A}, d)$  is exact if and only if  $F(\mathcal{M}_1) \rightarrow F(\mathcal{M}_2) \rightarrow F(\mathcal{M}_3)$  is exact in  $Mod(\mathcal{A})$ . Hence we conclude that (3) holds as filtered colimits are exact in  $Mod(\mathcal{A})$ .  $\square$

Combining Lemmas 13.2 and 4.2 we find that there is an exact and faithful functor

$$Mod(\mathcal{A}, d) \longrightarrow \text{Comp}(\mathcal{O})$$

of abelian categories. For a differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  the cohomology  $\mathcal{O}$ -modules, denoted  $H^i(\mathcal{M})$ , are defined as the cohomology of the complex of  $\mathcal{O}$ -modules corresponding to  $\mathcal{M}$ . Therefore, a short exact sequence  $0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow 0$  of differential graded  $\mathcal{A}$ -modules gives rise to a long exact sequence

$$\text{0FRK} \quad (13.2.1) \quad H^n(\mathcal{K}) \rightarrow H^n(\mathcal{L}) \rightarrow H^n(\mathcal{M}) \rightarrow H^{n+1}(\mathcal{K})$$

of cohomology modules, see Homology, Lemma 13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  is *acyclic* if  $H^k(\mathcal{M}) = 0$  for all  $k \in \mathbf{Z}$ . We say that a homomorphism  $\mathcal{M} \rightarrow \mathcal{N}$  of differential graded  $\mathcal{A}$ -modules is a *quasi-isomorphism* if it induces isomorphisms  $H^k(\mathcal{M}) \rightarrow H^k(\mathcal{N})$  for all  $k \in \mathbf{Z}$ . And so on and so forth.

#### 14. The differential graded category of modules

$\text{0FRL}$  This section is the analogue of Differential Graded Algebra, Example 26.8. For our conventions on differential graded categories, please see Differential Graded Algebra, Section 26.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . We will construct a differential graded category

$$Mod^{dg}(\mathcal{A}, d)$$

over  $R = \Gamma(\mathcal{C}, \mathcal{O})$  whose associated category of complexes is the category of differential graded  $\mathcal{A}$ -modules:

$$Mod(\mathcal{A}, d) = \text{Comp}(Mod^{dg}(\mathcal{A}, d))$$

As objects of  $Mod^{dg}(\mathcal{A}, d)$  we take right differential graded  $\mathcal{A}$ -modules, see Section 13. Given differential graded  $\mathcal{A}$ -modules  $\mathcal{L}$  and  $\mathcal{M}$  we set

$$\text{Hom}_{Mod^{dg}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M}) = \text{Hom}_{Mod^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \text{Hom}^n(\mathcal{L}, \mathcal{M})$$

as a graded  $R$ -module, see Section 5. In other words, the  $n$ th graded piece  $\text{Hom}^n(\mathcal{L}, \mathcal{M})$  is the  $R$ -module of right  $\mathcal{A}$ -module maps homogeneous of degree  $n$ . For an element  $f \in \text{Hom}^n(\mathcal{L}, \mathcal{M})$  we set

$$d(f) = d_{\mathcal{M}} \circ f - (-1)^n f \circ d_{\mathcal{L}}$$

To make sense of this we think of  $d_{\mathcal{M}}$  and  $d_{\mathcal{L}}$  as graded  $\mathcal{O}$ -module maps and we use composition of graded  $\mathcal{O}$ -module maps. It is clear that  $d(f)$  is homogeneous of

degree  $n+1$  as a graded  $\mathcal{O}$ -module map, and it is  $\mathcal{A}$ -linear because for homogeneous local sections  $x$  and  $a$  of  $\mathcal{M}$  and  $\mathcal{A}$  we have

$$\begin{aligned} d(f)(xa) &= d_{\mathcal{M}}(f(x)a) - (-1)^n f(d_{\mathcal{L}}(xa)) \\ &= d_{\mathcal{M}}(f(x))a + (-1)^{\deg(x)+n} f(x)d(a) - (-1)^n f(d_{\mathcal{L}}(x))a - (-1)^{n+\deg(x)} f(x)d(a) \\ &= d(f)(x)a \end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on  $\text{Hom}$ ).

For differential graded  $\mathcal{A}$ -modules  $\mathcal{K}$ ,  $\mathcal{L}$ ,  $\mathcal{M}$  we have already defined the composition

$$\text{Hom}^m(\mathcal{L}, \mathcal{M}) \times \text{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

in Section 5 by the usual composition of maps of sheaves. This defines a map of differential graded modules

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M}) \otimes_R \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}, \mathcal{M})$$

as required in Differential Graded Algebra, Definition 26.1 because

$$\begin{aligned} d(g \circ f) &= d_{\mathcal{M}} \circ g \circ f - (-1)^{n+m} g \circ f \circ d_{\mathcal{K}} \\ &= (d_{\mathcal{M}} \circ g - (-1)^m g \circ d_{\mathcal{L}}) \circ f + (-1)^m g \circ (d_{\mathcal{L}} \circ f - (-1)^n f \circ d_{\mathcal{K}}) \\ &= d(g) \circ f + (-1)^m g \circ d(f) \end{aligned}$$

if  $f$  has degree  $n$  and  $g$  has degree  $m$  as desired.

### 15. Tensor product for sheaves of differential graded modules

0FRM This section is the analogue of part of Differential Graded Algebra, Section 12.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right differential graded  $\mathcal{A}$ -module and let  $\mathcal{N}$  be a left differential graded  $\mathcal{A}$ -module. In this situation we define the *tensor product*  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  as follows. As a graded  $\mathcal{O}$ -module it is given by the construction in Section 6. It comes endowed with a differential

$$d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}} : (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^n \longrightarrow (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^{n+1}$$

defined by the rule that

$$d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}(x \otimes y) = d_{\mathcal{M}}(x) \otimes y + (-1)^{\deg(x)} x \otimes d_{\mathcal{N}}(y)$$

for homogeneous local sections  $x$  and  $y$  of  $\mathcal{M}$  and  $\mathcal{N}$ . To see that this is well defined we have to show that  $d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$  annihilates elements of the form  $xa \otimes y - x \otimes ay$  for homogeneous local sections  $x, a, y$  of  $\mathcal{M}, \mathcal{A}, \mathcal{N}$ . We compute

$$\begin{aligned} &d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}(xa \otimes y - x \otimes ay) \\ &= d_{\mathcal{M}}(xa) \otimes y + (-1)^{\deg(x)+\deg(a)} xa \otimes d_{\mathcal{N}}(y) - d_{\mathcal{M}}(x) \otimes ay - (-1)^{\deg(x)} x \otimes d_{\mathcal{N}}(ay) \\ &= d_{\mathcal{M}}(x)a \otimes y + (-1)^{\deg(x)} xd(a) \otimes y + (-1)^{\deg(x)+\deg(a)} xa \otimes d_{\mathcal{N}}(y) \\ &\quad - d_{\mathcal{M}}(x) \otimes ay - (-1)^{\deg(x)} x \otimes d(a)y - (-1)^{\deg(x)+\deg(a)} x \otimes ad_{\mathcal{N}}(y) \end{aligned}$$

then we observe that the elements

$$d_{\mathcal{M}}(x)a \otimes y - d_{\mathcal{M}}(x) \otimes ay, \quad xd(a) \otimes y - x \otimes d(a)y, \quad \text{and} \quad xa \otimes d_{\mathcal{N}}(y) - x \otimes ad_{\mathcal{N}}(y)$$

map to zero in  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  and we conclude. We omit the verification that  $d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}} \circ d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}} = 0$ .

If we fix the left differential graded  $\mathcal{A}$ -module  $\mathcal{N}$  we obtain a functor

$$- \otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}(\mathcal{A}, d) \longrightarrow \text{Comp}(\mathcal{O})$$

where on the right hand side we have the category of complexes of  $\mathcal{O}$ -modules. This can be upgraded to a functor of differential graded categories

$$- \otimes_{\mathcal{A}} \mathcal{N} : \text{Mod}^{dg}(\mathcal{A}, d) \longrightarrow \text{Comp}^{dg}(\mathcal{O})$$

On underlying graded objects, we send a homomorphism  $f : \mathcal{M} \rightarrow \mathcal{M}'$  of degree  $n$  to the degree  $n$  map  $f \otimes \text{id}_{\mathcal{N}} : \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$ , because this is what we did in Section 6. To show that this works, we have to verify that the map

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{M}, \mathcal{M}') \longrightarrow \text{Hom}_{\text{Comp}^{dg}(\mathcal{O})}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N})$$

is compatible with differentials. To see this for  $f$  as above we have to show that

$$(d_{\mathcal{M}'} \circ f - (-1)^n f \circ d_{\mathcal{M}}) \otimes \text{id}_{\mathcal{N}}$$

is equal to

$$d_{\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}} \circ (f \otimes \text{id}_{\mathcal{N}}) - (-1)^n (f \otimes \text{id}_{\mathcal{N}}) \circ d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$

Let us compute the effect of these operators on a local section of the form  $x \otimes y$  with  $x$  and  $y$  homogeneous local sections of  $\mathcal{M}$  and  $\mathcal{N}$ . For the first we obtain

$$(d_{\mathcal{M}'}(f(x)) - (-1)^n f(d_{\mathcal{M}}(x))) \otimes y$$

and for the second we obtain

$$\begin{aligned} & d_{\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}}(f(x) \otimes y) - (-1)^n (f \otimes \text{id}_{\mathcal{N}})(d_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}(x \otimes y)) \\ &= d_{\mathcal{M}'}(f(x)) \otimes y + (-1)^{\deg(x)+n} f(x) \otimes d_{\mathcal{N}}(y) \\ &\quad - (-1)^n f(d_{\mathcal{M}}(x)) \otimes y - (-1)^n (-1)^{\deg(x)} f(x) \otimes d_{\mathcal{N}}(y) \end{aligned}$$

which is indeed the same local section.

## 16. Internal hom for sheaves of differential graded modules

0FRN We are going to need the sheafified version of the construction in Section 14. Let  $(\mathcal{C}, \mathcal{O})$ ,  $\mathcal{A}$ ,  $\mathcal{M}$ ,  $\mathcal{L}$  be as in Section 14. Then we define

$$\text{Hom}_{\mathcal{A}}^{dg}(\mathcal{M}, \mathcal{L}) = \text{Hom}_{\mathcal{A}}^{gr}(\mathcal{M}, \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}^n(\mathcal{M}, \mathcal{L})$$

as a graded  $\mathcal{O}$ -module, see Section 7. In other words, a section  $f$  of the  $n$ th graded piece  $\text{Hom}_{\mathcal{A}}^n(\mathcal{L}, \mathcal{M})$  over  $U$  is a map of right  $\mathcal{A}_U$ -module map  $\mathcal{L}|_U \rightarrow \mathcal{M}|_U$  homogeneous of degree  $n$ . For such  $f$  we set

$$d(f) = d_{\mathcal{M}}|_U \circ f - (-1)^n f \circ d_{\mathcal{L}}|_U$$

To make sense of this we think of  $d_{\mathcal{M}}|_U$  and  $d_{\mathcal{L}}|_U$  as graded  $\mathcal{O}_U$ -module maps and we use composition of graded  $\mathcal{O}_U$ -module maps. It is clear that  $d(f)$  is homogeneous of degree  $n+1$  as a graded  $\mathcal{O}_U$ -module map. Using the exact same computation as in Section 14 we see that  $d(f)$  is  $\mathcal{A}_U$ -linear.

As in Section 14 there is a composition map

$$\text{Hom}_{\mathcal{A}}^{dg}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \text{Hom}_{\mathcal{A}}^{dg}(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}_{\mathcal{A}}^{dg}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of differential graded  $\mathcal{O}$ -modules defined in Section 15. This map is given by the composition map

$$\text{Hom}^m(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \text{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally). Using the exact same computation as in Section 14 on local sections we see that the composition map is a morphism of differential graded  $\mathcal{O}$ -modules.

With these definitions we have

$$\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{dg}}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om_{\mathcal{A}}^{\mathrm{dg}}(\mathcal{L}, \mathcal{M}))$$

as graded  $R$ -modules compatible with composition.

### 17. Sheaves of differential graded bimodules and tensor-hom adjunction

0FRP This section is the analogue of part of Differential Graded Algebra, Section 12.

0FRQ **Definition 17.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A *differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule* is given by a complex  $\mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules endowed with  $\mathcal{O}$ -bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \rightarrow \mathcal{M}^{n+m}, \quad (x, b) \mapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \rightarrow \mathcal{M}^{n+m}, \quad (a, x) \mapsto ax$$

called the multiplication maps with the following properties

- (1) multiplication satisfies  $a(a'x) = (aa')x$  and  $(xb)b' = x(bb')$ ,
- (2)  $(ax)b = a(xb)$ ,
- (3)  $d(ax) = d(a)x + (-1)^{\mathrm{deg}(a)}ad(x)$  and  $d(xb) = d(x)b + (-1)^{\mathrm{deg}(x)}xd(b)$ ,
- (4) the identity section 1 of  $\mathcal{A}^0$  acts as the identity by multiplication, and
- (5) the identity section 1 of  $\mathcal{B}^0$  acts as the identity by multiplication.

We often denote such a structure  $\mathcal{M}$  and sometimes we write  ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ . A *homomorphism of differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodules*  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a map of complexes  $f : \mathcal{M}^\bullet \rightarrow \mathcal{N}^\bullet$  of  $\mathcal{O}$ -modules compatible with the multiplication maps.

Given a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule  $\mathcal{M}$  and an object  $U \in \mathrm{Ob}(\mathcal{C})$  we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a differential graded  $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodule.

Observe that a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule  $\mathcal{M}$  is the same thing as a right differential graded  $\mathcal{B}$ -module which is also a left differential graded  $\mathcal{A}$ -module such that the grading and differentials agree and such that the  $\mathcal{A}$ -module structure commutes with the  $\mathcal{B}$ -module structure. Here is a precise statement.

0FRR **Lemma 17.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{N}$  be a right differential graded  $\mathcal{B}$ -module. There is a 1-to-1 correspondence between  $(\mathcal{A}, \mathcal{B})$ -bimodule structures on  $\mathcal{N}$  compatible with the given differential graded  $\mathcal{B}$ -module structure and homomorphisms*

$$\mathcal{A} \longrightarrow \mathcal{H}om_{\mathcal{B}}^{\mathrm{dg}}(\mathcal{N}, \mathcal{N})$$

*of differential graded  $\mathcal{O}$ -algebras.*

**Proof.** Omitted. □



Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right differential graded  $\mathcal{A}$ -module and let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. In this case the differential graded tensor product defined in Section 15

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

is a right differential graded  $\mathcal{B}$ -module with multiplication maps as in Section 8. This construction defines a functor and a functor of graded categories

$$\otimes_{\mathcal{A}} \mathcal{N} : Mod(\mathcal{A}, d) \longrightarrow Mod(\mathcal{B}, d) \quad \text{and} \quad \otimes_{\mathcal{A}} \mathcal{N} : Mod^{dg}(\mathcal{A}, d) \longrightarrow Mod^{dg}(\mathcal{B}, d)$$

by sending homomorphisms of degree  $n$  from  $\mathcal{M} \rightarrow \mathcal{M}'$  to the induced map of degree  $n$  from  $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  to  $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$ .

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Let  $\mathcal{L}$  be a right differential graded  $\mathcal{B}$ -module. In this case the differential graded internal hom defined in Section 16

$$\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L})$$

is a right differential graded  $\mathcal{A}$ -module where the right graded  $\mathcal{A}$ -module structure is the one defined in Section 8. Another way to define the multiplication is the use the composition

$$\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L}) \otimes_{\mathcal{O}} \mathcal{A} \rightarrow \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L}) \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{N}) \rightarrow \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L})$$

where the first arrow comes from Lemma 17.2 and the second arrow is the composition of Section 16. Since these arrows are both compatible with differentials, we conclude that we indeed obtain a differential graded  $\mathcal{A}$ -module. This construction defines a functor and a functor of differential graded categories

$$\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, -) : Mod(\mathcal{B}, d) \longrightarrow Mod(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, -) : Mod^{dg}(\mathcal{B}, d) \longrightarrow Mod^{dg}(\mathcal{A}, d)$$

by sending homomorphisms of degree  $n$  from  $\mathcal{L} \rightarrow \mathcal{L}'$  to the induced map of degree  $n$  from  $\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L})$  to  $\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L}')$ .

**OFRS Lemma 17.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a right differential graded  $\mathcal{A}$ -module. Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Let  $\mathcal{L}$  be a right differential graded  $\mathcal{B}$ -module. With conventions as above we have*

$$\text{Hom}_{Mod^{dg}(\mathcal{B}, d)}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \text{Hom}_{Mod^{dg}(\mathcal{A}, d)}(\mathcal{M}, \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om_{\mathcal{A}}^{dg}(\mathcal{M}, \mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, \mathcal{L}))$$

functorially in  $\mathcal{M}, \mathcal{N}, \mathcal{L}$ .

**Proof.** Omitted. Hint: On the graded level we have seen this is true in Lemma 8.2. Thus it suffices to check the isomorphisms are compatible with differentials which can be done by a computation on the level of local sections.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  and  $\mathcal{B}$  be a sheaves of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . As a special case of the above, suppose we are given a homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  of differential graded  $\mathcal{O}$ -algebras. Then we obtain a functor and a functor of differential graded categories

$$\otimes_{\mathcal{A}, \varphi} \mathcal{B} : Mod(\mathcal{A}, d) \longrightarrow Mod(\mathcal{B}, d) \quad \text{and} \quad \otimes_{\mathcal{A}, \varphi} \mathcal{B} : Mod^{dg}(\mathcal{A}, d) \longrightarrow Mod^{dg}(\mathcal{B}, d)$$

On the other hand, we have the restriction functors

$$res_\varphi : Mod(\mathcal{B}, d) \longrightarrow Mod(\mathcal{A}, d) \quad \text{and} \quad res_\varphi : Mod^{dg}(\mathcal{B}, d) \longrightarrow Mod^{dg}(\mathcal{A}, d)$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write  ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$  for  $\mathcal{B}$  viewed as a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Then for any right differential graded  $\mathcal{B}$ -module  $\mathcal{L}$  we have

$$\mathcal{H}om_{\mathcal{B}}^{dg}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = res_\varphi(\mathcal{L})$$

as right differential graded  $\mathcal{A}$ -modules. Thus Lemma 8.2 tells us that we have a functorial isomorphism

$$\text{Hom}_{Mod^{dg}(\mathcal{B}, d)}(\mathcal{M} \otimes_{\mathcal{A}, \varphi} \mathcal{B}, \mathcal{L}) = \text{Hom}_{Mod^{dg}(\mathcal{A}, d)}(\mathcal{M}, res_\varphi(\mathcal{L}))$$

We usually drop the dependence on  $\varphi$  in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om_{\mathcal{A}}^{dg}(\mathcal{M}, \mathcal{L})$$

of graded  $\mathcal{O}$ -modules.

### 18. Pull and push for sheaves of differential graded modules

OFRT Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \rightarrow \mathcal{A}$$

of differential graded  $f^{-1}\mathcal{O}_{\mathcal{D}}$ -algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map  $\varphi : f^*\mathcal{B} \rightarrow \mathcal{A}$  of differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebras or equivalently  $\varphi$  can be viewed as a map

$$\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$$

of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. See Remark 12.2.

Let us define a functor

$$f_* : Mod(\mathcal{A}, d) \longrightarrow Mod(\mathcal{B}, d)$$

Given a differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  we define  $f_*\mathcal{M}$  to be the graded  $\mathcal{B}$ -module constructed in Section 9 with differential given by the maps  $f_*d : f_*\mathcal{M}^n \rightarrow f_*\mathcal{M}^{n+1}$ . The construction is clearly functorial in  $\mathcal{M}$  and we obtain our functor.

Let us define a functor

$$f^* : Mod(\mathcal{B}, d) \longrightarrow Mod(\mathcal{A}, d)$$

Given a differential graded  $\mathcal{B}$ -module  $\mathcal{N}$  we define  $f^*\mathcal{N}$  to be the graded  $\mathcal{A}$ -module constructed in Section 9. Recall that

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B}} \mathcal{A}$$

Since  $f^{-1}\mathcal{N}$  comes with the differentials  $f^{-1}d : f^{-1}\mathcal{N}^n \rightarrow f^{-1}\mathcal{N}^{n+1}$  we can view this tensor product as an example of the tensor product discussed in Section 17 which provides us with a differential. The construction is clearly functorial in  $\mathcal{N}$  and we obtain our functor  $f^*$ .

The functors  $f_*$  and  $f^*$  are readily enhanced to give functors of differential graded categories

$$f_* : \text{Mod}^{dg}(\mathcal{A}, d) \longrightarrow \text{Mod}^{dg}(\mathcal{B}, d) \quad \text{and} \quad f^* : \text{Mod}^{dg}(\mathcal{B}, d) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, d)$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogeneous morphisms of degree  $n$ .

0FRU **Lemma 18.1.** *In the situation above we have*

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{B}, d)}(\mathcal{N}, f_*\mathcal{M}) = \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(f^*\mathcal{N}, \mathcal{M})$$

**Proof.** Omitted. Hints: This is true for the underlying graded categories by Lemma 9.1. A calculation shows that these isomorphisms are compatible with differentials.  $\square$

### 19. Localization and sheaves of differential graded modules

0FRV Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$  and denote

$$j : (\text{Sh}(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (\text{Sh}(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 19). Below we will use the following fact: for  $\mathcal{O}_U$ -modules  $\mathcal{M}_i$ ,  $i = 1, 2$  and a  $\mathcal{O}$ -module  $\mathcal{A}$  there is a canonical map

$$j_! : \text{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \text{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have  $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$  by Modules on Sites, Lemma 27.9.

Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}$ -algebra. We will denote  $\mathcal{A}_U$  the restriction of  $\mathcal{A}$  to  $\mathcal{C}/U$ , in other words, we have  $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$ . In Section 18 we have constructed adjoint functors

$$j_* : \text{Mod}^{dg}(\mathcal{A}_U, d) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, d) \quad \text{and} \quad j^* : \text{Mod}^{dg}(\mathcal{A}, d) \longrightarrow \text{Mod}^{dg}(\mathcal{A}_U, d)$$

with  $j^*$  left adjoint to  $j_*$ . We claim there is in addition an exact functor

$$j_! : \text{Mod}^{dg}(\mathcal{A}_U, d) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, d)$$

right adjoint to  $j_*$ . Namely, given a differential graded  $\mathcal{A}_U$ -module  $\mathcal{M}$  we define  $j_!\mathcal{M}$  to be the graded  $\mathcal{A}$ -module constructed in Section 10 with differentials  $j_!d : j_!\mathcal{M}^n \rightarrow j_!\mathcal{M}^{n+1}$ . Given a homogeneous map  $f : \mathcal{M} \rightarrow \mathcal{M}'$  of degree  $n$  of differential graded  $\mathcal{A}_U$ -modules, we obtain a homogeneous map  $j_!f : j_!\mathcal{M} \rightarrow j_!\mathcal{M}'$  of degree  $n$  of differential graded  $\mathcal{A}$ -modules. We omit the straightforward verification that this construction is compatible with differentials. Thus we obtain our functor.

