

# COHOMOLOGY ON SITES

01FQ

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## 1. Introduction

01FR In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [AGV71], [God73] and [Ive86].

## 2. Topics

01FS Here are some topics that should be discussed in this chapter, and have not yet been written.

- (1) Cohomology of a sheaf of modules on a site is the same as the cohomology of the underlying abelian sheaf.
- (2) Hypercohomology on a site.
- (3) Ext-groups.
- (4) Ext sheaves.
- (5) Tor functors.
- (6) Higher direct images for a morphism of sites.
- (7) Derived pullback for morphisms between ringed sites.
- (8) Cup-product.
- (9) Group cohomology.
- (10) Comparison of group cohomology and cohomology on  $\mathcal{T}_G$ .
- (11) Čech cohomology on sites.
- (12) Čech to cohomology spectral sequence on sites.
- (13) Leray Spectral sequence for a morphism between ringed sites.
- (14) Etc, etc, etc.

## 3. Cohomology of sheaves

01FT Let  $\mathcal{C}$  be a site, see Sites, Definition 6.2. Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ . We know that the category of abelian sheaves on  $\mathcal{C}$  has enough injectives, see Injectives, Theorem 7.4. Hence we can choose an injective resolution  $\mathcal{F}[0] \rightarrow \mathcal{I}^\bullet$ . For any object  $U$  of the site  $\mathcal{C}$  we define

$$071C \quad (3.0.1) \quad H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the  $i$ th cohomology group of the abelian sheaf  $\mathcal{F}$  over the object  $U$ . In other words, these are the right derived functors of the functor  $\mathcal{F} \mapsto \mathcal{F}(U)$ . The family of functors  $H^i(U, -)$  forms a universal  $\delta$ -functor  $Ab(\mathcal{C}) \rightarrow Ab$ .

It sometimes happens that the site  $\mathcal{C}$  does not have a final object. In this case we define the *global sections* of a presheaf of sets  $\mathcal{F}$  over  $\mathcal{C}$  to be the set

$$071D \quad (3.0.2) \quad \Gamma(\mathcal{C}, \mathcal{F}) = \text{Mor}_{\text{PSh}(\mathcal{C})}(e, \mathcal{F})$$

where  $e$  is a final object in the category of presheaves on  $\mathcal{C}$ . In this case, given an abelian sheaf  $\mathcal{F}$  on  $\mathcal{C}$ , we define the  *$i$ th cohomology group of  $\mathcal{F}$  on  $\mathcal{C}$*  as follows

$$071E \quad (3.0.3) \quad H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

in other words, it is the  *$i$ th right derived functor* of the global sections functor. The family of functors  $H^i(\mathcal{C}, -)$  forms a universal  $\delta$ -functor  $Ab(\mathcal{C}) \rightarrow Ab$ .

Let  $f : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be a morphism of topoi, see Sites, Definition 15.1. With  $\mathcal{F}[0] \rightarrow \mathcal{I}^\bullet$  as above we define

$$071F \quad (3.0.4) \quad R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the  *$i$ th higher direct image of  $\mathcal{F}$* . These are the right derived functors of  $f_*$ . The family of functors  $R^i f_*$  forms a universal  $\delta$ -functor from  $Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$ .

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site, see Modules on Sites, Definition 6.1. Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module. We know that the category of  $\mathcal{O}$ -modules has enough injectives, see Injectives, Theorem 8.4. Hence we can choose an injective resolution  $\mathcal{F}[0] \rightarrow \mathcal{I}^\bullet$ . For any object  $U$  of the site  $\mathcal{C}$  we define

$$071G \quad (3.0.5) \quad H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the  *$i$ th cohomology group of  $\mathcal{F}$  over  $U$* . The family of functors  $H^i(U, -)$  forms a universal  $\delta$ -functor  $Mod(\mathcal{O}) \rightarrow Mod_{\mathcal{O}(U)}$ . Similarly

$$071H \quad (3.0.6) \quad H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

it the  *$i$ th cohomology group of  $\mathcal{F}$  on  $\mathcal{C}$* . The family of functors  $H^i(\mathcal{C}, -)$  forms a universal  $\delta$ -functor  $Mod(\mathcal{C}) \rightarrow Mod_{\Gamma(\mathcal{C}, \mathcal{O})}$ .

Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi, see Modules on Sites, Definition 7.1. With  $\mathcal{F}[0] \rightarrow \mathcal{I}^\bullet$  as above we define

$$071I \quad (3.0.7) \quad R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the  *$i$ th higher direct image of  $\mathcal{F}$* . These are the right derived functors of  $f_*$ . The family of functors  $R^i f_*$  forms a universal  $\delta$ -functor from  $Mod(\mathcal{O}) \rightarrow Mod(\mathcal{O}')$ .