0FRW **Lemma 19.1.** *In the situation above we have*

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(j_!\mathcal{M}, \mathcal{N}) = \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}_U, d)}(\mathcal{M}, j^*\mathcal{N})$$

**Proof.** Omitted. Hint: We have seen in Lemma 10.1 that the lemma is true on graded level. Thus all that needs to be checked is that the resulting isomorphism is compatible with differentials.  $\square$

0FRX **Lemma 19.2.** *In the situation above, let  $\mathcal{M}$  be a right differential graded  $\mathcal{A}_U$ -module and let  $\mathcal{N}$  be a left differential graded  $\mathcal{A}$ -module. Then*

$$j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

*as complexes of  $\mathcal{O}$ -modules functorially in  $\mathcal{M}$  and  $\mathcal{N}$ .*

**Proof.** As graded modules, this follows from Lemma 10.2. We omit the verification that this isomorphism is compatible with differentials.  $\square$

## 20. Shift functors on sheaves of differential graded modules

0FRY Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module. Let  $k \in \mathbf{Z}$ . We define the  $k$ th shift of  $\mathcal{M}$ , denoted  $\mathcal{M}[k]$ , as follows

- (1) as a graded  $\mathcal{A}$ -module we let  $\mathcal{M}[k]$  be as defined in Section 11,
- (2) the differential  $d_{\mathcal{M}[k]} : (\mathcal{M}[k])^n \rightarrow (\mathcal{M}[k])^{n+1}$  is defined to be  $(-1)^k d_{\mathcal{M}} : \mathcal{M}^{n+k} \rightarrow \mathcal{M}^{n+k+1}$ .

For a homomorphism  $f : \mathcal{L} \rightarrow \mathcal{M}$  of  $\mathcal{A}$ -modules homogeneous of degree  $n$ , we let  $f[k] : \mathcal{L}[k] \rightarrow \mathcal{M}[k]$  be given by the same component maps as  $f$ . Then  $f[k]$  is a homogeneous  $\mathcal{A}$ -module map of degree  $n$ . This gives a map

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M}) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{L}[k], \mathcal{M}[k])$$

compatible with differentials (it follows from the fact that the signs of the differentials of  $\mathcal{L}$  and  $\mathcal{M}$  are changed by the same amount). These choices are compatible with the choice in Differential Graded Algebra, Definition 4.3. It is clear that we have defined a functor

$$[k] : \mathrm{Mod}^{dg}(\mathcal{A}, d) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{A}, d)$$

of differential graded categories and that we have  $[k+l] = [k] \circ [l]$ .

We claim that the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M})[k]$$

defined in Section 11 on underlying graded modules is compatible with the differentials. To see this, suppose we have a right  $\mathcal{A}$ -module map  $f : \mathcal{L} \rightarrow \mathcal{M}[k]$  homogeneous of degree  $n$ ; this is an element of degree  $n$  of the LHS. Denote  $f' : \mathcal{L} \rightarrow \mathcal{M}$  the homogeneous  $\mathcal{A}$ -module map of degree  $n+k$  with the **same** component maps as  $f$ . By our conventions, this is the corresponding element of degree  $n$  of the RHS. By definition of the differential of LHS we obtain

$$d_{LHS}(f) = d_{\mathcal{M}[k]} \circ f - (-1)^n f \circ d_{\mathcal{L}} = (-1)^k d_{\mathcal{M}} \circ f - (-1)^n f \circ d_{\mathcal{L}}$$

and for the differential on the RHS we obtain

$$d_{RHS}(f') = (-1)^k (d_{\mathcal{M}} \circ f' - (-1)^{n+k} f' \circ d_{\mathcal{L}}) = (-1)^k d_{\mathcal{M}} \circ f' - (-1)^n f' \circ d_{\mathcal{L}}$$

These maps have the same component maps and the proof is complete.

## 21. The homotopy category

0FRZ This section is the analogue of Differential Graded Algebra, Section 5.

0FS0 **Definition 21.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $f, g : \mathcal{M} \rightarrow \mathcal{N}$  be homomorphisms of differential graded  $\mathcal{A}$ -modules. A *homotopy between  $f$  and  $g$*  is a graded  $\mathcal{A}$ -module map  $h : \mathcal{M} \rightarrow \mathcal{N}$  homogeneous of degree  $-1$  such that

$$f - g = d_{\mathcal{N}} \circ h + h \circ d_{\mathcal{M}}$$

If a homotopy exists, then we say  $f$  and  $g$  are *homotopic*.

In the situation of the definition, if we have maps  $a : \mathcal{K} \rightarrow \mathcal{M}$  and  $c : \mathcal{N} \rightarrow \mathcal{L}$  then we see that

$$\begin{array}{ccc} h \text{ is a homotopy} & \Rightarrow & c \circ h \circ a \text{ is a homotopy} \\ \text{between } f \text{ and } g & & \text{between } c \circ f \circ a \text{ and } c \circ g \circ a \end{array}$$

Thus we can define composition of homotopy classes of morphisms in  $Mod(\mathcal{A}, d)$ .

0FS1 **Definition 21.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The *homotopy category*, denoted  $K(Mod(\mathcal{A}, d))$ , is the category whose objects are the objects of  $Mod(\mathcal{A}, d)$  and whose morphisms are homotopy classes of homomorphisms of differential graded  $\mathcal{A}$ -modules.

The notation  $K(Mod(\mathcal{A}, d))$  is not standard but at least is consistent with the use of  $K(-)$  in other places of the Stacks project.

In Differential Graded Algebra, Definition 26.3 we have defined what we mean by the category of complexes  $Comp(\mathcal{S})$  and the homotopy category  $K(\mathcal{S})$  of a differential graded category  $\mathcal{S}$ . Applying this to the differential graded category  $Mod^{dg}(\mathcal{A}, d)$  we obtain

$$Mod(\mathcal{A}, d) = Comp(Mod^{dg}(\mathcal{A}, d))$$

(see discussion in Section 14) and we obtain

$$K(Mod(\mathcal{A}, d)) = K(Mod^{dg}(\mathcal{A}, d))$$

To see that this last equality is true, note that we have the equality

$$d_{\text{Hom}_{Mod^{dg}(\mathcal{A}, d)}(\mathcal{M}, \mathcal{N})}(h) = d_{\mathcal{N}} \circ h + h \circ d_{\mathcal{M}}$$

when  $h$  is as in Definition 21.1. We omit the details.

0FS2 **Lemma 21.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The homotopy category  $K(Mod(\mathcal{A}, d))$  has direct sums and products.*

**Proof.** Omitted. Hint: Just use the direct sums and products as in Lemma 13.2. This works because we saw that these functors commute with the forgetful functor to the category of graded  $\mathcal{A}$ -modules and because  $\prod$  and  $\bigoplus$  are exact functors on the category of families of abelian groups.  $\square$

## 22. Cones and triangles

0FS3 In this section we use the material from Differential Graded Algebra, Section 27 to conclude that the homotopy category of the category of differential graded  $\mathcal{A}$ -modules is a triangulated category.

0FS4 **Lemma 22.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The differential graded category  $Mod^{dg}(\mathcal{A}, d)$  satisfies axioms (A) and (B) of Differential Graded Algebra, Section 27.*

**Proof.** Suppose given differential graded  $\mathcal{A}$ -modules  $\mathcal{M}$  and  $\mathcal{N}$ . Consider the differential graded  $\mathcal{A}$ -module  $\mathcal{M} \oplus \mathcal{N}$  defined in the obvious manner. Then the coprojections  $i : \mathcal{M} \rightarrow \mathcal{M} \oplus \mathcal{N}$  and  $j : \mathcal{N} \rightarrow \mathcal{M} \oplus \mathcal{N}$  and the projections  $p : \mathcal{M} \oplus \mathcal{N} \rightarrow \mathcal{N}$  and  $q : \mathcal{M} \oplus \mathcal{N} \rightarrow \mathcal{M}$  are morphisms of differential graded  $\mathcal{A}$ -modules. Hence  $i, j, p, q$  are homogeneous of degree 0 and closed, i.e.,  $d(i) = 0$ , etc.

Thus this direct sum is a differential graded sum in the sense of Differential Graded Algebra, Definition 26.4. This proves axiom (A).

Axiom (B) was shown in Section 20.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Recall that a sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow 0$$

in  $\text{Mod}(\mathcal{A}, d)$  is called an *admissible short exact sequence* (in Differential Graded Algebra, Section 27) if it is split in  $\text{Mod}(\mathcal{A})$ . In other words, if it is split as a sequence of graded  $\mathcal{A}$ -modules. Denote  $s : \mathcal{N} \rightarrow \mathcal{L}$  and  $\pi : \mathcal{L} \rightarrow \mathcal{K}$  graded  $\mathcal{A}$ -module splittings. Combining Lemma 22.1 and Differential Graded Algebra, Lemma 27.1 we obtain a triangle

$$\mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow \mathcal{K}[1]$$

where the arrow  $\mathcal{N} \rightarrow \mathcal{K}[1]$  in the proof of Differential Graded Algebra, Lemma 27.1 is constructed as

$$\delta = \pi \circ d_{\text{Hom}_{\text{Mod}^{\text{dg}}(\mathcal{A}, d)}(\mathcal{L}, \mathcal{M})}(s) = \pi \circ d_{\mathcal{L}} \circ s - \pi \circ s \circ d_{\mathcal{N}} = \pi \circ d_{\mathcal{L}} \circ s$$

with apologies for the horrendous notation. In any case, we see that in our setting the boundary map  $\delta$  as constructed in Differential Graded Algebra, Lemma 27.1 agrees on underlying complexes of  $\mathcal{O}$ -modules with the usual boundary map used throughout the Stacks project for termwise split short exact sequences of complexes, see Derived Categories, Definition 9.9.

0FS5 **Definition 22.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $f : \mathcal{K} \rightarrow \mathcal{L}$  be a homomorphism of differential graded  $\mathcal{A}$ -modules. The *cone* of  $f$  is the differential graded  $\mathcal{A}$ -module  $C(f)$  defined as follows:

- (1) the underlying complex of  $\mathcal{O}$ -modules is the cone of the corresponding map  $f : \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  of complexes of  $\mathcal{A}$ -modules, i.e., we have  $C(f)^n = \mathcal{L}^n \oplus \mathcal{K}^{n+1}$  and differential

$$d_{C(f)} = \begin{pmatrix} d_{\mathcal{L}} & f \\ 0 & -d_{\mathcal{K}} \end{pmatrix}$$

- (2) the multiplication map

$$C(f)^n \times \mathcal{A}^m \rightarrow C(f)^{n+m}$$

is the direct sum of the multiplication map  $\mathcal{L}^n \times \mathcal{A}^m \rightarrow \mathcal{L}^{n+m}$  and the multiplication map  $\mathcal{K}^{n+1} \times \mathcal{A}^m \rightarrow \mathcal{K}^{n+1+m}$ .

It comes equipped with canonical homomorphisms of differential graded  $\mathcal{A}$ -modules  $i : \mathcal{L} \rightarrow C(f)$  and  $p : C(f) \rightarrow \mathcal{K}[1]$  induced by the obvious maps.

Observe that in the situation of the definition the sequence

$$0 \rightarrow \mathcal{L} \rightarrow C(f) \rightarrow \mathcal{K}[1] \rightarrow 0$$

is an admissible short exact sequence.

0FS6 **Lemma 22.3.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The differential graded category  $\text{Mod}^{\text{dg}}(\mathcal{A}, d)$  satisfies axiom (C) formulated in Differential Graded Algebra, Situation 27.2.

**Proof.** Let  $f : \mathcal{K} \rightarrow \mathcal{L}$  be a homomorphism of differential graded  $\mathcal{A}$ -modules. By the above we have an admissible short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow C(f) \rightarrow \mathcal{K}[1] \rightarrow 0$$

To finish the proof we have to show that the boundary map

$$\delta : \mathcal{K}[1] \rightarrow \mathcal{L}[1]$$

associated to this (see discussion above) is equal to  $f[1]$ . For the section  $s : \mathcal{K}[1] \rightarrow C(f)$  we use in degree  $n$  the embedding  $\mathcal{K}^{n+1} \rightarrow C(f)^n$ . Then in degree  $n$  the map  $\pi$  is given by the projections  $C(f)^n \rightarrow \mathcal{L}^n$ . Then finally we have to compute

$$\delta = \pi \circ d_{C(f)} \circ s$$

(see discussion above). In matrix notation this is equal to

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} d_{\mathcal{L}} & f \\ 0 & -d_{\mathcal{K}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f$$

as desired.  $\square$

At this point we know that all lemmas proved in Differential Graded Algebra, Section 27 are valid for the differential graded category  $Mod^{dg}(\mathcal{A}, d)$ . In particular, we have the following.

0FS7 **Proposition 22.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The homotopy category  $K(Mod(\mathcal{A}, d))$  is a triangulated category where*

- (1) *the shift functors are those constructed in Section 20,*
- (2) *the distinguished triangles are those triangles in  $K(Mod(\mathcal{A}, d))$  which are isomorphic as a triangle to a triangle*

$$\mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \xrightarrow{\delta} \mathcal{K}[1], \quad \delta = \pi \circ d_{\mathcal{L}} \circ s$$

*constructed from an admissible short exact sequence  $0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow 0$  in  $Mod(\mathcal{A}, d)$  above.*

**Proof.** Recall that  $K(Mod(\mathcal{A}, d)) = K(Mod^{dg}(\mathcal{A}, d))$ , see Section 21. Having said this, the proposition follows from Lemmas 22.1 and 22.3 and Differential Graded Algebra, Proposition 27.16.  $\square$

0FS8 **Remark 22.5.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $C = C(\text{id}_{\mathcal{A}})$  be the cone on the identity map  $\mathcal{A} \rightarrow \mathcal{A}$  viewed as a map of differential graded  $\mathcal{A}$ -modules. Then

$$\text{Hom}_{Mod(\mathcal{A}, d)}(C, \mathcal{M}) = \{(x, y) \in \Gamma(\mathcal{C}, \mathcal{M}^0) \times \Gamma(\mathcal{C}, \mathcal{M}^{-1}) \mid x = d(y)\}$$

where the map from left to right sends  $f$  to the pair  $(x, y)$  where  $x$  is the image of the global section  $(0, 1)$  of  $C^{-1} = \mathcal{A}^{-1} \oplus \mathcal{A}^0$  and where  $y$  is the image of the global section  $(1, 0)$  of  $C^0 = \mathcal{A}^0 \oplus \mathcal{A}^1$ .

0FS9 **Lemma 22.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a differential graded  $\mathcal{O}$ -algebra. The category  $Mod(\mathcal{A}, d)$  is a Grothendieck abelian category.*

**Proof.** By Lemma 13.2 and the definition of a Grothendieck abelian category (Injectives, Definition 10.1) it suffices to show that  $Mod(\mathcal{A}, d)$  has a generator. For every object  $U$  of  $\mathcal{C}$  we denote  $C_U$  the cone on the identity map  $\mathcal{A}_U \rightarrow \mathcal{A}_U$  as in Remark 22.5. We claim that

$$\mathcal{G} = \bigoplus_{k,U} j_{U!} C_U[k]$$

is a generator where the sum is over all objects  $U$  of  $\mathcal{C}$  and  $k \in \mathbf{Z}$ . Indeed, given a differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  if there are no nonzero maps from  $\mathcal{G}$  to  $\mathcal{M}$ , then we see that for all  $k$  and  $U$  we have

$$\begin{aligned} \text{Hom}_{Mod(\mathcal{A})}(j_{U!} C_U[k], \mathcal{M}) &= \text{Hom}_{Mod(\mathcal{A}_U)}(C_U[k], \mathcal{M}|_U) \\ &= \{(x, y) \in \mathcal{M}^{-k}(U) \times \mathcal{M}^{-k-1}(U) \mid x = d(y)\} \end{aligned}$$

is equal to zero. Hence  $\mathcal{M}$  is zero.  $\square$

### 23. Flat resolutions

0FSA This section is the analogue of Differential Graded Algebra, Section 20.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let us call a right differential graded  $\mathcal{A}$ -module  $\mathcal{P}$  *good* if

- (1) the functor  $\mathcal{N} \mapsto \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$  is exact on the category of graded left  $\mathcal{A}$ -modules,
- (2) if  $\mathcal{N}$  is an acyclic differential graded left  $\mathcal{A}$ -module, then  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$  is acyclic,
- (3) for any morphism  $(f, f^\#) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi and any differential graded  $\mathcal{O}'$ -algebra  $\mathcal{A}'$  and any map  $\varphi : f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $f^{-1}\mathcal{O}_{\mathcal{D}}$ -algebras we have properties (1) and (2) for the pullback  $f^*\mathcal{P}$  (Section 18) viewed as a differential graded  $\mathcal{A}'$ -module.

The first condition means that  $\mathcal{P}$  is flat as a right graded  $\mathcal{A}$ -module, the second condition means that  $\mathcal{P}$  is K-flat in the sense of Spaltenstein (see Cohomology on Sites, Section 17), and the third condition is that this holds after arbitrary base change.

Perhaps surprisingly, there are many good modules.

0FSB **Lemma 23.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $U \in \text{Ob}(\mathcal{C})$ . Then  $j_{!}\mathcal{A}_U$  is a good differential graded  $\mathcal{A}$ -module.*

**Proof.** Let  $\mathcal{N}$  be a left graded  $\mathcal{A}$ -module. By Lemma 10.2 we have

$$j_{!}\mathcal{A}_U \otimes_{\mathcal{A}} \mathcal{N} = j_{!}(\mathcal{A}_U \otimes_{\mathcal{A}_U} \mathcal{N}|_U) = j_{!}(\mathcal{N}_U)$$

as graded modules. Since both restriction to  $U$  and  $j_{!}$  are exact this proves condition (1). The same argument works for (2) using Lemma 19.2.

Consider a morphism  $(f, f^\#) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi, a differential graded  $\mathcal{O}'$ -algebra  $\mathcal{A}'$ , and a map  $\varphi : f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $f^{-1}\mathcal{O}$ -algebras. We have to show that

$$f^*j_{!}\mathcal{A}_U = f^{-1}j_{!}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$$

satisfies (1) and (2) for the ringed topoi  $(Sh(\mathcal{C}'), \mathcal{O}')$  endowed with the sheaf of differential graded  $\mathcal{O}'$ -algebras  $\mathcal{A}'$ . To prove this we may replace  $(Sh(\mathcal{C}), \mathcal{O})$  and  $(Sh(\mathcal{C}'), \mathcal{O}')$  by equivalent ringed topoi. Thus by Modules on Sites, Lemma 7.2



we may assume that  $f$  comes from a morphism of sites  $f : \mathcal{C} \rightarrow \mathcal{C}'$  given by the continuous functor  $u : \mathcal{C} \rightarrow \mathcal{C}'$ . In this case, set  $U' = u(U)$  and denote  $j' : Sh(\mathcal{C}'/U') \rightarrow Sh(\mathcal{C}')$  the corresponding localization morphism. We obtain a commutative square of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}'/U'), \mathcal{O}'_{U'}) & \xrightarrow{(j', (j')^\#)} & (Sh(\mathcal{C}'), \mathcal{O}') \\ (f', (f')^\#) \downarrow & & \downarrow (f, f^\#) \\ (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow{(j, j^\#)} & (Sh(\mathcal{C}), \mathcal{O}). \end{array}$$

and we have  $f'_*(j')^{-1} = j^{-1}f_*$ . See Modules on Sites, Lemma 20.1. By uniqueness of adjoints we obtain  $f^{-1}j_! = j'_!(f')^{-1}$ . Thus we obtain

$$\begin{aligned} f^*j_!\mathcal{A}_U &= f^{-1}j_!\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!(f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!((f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}|_{U'}} \mathcal{A}'|_{U'}) \\ &= j'_!\mathcal{A}'_{U'} \end{aligned}$$

The first equation is the definition of the pullback of  $j_!\mathcal{A}_U$  to a differential graded module over  $\mathcal{A}'$ . The second equation because  $f^{-1}j_! = j'_!(f')^{-1}$ . The third equation by Lemma 19.2 applied to the ringed site  $(\mathcal{C}', f^{-1}\mathcal{O})$  with sheaf of differential graded algebras  $f^{-1}\mathcal{A}$  and with differential graded modules  $(f')^{-1}\mathcal{A}_U$  on  $\mathcal{C}'/U'$  and  $\mathcal{A}'$  on  $\mathcal{C}'$ . The fourth equation holds because of course we have  $(f')^{-1}\mathcal{A}_U = f^{-1}\mathcal{A}|_{U'}$ . Hence we see that the pullback is another module of the same kind and we've proven conditions (1) and (2) for it above.  $\square$

0FSC **Lemma 23.2.** *et  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $0 \rightarrow \mathcal{P} \rightarrow \mathcal{P}' \rightarrow \mathcal{P}'' \rightarrow 0$  be an admissible short exact sequence of differential graded  $\mathcal{A}$ -modules. If two-out-of-three of these modules are good, so is the third.*

**Proof.** For condition (1) this is immediate as the sequence is a direct sum at the graded level. For condition (2) note that for any left differential graded  $\mathcal{A}$ -module, the sequence

$$0 \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N} \rightarrow 0$$

is an admissible short exact sequence of differential graded  $\mathcal{O}$ -modules (since forgetting the differential the tensor product is just taken in the category of graded modules). Hence if two out of three are exact as complexes of  $\mathcal{O}$ -modules, so is the third. Finally, the same argument shows that given a morphism  $(f, f^\#) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi, a differential graded  $\mathcal{O}'$ -algebra  $\mathcal{A}'$ , and a map  $\varphi : f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $f^{-1}\mathcal{O}$ -algebras we have that

$$0 \rightarrow f^*\mathcal{P} \rightarrow f^*\mathcal{P}' \rightarrow f^*\mathcal{P}'' \rightarrow 0$$

is an admissible short exact sequence of differential graded  $\mathcal{A}'$ -modules and the same argument as above applies here.  $\square$

0FSD **Lemma 23.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . An arbitrary direct sum of good differential graded  $\mathcal{A}$ -modules is good. A filtered colimit of good differential graded  $\mathcal{A}$ -modules is good.*

**Proof.** Omitted. Hint: direct sums and filtered colimits commute with tensor products and with pullbacks.  $\square$

0FSE **Lemma 23.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module. There exists a homomorphism  $\mathcal{P} \rightarrow \mathcal{M}$  of differential graded  $\mathcal{A}$ -modules with the following properties*

- (1)  $\mathcal{P} \rightarrow \mathcal{M}$  is surjective,
- (2)  $\text{Ker}(d_{\mathcal{P}}) \rightarrow \text{Ker}(d_{\mathcal{M}})$  is surjective, and
- (3)  $\mathcal{P}$  is good.

**Proof.** Consider triples  $(U, k, x)$  where  $U$  is an object of  $\mathcal{C}$ ,  $k \in \mathbf{Z}$ , and  $x$  is a section of  $\mathcal{M}^k$  over  $U$  with  $d_{\mathcal{M}}(x) = 0$ . Then we obtain a unique morphism of differential graded  $\mathcal{A}_U$ -modules  $\varphi_x : \mathcal{A}_U[-k] \rightarrow \mathcal{M}|_U$  mapping 1 to  $x$ . This is adjoint to a morphism  $\psi_x : j_{U!}\mathcal{A}_U[-k] \rightarrow \mathcal{M}$ . Observe that  $1 \in \mathcal{A}_U(U)$  corresponds to a section  $1 \in j_{U!}\mathcal{A}_U[-k](U)$  of degree  $k$  whose differential is zero and which is mapped to  $x$  by  $\psi_x$ . Thus if we consider the map

$$\bigoplus_{(U, k, x)} j_{U!}\mathcal{A}_U[-k] \longrightarrow \mathcal{M}$$

then we will have conditions (2) and (3). Namely, the objects  $j_{U!}\mathcal{A}_U[-k]$  are good (Lemma 23.1) and any direct sum of good objects is good (Lemma 23.3).

Next, consider triples  $(U, k, x)$  where  $U$  is an object of  $\mathcal{C}$ ,  $k \in \mathbf{Z}$ , and  $x$  is a section of  $\mathcal{M}^k$  (not necessarily annihilated by the differential). Then we can consider the cone  $C_U$  on the identity map  $\mathcal{A}_U \rightarrow \mathcal{A}_U$  as in Remark 22.5. The element  $x$  will determine a map  $\varphi_x : C_U[-k-1] \rightarrow \mathcal{A}_U$ , see Remark 22.5. Now, since we have an admissible short exact sequence

$$0 \rightarrow \mathcal{A}_U \rightarrow C_U \rightarrow \mathcal{A}_U[1] \rightarrow 0$$

we conclude that  $j_{U!}C_U$  is a good module by Lemma 23.2 and the already used Lemma 23.1. As above we conclude that the direct sum of the maps  $\psi_x : j_{U!}C_U \rightarrow \mathcal{M}$  adjoint to the  $\varphi_x$

$$\bigoplus_{(U, k, x)} j_{U!}C_U \longrightarrow \mathcal{M}$$

is surjective. Taking the direct sum with the map produced in the first paragraph we conclude.  $\square$

0FSF **Remark 23.5.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. A *sheaf of graded sets* on  $\mathcal{C}$  is a sheaf of sets  $\mathcal{S}$  endowed with a map  $\text{deg} : \mathcal{S} \rightarrow \mathbf{Z}$  of sheaves of sets. Let us denote  $\mathcal{O}[\mathcal{S}]$  the graded  $\mathcal{O}$ -module which is the free  $\mathcal{O}$ -module on the graded sheaf of sets  $\mathcal{S}$ . More precisely, the  $n$ th graded part of  $\mathcal{O}[\mathcal{S}]$  is the sheafification of the rule

$$U \longmapsto \bigoplus_{s \in \mathcal{S}(U), \text{deg}(s)=n} s \cdot \mathcal{O}(U)$$

With zero differential we also may consider this as a differential graded  $\mathcal{O}$ -module. Let  $\mathcal{A}$  be a sheaf of graded  $\mathcal{O}$ -algebras. Then we similarly define  $\mathcal{A}[\mathcal{S}]$  to be the graded  $\mathcal{A}$ -module whose  $n$ th graded part is the sheafification of the rule

$$U \longmapsto \bigoplus_{s \in \mathcal{S}(U)} s \cdot \mathcal{A}^{n-\text{deg}(s)}(U)$$

If  $\mathcal{A}$  is a differential graded  $\mathcal{O}$ -algebra, then we turn this into a differential graded  $\mathcal{O}$ -module by setting  $d(s) = 0$  for all  $s \in \mathcal{S}(U)$  and sheafifying.

0FSG **Lemma 23.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a differential graded  $\mathcal{A}$ -algebra. Let  $\mathcal{S}$  be a sheaf of graded sets on  $\mathcal{C}$ . Then the free graded module  $\mathcal{A}[\mathcal{S}]$  on  $\mathcal{S}$  endowed with differential as in Remark 23.5 is a good differential graded  $\mathcal{A}$ -module.*

**Proof.** Let  $\mathcal{N}$  be a left graded  $\mathcal{A}$ -module. Then we have

$$\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N} = \mathcal{O}[\mathcal{S}] \otimes_{\mathcal{O}} \mathcal{N} = \mathcal{N}[\mathcal{S}]$$

where  $\mathcal{N}[\mathcal{S}]$  is the graded  $\mathcal{O}$ -module whose degree  $n$  part is the sheaf associated to the presheaf

$$U \mapsto \bigoplus_{s \in \mathcal{S}(U)} s \cdot \mathcal{N}^{n - \deg(s)}(U)$$

It is clear that  $\mathcal{N} \rightarrow \mathcal{N}[\mathcal{S}]$  is an exact functor, hence  $\mathcal{A}[\mathcal{S}]$  is flat as a graded  $\mathcal{A}$ -module. Next, suppose that  $\mathcal{N}$  is a differential graded left  $\mathcal{A}$ -module. Then we have

$$H^*(\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N}) = H^*(\mathcal{O}[\mathcal{S}] \otimes_{\mathcal{O}} \mathcal{N})$$

as graded sheaves of  $\mathcal{O}$ -modules, which by the flatness (over  $\mathcal{O}$ ) is equal to

$$H^*(\mathcal{N})[\mathcal{S}]$$

as a graded  $\mathcal{O}$ -module. Hence if  $\mathcal{N}$  is acyclic, then  $\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N}$  is acyclic.

Finally, consider a morphism  $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi, a differential graded  $\mathcal{O}'$ -algebra  $\mathcal{A}'$ , and a map  $\varphi : f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $f^{-1}\mathcal{O}$ -algebras. Then it is straightforward to see that

$$f^*\mathcal{A}[\mathcal{S}] = \mathcal{A}'[f^{-1}\mathcal{S}]$$

which finishes the proof that our module is good.  $\square$

0FSH **Lemma 23.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module. There exists a homomorphism  $\mathcal{P} \rightarrow \mathcal{M}$  of differential graded  $\mathcal{A}$ -modules with the following properties*

- (1)  $\mathcal{P} \rightarrow \mathcal{M}$  is a quasi-isomorphism, and
- (2)  $\mathcal{P}$  is good.

**First proof.** Let  $\mathcal{S}_0$  be the sheaf of graded sets (Remark 23.5) whose degree  $n$  part is  $\text{Ker}(d_{\mathcal{M}}^n)$ . Consider the homomorphism of differential graded modules

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{S}_0] \longrightarrow \mathcal{M}$$

where the left hand side is as in Remark 23.5 and the map sends a local section  $s$  of  $\mathcal{S}_0$  to the corresponding local section of  $\mathcal{M}^{\deg(s)}$  (which is in the kernel of the differential, so our map is a map of differential graded modules indeed). By construction the induced maps on cohomology sheaves  $H^n(\mathcal{P}_0) \rightarrow H^n(\mathcal{M})$  are surjective. We are going to inductively construct maps

$$\mathcal{P}_0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \rightarrow \dots \rightarrow \mathcal{M}$$

Observe that of course  $H^*(\mathcal{P}_i) \rightarrow H^*(\mathcal{M})$  will be surjective for all  $i$ . Given  $\mathcal{P}_i \rightarrow \mathcal{M}$  denote  $\mathcal{S}_{i+1}$  the sheaf of graded sets whose degree  $n$  part is

$$\text{Ker}(d_{\mathcal{P}_i}^{n+1}) \times_{\mathcal{M}^{n+1}, d} \mathcal{M}^n$$

Then we set

$$\mathcal{P}_{i+1} = \mathcal{P}_i \oplus \mathcal{A}[\mathcal{S}_{i+1}]$$

as graded  $\mathcal{A}$ -module with differential and map to  $\mathcal{M}$  defined as follows

- (1) for local sections of  $\mathcal{P}_i$  use the differential on  $\mathcal{P}_i$  and the given map to  $\mathcal{M}$ ,

- (2) for a local section  $s = (p, m)$  of  $\mathcal{S}_{i+1}$  we set  $d(s)$  equal to  $p$  viewed as a section of  $\mathcal{P}_i$  of degree  $\deg(s) + 1$  and we map  $s$  to  $m$  in  $\mathcal{M}$ , and
- (3) extend the differential uniquely so that the Leibniz rule holds.

This makes sense because  $d(m)$  is the image of  $p$  and  $d(p) = 0$ . Finally, we set  $\mathcal{P} = \text{colim } \mathcal{P}_i$  with the induced map to  $\mathcal{M}$ .

The map  $\mathcal{P} \rightarrow \mathcal{M}$  is a quasi-isomorphism: we have  $H^n(\mathcal{P}) = \text{colim } H^n(\mathcal{P}_i)$  and for each  $i$  the map  $H^n(\mathcal{P}_i) \rightarrow H^n(\mathcal{M})$  is surjective with kernel annihilated by the map  $H^n(\mathcal{P}_i) \rightarrow H^n(\mathcal{P}_{i+1})$  by construction. Each  $\mathcal{P}_i$  is good because  $\mathcal{P}_0$  is good by Lemma 23.6 and each  $\mathcal{P}_{i+1}$  is in the middle of the admissible short exact sequence  $0 \rightarrow \mathcal{P}_i \rightarrow \mathcal{P}_{i+1} \rightarrow \mathcal{A}[\mathcal{S}_{i+1}] \rightarrow 0$  whose outer terms are good by induction. Hence  $\mathcal{P}_{i+1}$  is good by Lemma 23.2. Finally, we conclude that  $\mathcal{P}$  is good by Lemma 23.3.  $\square$

**Second proof.** We urge the reader to read the proof of Differential Graded Algebra, Lemma 20.4 before reading this proof. Set  $\mathcal{M} = \mathcal{M}_0$ . We inductively choose short exact sequences

$$0 \rightarrow \mathcal{M}_{i+1} \rightarrow \mathcal{P}_i \rightarrow \mathcal{M}_i \rightarrow 0$$

where the maps  $\mathcal{P}_i \rightarrow \mathcal{M}_i$  are chosen as in Lemma 23.4. This gives a “resolution”

$$\dots \rightarrow \mathcal{P}_2 \xrightarrow{f_2} \mathcal{P}_1 \xrightarrow{f_1} \mathcal{P}_0 \rightarrow \mathcal{M} \rightarrow 0$$

Then we let  $\mathcal{P}$  be the differential graded  $\mathcal{A}$ -module defined as follows

- (1) as a graded  $\mathcal{A}$ -module we set  $\mathcal{P} = \bigoplus_{a \leq 0} \mathcal{P}_{-a}[-a]$ , i.e., the degree  $n$  part is given by  $\mathcal{P}^n = \bigoplus_{a+b=n} \mathcal{P}_{-a}^b$ ,
- (2) the differential on  $\mathcal{P}$  is as in the construction of the total complex associated to a double complex given by

$$d_{\mathcal{P}}(x) = f_{-a}(x) + (-1)^a d_{\mathcal{P}_{-a}}(x)$$

for  $x$  a local section of  $\mathcal{P}_{-a}^b$ .

With these conventions  $\mathcal{P}$  is indeed a differential graded  $\mathcal{A}$ -module; we omit the details. There is a map  $\mathcal{P} \rightarrow \mathcal{M}$  of differential graded  $\mathcal{A}$ -modules which is zero on the summands  $\mathcal{P}_{-a}[-a]$  for  $a < 0$  and the given map  $\mathcal{P}_0 \rightarrow \mathcal{M}$  for  $a = 0$ . Observe that we have

$$\mathcal{P} = \text{colim}_i F_i \mathcal{P}$$

where  $F_i \mathcal{P} \subset \mathcal{P}$  is the differential graded  $\mathcal{A}$ -submodule whose underlying graded  $\mathcal{A}$ -module is

$$F_i \mathcal{P} = \bigoplus_{i \geq -a \geq 0} \mathcal{P}_{-a}[-a]$$

It is immediate that the maps

$$0 \rightarrow F_1 \mathcal{P} \rightarrow F_2 \mathcal{P} \rightarrow F_3 \mathcal{P} \rightarrow \dots \rightarrow \mathcal{P}$$

are all admissible monomorphisms and we have admissible short exact sequences

$$0 \rightarrow F_i \mathcal{P} \rightarrow F_{i+1} \mathcal{P} \rightarrow \mathcal{P}_{i+1}[i+1] \rightarrow 0$$

By induction and Lemma 23.2 we find that  $F_i \mathcal{P}$  is a good differential graded  $\mathcal{A}$ -module. Since  $\mathcal{P} = \text{colim } F_i \mathcal{P}$  we find that  $\mathcal{P}$  is good by Lemma 23.3.

Finally, we have to show that  $\mathcal{P} \rightarrow \mathcal{M}$  is a quasi-isomorphism. If  $\mathcal{C}$  has enough points, then this follows from the elementary Homology, Lemma 26.2 by checking on stalks. In general, we can argue as follows (this proof is far too long — there is an alternative argument by working with local sections as in the elementary proof but

it is also rather long). Since filtered colimits are exact on the category of abelian sheaves, we have

$$H^d(\mathcal{P}) = \operatorname{colim} H^d(F_i \mathcal{P})$$

We claim that for each  $i \geq 0$  and  $d \in \mathbf{Z}$  we have (a) a short exact sequence

$$0 \rightarrow H^d(\mathcal{M}_{i+1}[i]) \rightarrow H^d(F_i \mathcal{P}) \rightarrow H^d(\mathcal{M}) \rightarrow 0$$

where the second arrow comes from  $F_i \mathcal{P} \rightarrow \mathcal{P} \rightarrow \mathcal{M}$  and (b) the composition

$$H^d(\mathcal{M}_{i+1}[i]) \rightarrow H^d(F_i \mathcal{P}) \rightarrow H^d(F_{i+1} \mathcal{P})$$

is zero. It is clear that the claim suffices to finish the proof.

Proof of the claim. For any  $i \geq 0$  there is a map  $\mathcal{M}_{i+1}[i] \rightarrow F_i \mathcal{P}$  coming from the inclusion of  $\mathcal{M}_{i+1}$  into  $\mathcal{P}_i$  as the kernel of  $f_i$ . Consider the short exact sequence

$$0 \rightarrow \mathcal{M}_{i+1}[i] \rightarrow F_i \mathcal{P} \rightarrow C_i \rightarrow 0$$

of complexes of  $\mathcal{O}$ -modules defining  $C_i$ . Observe that  $C_0 = \mathcal{M}_0 = \mathcal{M}$ . Also, observe that  $C_i$  is the total complex associated to the double complex  $C_i^{\bullet, \bullet}$  with columns

$$\mathcal{M}_i = \mathcal{P}_i / \mathcal{M}_{i+1}, \mathcal{P}_{i-1}, \dots, \mathcal{P}_0$$

in degree  $-i, -i+1, \dots, 0$ . There is a map of double complexes  $C_i^{\bullet, \bullet} \rightarrow C_{i-1}^{\bullet, \bullet}$  which is 0 on the column in degree  $-i$ , is the surjection  $\mathcal{P}_{i-1} \rightarrow \mathcal{M}_{i-1}$  in degree  $-i+1$ , and is the identity on the other columns. Hence there are maps of complexes

$$C_i \longrightarrow C_{i-1}$$

These maps are surjective quasi-isomorphisms because the kernel is the total complex on the double complex with columns  $\mathcal{M}_i, \mathcal{M}_i$  in degrees  $-i, -i+1$  and the identity map between these two columns. Using the resulting identifications  $H^d(C_i) = H^d(C_{i-1}) = \dots = H^d(\mathcal{M})$  this already shows we get a long exact sequence

$$H^d(\mathcal{M}_{i+1}[i]) \rightarrow H^d(F_i \mathcal{P}) \rightarrow H^d(\mathcal{M}) \rightarrow H^{d+1}(\mathcal{M}_{i+1}[i])$$

from the short exact sequence of complexes above. However, we also have the commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}_{i+2}[i+1] & \xrightarrow{a} & T_{i+1} & \longrightarrow & F_{i+1} \mathcal{P} & \longrightarrow & C_{i+1} \\ & & \uparrow b & & \uparrow & & \downarrow \\ & & \mathcal{M}_{i+1}[i] & \longrightarrow & F_i \mathcal{P} & \longrightarrow & C_i \end{array}$$

where  $T_{i+1}$  is the total complex on the double complex with columns  $\mathcal{P}_{i+1}, \mathcal{M}_{i+1}$  placed in degrees  $-i-1$  and  $-i$ . In other words,  $T_{i+1}$  is a shift of the cone on the map  $\mathcal{P}_{i+1} \rightarrow \mathcal{M}_{i+1}$  and we find that  $a$  is a quasi-isomorphism and the map  $a^{-1} \circ b$  is a shift of the third map of the distinguished triangle in  $D(\mathcal{O})$  associated to the short exact sequence

$$0 \rightarrow \mathcal{M}_{i+2} \rightarrow \mathcal{P}_{i+1} \rightarrow \mathcal{M}_{i+1} \rightarrow 0$$

The map  $H^d(\mathcal{P}_{i+1}) \rightarrow H^d(\mathcal{M}_{i+1})$  is surjective because we chose our maps such that  $\operatorname{Ker}(d_{\mathcal{P}_{i+1}}) \rightarrow \operatorname{Ker}(d_{\mathcal{M}_{i+1}})$  is surjective. Thus we see that  $a^{-1} \circ b$  is zero on cohomology sheaves. This proves part (b) of the claim. Since  $T_{i+1}$  is the kernel

of the surjective map of complexes  $F_{i+1}\mathcal{P} \rightarrow C_i$  we find a map of long exact cohomology sequences

$$\begin{array}{ccccccc} H^d(T_{i+1}) & \longrightarrow & H^d(F_{i+1}\mathcal{P}) & \longrightarrow & H^d(\mathcal{M}) & \longrightarrow & H^{d+1}(T_{i+1}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^d(\mathcal{M}_{i+1}[i]) & \longrightarrow & H^d(F_i\mathcal{P}) & \longrightarrow & H^d(\mathcal{M}) & \longrightarrow & H^{d+1}(\mathcal{M}_{i+1}[i]) \end{array}$$

Here we know, by the discussion above, that the vertical maps on the outside are zero. Hence the maps  $H^d(F_{i+1}\mathcal{P}) \rightarrow H^d(\mathcal{M})$  are surjective and part (a) of the claim follows. More precisely, the claim follows for  $i > 0$  and we leave the claim for  $i = 0$  to the reader (actually it suffices to prove the claim for all  $i \gg 0$  in order to get the lemma).  $\square$

0FSI **Lemma 23.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{P}$  be a good acyclic right differential graded  $\mathcal{A}$ -module.*

- (1) *for any differential graded left  $\mathcal{A}$ -module  $\mathcal{N}$  the tensor product  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$  is acyclic,*
- (2) *for any morphism  $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi and any differential graded  $\mathcal{O}'$ -algebra  $\mathcal{A}'$  and any map  $\varphi : f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $f^{-1}\mathcal{O}$ -algebras the pullback  $f^*\mathcal{P}$  is acyclic and good.*

**Proof.** Proof of (1). By Lemma 23.7 we can choose a good left differential graded  $\mathcal{Q}$  and a quasi-isomorphism  $\mathcal{Q} \rightarrow \mathcal{N}$ . Then  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q}$  is acyclic because  $\mathcal{Q}$  is good. Let  $\mathcal{N}'$  be the cone on the map  $\mathcal{Q} \rightarrow \mathcal{N}$ . Then  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$  is acyclic because  $\mathcal{P}$  is good and because  $\mathcal{N}'$  is acyclic (as the cone on a quasi-isomorphism). We have a distinguished triangle

$$\mathcal{Q} \rightarrow \mathcal{N} \rightarrow \mathcal{N}' \rightarrow \mathcal{Q}[1]$$

in  $K(\text{Mod}(\mathcal{A}, d))$  by our construction of the triangulated structure. Since  $\mathcal{P} \otimes_{\mathcal{A}} -$  sends distinguished triangles to distinguished triangles, we obtain a distinguished triangle

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q} \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}' \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{Q}[1]$$

in  $K(\text{Mod}(\mathcal{O}))$ . Thus we conclude.

Proof of (2). Observe that  $f^*\mathcal{P}$  is good by our definition of good modules. Recall that  $f^*\mathcal{P} = f^{-1}\mathcal{P} \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$ . Then  $f^{-1}\mathcal{P}$  is a good acyclic (because  $f^{-1}$  is exact) differential graded  $f^{-1}\mathcal{A}$ -module. Hence we see that  $f^*\mathcal{P}$  is acyclic by part (1).  $\square$

## 24. The differential graded hull of a graded module

0FSJ The differential graded hull of a graded module  $\mathcal{N}$  is the result of applying the functor  $G$  in the following lemma.

0FSK **Lemma 24.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The forgetful functor  $F : \text{Mod}(\mathcal{A}, d) \rightarrow \text{Mod}(\mathcal{A})$  has a left adjoint  $G : \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{A}, d)$ .*

**Proof.** To prove the existence of  $G$  we can use the adjoint functor theorem, see Categories, Theorem 25.3 (observe that we have switched the roles of  $F$  and  $G$ ). The exactness conditions on  $F$  are satisfied by Lemma 13.2. The set theoretic

condition can be seen as follows: suppose given a graded  $\mathcal{A}$ -module  $\mathcal{N}$ . Then for any map

$$\varphi : \mathcal{N} \longrightarrow F(\mathcal{M})$$

we can consider the smallest differential graded  $\mathcal{A}$ -submodule  $\mathcal{M}' \subset \mathcal{M}$  with  $\text{Im}(\varphi) \subset F(\mathcal{M}')$ . It is clear that  $\mathcal{M}'$  is the image of the map of graded  $\mathcal{A}$ -modules

$$\mathcal{N} \oplus \mathcal{N}[-1] \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{M}$$

defined by

$$(n, \sum n_i \otimes a_i) \longmapsto \varphi(n) + \sum d(\varphi(n_i))a_i$$

because the image of this map is easily seen to be a differential graded submodule of  $\mathcal{M}$ . Thus the number of possible isomorphism classes of these  $\mathcal{M}'$  is bounded and we conclude.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module and suppose we have a short exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow F(\mathcal{M}) \rightarrow \mathcal{N}' \rightarrow 0$$

in  $\text{Mod}(\mathcal{A})$ . Then we obtain a canonical graded  $\mathcal{A}$ -module homomorphism

$$\bar{d} : \mathcal{N} \rightarrow \mathcal{N}'[1]$$

as follows: given a local section  $x$  of  $\mathcal{N}$  denote  $\bar{d}(x)$  the image in  $\mathcal{N}'$  of  $d_{\mathcal{M}}(x)$  when  $x$  is viewed as a local section of  $\mathcal{M}$ .

**0FSL Lemma 24.2.** *The functors  $F, G$  of Lemma 24.1 have the following properties. Given a graded  $\mathcal{A}$ -module  $\mathcal{N}$  we have*

- (1) *the counit  $\mathcal{N} \rightarrow F(G(\mathcal{N}))$  is injective,*
- (2) *the map  $\bar{d} : \mathcal{N} \rightarrow \text{Coker}(\mathcal{N} \rightarrow F(G(\mathcal{N}))) [1]$  is an isomorphism, and*
- (3)  *$G(\mathcal{N})$  is an acyclic differential graded  $\mathcal{A}$ -module.*

**Proof.** We observe that property (3) is a consequence of properties (1) and (2). Namely, if  $s$  is a nonzero local section of  $F(G(\mathcal{N}))$  with  $d(s) = 0$ , then  $s$  cannot be in the image of  $\mathcal{N} \rightarrow F(G(\mathcal{N}))$ . Hence we can write the image  $\bar{s}$  of  $s$  in the cokernel as  $\bar{d}(s')$  for some local section  $s'$  of  $\mathcal{N}$ . Then we see that  $s = d(s')$  because the difference  $s - d(s')$  is still in the kernel of  $d$  and is contained in the image of the counit.

Let us write temporarily  $\mathcal{A}_{gr}$ , respectively  $\mathcal{A}_{dg}$  the sheaf  $\mathcal{A}$  viewed as a (right) graded module over itself, respectively as a (right) differential graded module over itself. The most important case of the lemma is to understand what is  $G(\mathcal{A}_{gr})$ . Of course  $G(\mathcal{A}_{gr})$  is the object of  $\text{Mod}(\mathcal{A}, d)$  representing the functor

$$\mathcal{M} \longmapsto \text{Hom}_{\text{Mod}(\mathcal{A})}(\mathcal{A}_{gr}, F(\mathcal{M})) = \Gamma(\mathcal{C}, \mathcal{M})$$

By Remark 22.5 we see that this functor is represented by  $C[-1]$  where  $C$  is the cone on the identity of  $\mathcal{A}_{dg}$ . We have a short exact sequence

$$0 \rightarrow \mathcal{A}_{dg}[-1] \rightarrow C[-1] \rightarrow \mathcal{A}_{dg} \rightarrow 0$$

in  $\text{Mod}(\mathcal{A}, d)$  which is split by the counit  $\mathcal{A}_{gr} \rightarrow F(C[-1])$  in  $\text{Mod}(\mathcal{A})$ . Thus  $G(\mathcal{A}_{gr})$  satisfies properties (1) and (2).

Let  $U$  be an object of  $\mathcal{C}$ . Denote  $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$  the localization morphism. Denote  $\mathcal{A}_U$  the restriction of  $\mathcal{A}$  to  $U$ . We will use the notation  $\mathcal{A}_{U,gr}$  to denote  $\mathcal{A}_U$  viewed

as a graded  $\mathcal{A}_U$ -module. Denote  $F_U : Mod(\mathcal{A}_U, d) \rightarrow Mod(\mathcal{A}_U)$  the forgetful functor and denote  $G_U$  its adjoint. Then we have the commutative diagrams

$$\begin{array}{ccc} Mod(\mathcal{A}, d) & \xrightarrow{F} & Mod(\mathcal{A}) \\ j_U^* \downarrow & & \downarrow j_U^* \\ Mod(\mathcal{A}_U, d) & \xrightarrow{F_U} & Mod(\mathcal{A}_U) \end{array} \quad \text{and} \quad \begin{array}{ccc} Mod(\mathcal{A}_U, d) & \xrightarrow{F_U} & Mod(\mathcal{A}_U) \\ j_{U!} \downarrow & & \downarrow j_{U!} \\ Mod(\mathcal{A}, d) & \xrightarrow{F} & Mod(\mathcal{A}) \end{array}$$

by the construction of  $j_U^*$  and  $j_{U!}$  in Sections 9, 18, 10, and 19. By uniqueness of adjoints we obtain  $j_{U!} \circ G_U = G \circ j_{U!}$ . Since  $j_{U!}$  is an exact functor, we see that the properties (1) and (2) for the counit  $\mathcal{A}_{U,gr} \rightarrow F_U(G_U(\mathcal{A}_{U,gr}))$  which we've seen in the previous part of the proof imply properties (1) and (2) for the counit  $j_{U!}\mathcal{A}_{U,gr} \rightarrow F(G(j_{U!}\mathcal{A}_{U,gr})) = j_{U!}F_U(G_U(\mathcal{A}_{U,gr}))$ .

In the proof of Lemma 11.1 we have seen that any object of  $Mod(\mathcal{A})$  is a quotient of a direct sum of copies of  $j_{U!}\mathcal{A}_{U,gr}$ . Since  $G$  is a left adjoint, we see that  $G$  commutes with direct sums. Thus properties (1) and (2) hold for direct sums of objects for which they hold. Thus we see that every object  $\mathcal{N}$  of  $Mod(\mathcal{A})$  fits into an exact sequence

$$\mathcal{N}_1 \rightarrow \mathcal{N}_0 \rightarrow \mathcal{N} \rightarrow 0$$

such that (1) and (2) hold for  $\mathcal{N}_1$  and  $\mathcal{N}_0$ . We leave it to the reader to deduce (1) and (2) for  $\mathcal{N}$  using that  $G$  is right exact.  $\square$

## 25. K-injective differential graded modules

0FSM This section is the analogue of Injectives, Section 12 in the setting of sheaves of differential graded modules over a sheaf of differential graded algebras.

0FSN **Lemma 25.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a sheaf of graded algebras on  $(\mathcal{C}, \mathcal{O})$ . There exists a set  $T$  and for each  $t \in T$  an injective map  $\mathcal{N}_t \rightarrow \mathcal{N}'_t$  of graded  $\mathcal{A}$ -modules such that an object  $\mathcal{I}$  of  $Mod(\mathcal{A})$  is injective if and only if for every solid diagram*

$$\begin{array}{ccc} \mathcal{N}_t & \longrightarrow & \mathcal{I} \\ \downarrow & & \nearrow \text{dotted} \\ \mathcal{N}'_t & & \end{array}$$

*a dotted arrow exists in  $Mod(\mathcal{A})$  making the diagram commute.*

**Proof.** This is true in any Grothendieck abelian category, see Injectives, Lemma 11.6. By Lemma 11.1 the category  $Mod(\mathcal{A})$  is a Grothendieck abelian category.  $\square$

0FSP **Definition 25.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A differential graded  $\mathcal{A}$ -module  $\mathcal{I}$  is said to be *graded injective*<sup>2</sup> if  $\mathcal{M}$  viewed as a graded  $\mathcal{A}$ -module is an injective object of the category  $Mod(\mathcal{A})$  of graded  $\mathcal{A}$ -modules.

0FSQ **Remark 25.3.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{I}$  be a graded injective differential graded  $\mathcal{A}$ -module. Let

$$0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3 \rightarrow 0$$

<sup>2</sup>This may be nonstandard terminology.



be a short exact sequence of differential graded  $\mathcal{A}$ -modules. Since  $\mathcal{I}$  is graded injective we obtain a short exact sequence of complexes

$$0 \rightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},d)}(\mathcal{M}_3, \mathcal{I}) \rightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},d)}(\mathcal{M}_2, \mathcal{I}) \rightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},d)}(\mathcal{M}_1, \mathcal{I}) \rightarrow 0$$

of  $\Gamma(\mathcal{C}, \mathcal{O})$ -modules. Taking cohomology we obtain a long exact sequence

$$\begin{array}{ccc} \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_3, \mathcal{I}) & & \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_3, \mathcal{I})[1] \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_2, \mathcal{I}) & & \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_2, \mathcal{I})[1] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_1, \mathcal{I}) & & \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A},d))}(\mathcal{M}_1, \mathcal{I})[1] \end{array}$$

of groups of homomorphisms in the homotopy category. The point is that we get this even though we didn't assume that our short exact sequence is admissible (so the short exact sequence in general does not define a distinguished triangle in the homotopy category).

**0FSR Lemma 25.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $T$  be a set and for each  $t \in T$  let  $\mathcal{I}_t$  be a graded injective differential graded  $\mathcal{A}$ -module. Then  $\prod \mathcal{I}_t$  is a graded injective differential graded  $\mathcal{A}$ -module.*

**Proof.** This is true because products of injectives are injectives, see Homology, Lemma 27.3, and because products in  $\mathrm{Mod}(\mathcal{A}, d)$  are compatible with products in  $\mathrm{Mod}(\mathcal{A})$  via the forgetful functor.  $\square$

**0FSS Lemma 25.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . There exists a set  $T$  and for each  $t \in T$  an injective map  $\mathcal{M}_t \rightarrow \mathcal{M}'_t$  of acyclic differential graded  $\mathcal{A}$ -modules such that for an object  $\mathcal{I}$  of  $\mathrm{Mod}(\mathcal{A}, d)$  the following are equivalent*

- (1)  $\mathcal{I}$  is graded injective, and
- (2) for every solid diagram

$$\begin{array}{ccc} \mathcal{M}_t & \longrightarrow & \mathcal{I} \\ \downarrow & & \nearrow \\ \mathcal{M}'_t & & \end{array}$$

a dotted arrow exists in  $\mathrm{Mod}(\mathcal{A}, d)$  making the diagram commute.

**Proof.** Let  $T$  and  $\mathcal{N}_t \rightarrow \mathcal{N}'_t$  be as in Lemma 25.1. Denote  $F : \mathrm{Mod}(\mathcal{A}, d) \rightarrow \mathrm{Mod}(\mathcal{A})$  the forgetful functor. Let  $G$  be the left adjoint functor to  $F$  as in Lemma 24.1. Set

$$\mathcal{M}_t = G(\mathcal{N}_t) \rightarrow G(\mathcal{N}'_t) = \mathcal{M}'_t$$

This is an injective map of acyclic differential graded  $\mathcal{A}$ -modules by Lemma 24.2. Since  $G$  is the left adjoint to  $F$  we see that there exists a dotted arrow in the

diagram

$$\begin{array}{ccc} \mathcal{M}_t & \longrightarrow & \mathcal{I} \\ \downarrow & \nearrow \text{dotted} & \\ \mathcal{M}'_t & & \end{array}$$

if and only if there exists a dotted arrow in the diagram

$$\begin{array}{ccc} \mathcal{N}_t & \longrightarrow & F(\mathcal{I}) \\ \downarrow & \nearrow \text{dotted} & \\ \mathcal{N}'_t & & \end{array}$$

Hence the result follows from the choice of our collection of arrows  $\mathcal{N}_t \rightarrow \mathcal{N}'_t$ .  $\square$

**Lemma 25.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . There exists a set  $S$  and for each  $s$  an acyclic differential graded  $\mathcal{A}$ -module  $\mathcal{M}_s$  such that for every nonzero acyclic differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  there is an  $s \in S$  and an injective map  $\mathcal{M}_s \rightarrow \mathcal{M}$  in  $\text{Mod}(\mathcal{A}, d)$ .*

**Proof.** Before we start recall that our conventions guarantee the site  $\mathcal{C}$  has a set of objects and morphisms and a set  $\text{Cov}(\mathcal{C})$  of coverings. If  $\mathcal{F}$  is a differential graded  $\mathcal{A}$ -module, let us define  $|\mathcal{F}|$  to be the sum of the cardinality of

$$\coprod_{(U, n)} \mathcal{F}^n(U)$$

as  $U$  ranges over the objects of  $\mathcal{C}$  and  $n \in \mathbf{Z}$ . Choose an infinite cardinal  $\kappa$  bigger than the cardinals  $|\text{Ob}(\mathcal{C})|$ ,  $|\text{Arrows}(\mathcal{C})|$ ,  $|\text{Cov}(\mathcal{C})|$ ,  $\sup |I|$  for  $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})$ , and  $|\mathcal{A}|$ .

Let  $\mathcal{F} \subset \mathcal{M}$  be an inclusion of differential graded  $\mathcal{A}$ -modules. Suppose given a set  $K$  and for each  $k \in K$  a triple  $(U_k, n_k, x_k)$  consisting of an object  $U_k$  of  $\mathcal{C}$ , integer  $n_k$ , and a section  $x_k \in \mathcal{M}^{n_k}(U_k)$ . Then we can consider the smallest differential graded  $\mathcal{A}$ -submodule  $\mathcal{F}' \subset \mathcal{M}$  containing  $\mathcal{F}$  and the sections  $x_k$  for  $k \in K$ . We can describe

$$(\mathcal{F}')^n(U) \subset \mathcal{M}^n(U)$$

as the set of elements  $x \in \mathcal{M}^n(U)$  such that there exists  $\{f_i : U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})$  such that for each  $i \in I$  there is a finite set  $T_i$  and morphisms  $g_t : U_i \rightarrow U_{k_t}$

$$f_i^* x = y_i + \sum_{t \in T_i} a_{it} g_t^* x_{k_t} + b_{it} g_t^* d(x_{k_t})$$

for some section  $y_i \in \mathcal{F}^n(U)$  and sections  $a_{it} \in \mathcal{A}^{n-n_{k_t}}(U_i)$  and  $b_{it} \in \mathcal{A}^{n-n_{k_t}-1}(U_i)$ . (Details omitted; hints: these sections are certainly in  $\mathcal{F}'$  and you show conversely that this rule defines a differential graded  $\mathcal{A}$ -submodule.) It follows from this description that  $|\mathcal{F}'| \leq \max(|\mathcal{F}|, |K|, \kappa)$ .

Let  $\mathcal{M}$  be a nonzero acyclic differential graded  $\mathcal{A}$ -module. Then we can find an integer  $n$  and a nonzero section  $x$  of  $\mathcal{M}^n$  over some object  $U$  of  $\mathcal{C}$ . Let

$$\mathcal{F}_0 \subset \mathcal{M}$$

be the smallest differential graded  $\mathcal{A}$ -submodule containing  $x$ . By the previous paragraph we have  $|\mathcal{F}_0| \leq \kappa$ . By induction, given  $\mathcal{F}_0, \dots, \mathcal{F}_n$  define  $\mathcal{F}_{n+1}$  as follows. Consider the set

$$L = \{(U, n, x)\} \{U_i \rightarrow U\}_{i \in I}, (x_i)_{i \in I}\}$$

of triples where  $U$  is an object of  $\mathcal{C}$ ,  $n \in \mathbf{Z}$ , and  $x \in \mathcal{F}_n(U)$  with  $d(x) = 0$ . Since  $\mathcal{M}$  is acyclic for each triple  $l = (U_l, n_l, x_l) \in L$  we can choose  $\{(U_{l,i} \rightarrow U_l)_{i \in I_l} \in \text{Cov}(\mathcal{C})$  and  $x_{l,i} \in \mathcal{M}^{n_l-1}(U_{l,i})$  such that  $d(x_{l,i}) = x|_{U_{l,i}}$ . Then we set

$$K = \{(U_{l,i}, n_l - 1, x_{l,i}) \mid l \in L, i \in I_l\}$$

and we let  $\mathcal{F}_{n+1}$  be the smallest differential graded  $\mathcal{A}$ -submodule of  $\mathcal{M}$  containing  $\mathcal{F}_n$  and the sections  $x_{l,i}$ . Since  $|K| \leq \max(\kappa, |\mathcal{F}_n|)$  we conclude that  $|\mathcal{F}_{n+1}| \leq \kappa$  by induction.

By construction the inclusion  $\mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$  induces the zero map on cohomology sheaves. Hence we see that  $\mathcal{F} = \bigcup \mathcal{F}_n$  is a nonzero acyclic submodule with  $|\mathcal{F}| \leq \kappa$ . Since there is only a set of isomorphism classes of differential graded  $\mathcal{A}$ -modules  $\mathcal{F}$  with  $|\mathcal{F}|$  bounded, we conclude.  $\square$

0FSU **Definition 25.7.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . A differential graded  $\mathcal{A}$ -module  $\mathcal{I}$  is *K-injective* if for every acyclic differential graded  $\mathcal{M}$  we have

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{I}) = 0$$

Please note the similarity with Derived Categories, Definition 31.1.

0FSV **Lemma 25.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $T$  be a set and for each  $t \in T$  let  $\mathcal{I}_t$  be a K-injective differential graded  $\mathcal{A}$ -module. Then  $\prod \mathcal{I}_t$  is a K-injective differential graded  $\mathcal{A}$ -module.*

**Proof.** Let  $\mathcal{K}$  be an acyclic differential graded  $\mathcal{A}$ -module. Then we have

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}, \prod_{t \in T} \mathcal{I}_t) = \prod_{t \in T} \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}, \mathcal{I}_t)$$

because taking products in  $\text{Mod}(\mathcal{A}, d)$  commutes with the forgetful functor to graded  $\mathcal{A}$ -modules. Since taking products is an exact functor on the category of abelian groups we conclude.  $\square$

0FSW **Lemma 25.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{I}$  be a K-injective and graded injective object of  $\text{Mod}(\mathcal{A}, d)$ . For every solid diagram in  $\text{Mod}(\mathcal{A}, d)$*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a} & \mathcal{I} \\ \downarrow b & \nearrow \text{dotted} & \\ \mathcal{M}' & & \end{array}$$

where  $b$  is injective and  $\mathcal{M}$  is acyclic a dotted arrow exists making the diagram commute.

**Proof.** Since  $\mathcal{M}$  is acyclic and  $\mathcal{I}$  is K-injective, there exists a graded  $\mathcal{A}$ -module map  $h : \mathcal{M} \rightarrow \mathcal{I}$  of degree  $-1$  such that  $a = d(h)$ . Since  $\mathcal{I}$  is graded injective and  $b$  is injective, there exists a graded  $\mathcal{A}$ -module map  $h' : \mathcal{M}' \rightarrow \mathcal{I}$  of degree  $-1$  such that  $h = h' \circ b$ . Then we can take  $a' = d(h')$  as the dotted arrow.  $\square$

0FSX **Lemma 25.10.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{I}$  be a  $K$ -injective and graded injective object of  $\text{Mod}(\mathcal{A}, d)$ . For every solid diagram in  $\text{Mod}(\mathcal{A}, d)$*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a} & \mathcal{I} \\ \downarrow b & \nearrow \text{dotted} & \\ \mathcal{M}' & & \end{array}$$

where  $b$  is a quasi-isomorphism a dotted arrow exists making the diagram commute up to homotopy.

**Proof.** After replacing  $\mathcal{M}'$  by the direct sum of  $\mathcal{M}'$  and the cone on the identity on  $\mathcal{M}$  (which is acyclic) we may assume  $b$  is also injective. Then the cokernel  $\mathcal{Q}$  of  $b$  is acyclic. Thus we see that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{Q}, \mathcal{I}) = \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{Q}, \mathcal{I})[1] = 0$$

as  $\mathcal{I}$  is  $K$ -injective. As  $\mathcal{I}$  is graded injective by Remark 25.3 we see that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}', \mathcal{I}) \longrightarrow \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{I})$$

is bijective and the proof is complete.  $\square$

0FSY **Lemma 25.11.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . There exists a set  $R$  and for each  $r \in R$  an injective map  $\mathcal{M}_r \rightarrow \mathcal{M}'_r$  of acyclic differential graded  $\mathcal{A}$ -modules such that for an object  $\mathcal{I}$  of  $\text{Mod}(\mathcal{A}, d)$  the following are equivalent*

- (1)  $\mathcal{I}$  is  $K$ -injective and graded injective, and
- (2) for every solid diagram

$$\begin{array}{ccc} \mathcal{M}_r & \xrightarrow{\quad} & \mathcal{I} \\ \downarrow & \nearrow \text{dotted} & \\ \mathcal{M}'_r & & \end{array}$$

a dotted arrow exists in  $\text{Mod}(\mathcal{A}, d)$  making the diagram commute.

**Proof.** Let  $T$  and  $\mathcal{M}_t \rightarrow \mathcal{M}'_t$  be as in Lemma 25.5. Let  $S$  and  $\mathcal{M}_s$  be as in Lemma 25.6. Choose an injective map  $\mathcal{M}_s \rightarrow \mathcal{M}'_s$  of acyclic differential graded  $\mathcal{A}$ -modules which is homotopic to zero. This is possible because we may take  $\mathcal{M}'_s$  to be the cone on the identity; in that case it is even true that the identity on  $\mathcal{M}'_s$  is homotopic to zero, see Differential Graded Algebra, Lemma 27.4 which applies by the discussion in Section 22. We claim that  $R = T \amalg S$  with the given maps works.

The implication (1)  $\Rightarrow$  (2) holds by Lemma 25.9.

Assume (2). First, by Lemma 25.5 we see that  $\mathcal{I}$  is graded injective. Next, let  $\mathcal{M}$  be an acyclic differential graded  $\mathcal{A}$ -module. We have to show that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{I}) = 0$$

The proof will be exactly the same as the proof of Injectives, Lemma 12.3.

We are going to construct by induction on the ordinal  $\alpha$  an acyclic differential graded submodule  $\mathcal{K}_\alpha \subset \mathcal{M}$  as follows. For  $\alpha = 0$  we set  $\mathcal{K}_0 = 0$ . For  $\alpha > 0$  we proceed as follows:

- (1) If  $\alpha = \beta + 1$  and  $\mathcal{K}_\beta = \mathcal{M}$  then we choose  $\mathcal{K}_\alpha = \mathcal{K}_\beta$ .
- (2) If  $\alpha = \beta + 1$  and  $\mathcal{K}_\beta \neq \mathcal{M}$  then  $\mathcal{M}/\mathcal{K}_\beta$  is a nonzero acyclic differential graded  $\mathcal{A}$ -module. We choose a differential graded  $\mathcal{A}$  submodule  $\mathcal{N}_\alpha \subset \mathcal{M}/\mathcal{K}_\beta$  isomorphic to  $\mathcal{M}_s$  for some  $s \in S$ , see Lemma 25.6. Finally, we let  $\mathcal{K}_\alpha \subset \mathcal{M}$  be the inverse image of  $\mathcal{N}_\alpha$ .
- (3) If  $\alpha$  is a limit ordinal we set  $\mathcal{K}_\beta = \text{colim } \mathcal{K}_\alpha$ .

It is clear that  $\mathcal{M} = \mathcal{K}_\alpha$  for a suitably large ordinal  $\alpha$ . We will prove that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{K}_\alpha, \mathcal{I})$$

is zero by transfinite induction on  $\alpha$ . It holds for  $\alpha = 0$  since  $\mathcal{K}_0$  is zero. Suppose it holds for  $\beta$  and  $\alpha = \beta + 1$ . In case (1) of the list above the result is clear. In case (2) there is a short exact sequence

$$0 \rightarrow \mathcal{K}_\beta \rightarrow \mathcal{K}_\alpha \rightarrow \mathcal{N}_\alpha \rightarrow 0$$

By Remark 25.3 and since we've seen that  $\mathcal{I}$  is graded injective, we obtain an exact sequence

$$\text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{K}_\beta, \mathcal{I}) \rightarrow \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{K}_\alpha, \mathcal{I}) \rightarrow \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{N}_\alpha, \mathcal{I})$$

By induction the term on the left is zero. By assumption (2) the term on the right is zero: any map  $\mathcal{M}_s \rightarrow \mathcal{I}$  factors through  $\mathcal{M}'_s$  and hence is homotopic to zero. Thus the middle group is zero too. Finally, suppose that  $\alpha$  is a limit ordinal. Because we also have  $\mathcal{K}_\alpha = \text{colim } \mathcal{K}_\alpha$  as graded  $\mathcal{A}$ -modules we see that

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}_\alpha, \mathcal{I}) = \lim_{\beta < \alpha} \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, d)}(\mathcal{K}_\beta, \mathcal{I})$$

as complexes of abelian groups. The cohomology groups of these complexes compute morphisms in  $K(\text{Mod}(\mathcal{A}, d))$  between shifts. The transition maps in the system of complexes are surjective by Remark 25.3 because  $\mathcal{I}$  is graded injective. Moreover, for a limit ordinal  $\beta \leq \alpha$  we have equality of limit and value. Thus we may apply Homology, Lemma 31.8 to conclude.  $\square$

0FSZ **Lemma 25.12.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $R$  be a set and for each  $r \in R$  let an injective map  $\mathcal{M}_r \rightarrow \mathcal{M}'_r$  of acyclic differential graded  $\mathcal{A}$ -modules be given. There exists a functor  $M : \text{Mod}(\mathcal{A}, d) \rightarrow \text{Mod}(\mathcal{A}, d)$  and a natural transformation  $j : id \rightarrow M$  such that*

- (1)  $j_{\mathcal{M}} : \mathcal{M} \rightarrow M(\mathcal{M})$  is injective and a quasi-isomorphism,
- (2) for every solid diagram

$$\begin{array}{ccc} \mathcal{M}_r & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow j_{\mathcal{M}} \\ \mathcal{M}'_r & \cdots \longrightarrow & M(\mathcal{M}) \end{array}$$

a dotted arrow exists in  $\text{Mod}(\mathcal{A}, d)$  making the diagram commute.

**Proof.** We define  $M(\mathcal{M})$  as the pushout in the following diagram

$$\begin{array}{ccc} \bigoplus_{(r, \varphi)} \mathcal{M}_r & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \bigoplus_{(r, \varphi)} \mathcal{M}'_r & \longrightarrow & M(\mathcal{M}) \end{array}$$

where the direct sum is over all pairs  $(r, \varphi)$  with  $r \in R$  and  $\varphi \in \text{Hom}_{\text{Mod}(\mathcal{A}, d)}(\mathcal{M}_r, \mathcal{M})$ . Since the pushout of an injective map is injective, we see that  $\mathcal{M} \rightarrow M(\mathcal{M})$  is injective. Since the cokernel of the left vertical arrow is acyclic, we see that the (isomorphic) cokernel of  $\mathcal{M} \rightarrow M(\mathcal{M})$  is acyclic, hence  $\mathcal{M} \rightarrow M(\mathcal{M})$  is a quasi-isomorphism. Property (2) holds by construction. We omit the verification that this procedure can be turned into a functor.  $\square$

0FT0 **Theorem 25.13.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . For every differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  there exists a quasi-isomorphism  $\mathcal{M} \rightarrow \mathcal{I}$  where  $\mathcal{I}$  is a graded injective and  $K$ -injective differential graded  $\mathcal{A}$ -module. Moreover, the construction is functorial in  $\mathcal{M}$ .*

**Proof.** Let  $R$  and  $\mathcal{M}_r \rightarrow \mathcal{M}'_r$  be a set of morphisms of  $\text{Mod}(\mathcal{A}, d)$  found in Lemma 25.11. Let  $M$  with transformation  $\text{id} \rightarrow M$  be as constructed in Lemma 25.12 using  $R$  and  $\mathcal{M}_r \rightarrow \mathcal{M}'_r$ . Using transfinite recursion we define a sequence of functors  $M_\alpha$  and natural transformations  $M_\beta \rightarrow M_\alpha$  for  $\alpha < \beta$  by setting

- (1)  $M_0 = \text{id}$ ,
- (2)  $M_{\alpha+1} = M \circ M_\alpha$  with natural transformation  $M_\beta \rightarrow M_{\alpha+1}$  for  $\beta < \alpha + 1$  coming from the already constructed  $M_\beta \rightarrow M_\alpha$  and the maps  $M_\alpha \rightarrow M \circ M_\alpha$  coming from  $\text{id} \rightarrow M$ , and
- (3)  $M_\alpha = \text{colim}_{\beta < \alpha} M_\beta$  if  $\alpha$  is a limit ordinal with the coprojections as transformations  $M_\beta \rightarrow M_\alpha$  for  $\alpha < \beta$ .

Observe that for every differential graded  $\mathcal{A}$ -module the maps  $\mathcal{M} \rightarrow M_\beta(\mathcal{M}) \rightarrow M_\alpha(\mathcal{M})$  are injective quasi-isomorphisms (as filtered colimits are exact).

Recall that  $\text{Mod}(\mathcal{A}, d)$  is a Grothendieck abelian category. Thus by Injectives, Proposition 11.5 (applied to the direct sum of  $\mathcal{M}_r$  for all  $r \in R$ ) there is a limit ordinal  $\alpha$  such that  $\mathcal{M}_r$  is  $\alpha$ -small with respect to injections for every  $r \in R$ . We claim that  $\mathcal{M} \rightarrow M_\alpha(\mathcal{M})$  is the desired functorial embedding of  $\mathcal{M}$  into a graded injective  $K$ -injective module.

Namely, any map  $\mathcal{M}_r \rightarrow M_\alpha(\mathcal{M})$  factors through  $M_\beta(\mathcal{M})$  for some  $\beta < \alpha$ . However, by the construction of  $M$  we see that this means that  $\mathcal{M}_r \rightarrow M_{\beta+1}(\mathcal{M}) = M(M_\beta(\mathcal{M}))$  factors through  $\mathcal{M}'_r$ . Since  $M_\beta(\mathcal{M}) \subset M_{\beta+1}(\mathcal{M}) \subset M_\alpha(\mathcal{M})$  we get the desired factorization into  $M_\alpha(\mathcal{M})$ . We conclude by our choice of  $R$  and  $\mathcal{M}_r \rightarrow \mathcal{M}'_r$  in Lemma 25.11.  $\square$

## 26. The derived category

0FT1 This section is the analogue of Differential Graded Algebra, Section 22.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . We will construct the derived category  $D(\mathcal{A}, d)$  by inverting the quasi-isomorphisms in  $K(\text{Mod}(\mathcal{A}, d))$ .

0FT2 **Lemma 26.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The functor  $H^0 : \text{Mod}(\mathcal{A}, d) \rightarrow \text{Mod}(\mathcal{O})$  of Section 13 factors through a functor*

$$H^0 : K(\text{Mod}(\mathcal{A}, d)) \rightarrow \text{Mod}(\mathcal{O})$$

*which is homological in the sense of Derived Categories, Definition 3.5.*

**Proof.** It follows immediately from the definitions that there is a commutative diagram

$$\begin{array}{ccc} \text{Mod}(\mathcal{A}, d) & \longrightarrow & K(\text{Mod}(\mathcal{A}, d)) \\ \downarrow & & \downarrow \\ \text{Comp}(\mathcal{O}) & \longrightarrow & K(\text{Mod}(\mathcal{O})) \end{array}$$

Since  $H^0(\mathcal{M})$  is defined as the zeroth cohomology sheaf of the underlying complex of  $\mathcal{O}$ -modules of  $\mathcal{M}$  the lemma follows from the case of complexes of  $\mathcal{O}$ -modules which is a special case of Derived Categories, Lemma 11.1.  $\square$

0FT3 **Lemma 26.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The full subcategory  $\text{Ac}$  of the homotopy category  $K(\text{Mod}(\mathcal{A}, d))$  consisting of acyclic modules is a strictly full saturated triangulated subcategory of  $K(\text{Mod}(\mathcal{A}, d))$ .*

**Proof.** Of course an object  $\mathcal{M}$  of  $K(\text{Mod}(\mathcal{A}, d))$  is in  $\text{Ac}$  if and only if  $H^i(\mathcal{M}) = H^0(\mathcal{M}[i])$  is zero for all  $i$ . The lemma follows from this, Lemma 26.1, and Derived Categories, Lemma 6.3. See also Derived Categories, Definitions 6.1 and 3.4 and Lemma 4.16.  $\square$

0FT4 **Lemma 26.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Consider the subclass  $\text{Qis} \subset \text{Arrows}(K(\text{Mod}(\mathcal{A}, d)))$  consisting of quasi-isomorphisms. This is a saturated multiplicative system compatible with the triangulated structure on  $K(\text{Mod}(\mathcal{A}, d))$ .*

**Proof.** Observe that if  $f, g : \mathcal{M} \rightarrow \mathcal{N}$  are morphisms of  $\text{Mod}(\mathcal{A}, d)$  which are homotopic, then  $f$  is a quasi-isomorphism if and only if  $g$  is a quasi-isomorphism. Namely, the maps  $H^i(f) = H^0(f[i])$  and  $H^i(g) = H^0(g[i])$  are the same by Lemma 26.1. Thus it is unambiguous to say that a morphism of the homotopy category  $K(\text{Mod}(\mathcal{A}, d))$  is a quasi-isomorphism. For definitions of “multiplicative system”, “saturated”, and “compatible with the triangulated structure” see Derived Categories, Definition 5.1 and Categories, Definitions 27.1 and 27.20.

To actually prove the lemma consider the composition of exact functors of triangulated categories

$$K(\text{Mod}(\mathcal{A}, d)) \longrightarrow K(\text{Mod}(\mathcal{O})) \longrightarrow D(\mathcal{O})$$

and observe that a morphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  of  $K(\text{Mod}(\mathcal{A}, d))$  is in  $\text{Qis}$  if and only if it maps to an isomorphism in  $D(\mathcal{O})$ . Thus the lemma follows from Derived Categories, Lemma 5.4.  $\square$

In the situation of Lemma 26.3 we can apply Derived Categories, Proposition 5.6 to obtain an exact functor of triangulated categories

$$Q : K(\text{Mod}(\mathcal{A}, d)) \longrightarrow \text{Qis}^{-1}K(\text{Mod}(\mathcal{A}, d))$$

However, as  $\text{Mod}(\mathcal{A}, d)$  is a “big” category, i.e., its objects form a proper class, it isn’t immediately clear that given  $\mathcal{M}$  and  $\mathcal{N}$  the construction of  $\text{Qis}^{-1}K(\text{Mod}(\mathcal{A}, d))$  produces a *set*

$$\text{Mor}_{\text{Qis}^{-1}K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{N})$$

of morphisms. However, this is true thanks to our construction of K-injective complexes. Namely, by Theorem 25.13 we can choose a quasi-isomorphism  $s_0 : \mathcal{N} \rightarrow \mathcal{I}$  where  $\mathcal{I}$  is a graded injective and K-injective differential graded  $\mathcal{A}$ -module.

Next, recall that elements of the displayed set are equivalence classes of pairs  $(f : \mathcal{M} \rightarrow \mathcal{N}', s : \mathcal{N} \rightarrow \mathcal{N}')$  where  $f$  is an arbitrary morphism of  $K(\text{Mod}(\mathcal{A}, d))$  and  $s$  is a quasi-isomorphism, see the description of the left calculus of fractions in Categories, Section 27. By Lemma 25.10 we can choose the dotted arrow

$$\begin{array}{ccc} \mathcal{M} & & \mathcal{N} \\ & \searrow f & \swarrow s \quad \searrow s_0 \\ & \mathcal{N}' & \dashrightarrow \mathcal{I} \end{array}$$

making the diagram commute (in the homotopy category). Thus the pair  $(f, s)$  is equivalent to the pair  $(s' \circ f, s_0)$  and we find that the collection of equivalence classes forms a set.

0FT5 **Definition 26.4.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\text{Qis}$  be as in Lemma 26.3. The *derived category of  $(\mathcal{A}, d)$*  is the triangulated category

$$D(\mathcal{A}, d) = \text{Qis}^{-1}K(\text{Mod}(\mathcal{A}, d))$$

discussed in more detail above.

We prove some facts about this construction.

0FT6 **Lemma 26.5.** *In Definition 26.4 the kernel of the localization functor  $Q : K(\text{Mod}(\mathcal{A}, d)) \rightarrow D(\mathcal{A}, d)$  is the category  $\text{Ac}$  of Lemma 26.2.*

**Proof.** This is immediate from Derived Categories, Lemma 5.9 and the fact that  $0 \rightarrow \mathcal{M}$  is a quasi-isomorphism if and only if  $\mathcal{M}$  is acyclic.  $\square$

0FT7 **Lemma 26.6.** *In Definition 26.4 the functor  $H^0 : K(\text{Mod}(\mathcal{A}, d)) \rightarrow \text{Mod}(\mathcal{O})$  factors through a homological functor  $H^0 : D(\mathcal{A}, d) \rightarrow \text{Mod}(\mathcal{O})$ .*

**Proof.** Follows immediately from Derived Categories, Lemma 5.7.  $\square$

Here is the promised lemma computing morphism sets in the derived category.

0FT8 **Lemma 26.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}$  and  $\mathcal{N}$  be differential graded  $\mathcal{A}$ -modules. Let  $\mathcal{N} \rightarrow \mathcal{I}$  be a quasi-isomorphism with  $\mathcal{I}$  a graded injective and  $K$ -injective differential graded  $\mathcal{A}$ -module. Then*

$$\text{Hom}_{D(\mathcal{A}, d)}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{I})$$

**Proof.** Since  $\mathcal{N} \rightarrow \mathcal{I}$  is a quasi-isomorphism we see that

$$\text{Hom}_{D(\mathcal{A}, d)}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{D(\mathcal{A}, d)}(\mathcal{M}, \mathcal{I})$$

In the discussion preceding Definition 26.4 we found, using Lemma 25.10, that any morphism  $\mathcal{M} \rightarrow \mathcal{I}$  in  $D(\mathcal{A}, d)$  can be represented by a morphism  $f : \mathcal{M} \rightarrow \mathcal{I}$  in  $K(\text{Mod}(\mathcal{A}, d))$ . Now, if  $f, f' : \mathcal{M} \rightarrow \mathcal{I}$  are two morphism in  $K(\text{Mod}(\mathcal{A}, d))$ , then they define the same morphism in  $D(\mathcal{A}, d)$  if and only if there exists a quasi-isomorphism  $g : \mathcal{I} \rightarrow \mathcal{K}$  in  $K(\text{Mod}(\mathcal{A}, d))$  such that  $g \circ f = g \circ f'$ , see Categories, Lemma 27.6. However, by Lemma 25.10 there exists a map  $h : \mathcal{K} \rightarrow \mathcal{I}$  such that  $h \circ g = \text{id}_{\mathcal{I}}$  in  $K(\text{Mod}(\mathcal{A}, d))$ . Thus  $g \circ f = g \circ f'$  implies  $f = f'$  and the proof is complete.  $\square$



0FT9 **Lemma 26.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Then*

- (1)  $D(\mathcal{A}, d)$  has both direct sums and products,
- (2) direct sums are obtained by taking direct sums of differential graded  $\mathcal{A}$ -modules,
- (3) products are obtained by taking products of  $K$ -injective differential graded modules.

**Proof.** We will use that  $Mod(\mathcal{A}, d)$  is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in  $K(Mod(\mathcal{A}, d))$ . See Lemmas 13.2 and 21.3.

Let  $\mathcal{M}_j$  be a family of differential graded  $\mathcal{A}$ -modules. Consider the direct sum  $\mathcal{M} = \bigoplus \mathcal{M}_j$  as a differential graded  $\mathcal{A}$ -module. For a differential graded  $\mathcal{A}$ -module  $\mathcal{N}$  choose a quasi-isomorphism  $\mathcal{N} \rightarrow \mathcal{I}$  where  $\mathcal{I}$  is graded injective and  $K$ -injective as a differential graded  $\mathcal{A}$ -module. See Theorem 25.13. Using Lemma 26.7 we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A}, d)}(\mathcal{M}, \mathcal{N}) &= \mathrm{Hom}_{K(\mathcal{A}, d)}(\mathcal{M}, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{K(\mathcal{A}, d)}(\mathcal{M}_j, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{D(\mathcal{A}, d)}(\mathcal{M}_j, \mathcal{I}) \end{aligned}$$

whence the existence of direct sums in  $D(\mathcal{A}, d)$  as given in part (2) of the lemma.

Let  $\mathcal{M}_j$  be a family of differential graded  $\mathcal{A}$ -modules. For each  $j$  choose a quasi-isomorphism  $\mathcal{M} \rightarrow \mathcal{I}_j$  where  $\mathcal{I}_j$  is graded injective and  $K$ -injective as a differential graded  $\mathcal{A}$ -module. Consider the product  $\mathcal{I} = \prod \mathcal{I}_j$  of differential graded  $\mathcal{A}$ -modules. By Lemmas 25.8 and 25.4 we see that  $\mathcal{I}$  is graded injective and  $K$ -injective as a differential graded  $\mathcal{A}$ -module. For a differential graded  $\mathcal{A}$ -module  $\mathcal{N}$  using Lemma 26.7 we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A}, d)}(\mathcal{N}, \mathcal{I}) &= \mathrm{Hom}_{K(\mathcal{A}, d)}(\mathcal{N}, \mathcal{I}) \\ &= \prod \mathrm{Hom}_{K(\mathcal{A}, d)}(\mathcal{N}, \mathcal{I}_j) \\ &= \prod \mathrm{Hom}_{D(\mathcal{A}, d)}(\mathcal{N}, \mathcal{M}_j) \end{aligned}$$

whence the existence of products in  $D(\mathcal{A}, d)$  as given in part (3) of the lemma.  $\square$

## 27. The canonical delta-functor

0FTA Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Consider the functor  $Mod(\mathcal{A}, d) \rightarrow K(Mod(\mathcal{A}, d))$ . This functor is **not** a  $\delta$ -functor in general. However, it turns out that the functor  $Mod(\mathcal{A}, d) \rightarrow D(\mathcal{A}, d)$  is a  $\delta$ -functor. In order to see this we have to define the morphisms  $\delta$  associated to a short exact sequence

$$0 \rightarrow \mathcal{K} \xrightarrow{a} \mathcal{L} \xrightarrow{b} \mathcal{M} \rightarrow 0$$

in the abelian category  $Mod(\mathcal{A}, d)$ . Consider the cone  $C(a)$  of the morphism  $a$  together with its canonical morphisms  $i : \mathcal{L} \rightarrow C(a)$  and  $p : C(a) \rightarrow \mathcal{K}[1]$ , see Definition 22.2. There is a homomorphism of differential graded  $\mathcal{A}$ -modules

$$q : C(a) \rightarrow \mathcal{M}$$

by Differential Graded Algebra, Lemma 27.3 (which we may use by the discussion in Section 22) applied to the diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{a} & \mathcal{L} \\ \downarrow & & \downarrow b \\ 0 & \longrightarrow & \mathcal{M} \end{array}$$

The map  $q$  is a quasi-isomorphism for example because this is true in the category of morphisms of complexes of  $\mathcal{O}$ -modules, see discussion in Derived Categories, Section 12. According to Differential Graded Algebra, Lemma 27.13 (which we may use by the discussion in Section 22) the triangle

$$(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$$

is a distinguished triangle in  $K(\text{Mod}(\mathcal{A}, d))$ . As the localization functor  $K(\text{Mod}(\mathcal{A}, d)) \rightarrow D(\mathcal{A}, d)$  is exact we see that  $(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$  is a distinguished triangle in  $D(\mathcal{A}, d)$ . Since  $q$  is a quasi-isomorphism we see that  $q$  is an isomorphism in  $D(\mathcal{A}, d)$ . Hence we deduce that

$$(\mathcal{K}, \mathcal{L}, \mathcal{M}, a, b, -p \circ q^{-1})$$

is a distinguished triangle of  $D(\mathcal{A}, d)$ . This suggests the following lemma.

0FTB **Lemma 27.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . The localization functor  $\text{Mod}(\mathcal{A}, d) \rightarrow D(\mathcal{A}, d)$  has the natural structure of a  $\delta$ -functor, with*

$$\delta_{\mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{M}} = -p \circ q^{-1}$$

with  $p$  and  $q$  as explained above.

**Proof.** We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 3.6. This follows from Differential Graded Algebra, Lemma 27.3 (which we may use by the discussion in Section 22) with a bit of work. Compare with Derived Categories, Lemma 12.1.  $\square$

0FTC **Lemma 27.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Let  $\mathcal{M}_n$  be a system of differential graded  $\mathcal{A}$ -modules. Then the derived colimit  $\text{hocolim} \mathcal{M}_n$  in  $D(\mathcal{A}, d)$  is represented by the differential graded module  $\text{colim} \mathcal{M}_n$ .*

**Proof.** Set  $\mathcal{M} = \text{colim} \mathcal{M}_n$ . We have an exact sequence of differential graded  $\mathcal{A}$ -modules

$$0 \rightarrow \bigoplus \mathcal{M}_n \rightarrow \bigoplus \mathcal{M}_n \rightarrow \mathcal{M} \rightarrow 0$$

by Derived Categories, Lemma 33.6 (applied the underlying complexes of  $\mathcal{O}$ -modules). The direct sums are direct sums in  $D(\mathcal{A}, d)$  by Lemma 26.8. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 27.1).  $\square$

## 28. Derived pullback

0FTD Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \rightarrow \mathcal{A}$$

of differential graded  $f^{-1}\mathcal{O}_{\mathcal{D}}$ -algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map  $\varphi : f^*\mathcal{B} \rightarrow \mathcal{A}$  of differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebras or equivalently  $\varphi$  can be viewed as a map

$$\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$$

of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. See Remark 12.2.

In addition to the above, let  $\mathcal{A}'$  be a second differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra and let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{A}')$ -bimodule. In this setting we can consider the functor

$$Mod(\mathcal{B}, d) \longrightarrow Mod(\mathcal{A}', d), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

Observe that this extends to a functor

$$Mod^{dg}(\mathcal{B}, d) \longrightarrow Mod^{dg}(\mathcal{A}', d), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

of differential graded categories by the discussion in Sections 18 and 17. It follows formally that we also obtain an exact functor

0FTE (28.0.1) 
$$K(Mod(\mathcal{B}, d)) \longrightarrow K(Mod(\mathcal{A}', d)), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

of triangulated categories.

0FTF **Lemma 28.1.** *In the situation above, the functor (28.0.1) composed with the localization functor  $K(Mod(\mathcal{A}', d)) \rightarrow D(\mathcal{A}', d)$  has a left derived extension  $D(\mathcal{B}, d) \rightarrow D(\mathcal{A}', d)$  whose value on a good right differential graded  $\mathcal{B}$ -module  $\mathcal{P}$  is  $f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ .*

**Proof.** Recall that for any (right) differential graded  $\mathcal{B}$ -module  $\mathcal{M}$  there exists a quasi-isomorphism  $\mathcal{P} \rightarrow \mathcal{M}$  with  $\mathcal{P}$  a good differential graded  $\mathcal{B}$ -module. See Lemma 23.7. Hence by Derived Categories, Lemma 14.15 it suffices to show that given a quasi-isomorphism  $\mathcal{P} \rightarrow \mathcal{P}'$  of good differential graded  $\mathcal{B}$ -modules the induced map

$$f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \longrightarrow f^*\mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N}$$

is a quasi-isomorphism. The cone  $\mathcal{P}''$  on  $\mathcal{P} \rightarrow \mathcal{P}'$  is a good differential graded  $\mathcal{A}$ -module by Lemma 23.2. Since we have a distinguished triangle

$$\mathcal{P} \rightarrow \mathcal{P}' \rightarrow \mathcal{P}'' \rightarrow \mathcal{P}[1]$$

in  $K(Mod(\mathcal{B}, d))$  we obtain a distinguished triangle

$$f^*\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow f^*\mathcal{P}' \otimes_{\mathcal{A}} \mathcal{N} \rightarrow f^*\mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N} \rightarrow f^*\mathcal{P}[1] \otimes_{\mathcal{A}} \mathcal{N}$$

in  $K(Mod(\mathcal{A}', d))$ . By Lemma 23.8 the differential graded module  $f^*\mathcal{P}'' \otimes_{\mathcal{A}} \mathcal{N}$  is acyclic and the proof is complete.  $\square$

0FTG **Definition 28.2.** Derived tensor product and derived pullback.

- (1) Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{B}$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. The functor  $D(\mathcal{A}, d) \rightarrow D(\mathcal{B}, d)$  constructed in Lemma 28.1 is called the *derived tensor product* and denoted  $- \otimes_{\mathcal{A}}^L \mathcal{N}$ .

- (2) Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Let  $\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$  be a homomorphism of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. The functor  $D(\mathcal{B}, d) \rightarrow D(\mathcal{A}, d)$  constructed in Lemma 28.1 is called *derived pullback* and denote  $Lf^*$ .

With this language in place we can express some obvious compatibilities.

0FTH **Lemma 28.3.** *In Lemma 28.1 the functor  $D(\mathcal{B}, d) \rightarrow D(\mathcal{A}', d)$  is equal to  $\mathcal{M} \mapsto Lf^*\mathcal{M} \otimes_{\mathcal{A}}^L \mathcal{N}$ .*

**Proof.** Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because  $f^*\mathcal{P}$  is a good differential graded  $\mathcal{A}$ -module if  $\mathcal{P}$  is a good differential graded  $\mathcal{B}$ -module.  $\square$

0FTI **Lemma 28.4.** *Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  and  $(g, g^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}''), \mathcal{O}'')$  be morphisms of ringed topoi. Let  $\mathcal{A}$ ,  $\mathcal{A}'$ , and  $\mathcal{A}''$  be a differential graded  $\mathcal{O}$ -algebra,  $\mathcal{O}'$ -algebra, and  $\mathcal{O}''$ -algebra. Let  $\varphi : \mathcal{A}' \rightarrow f_*\mathcal{A}$  and  $\varphi' : \mathcal{A}'' \rightarrow g_*\mathcal{A}'$  be a homomorphism of differential graded  $\mathcal{O}'$ -algebras and  $\mathcal{O}''$ -algebras. Then we have  $L(g \circ f)^* = Lf^* \circ Lg^* : D(\mathcal{A}'', d) \rightarrow D(\mathcal{A}, d)$ .*

**Proof.** Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because  $f^*\mathcal{P}$  is a good differential graded  $\mathcal{A}'$ -module if  $\mathcal{P}$  is a good differential graded  $\mathcal{A}$ -module.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{B}$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N} \rightarrow \mathcal{N}'$  be a homomorphism of differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodules. Then we obtain canonical maps

$$t : \mathcal{M} \otimes_{\mathcal{A}}^L \mathcal{N} \longrightarrow \mathcal{M} \otimes_{\mathcal{A}}^L \mathcal{N}'$$

functorial in  $\mathcal{M}$  in  $D(\mathcal{A}, d)$  which define a natural transformation between exact functors  $D(\mathcal{A}, d) \rightarrow D(\mathcal{B}, d)$  of triangulated categories. The value of  $t$  on a good differential graded  $\mathcal{A}$ -module  $\mathcal{P}$  is the obvious map

$$\mathcal{P} \otimes_{\mathcal{A}}^L \mathcal{N} = \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}' = \mathcal{P} \otimes_{\mathcal{A}}^L \mathcal{N}'$$

0FTJ **Lemma 28.5.** *In the situation above, if  $\mathcal{N} \rightarrow \mathcal{N}'$  is an isomorphism on cohomology sheaves, then  $t$  is an isomorphism of functors  $(- \otimes_{\mathcal{A}}^L \mathcal{N}) \rightarrow (- \otimes_{\mathcal{A}}^L \mathcal{N}')$ .*

**Proof.** It is enough to show that  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$  is an isomorphism on cohomology sheaves for any good differential graded  $\mathcal{A}$ -module  $\mathcal{P}$ . To do this, let  $\mathcal{N}''$  be the cone on the map  $\mathcal{N} \rightarrow \mathcal{N}'$  as a left differential graded  $\mathcal{A}$ -module, see Definition 22.2. (To be sure,  $\mathcal{N}''$  is a bimodule too but we don't need this.) By functoriality of the tensor construction (it is a functor of differential graded categories) we see that  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}''$  is the cone (as a complex of  $\mathcal{O}$ -modules) on the map  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}'$ . Hence it suffices to show that  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}''$  is acyclic. This follows from the fact that  $\mathcal{P}$  is good and the fact that  $\mathcal{N}''$  is acyclic as a cone on a quasi-isomorphism.  $\square$

0FTK **Lemma 28.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{B}$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. If  $\mathcal{N}$  is good as a left differential graded  $\mathcal{A}$ -module, then we have  $\mathcal{M} \otimes_{\mathcal{A}}^L \mathcal{N} = \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  for all differential graded  $\mathcal{A}$ -modules  $\mathcal{M}$ .*

**Proof.** Let  $\mathcal{P} \rightarrow \mathcal{M}$  be a quasi-isomorphism where  $\mathcal{P}$  is a good (right) differential graded  $\mathcal{A}$ -module. To prove the lemma we have to show that  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  is a quasi-isomorphism. The cone  $C$  on the map  $\mathcal{P} \rightarrow \mathcal{M}$  is an acyclic right differential graded  $\mathcal{A}$ -module. Hence  $C \otimes_{\mathcal{A}} \mathcal{N}$  is acyclic as  $\mathcal{N}$  is assumed good as a left differential graded  $\mathcal{A}$ -module. Since  $C \otimes_{\mathcal{A}} \mathcal{N}$  is the cone on the maps  $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$  as a complex of  $\mathcal{O}$ -modules we conclude.  $\square$

0FTL **Lemma 28.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{A}', \mathcal{A}''$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  and  $\mathcal{N}'$  be a differential graded  $(\mathcal{A}, \mathcal{A}')$ -bimodule and  $(\mathcal{A}', \mathcal{A}'')$ -bimodule. Assume that the canonical map*

$$\mathcal{N} \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

*in  $D(\mathcal{A}'', d)$  is a quasi-isomorphism. Then we have*

$$(\mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{N}) \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}' = \mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$$

*as functors  $D(\mathcal{A}, d) \rightarrow D(\mathcal{A}'', d)$ .*

**Proof.** Choose a good differential graded  $\mathcal{A}$ -module  $\mathcal{P}$  and a quasi-isomorphism  $\mathcal{P} \rightarrow \mathcal{M}$ , see Lemma 23.7. Then

$$\mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}') = \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

and we have

$$(\mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{N}) \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}' = (\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}'$$

Thus we have to show the canonical map

$$(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}' \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

is a quasi-isomorphism. Choose a quasi-isomorphism  $\mathcal{Q} \rightarrow \mathcal{N}'$  where  $\mathcal{Q}$  is a good left differential graded  $\mathcal{A}'$ -module (Lemma 23.7). By Lemma 28.6 the map above as a map in the derived category of  $\mathcal{O}$ -modules is the map

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

Since  $\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \rightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$  is a quasi-isomorphism by assumption and  $\mathcal{P}$  is a good differential graded  $\mathcal{A}$ -module this map is an quasi-isomorphism by Lemma 28.5 (the left and right hand side compute  $\mathcal{P} \otimes_{\mathcal{A}}^{\mathbf{L}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q})$  and  $\mathcal{P} \otimes_{\mathcal{A}}^{\mathbf{L}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$  or you can just repeat the argument in the proof of the lemma).  $\square$

## 29. Derived pushforward

0FTM The existence of enough K-injective guarantees that we can take the right derived functor of any exact functor on the homotopy category.

0FTN **Lemma 29.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{A}, d)$  be a sheaf of differential graded algebras on  $(\mathcal{C}, \mathcal{O})$ . Then any exact functor*

$$T : K(\text{Mod}(\mathcal{A}, d)) \longrightarrow \mathcal{D}$$

*of triangulated categories has a right derived extension  $RT : D(\mathcal{A}, d) \rightarrow \mathcal{D}$  whose value on a graded injective and K-injective differential graded  $\mathcal{A}$ -module  $\mathcal{I}$  is  $T(\mathcal{I})$ .*

**Proof.** By Theorem 25.13 for any (right) differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  there exists a quasi-isomorphism  $\mathcal{M} \rightarrow \mathcal{I}$  where  $\mathcal{I}$  is a graded injective and K-injective differential graded  $\mathcal{A}$ -module. Hence by Derived Categories, Lemma 14.15 it suffices to show that given a quasi-isomorphism  $\mathcal{I} \rightarrow \mathcal{I}'$  of differential graded  $\mathcal{A}$ -modules

which are both graded injective and K-injective then  $T(\mathcal{I}) \rightarrow T(\mathcal{I}')$  is an isomorphism. This is true because the map  $\mathcal{I} \rightarrow \mathcal{I}'$  is an isomorphism in  $K(\text{Mod}(\mathcal{A}, d))$  as follows for example from Lemma 26.7 (or one can deduce it from Lemma 25.10).  $\square$

There are a number of functors we have already seen to which this applies. Here are two examples.

0FTP **Definition 29.2.** Derived internal hom and derived pushforward.

- (1) Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{B}$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. The right derived extension

$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, d) \longrightarrow D(\mathcal{A}, d)$$

of the internal hom functor  $\mathcal{H}om_{\mathcal{B}}^{dg}(\mathcal{N}, -)$  is called *derived internal hom*.

- (2) Let  $(f, f^{\sharp}) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Let  $\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$  be a homomorphism of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. The right derived extension

$$Rf_* : D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d)$$

of the pushforward  $f_*$  is called *derived pushforward*.

It turns out that  $Rf_* : D(\mathcal{A}, d) \rightarrow D(\mathcal{B}, d)$  agrees with derived pushforward on underlying complexes of  $\mathcal{O}$ -modules, see Lemma 29.8.

These functors are the adjoints of derived pullback and derived tensor product.

0FTQ **Lemma 29.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}, \mathcal{B}$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{B})$ -bimodule. Then*

$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, d) \longrightarrow D(\mathcal{A}, d)$$

*is right adjoint to*

$$- \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{N} : D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d)$$

**Proof.** This follows from Derived Categories, Lemma 30.1 and Lemma 17.3.  $\square$

0FTR **Lemma 29.4.** *Let  $(f, f^{\sharp}) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Let  $\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$  be a homomorphism of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. Then*

$$Rf_* : D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d)$$

*is right adjoint to*

$$Lf^* : D(\mathcal{B}, d) \longrightarrow D(\mathcal{A}, d)$$

**Proof.** This follows from Derived Categories, Lemma 30.1 and Lemma 18.1.  $\square$

Next, we discuss what happens in the situation considered in Section 28.

Let  $(f, f^{\sharp}) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \rightarrow \mathcal{A}$$

of differential graded  $f^{-1}\mathcal{O}_{\mathcal{D}}$ -algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map  $\varphi : f^*\mathcal{B} \rightarrow \mathcal{A}$  of differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebras or equivalently  $\varphi$  can be viewed as a map

$$\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$$

of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. See Remark 12.2.

In addition to the above, let  $\mathcal{A}'$  be a second differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra and let  $\mathcal{N}$  be a differential graded  $(\mathcal{A}, \mathcal{A}')$ -bimodule. In this setting we can consider the functor

$$\text{Mod}(\mathcal{A}', d) \longrightarrow \text{Mod}(\mathcal{B}, d), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om_{\mathcal{A}'}^{dg}(\mathcal{N}, \mathcal{M})$$

Observe that this extends to a functor

$$\text{Mod}^{dg}(\mathcal{A}', d) \longrightarrow \text{Mod}^{dg}(\mathcal{B}, d), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om_{\mathcal{A}'}^{dg}(\mathcal{N}, \mathcal{M})$$

of differential graded categories by the discussion in Sections 18 and 17. It follows formally that we also obtain an exact functor

$$\text{0FTS} \quad (29.4.1) \quad K(\text{Mod}(\mathcal{A}', d)) \longrightarrow K(\text{Mod}(\mathcal{B}, d)), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om_{\mathcal{A}'}^{dg}(\mathcal{N}, \mathcal{M})$$

of triangulated categories.

**0FTT Lemma 29.5.** *In the situation above, denote  $RT : D(\mathcal{A}', d) \rightarrow D(\mathcal{B}, d)$  the right derived extension of (29.4.1). Then we have*

$$RT(\mathcal{M}) = Rf_* R\mathcal{H}om(\mathcal{N}, \mathcal{M})$$

*functorially in  $\mathcal{M}$ .*

**Proof.** By Lemmas 17.3 and 18.1 the functor (29.4.1) is right adjoint to the functor (28.0.1). By Derived Categories, Lemma 30.1 the functor  $RT$  is right adjoint to the functor of Lemma 28.1 which is equal to  $Lf^*(-) \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{N}$  by Lemma 28.3. By Lemmas 29.3 and 29.4 the functor  $Lf^*(-) \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{N}$  is left adjoint to  $Rf_* R\mathcal{H}om(\mathcal{N}, -)$ . Thus we conclude by uniqueness of adjoints.  $\square$

**0FTU Lemma 29.6.** *Let  $(f, f^{\sharp}) : (\text{Sh}(\mathcal{C}), \mathcal{O}) \rightarrow (\text{Sh}(\mathcal{C}'), \mathcal{O}')$  and  $(g, g^{\sharp}) : (\text{Sh}(\mathcal{C}'), \mathcal{O}') \rightarrow (\text{Sh}(\mathcal{C}''), \mathcal{O}'')$  be morphisms of ringed topoi. Let  $\mathcal{A}$ ,  $\mathcal{A}'$ , and  $\mathcal{A}''$  be a differential graded  $\mathcal{O}$ -algebra,  $\mathcal{O}'$ -algebra, and  $\mathcal{O}''$ -algebra. Let  $\varphi : \mathcal{A}' \rightarrow f_*\mathcal{A}$  and  $\varphi' : \mathcal{A}'' \rightarrow g_*\mathcal{A}'$  be a homomorphism of differential graded  $\mathcal{O}'$ -algebras and  $\mathcal{O}''$ -algebras. Then we have  $R(g \circ f)_* = Rg_* \circ Rf_* : D(\mathcal{A}, d) \rightarrow D(\mathcal{A}'', d)$ .*

**Proof.** Follows from Lemmas 28.4 and 29.4 and uniqueness of adjoints.  $\square$

**0FTV Lemma 29.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$ ,  $\mathcal{A}'$ ,  $\mathcal{A}''$  be differential graded  $\mathcal{O}$ -algebras. Let  $\mathcal{N}$  and  $\mathcal{N}'$  be a differential graded  $(\mathcal{A}, \mathcal{A}')$ -bimodule and  $(\mathcal{A}', \mathcal{A}'')$ -bimodule. Assume that the canonical map*

$$\mathcal{N} \otimes_{\mathcal{A}'}^{\mathbf{L}} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

*in  $D(\mathcal{A}'', d)$  is a quasi-isomorphism. Then we have*

$$R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}', -) = R\mathcal{H}om_{\mathcal{A}'}(\mathcal{N}, R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N}', -))$$

*as functors  $D(\mathcal{A}'', d) \rightarrow D(\mathcal{A}, d)$ .*

**Proof.** Follows from Lemmas 28.7 and 29.3 and uniqueness of adjoints.  $\square$

0FTW **Lemma 29.8.** *Let  $(f, f^\#) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra. Let  $\mathcal{B}$  be a differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebra. Let  $\varphi : \mathcal{B} \rightarrow f_*\mathcal{A}$  be a homomorphism of differential graded  $\mathcal{O}_{\mathcal{D}}$ -algebras. The diagram*

$$\begin{array}{ccc} D(\mathcal{A}, d) & \xrightarrow{\text{forget}} & D(\mathcal{O}_{\mathcal{C}}) \\ Rf_* \downarrow & & \downarrow Rf_* \\ D(\mathcal{B}, d) & \xrightarrow{\text{forget}} & D(\mathcal{O}_{\mathcal{D}}) \end{array}$$

*commutes.*

**Proof.** Besides identifying some categories, this lemma follows immediately from Lemma 29.6.

We may view  $(\mathcal{O}_{\mathcal{C}}, 0)$  as a differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebra by placing  $\mathcal{O}_{\mathcal{C}}$  in degree 0 and endowing it with the zero differential. It is clear that we have

$$\text{Mod}(\mathcal{O}_{\mathcal{C}}, 0) = \text{Comp}(\mathcal{O}_{\mathcal{C}}) \quad \text{and} \quad D(\mathcal{O}_{\mathcal{C}}, 0) = D(\mathcal{O}_{\mathcal{C}})$$

Via this identification the forgetful functor  $\text{Mod}(\mathcal{A}, d) \rightarrow \text{Comp}(\mathcal{O}_{\mathcal{C}})$  is the “push-forward”  $\text{id}_{\mathcal{C},*}$  defined in Section 18 corresponding to the identity morphism  $\text{id}_{\mathcal{C}} : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{C}, \mathcal{O}_{\mathcal{C}})$  of ringed topoi and the map  $(\mathcal{O}_{\mathcal{C}}, 0) \rightarrow (\mathcal{A}, d)$  of differential graded  $\mathcal{O}_{\mathcal{C}}$ -algebras. Since  $\text{id}_{\mathcal{C},*}$  is exact, we immediately see that

$$\text{Rid}_{\mathcal{C},*} = \text{forget} : D(\mathcal{A}, d) \longrightarrow D(\mathcal{O}_{\mathcal{C}}, 0) = D(\mathcal{O}_{\mathcal{C}})$$

The exact same reasoning shows that

$$\text{Rid}_{\mathcal{D},*} = \text{forget} : D(\mathcal{B}, d) \longrightarrow D(\mathcal{O}_{\mathcal{D}}, 0) = D(\mathcal{O}_{\mathcal{D}})$$

Moreover, the construction of  $Rf_* : D(\mathcal{O}_{\mathcal{C}}) \rightarrow D(\mathcal{O}_{\mathcal{D}})$  of Cohomology on Sites, Section 19 agrees with the construction of  $Rf_* : D(\mathcal{O}_{\mathcal{C}}, 0) \rightarrow D(\mathcal{O}_{\mathcal{D}}, 0)$  in Definition 29.2 as both functors are defined as the right derived extension of pushforward on underlying complexes of modules. By Lemma 29.6 we see that both  $Rf_* \circ \text{Rid}_{\mathcal{C},*}$  and  $\text{Rid}_{\mathcal{D},*} \circ Rf_*$  are the derived functors of  $f_* \circ \text{forget} = \text{forget} \circ f_*$  and hence equal by uniqueness of adjoints.  $\square$

0FTX **Lemma 29.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{A}$  be a differential graded  $\mathcal{O}$ -algebra. Let  $\mathcal{M}$  be a differential graded  $\mathcal{A}$ -module. Let  $n \in \mathbf{Z}$ . We have*

$$H^n(\mathcal{C}, \mathcal{M}) = \text{Hom}_{D(\mathcal{A}, d)}(\mathcal{A}, \mathcal{M}[n])$$

*where on the left hand side we have the cohomology of  $\mathcal{M}$  viewed as a complex of  $\mathcal{O}$ -modules.*

**Proof.** To prove the formula, observe that

$$R\Gamma(\mathcal{C}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{I})$$

where  $\mathcal{M} \rightarrow \mathcal{I}$  is a quasi-isomorphism to a graded injective and K-injective differential graded  $\mathcal{A}$ -module  $\mathcal{I}$  (combine Lemmas 29.1 and 29.8). By Lemma 26.7 we have

$$\text{Hom}_{D(\mathcal{A}, d)}(\mathcal{A}, \mathcal{M}[n]) = \text{Hom}_{K(\text{Mod}(\mathcal{A}, d))}(\mathcal{M}, \mathcal{I}[n]) = H^0(\Gamma(\mathcal{C}, \mathcal{I}[n])) = H^n(\Gamma(\mathcal{C}, \mathcal{I}))$$

Combining these two results we obtain our equality.  $\square$



### 30. Equivalences of derived categories

0FTY This section is the analogue of Differential Graded Algebra, Section 37.

0FTZ **Lemma 30.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a homomorphism of differential graded  $\mathcal{O}$ -algebras which induces an isomorphism on cohomology sheaves, then*

$$D(\mathcal{A}, d) \longrightarrow D(\mathcal{B}, d), \quad \mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B}$$

*is an equivalence of categories.*

**Proof.** Recall that the restriction functor

$$\text{Mod}^{dg}(\mathcal{B}, d) \rightarrow \text{Mod}^{dg}(\mathcal{A}, d), \quad \mathcal{N} \mapsto \text{res}_{\varphi} \mathcal{N}$$

is a right adjoint to

$$\text{Mod}^{dg}(\mathcal{A}, d) \rightarrow \text{Mod}^{dg}(\mathcal{B}, d), \quad \mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$$

See Section 17. Since restriction sends quasi-isomorphisms to quasi-isomorphisms, we see that it trivially has a left derived extension (given by restriction). This functor will be right adjoint to  $-\otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B}$  by Derived Categories, Lemma 30.1. The adjunction map

$$\mathcal{M} \rightarrow \text{res}_{\varphi}(\mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B})$$

is an isomorphism in  $D(\mathcal{A}, d)$  by our assumption that  $\mathcal{A} \rightarrow \mathcal{B}$  is a quasi-isomorphism of (left) differential graded  $\mathcal{A}$ -modules. In particular, the functor of the lemma is fully faithful, see Categories, Lemma 24.4. It is clear that the kernel of the restriction functor  $D(\mathcal{B}, d) \rightarrow D(\mathcal{A}, d)$  is zero. Thus we conclude by Derived Categories, Lemma 7.2.  $\square$

### 31. Resolutions of differential graded algebras

0FU0 This section is the analogue of Differential Graded Algebra, Section 38.

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. As in Remark 23.5 consider a sheaf of graded sets  $\mathcal{S}$  on  $\mathcal{C}$ . Let us think of the  $r$ -fold self product  $\mathcal{S} \times \dots \times \mathcal{S}$  as a sheaf of graded sets with the rule  $\deg(s_1 \cdot \dots \cdot s_r) = \sum \deg(s_i)$ . Here given local sections  $s_i \in \mathcal{S}(U)$ ,  $i = 1, \dots, r$  we use  $s_1 \cdot \dots \cdot s_r$  to denote the corresponding section of  $\mathcal{S} \times \dots \times \mathcal{S}$  over  $U$ . Let us denote  $\mathcal{O}\langle \mathcal{S} \rangle$  the free graded  $\mathcal{O}$ -algebra on  $\mathcal{S}$ . More precisely, we set

$$\mathcal{O}\langle \mathcal{S} \rangle = \mathcal{O} \oplus \bigoplus_{r \geq 1} \mathcal{O}[\mathcal{S} \times \dots \times \mathcal{S}]$$

with notation as in Remark 23.5. This becomes a sheaf of graded  $\mathcal{O}$ -algebras by concatenation

$$(s_1 \cdot \dots \cdot s_r)(s'_1 \cdot \dots \cdot s'_{r'}) = s_1 \cdot \dots \cdot s_r \cdot s'_1 \cdot \dots \cdot s'_{r'}$$

We may endow  $\mathcal{O}\langle \mathcal{S} \rangle$  with a differential by setting  $d(s) = 0$  for all local sections  $s$  of  $\mathcal{S}$  and extending uniquely using the Leibniz rule although it is important to also consider other differentials.

Indeed, suppose that we are given a system of the following kind

- (1) for  $i = 0, 1, 2, \dots$  sheaves of graded sets  $\mathcal{S}_i$ ,
- (2) for  $i = 0, 1, 2, \dots$  maps

$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i = \mathcal{O}\langle \mathcal{S}_0 \amalg \dots \amalg \mathcal{S}_i \rangle$$

of sheaves of graded sets of degree 1 whose image is contained in the kernel of the inductively defined differential on the target.

More precisely, we first set  $\mathcal{A}_0 = \mathcal{O}\langle \mathcal{S}_0 \rangle$  and we endow it with the unique differential satisfying the Leibniz rule where  $d(s) = 0$  for any local section  $s$  of  $\mathcal{S}$ . By induction, assume given a differential  $d$  on  $\mathcal{A}_i$ . Then we extend it to the unique differential on  $\mathcal{A}_{i+1}$  satisfying the Leibniz rule and with

$$d(s) = \delta(s)$$

where  $\delta(s) = \delta_j(s)$  if  $s$  is in the summand  $\mathcal{S}_j$  of  $\mathcal{S}_0 \amalg \dots \amalg \mathcal{S}_{i+1}$ . This makes sense exactly because  $\delta(s)$  is in the kernel of the inductively defined differential.

0FU1 **Lemma 31.1.** *In the situation above the differential graded  $\mathcal{O}$ -algebra*

$$\mathcal{A} = \operatorname{colim} \mathcal{A}_i$$

*has the following property: for any morphism  $(f, f^\#) : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  of ringed topoi, the pullback  $f^*\mathcal{A}$  is flat as a graded  $\mathcal{O}'$ -module and is K-flat as a complex of  $\mathcal{O}'$ -modules.*

**Proof.** Observe that  $f^*\mathcal{A} = \operatorname{colim} f^*\mathcal{A}_i$  and that

$$f^*\mathcal{A}_i = \mathcal{O}'\langle f^{-1}\mathcal{S}_0 \amalg \dots \amalg f^{-1}\mathcal{S}_i \rangle$$

with differential given by the inductive procedure above using  $f^{-1}\delta_{i+1}$ . Thus it suffices to prove that  $\mathcal{A}$  is flat as a graded  $\mathcal{O}$ -module and is K-flat as a complex of  $\mathcal{O}$ -modules. For this it suffices to prove that each  $\mathcal{A}_i$  is flat as a graded  $\mathcal{O}$ -module and is K-flat as a complex of  $\mathcal{O}$ -modules, compare with Lemma 23.3.

For  $i \geq 1$  write  $\mathcal{S} = \mathcal{S}_0 \amalg \dots \amalg \mathcal{S}_i$  so that we have  $\mathcal{A}_i = \mathcal{O}\langle \mathcal{S} \rangle$  as a graded  $\mathcal{O}$ -algebra. We are going to construct a filtration of this algebra by differential graded  $\mathcal{O}$ -submodules.

Set  $W = \mathbf{Z}_{\geq 0}^{i+1}$  considered with lexicographical ordering. Namely, given  $w = (w_0, \dots, w_i)$  and  $w' = (w'_0, \dots, w'_i)$  in  $W$  we say

$$w > w' \Leftrightarrow \exists j, 0 \leq j \leq i : w_j = w'_j, w_{j-1} = w'_{j-1}, \dots, w_{j+1} = w'_{j+1}, w_j > w'_j$$

and so on. Suppose given a section  $s = s_1 \cdot \dots \cdot s_r$  of  $\mathcal{S} \times \dots \times \mathcal{S}$  over  $U$ . We say that the *weight of  $s$  is defined* if we have  $s_a \in \mathcal{S}_{j_a}(U)$  for a unique  $0 \leq j_a \leq i$ . In this case we define the weight

$$w(s) = (w_0(s), \dots, w_i(s)) \in W, \quad w_j(s) = |\{a \mid j_a = j\}|$$

The weight of any section of  $\mathcal{S} \times \dots \times \mathcal{S}$  is defined locally. The reader checks easily that we obtain a disjoint union decomposition

$$\mathcal{S} \times \dots \times \mathcal{S} = \coprod_{w \in W} (\mathcal{S} \times \dots \times \mathcal{S})_w$$

into the subsheaves of sections of a given weight. Of course only  $w \in W$  with  $\sum_{0 \leq j \leq i} w_j = r$  show up for a given  $r$ . We correspondingly obtain a decomposition

$$\mathcal{A}_i = \mathcal{O} \oplus \bigoplus_{r \geq 1} \bigoplus_{w \in W} \mathcal{O}[(\mathcal{S} \times \dots \times \mathcal{S})_w]$$

The rest of the proof relies on the following trivial observation: given  $r, w$  and local section  $s = s_1 \cdot \dots \cdot s_r$  of  $(\mathcal{S} \times \dots \times \mathcal{S})_w$  we have

$$d(s) \text{ is a local section of } \mathcal{O} \oplus \bigoplus_{r' \geq 1} \bigoplus_{w' \in W, w' < w} \mathcal{O}[(\mathcal{S} \times \dots \times \mathcal{S})_{w'}]$$

The reason is that in each of the expressions

$$(-1)^{\deg(s_1) + \dots + \deg(s_{a-1})} s_1 \cdot \dots \cdot s_{a-1} \cdot \delta(s_a) \cdot s_{a+1} \cdot \dots \cdot s_r$$

whose sum give the element  $d(s)$  the element  $\delta(s_a)$  is locally a  $\mathcal{O}$ -linear combination of elements  $s'_1 \cdot \dots \cdot s'_{r'}$  with  $s'_{a'}$  in  $\mathcal{S}_{j'_{a'}}$  for some  $0 \leq j'_{a'} < j_a$  where  $j_a$  is such that  $s_a$  is section of  $\mathcal{S}_{j_a}$ .

What this means is the following. Suppose for  $w \in W$  we set

$$F_w \mathcal{A}_i = \mathcal{O} \oplus \bigoplus_{r \geq 1} \bigoplus_{w' \in W, w' \leq w} \mathcal{O}[(\mathcal{S} \times \dots \times \mathcal{S})_{w'}]$$

By the observation above this is a differential graded  $\mathcal{O}$ -submodule. We get admissible short exact sequences

$$0 \rightarrow \operatorname{colim}_{w' < w} F_{w'} \mathcal{A}_i \rightarrow F_w \mathcal{A}_i \rightarrow \bigoplus_{r \geq 1} \mathcal{O}[(\mathcal{S} \times \dots \times \mathcal{S})_w] \rightarrow 0$$

of differential graded  $\mathcal{A}$ -modules where the differential on the right hand side is zero.

Now we finish the proof by transfinite induction over the ordered set  $W$ . The differential graded complex  $F_0 \mathcal{A}_0$  is the summand  $\mathcal{O}$  and this is  $K$ -flat and graded flat. For  $w \in W$  if the result is true for  $F_{w'} \mathcal{A}_i$  for  $w' < w$ , then by Lemmas 23.3, 23.2, and 23.6 we obtain the result for  $w$ . Finally, we have  $\mathcal{A}_i = \operatorname{colim}_{w \in W} F_w \mathcal{A}_i$  and we conclude.  $\square$

0FU2 **Lemma 31.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{B}, d)$  be a differential graded  $\mathcal{O}$ -algebra. There exists a quasi-isomorphism of differential graded  $\mathcal{O}$ -algebras  $(\mathcal{A}, d) \rightarrow (\mathcal{B}, d)$  such that  $\mathcal{A}$  is graded flat and  $K$ -flat as a complex of  $\mathcal{O}$ -modules and such that the same is true after pullback by any morphism of ringed topoi.*

**Proof.** The proof is exactly the same as the first proof of Lemma 23.7 but now working with free graded algebras instead of free graded modules.

We will construct  $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$  as in Lemma 31.1 by constructing

$$\mathcal{A}_0 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \dots \rightarrow \mathcal{B}$$

Let  $\mathcal{S}_0$  be the sheaf of graded sets (Remark 23.5) whose degree  $n$  part is  $\operatorname{Ker}(d_{\mathcal{B}}^n)$ . Consider the homomorphism of differential graded modules

$$\mathcal{A}_0 = \mathcal{O}\langle \mathcal{S}_0 \rangle \longrightarrow \mathcal{B}$$

where map sends a local section  $s$  of  $\mathcal{S}_0$  to the corresponding local section of  $\mathcal{A}^{\deg(s)}$  (which is in the kernel of the differential, so our map is a map of differential graded algebras indeed). By construction the induced maps on cohomology sheaves  $H^n(\mathcal{A}_0) \rightarrow H^n(\mathcal{B})$  are surjective and hence the same will remain true for all  $i$ .

Induction step of the construction. Given  $\mathcal{A}_i \rightarrow \mathcal{B}$  denote  $\mathcal{S}_{i+1}$  the sheaf of graded sets whose degree  $n$  part is

$$\operatorname{Ker}(d_{\mathcal{A}_i}^{n+1}) \times_{\mathcal{B}^{n+1, d}} \mathcal{B}^n$$

This comes equipped with a canonical map

$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i$$

whose image is contained in the kernel of  $d_{\mathcal{A}_i}$  by construction. Hence  $\mathcal{A}_{i+1} = \mathcal{O}\langle \mathcal{S}_0 \amalg \dots \amalg \mathcal{S}_{i+1} \rangle$  has a differential extending the differential on  $\mathcal{A}_i$ , see discussion at the start of this section. The map from  $\mathcal{A}_{i+1}$  to  $\mathcal{B}$  is the unique map of graded algebras which restricts to the given map on  $\mathcal{A}_i$  and sends a local section  $s = (a, b)$

of  $\mathcal{S}_{i+1}$  to  $b$  in  $\mathcal{B}$ . This is compatible with differentials exactly because  $d(b)$  is the image of  $a$  in  $\mathcal{B}$ .

The map  $\mathcal{A} \rightarrow \mathcal{B}$  is a quasi-isomorphism: we have  $H^n(\mathcal{A}) = \text{colim } H^n(\mathcal{A}_i)$  and for each  $i$  the map  $H^n(\mathcal{A}_i) \rightarrow H^n(\mathcal{B})$  is surjective with kernel annihilated by the map  $H^n(\mathcal{A}_i) \rightarrow H^n(\mathcal{A}_{i+1})$  by construction. Finally, the flatness condition for  $\mathcal{A}$  where shown in Lemma 31.1.  $\square$

### 32. Miscellany

0FU3 Let  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. Let  $\mathcal{A}$  be a sheaf of differential graded  $\mathcal{O}$ -algebras. Using the composition<sup>3</sup>

$$\mathcal{A} \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{A}$$

and the relative cup product (see Cohomology on Sites, Remark 19.7 and Section 33) we obtain a multiplication<sup>4</sup>

$$\mu : Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \longrightarrow Rf_* \mathcal{A}$$

in  $D(\mathcal{O}')$ . This multiplication is associative in the sense that the diagram

$$\begin{array}{ccc} Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} & \xrightarrow{\mu \otimes 1} & Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \\ \downarrow 1 \otimes \mu & & \downarrow \mu \\ Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} & \xrightarrow{\mu} & Rf_* \mathcal{A} \end{array}$$

commutes in  $D(\mathcal{O}')$ ; this follows from Cohomology on Sites, Lemma 33.2. In exactly the same way, given a right differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  we obtain a multiplication

$$\mu_{\mathcal{M}} : Rf_* \mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \longrightarrow Rf_* \mathcal{M}$$

in  $D(\mathcal{O}')$ . This multiplication is compatible with  $\mu$  above in the sense that the diagram

$$\begin{array}{ccc} Rf_* \mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} & \xrightarrow{\mu_{\mathcal{M}} \otimes 1} & Rf_* \mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} \\ \downarrow 1 \otimes \mu & & \downarrow \mu_{\mathcal{M}} \\ Rf_* \mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_* \mathcal{A} & \xrightarrow{\mu_{\mathcal{M}}} & Rf_* \mathcal{M} \end{array}$$

commutes in  $D(\mathcal{O}')$ ; again this follows from Cohomology on Sites, Lemma 33.2.

A particular example of the above is when one takes  $f$  to be the morphism to the punctual topos  $Sh(pt)$ . In that case  $\mu$  is just the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{A}) \otimes_{\Gamma(\mathcal{C}, \mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{A}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

and similarly  $\mu_{\mathcal{M}}$  is the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{M}) \otimes_{\Gamma(\mathcal{C}, \mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{M}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

<sup>3</sup>It would be more precise to write  $F(\mathcal{A}) \otimes_{\mathcal{O}}^{\mathbf{L}} F(\mathcal{A}) \rightarrow F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \rightarrow F(\mathcal{A})$  were  $F$  denotes the forgetful functor to complexes of  $\mathcal{O}$ -modules. Also, note that  $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}$  indicates the tensor product of Section 15 so that  $F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) = \text{Tot}(F(\mathcal{A}) \otimes_{\mathcal{O}} F(\mathcal{A}))$ . The first arrow of the sequence is the canonical map from the derived tensor product of two complexes of  $\mathcal{O}$ -modules to the usual tensor product of complexes of  $\mathcal{O}$ -modules.

<sup>4</sup>Here and below  $Rf_* : D(\mathcal{O}) \rightarrow D(\mathcal{O}')$  is the derived functor studied in Cohomology on Sites, Section 19 ff.

In general, via the identifications

$$R\Gamma(\mathcal{C}, \mathcal{A}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{A}) \quad \text{and} \quad R\Gamma(\mathcal{C}, \mathcal{M}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{M})$$

of Cohomology on Sites, Remark 14.4 the map  $\mu_{\mathcal{M}}$  induces the cup product on cohomology. To see this use Cohomology on Sites, Lemma 33.4 where the second morphism of topoi is the morphism from  $Sh(\mathcal{C}')$  to the punctual topos as above.

If  $\mathcal{M}_1 \rightarrow \mathcal{M}_2$  is a homomorphism of right differential graded  $\mathcal{A}$ -modules, then the diagram

$$\begin{array}{ccc} Rf_*\mathcal{M}_1 \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_1}} & Rf_*\mathcal{M}_1 \\ \downarrow & & \downarrow \\ Rf_*\mathcal{M}_2 \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_2}} & Rf_*\mathcal{M}_2 \end{array}$$

commutes in  $D(\mathcal{O}')$ ; this follows from the fact that the relative cup product is functorial. Suppose we have a short exact sequence

$$0 \rightarrow \mathcal{M}_1 \xrightarrow{a} \mathcal{M}_2 \rightarrow \mathcal{M}_3 \rightarrow 0$$

of right differential graded  $\mathcal{A}$ -modules. Then we claim that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{M}_3 \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_3}} & Rf_*\mathcal{M}_3 \\ Rf_*\delta \otimes \text{id} \downarrow & & \downarrow Rf_*\delta \\ Rf_*\mathcal{M}_1[1] \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}_1[1]}} & Rf_*\mathcal{M}_1[1] \end{array}$$

commutes in  $D(\mathcal{O}')$  where  $\delta : \mathcal{M}_3 \rightarrow \mathcal{M}_1[1]$  is the morphism of  $D(\mathcal{O})$  coming from the given short exact sequence (see Derived Categories, Section 12). This is clear if our sequence is split as a sequence of graded right  $\mathcal{A}$ -modules, because in this case  $\delta$  can be represented by a map of right  $\mathcal{A}$ -modules and the discussion above applies. In general we argue using the cone on  $a$  and the diagram

$$\begin{array}{ccccccc} \mathcal{M}_1 & \xrightarrow{a} & \mathcal{M}_2 & \xrightarrow{i} & C(a) & \xrightarrow{-p} & \mathcal{M}_1[1] \\ \downarrow & & \downarrow & & \downarrow q & & \downarrow \\ \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M}_3 & \xrightarrow{\delta} & \mathcal{M}_1[1] \end{array}$$

where the right square is commutative in  $D(\mathcal{O})$  by the definition of  $\delta$  in Derived Categories, Lemma 12.1. Now the cone  $C(a)$  has the structure of a right differential graded  $\mathcal{A}$ -module such that  $i, p, q$  are homomorphisms of right differential graded  $\mathcal{A}$ -modules, see Definition 22.2. Hence by the above we know that the corresponding diagrams commute for the morphisms  $q$  and  $-p$ . Since  $q$  is an isomorphism in  $D(\mathcal{O})$  we conclude the same is true for  $\delta$  as desired.

In the situation above given a right differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  let

$$\xi \in H^n(\mathcal{C}, \mathcal{M})$$

In other words,  $\xi$  is a degree  $n$  cohomology class in the cohomology of  $\mathcal{M}$  viewed as a complex of  $\mathcal{O}$ -modules. By Lemma 29.9 we can construct maps

$$x : \mathcal{A} \rightarrow \mathcal{M}'[n] \quad \text{and} \quad s : \mathcal{M} \rightarrow \mathcal{M}'$$

of right differential graded  $\mathcal{A}$ -modules where  $s$  is a quasi-isomorphism and such that  $\xi$  is the image of  $1 \in H^0(\mathcal{C}, \mathcal{A})$  via the morphism  $s[n]^{-1} \circ x$  in the derived

category  $D(\mathcal{A}, d)$  and a fortiori in the derived category  $D(\mathcal{O})$ . It follows that the corresponding map

$$\xi' = (s[n])^{-1} \circ x : \mathcal{A} \longrightarrow \mathcal{M}[n]$$

in  $D(\mathcal{O})$  is uniquely characterized by the following two properties

- (1)  $\xi'$  can be lifted to a morphism in  $D(\mathcal{A}, d)$ , and
- (2)  $\xi = \xi'(1)$  in  $H^0(\mathcal{C}, \mathcal{M}[n]) = H^n(\mathcal{C}, \mathcal{M})$ .

Using the compatibilities of  $x$  and  $s$  with the relative cup product discussed above it follows that for every<sup>5</sup> morphism of ringed topoi  $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  the derived pushforward

$$Rf_*\xi' : Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{M}[n]$$

of  $\xi'$  is compatible with the maps  $\mu$  and  $\mu_{\mathcal{M}[n]}$  constructed above in the sense that the diagram

$$\begin{array}{ccc} Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu} & Rf_*\mathcal{A} \\ Rf_*\xi' \otimes \text{id} \downarrow & & \downarrow Rf_*\xi' \\ Rf_*\mathcal{M}[n] \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} & \xrightarrow{\mu_{\mathcal{M}[n]}} & Rf_*\mathcal{M}[n] \end{array}$$

commutes in  $D(\mathcal{O}')$ . Using this compatibility for the map to the punctual topoi, we see in particular that

$$\begin{array}{ccc} R\Gamma(\mathcal{C}, \mathcal{A}) \otimes_{\Gamma(\mathcal{C}, \mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) & \longrightarrow & R\Gamma(\mathcal{C}, \mathcal{A}) \\ \xi' \otimes \text{id} \downarrow & & \downarrow \xi' \\ R\Gamma(\mathcal{C}, \mathcal{M}[n]) \otimes_{\Gamma(\mathcal{C}, \mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) & \longrightarrow & R\Gamma(\mathcal{C}, \mathcal{M}[n]) \end{array}$$

commutes. Combined with  $\xi'(1) = \xi$  this implies that the induced map on cohomology

$$\xi' : R\Gamma(\mathcal{C}, \mathcal{A}) \rightarrow R\Gamma(\mathcal{C}, \mathcal{M}[n]), \quad \eta \mapsto \xi \cup \eta$$

is given by left cup product by  $\xi$  as indicated.

### 33. Differential graded modules on a category

0GZ8 This section is the continuation of Cohomology on Sites, Section 43.

Let  $\mathcal{C}$  be a category. We think of  $\mathcal{C}$  as a site with the chaotic topology. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $(\mathcal{A}, d)$  be a sheaf of differential graded  $\mathcal{O}$ -algebras. In other words,  $\mathcal{O}$  is a presheaf of rings on the category  $\mathcal{C}$  and  $(\mathcal{A}, d)$  is a presheaf of differential graded  $\mathcal{O}$ -algebras on  $\mathcal{C}$ , see Categories, Definition 3.3.

0GZ9 **Definition 33.1.** In the situation above, we denote  $QC(\mathcal{A}, d)$  the full subcategory of  $D(\mathcal{A}, d)$  consisting of objects  $M$  such that for all  $U \rightarrow V$  in  $\mathcal{C}$  the canonical map

$$R\Gamma(V, M) \otimes_{\mathcal{A}(V)}^{\mathbf{L}} \mathcal{A}(U) \longrightarrow R\Gamma(U, M)$$

is an isomorphism in  $D(\mathcal{A}(U), d)$ .

0GZA **Lemma 33.2.** *In the situation above, the subcategory  $QC(\mathcal{A}, d)$  is a strictly full, saturated, triangulated subcategory of  $D(\mathcal{A}, d)$  preserved by arbitrary direct sums.*

<sup>5</sup>For example the identity morphism.

**Proof.** Let  $U$  be an object of  $\mathcal{C}$ . Since the topology on  $\mathcal{C}$  is chaotic, the functor  $\mathcal{F} \mapsto \mathcal{F}(U)$  is exact and commutes with direct sums. Hence the exact functor  $M \mapsto R\Gamma(U, M)$  is computed by representing  $K$  by any differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  and taking  $\mathcal{M}(U)$ . Thus  $R\Gamma(U, -)$  commutes with direct sums, see Lemma 26.8. Similarly, given a morphism  $U \rightarrow V$  of  $\mathcal{C}$  the derived tensor product functor  $- \otimes_{\mathcal{O}(A)}^{\mathbf{L}} \mathcal{A}(U) : D(\mathcal{A}(V)) \rightarrow D(\mathcal{A}(U))$  is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted.  $\square$

0GZB **Remark 33.3.** As above, let  $\mathcal{C}$  be a category viewed as a site with the chaotic topology, let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ , and let  $(\mathcal{A}, d)$  be a sheaf of differential graded  $\mathcal{O}$ -algebras. Then the analogue of Cohomology on Sites, Proposition 43.9 holds for  $QC(\mathcal{A}, d)$  with almost exactly the same proof:

- (1) any contravariant cohomological functor  $H : QC(\mathcal{A}, d) \rightarrow Ab$  which transforms direct sums into products is representable,
- (2) any exact functor  $F : QC(\mathcal{A}, d) \rightarrow \mathcal{D}$  of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and
- (3) the inclusion functor  $QC(\mathcal{A}, d) \rightarrow D(\mathcal{A}, d)$  has an exact right adjoint.

If we ever need this we will precisely formulate and prove this here.

Let  $u : \mathcal{C}' \rightarrow \mathcal{C}$  be a functor between categories. If we view  $\mathcal{C}$  and  $\mathcal{C}'$  as sites with the chaotic topology, then  $u$  is a continuous and cocontinuous functor. Hence we obtain a morphism  $g : Sh(\mathcal{C}') \rightarrow Sh(\mathcal{C})$  of topoi, see Sites, Lemma 21.1. Additionally, suppose given sheaves of rings  $\mathcal{O}$  on  $\mathcal{C}$  and  $\mathcal{O}'$  on  $\mathcal{C}'$  and a map  $g^\# : g^{-1}\mathcal{O} \rightarrow \mathcal{O}'$ . We denote the corresponding morphism of ringed topoi simply  $g : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$ , see Modules on Sites, Section 7. Finally, suppose that  $(\mathcal{A}, d)$  is a sheaf of differential graded  $\mathcal{O}$ -algebras and that  $(\mathcal{A}', d)$  is a sheaf of differential graded  $\mathcal{O}'$ -algebras and moreover that we are given a map  $\varphi : g^*\mathcal{A} \rightarrow \mathcal{A}'$  of differential graded  $\mathcal{O}'$ -algebras (see Section 18).

0GZC **Lemma 33.4.** *Let  $g : (Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{C}), \mathcal{O})$  and  $\varphi : g^*\mathcal{A} \rightarrow \mathcal{A}'$  be as above. Then the functor  $Lg^* : D(\mathcal{A}, d) \rightarrow D(\mathcal{A}', d)$  maps  $QC(\mathcal{A}, d)$  into  $QC(\mathcal{A}', d)$ .*

**Proof.** Let  $U' \in \text{Ob}(\mathcal{C}')$  with image  $U = u(U')$  in  $\mathcal{C}$ . Let  $pt$  denote the category with a single object and a single morphism. Denote  $(Sh(pt), \mathcal{O}'(U'))$  and  $(Sh(pt), \mathcal{O}(U))$  the ringed topoi as indicated endowed with the differential graded algebras  $\mathcal{A}'(U')$  and  $\mathcal{A}(U)$ . Of course we identify the derived category of differential graded modules on these with  $D(\mathcal{A}'(U'), d)$  and  $D(\mathcal{A}(U), d)$ . Then we have a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(pt), \mathcal{O}'(U')) & \xrightarrow{U'} & (Sh(\mathcal{C}'), \mathcal{O}') \\ \downarrow & & \downarrow g \\ (Sh(pt), \mathcal{O}(U)) & \xrightarrow{U} & (Sh(\mathcal{C}), \mathcal{O}) \end{array}$$

each endowed with corresponding differential graded algebras. Pullback along the lower horizontal morphism sends  $M$  in  $D(\mathcal{A}, d)$  to  $R\Gamma(U, K)$  viewed as an object in  $D(\mathcal{A}(U), d)$ . Pullback by the left vertical arrow sends  $M$  to  $M \otimes_{\mathcal{A}(U)}^{\mathbf{L}} \mathcal{A}'(U')$ . Going around the diagram either direction produces the same result (Lemma 28.4) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes_{\mathcal{A}(U)}^{\mathbf{L}} \mathcal{A}'(U')$$

Finally, let  $f' : U' \rightarrow V'$  be a morphism in  $\mathcal{C}'$  and denote  $f = u(f') : U = u(U') \rightarrow V = u(V')$  the image in  $\mathcal{C}$ . If  $K$  is in  $QC(\mathcal{A}, d)$  then we have

$$\begin{aligned} R\Gamma(V', Lg^*K) \otimes_{\mathcal{A}'(V')}^{\mathbf{L}} \mathcal{A}'(U') &= R\Gamma(V, K) \otimes_{\mathcal{A}(V)}^{\mathbf{L}} \mathcal{A}'(V') \otimes_{\mathcal{A}'(V')}^{\mathbf{L}} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes_{\mathcal{A}(V)}^{\mathbf{L}} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes_{\mathcal{A}(V)}^{\mathbf{L}} \mathcal{A}(U) \otimes_{\mathcal{A}(U)}^{\mathbf{L}} \mathcal{A}'(U') \\ &= R\Gamma(U, K) \otimes_{\mathcal{A}(U)}^{\mathbf{L}} \mathcal{A}'(U') \\ &= R\Gamma(U', Lg^*K) \end{aligned}$$

as desired. Here we have used the observation above both for  $U'$  and  $V'$ .  $\square$

### 34. Differential graded modules on a category, bis

0GZD We develop a few more results on the notion of quasi-coherent modules introduced in Section 33.

0GZE **Lemma 34.1.** *Let  $\mathcal{C}, \mathcal{O}, \mathcal{A}$  be as in Section 33. Let  $\mathcal{C}' \subset \mathcal{C}$  be a full subcategory with the following property: for every  $U \in \text{Ob}(\mathcal{C})$  the category  $U/\mathcal{C}'$  of arrows  $U \rightarrow U'$  is cofiltered. Denote  $\mathcal{O}', \mathcal{A}'$  the restrictions of  $\mathcal{O}, \mathcal{A}$  to  $\mathcal{C}'$ . Then restrictions induces an equivalence  $QC(\mathcal{A}, d) \rightarrow QC(\mathcal{A}', d)$ .*

**Proof.** We will construct a quasi-inverse of the functor. Namely, let  $M'$  be an object of  $QC(\mathcal{A}', d)$ . We may represent  $M'$  by a good differential graded module  $\mathcal{M}'$ , see Lemma 23.7. Then for every  $U' \in \text{Ob}(\mathcal{C}')$  the differential graded  $\mathcal{A}'(U')$ -module  $\mathcal{M}'(U')$  is K-flat and graded flat and for every morphism  $U'_1 \rightarrow U'_2$  of  $\mathcal{C}'$  the map

$$\mathcal{M}'(U'_2) \otimes_{\mathcal{A}'(U'_2)} \mathcal{A}'(U'_1) \longrightarrow \mathcal{M}'(U'_1)$$

is a quasi-isomorphism (as the source represents the derived tensor product). Consider the differential graded  $\mathcal{A}$ -module  $\mathcal{M}$  defined by the rule

$$\mathcal{M}(U) = \text{colim}_{U \rightarrow U' \in U/\mathcal{C}'} \mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

This is a filtered colimit of complexes by our assumption in the lemma. Since  $M'$  is in  $QC(\mathcal{A}', d)$  all the transition maps in the system are quasi-isomorphisms. Since filtered colimits are exact, we see that  $\mathcal{M}(U)$  in  $D(\mathcal{A}(U), d)$  is isomorphic to  $\mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$  for any morphism  $U \rightarrow U'$  with  $U' \in \text{Ob}(\mathcal{C}')$ .

We claim that  $\mathcal{M}$  is in  $QC(\mathcal{A}, d)$ : namely, given  $U \rightarrow V$  in  $\mathcal{C}$  we choose a map  $V \rightarrow V'$  with  $V' \in \text{Ob}(\mathcal{C}')$ . By the above we see that the map  $\mathcal{M}(V) \rightarrow \mathcal{M}(U)$  is identified with the map

$$\mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(V) \longrightarrow \mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(U)$$

Since  $\mathcal{M}'(V')$  is K-flat as differential graded  $\mathcal{A}'(V')$ -module, we conclude the claim is true.

The natural map  $\mathcal{M}|_{\mathcal{C}'} \rightarrow \mathcal{M}'$  is an isomorphism in  $D(\mathcal{A}', d)$  as follows immediately from the above.

Conversely, if we have an object  $E$  of  $QC(\mathcal{A}, d)$ , then we represent it by a good differential graded module  $\mathcal{E}$ . Setting  $\mathcal{M}' = \mathcal{E}|_{\mathcal{C}'}$  (this is another good differential graded module) we see that there is a map

$$\mathcal{E} \rightarrow \mathcal{M}$$



which over  $U$  in  $\mathcal{C}$  is given by the map

$$\mathcal{E}(U) \longrightarrow \operatorname{colim}_{U \rightarrow U' \in \mathcal{C}'} \mathcal{E}(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

which is a quasi-isomorphism by the same reason. Thus restriction and the construction above are quasi-inverse functors as desired.  $\square$

0GZF **Lemma 34.2.** *Let  $\mathcal{C}, \mathcal{O}$  be as in Section 33. Let  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism of differential graded  $\mathcal{O}$ -algebras which induces an isomorphism on cohomology sheaves, then the equivalence  $D(\mathcal{A}, d) \rightarrow D(\mathcal{B}, d)$  of Lemma 30.1 induces an equivalence  $QC(\mathcal{A}, d) \rightarrow QC(\mathcal{B}, d)$ .*

**Proof.** It suffices to show the following: given a morphism  $U \rightarrow V$  of  $\mathcal{C}$  and  $M$  in  $D(\mathcal{A}, d)$  the following are equivalent

- (1)  $R\Gamma(V, M) \otimes_{\mathcal{A}(V)}^{\mathbf{L}} \mathcal{A}(U) \rightarrow \Gamma(U, M)$  is an isomorphism in  $D(\mathcal{A}(U), d)$ , and
- (2)  $R\Gamma(V, M \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B}) \otimes_{\mathcal{B}(V)}^{\mathbf{L}} \mathcal{B}(U) \rightarrow \Gamma(U, M \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B})$  is an isomorphism in  $D(\mathcal{B}(U), d)$ .

Since the topology on  $\mathcal{C}$  is chaotic, this simply boils down to fact that  $\mathcal{A}(U) \rightarrow \mathcal{B}(U)$  and  $\mathcal{A}(V) \rightarrow \mathcal{B}(V)$  are quasi-isomorphisms. Details omitted.  $\square$

### 35. Inverse systems of differential graded algebras

0GZG In this section we consider the following special case of the situation discussed in Section 33:

- (1)  $\mathcal{C}$  is the category  $\mathbf{N}$  with a unique morphism  $i \rightarrow j$  if and only if  $i \leq j$ ,
- (2)  $\mathcal{O}$  is the constant (pre)sheaf of rings with value a given ring  $R$ .

In this setting a sheaf  $\mathcal{A}$  of differential graded  $\mathcal{O}$ -algebras is the same thing as an inverse system  $(A_n)$  of differential graded  $R$ -algebras. A sheaf  $\mathcal{M}$  of differential graded  $\mathcal{A}$ -modules is the same thing as an inverse system  $(M_n)$  where  $M_n$  is a differential graded  $A_n$ -module and the transition maps  $M_{n+1} \rightarrow M_n$  are  $A_{n+1}$ -module maps.

Suppose that  $\mathcal{B} = (B_n)$  is a second inverse system of differential graded  $R$ -algebras. Given a morphism  $\varphi : (A_n) \rightarrow (B_n)$  of pro-objects we will construct an exact functor from  $QC(\mathcal{A}, d)$  to  $QC(\mathcal{B}, d)$ . Namely, according to Categories, Example 22.6 the morphism  $\varphi$  is given by a sequence  $\dots \geq m(3) \geq m(2) \geq m(1)$  of integers and a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & A_{m(3)} & \longrightarrow & A_{m(2)} & \longrightarrow & A_{m(1)} \\ & & \downarrow \varphi_3 & & \downarrow \varphi_2 & & \downarrow \varphi_1 \\ \dots & \longrightarrow & B_3 & \longrightarrow & B_2 & \longrightarrow & B_1 \end{array}$$

of differential graded  $R$ -algebras. Then given a good sheaf of differential graded  $\mathcal{A}$ -modules  $\mathcal{M} = (M_n)$  representing an object of  $QC(\mathcal{A}, d)$  we can set

$$N_n = M_{m(n)} \otimes_{A_{m(n)}} B_n$$

This inverse system determines an object of  $QC(\mathcal{B}, d)$  because the  $A_{m(n)}$ -modules  $M_{m(n)}$  are  $K$ -flat; details omitted. We also leave it to the reader to show that the resulting functor is independent of the choices made in its construction.

0GZH **Lemma 35.1.** *In the situation above, suppose that  $\mathcal{A} = (A_n)$  and  $\mathcal{B} = (B_n)$  are inverse systems of differential graded  $R$ -algebras. If  $\varphi : (A_n) \rightarrow (B_n)$  is an isomorphism of pro-objects, then the functor  $QC(\mathcal{A}, d) \rightarrow QC(\mathcal{B}, d)$  constructed above is an equivalence.*

**Proof.** Let  $\psi : (B_n) \rightarrow (A_n)$  be a morphism of pro-objects which is inverse to  $\varphi$ . According to the discussion in Categories, Example 22.6 we may assume that  $\varphi$  is given by a system of maps as above and  $\psi$  is given  $n(1) < n(2) < \dots$  and a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & B_{n(3)} & \longrightarrow & B_{n(2)} & \longrightarrow & B_{n(1)} \\ & & \downarrow \psi_3 & & \downarrow \psi_2 & & \downarrow \psi_1 \\ \dots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 \end{array}$$

of differential graded  $R$ -algebras. Since  $\varphi \circ \psi = \text{id}$  we may, after possibly increasing the values of the functions  $n(\cdot)$  and  $m(\cdot)$  assume that  $B_{n(m(i))} \rightarrow A_{m(i)} \rightarrow B_i$  is the identity. It follows that the composition of the functors

$$QC(\mathcal{B}, d) \rightarrow QC(\mathcal{A}, d) \rightarrow QC(\mathcal{B}, d)$$

sends a good sheaf of differential graded  $\mathcal{B}$ -modules  $\mathcal{N} = (N_n)$  to the inverse system  $\mathcal{N}' = (N'_i)$  with values

$$N'_i = N_{n(m(i))} \otimes_{B_{n(m(i))}} B_i$$

which is canonically quasi-isomorphic to  $\mathcal{N}$  exactly because  $\mathcal{N}$  is an object of  $QC(\mathcal{B}, d)$  and because  $N_j$  is a  $K$ -flat differential graded module for all  $j$ . Since the same is true for the composition the other way around we conclude.  $\square$

Let  $\mathcal{C} = \mathbf{N}$  and  $\mathcal{O}$  the constant sheaf with value a ring  $R$  and let  $\mathcal{A}$  be given by an inverse system  $(A_n)$  of differential graded  $R$ -algebras. Suppose given two left differential graded  $\mathcal{A}$ -modules  $\mathcal{N}$  and  $\mathcal{N}'$  given by inverse systems  $(N_n)$  and  $(N'_n)$ . Thus each  $N_n$  and  $N'_n$  is a left differential graded  $A_n$ -module. Let us temporarily say that  $(N_n)$  and  $(N'_n)$  are *pro-isomorphic in the derived category* if there exist a sequence of integers

$$1 = n_0 < n_1 < n_2 < n_3 < \dots$$

and maps

$$N_{n_{2i}} \rightarrow N'_{n_{2i-1}} \quad \text{in} \quad D(A_{n_{2i}}^{opp}, d)$$

and

$$N'_{n_{2i+1}} \rightarrow N'_{n_{2i}} \quad \text{in} \quad D(A_{n_{2i+1}}^{opp}, d)$$

such that the compositions  $N_{n_{2i}} \rightarrow N_{n_{2i-2}}$  and  $N'_{n_{2i+1}} \rightarrow N'_{n_{2i-1}}$  are given by the transition maps of the respective systems.

0GZI **Lemma 35.2.** *If  $(N_n)$  and  $(N'_n)$  are pro-isomorphic in the derived category as defined above, then for every object  $(M_n)$  of  $D(\mathbf{N}, \mathcal{A})$  we have*

$$R \lim(M_n \otimes_{A_n}^{\mathbf{L}} N_n) = R \lim(M_n \otimes_{A_n}^{\mathbf{L}} N'_n)$$

in  $D(R)$ .

**Proof.** The assumption implies that the inverse system  $(M_n \otimes_{A_n}^{\mathbf{L}} N_n)$  of  $D(R)$  is pro-isomorphic (in the usual sense) to the inverse system  $(M_n \otimes_{A_n}^{\mathbf{L}} N'_n)$  of  $D(R)$ . Hence the result follows from the fact that taking  $R \lim$  is well defined for inverse systems in the derived category, see discussion in More on Algebra, Section 87.  $\square$

0GZJ **Lemma 35.3.** *Let  $R$  be a ring. Let  $f_1, \dots, f_r \in R$ . Let  $K_n$  be the Koszul complex on  $f_1^n, \dots, f_r^n$  viewed as a differential graded  $R$ -algebra. Let  $(M_n)$  be an object of  $D(\mathbf{N}, (K_n))$ . Then for any  $t \geq 1$  we have* This is a variant of [BS13, Lemma 3.5.4]

$$R\lim(M_n \otimes_R^{\mathbf{L}} K_t) = R\lim(M_n \otimes_{K_n}^{\mathbf{L}} K_t)$$

in  $D(R)$ .

**Proof.** We fix  $t \geq 1$ . For  $n \geq t$  let us denote  ${}_n K_t$  the differential graded  $R$ -algebra  $K_t$  viewed as a left differential graded  $K_n$ -module. Observe that

$$M_n \otimes_R^{\mathbf{L}} K_t = M_n \otimes_{K_n}^{\mathbf{L}} (K_n \otimes_R^{\mathbf{L}} K_t) = M_n \otimes_{K_n}^{\mathbf{L}} (K_n \otimes_R K_t)$$

Hence by Lemma 35.2 it suffices to show that  $({}_n K_t)$  and  $(K_n \otimes_R K_t)$  are pro-isomorphic in the derived category. The multiplication maps

$$K_n \otimes_R K_t \longrightarrow {}_n K_t$$

are maps of left differential graded  $K_n$ -modules. Thus to finish the proof it suffices to show that for all  $n \geq 1$  there exists an  $N > n$  and a map

$${}_N K_t \longrightarrow {}_N K_n \otimes_R K_t$$

in  $D(K_N^{opp}, \mathbf{d})$  whose composition with the multiplication map is the transition map (in either direction). This is done in Divided Power Algebra, Lemma 12.4 by an explicit construction.  $\square$

0GZK **Proposition 35.4.** *Let  $R$  be a Noetherian ring. Let  $I \subset R$  be an ideal. The following three categories are canonically equivalent:*

- (1) *Let  $\mathcal{A}$  be the sheaf of  $R$ -algebras on  $\mathbf{N}$  corresponding to the inverse system of  $R$ -algebras  $A_n = R/I^n$ . The category  $QC(\mathcal{A})$ .*
- (2) *Choose generators  $f_1, \dots, f_r$  of  $I$ . Let  $\mathcal{B}$  be the sheaf of differential graded  $R$ -algebras on  $\mathbf{N}$  corresponding to the inverse system of Koszul algebras on  $f_1^n, \dots, f_r^n$ . The category  $QC(\mathcal{B})$ .*
- (3) *The full subcategory  $D_{comp}(R, I) \subset D(R)$  of derived complete objects, see More on Algebra, Definition 91.4 and text following.*

**Proof.** Consider the obvious morphism  $f : (Sh(\mathbf{N}), \mathcal{A}) \rightarrow (Sh(pt), R)$  of ringed topoi and let us consider the adjoint functors  $Lf^*$  and  $Rf_*$ . The first restricts to a functor

$$F : D_{comp}(R, I) \longrightarrow QC(\mathcal{A})$$

which sends an object  $K$  of  $D_{comp}(R, I)$  represented by a  $K$ -flat complex  $K^\bullet$  to the object  $(K^\bullet \otimes_R R/I^n)$  of  $QC(\mathcal{A})$ . The second restricts to a functor

$$G : QC(\mathcal{A}) \longrightarrow D_{comp}(R, I)$$

which sends an object  $(M_n^\bullet)$  of  $QC(\mathcal{A})$  to  $R\lim M_n^\bullet$ . The output is derived complete for example by More on Algebra, Lemma 91.14. Also, it follows from More on Algebra, Proposition 94.2 that  $G \circ F = \text{id}$ . Thus to see that  $F$  and  $G$  are quasi-inverse equivalences it suffices to see that the kernel of  $G$  is zero (see Derived Categories, Lemma 7.2). However, it does not appear easy to show this directly!

In this paragraph we will show that  $QC(\mathcal{A})$  and  $QC(\mathcal{B})$  are equivalent. Write  $\mathcal{B} = (B_n)$  where  $B_n$  is the Koszul complex viewed as a cochain complex in degrees  $-r, -r+1, \dots, 0$ . By Divided Power Algebra, Remark 12.2 (but with chain complexes turned into cochain complexes) we can find  $1 < n_1 < n_2 < \dots$  and maps of

differential graded  $R$ -algebras  $B_{n_i} \rightarrow E_i \rightarrow R/(f_1^{n_i}, \dots, f_r^{n_i})$  and  $E_i \rightarrow B_{n_{i-1}}$  such that

$$\begin{array}{ccccccc} B_{n_1} & \longleftarrow & B_{n_2} & \longleftarrow & B_{n_3} & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ E_1 & \longleftarrow & E_2 & \longleftarrow & E_3 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ B_1 & \longleftarrow & B_{n_1} & \longleftarrow & B_{n_2} & \longleftarrow & \dots \end{array}$$

is a commutative diagram of differential graded  $R$ -algebras and such that  $E_i \rightarrow R/(f_1^{n_i}, \dots, f_r^{n_i})$  is a quasi-isomorphism. We conclude

- (1) there is an equivalence between  $QC(\mathcal{B})$  and  $QC((E_i))$ ,
- (2) there is an equivalence between  $QC((E_i))$  and  $QC((R/(f_1^{n_i}, \dots, f_r^{n_i})))$ ,
- (3) there is an equivalence between  $QC((R/(f_1^{n_i}, \dots, f_r^{n_i})))$  and  $QC(\mathcal{A})$ .

Namely, for (1) we can apply Lemma 35.1 to the diagram above which shows that  $(E_i)$  and  $(B_n)$  are pro-isomorphic. For (2) we can apply Lemma 34.2 to the inverse system of quasi-isomorphisms  $E_i \rightarrow R/(f_1^{n_i}, \dots, f_r^{n_i})$ . For (3) we can apply Lemma 35.1 and the elementary fact that the inverse systems  $(R/I^n)$  and  $(R/(f_1^{n_i}, \dots, f_r^{n_i}))$  are pro-isomorphic.

Exactly as in the first paragraph of the proof we can define adjoint functors<sup>6</sup>

$$F' : D_{comp}(R, I) \longrightarrow QC(\mathcal{B}) \quad \text{and} \quad G' : QC(\mathcal{B}) \longrightarrow D_{comp}(R, I).$$

The first sends an object  $K$  of  $D_{comp}(R, I)$  represented by a  $K$ -flat complex  $K^\bullet$  to the object  $(K^\bullet \otimes_R B_n)$  of  $QC(\mathcal{B})$ . The second sends an object  $(M_n)$  of  $QC(\mathcal{B})$  to  $R \lim M_n$ . Arguing as above it suffices to show that the kernel of  $G'$  is zero. So let  $\mathcal{M} = (M_n)$  be a good sheaf of differential graded modules over  $\mathcal{B}$  which represents an object of  $QC(\mathcal{B})$  in the kernel of  $G'$ . Then

$$0 = R \lim M_n \Rightarrow 0 = (R \lim M_n) \otimes_R^L B_t = R \lim (M_n \otimes_R^L B_t)$$

By Lemma 35.3 we have  $R \lim (M_n \otimes_R^L B_t) = R \lim (M_n \otimes_{B_n}^L B_t)$ . Since  $(M_n)$  is an object of  $QC(\mathcal{B})$  we see that the inverse system  $M_n \otimes_{B_n}^L B_t$  is eventually constant with value  $M_t$ . Hence  $M_t = 0$  as desired.  $\square$

0H1E **Remark 35.5.** Let  $R$  be a ring and let  $f_1, \dots, f_r \in R$  be a sequence of elements generating an ideal  $I$ . Let  $K_n$  be the Koszul complex on  $f_1^n, \dots, f_r^n$  viewed as a differential graded  $R$ -algebra. We say  $f_1, \dots, f_r$  is a *weakly proregular sequence* if for all  $n$  there is an  $m > n$  such that  $K_m \rightarrow K_n$  induces the zero map on cohomology except in degree 0. If so, then the arguments in the proof of Proposition 35.4 continue to work even when  $R$  is not Noetherian. In particular we see that  $QC(\{R/I^n\})$  is equivalent as an  $R$ -linear triangulated category to the category  $D_{comp}(R, I)$  of derived complete objects, provided  $I$  can be generated by a weakly proregular sequence. If the need arises, we will precisely state and prove this here.

<sup>6</sup>It can be shown that these functors are, via the equivalences above, compatible with  $F$  and  $G$  defined before.

**36. Other chapters**

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  - (2) Conventions
  - (3) Set Theory
  - (4) Categories
  - (5) Topology
  - (6) Sheaves on Spaces
  - (7) Sites and Sheaves
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  - (22) Differential Graded Algebra
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  - (26) Schemes
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  - (32) Limits of Schemes
  - (33) Varieties
  - (34) Topologies on Schemes
  - (35) Descent
  - (36) Derived Categories of Schemes
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  - (40) More on Groupoid Schemes
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- Topics in Scheme Theory
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- (54) Resolution of Surfaces
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(87) Formal Algebraic Spaces	(103) Cohomology of Algebraic Stacks
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(102) Limits of Algebraic Stacks	

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