#### 4. Derived functors

071J We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that  $(\mathcal{C}, \mathcal{O})$  is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 8.1 and Definition 11.3. By Derived Categories, Remark 24.3 there exists a resolution functor

$$j = j_{(\mathcal{C}, \mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where  $\mathcal{I}$  is the strictly full additive subcategory of  $Mod(\mathcal{O})$  which consists of injective  $\mathcal{O}$ -modules. For any left exact functor  $F : Mod(\mathcal{O}) \rightarrow \mathcal{B}$  into any abelian category

$\mathcal{B}$  we will denote  $RF$  the right derived functor of Derived Categories, Section 20 constructed using the resolution functor  $j$  just described:

$$05U5 \quad (4.0.1) \quad RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 25.1 for notation. Note that we may think of  $RF$  as defined on  $Mod(\mathcal{O})$ ,  $Comp^+(Mod(\mathcal{O}))$ , or  $K^+(\mathcal{O})$  depending on the situation. According to Derived Categories, Definition 17.2 we obtain the  $i$ th right derived functor

$$05U6 \quad (4.0.2) \quad R^i F = H^i \circ RF : Mod(\mathcal{O}) \longrightarrow \mathcal{B}$$

so that  $R^0 F = F$  and  $\{R^i F, \delta\}_{i \geq 0}$  is universal  $\delta$ -functor, see Derived Categories, Lemma 20.4.

Here are two special cases of this construction. Given a ring  $R$  we write  $K(R) = K(Mod_R)$  and  $D(R) = D(Mod_R)$  and similarly for the bounded versions. For any object  $U$  of  $\mathcal{C}$  have a left exact functor  $\Gamma(U, -) : Mod(\mathcal{O}) \longrightarrow Mod_{\mathcal{O}(U)}$  which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that  $H^i(U, -) = R^i \Gamma(U, -)$  is compatible with (3.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (3.0.6). If  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  is a morphism of ringed topoi then we get a left exact functor  $f_* : Mod(\mathcal{O}) \rightarrow Mod(\mathcal{O}')$  which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \rightarrow D^+(\mathcal{O}')$$

The  $i$ th cohomology sheaf of  $Rf_* \mathcal{F}^\bullet$  is denoted  $R^i f_* \mathcal{F}^\bullet$  and called the  $i$ th *higher direct image* in accordance with (3.0.7). The displayed functors above are exact functor of derived categories.

## 5. First cohomology and torsors

03AG

03AH **Definition 5.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}$  be a sheaf of (possibly non-commutative) groups on  $\mathcal{C}$ . A *pseudo torsor*, or more precisely a *pseudo  $\mathcal{G}$ -torsor*, is a sheaf of sets  $\mathcal{F}$  on  $\mathcal{C}$  endowed with an action  $\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$  such that

- (1) whenever  $\mathcal{F}(U)$  is nonempty the action  $\mathcal{G}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$  is simply transitive.

A *morphism of pseudo  $\mathcal{G}$ -torsors*  $\mathcal{F} \rightarrow \mathcal{F}'$  is simply a morphism of sheaves of sets compatible with the  $\mathcal{G}$ -actions. A *torsor*, or more precisely a  *$\mathcal{G}$ -torsor*, is a pseudo  $\mathcal{G}$ -torsor such that in addition

- (2) for every  $U \in \text{Ob}(\mathcal{C})$  there exists a covering  $\{U_i \rightarrow U\}_{i \in I}$  of  $U$  such that  $\mathcal{F}(U_i)$  is nonempty for all  $i \in I$ .

A *morphism of  $\mathcal{G}$ -torsors* is simply a morphism of pseudo  $\mathcal{G}$ -torsors. The *trivial  $\mathcal{G}$ -torsor* is the sheaf  $\mathcal{G}$  endowed with the obvious left  $\mathcal{G}$ -action.

It is clear that a morphism of torsors is automatically an isomorphism.

03AI **Lemma 5.2.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}$  be a sheaf of (possibly non-commutative) groups on  $\mathcal{C}$ . A  $\mathcal{G}$ -torsor  $\mathcal{F}$  is trivial if and only if  $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$ .

**Proof.** Omitted.  $\square$

03AJ **Lemma 5.3.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{H}$  be an abelian sheaf on  $\mathcal{C}$ . There is a canonical bijection between the set of isomorphism classes of  $\mathcal{H}$ -torsors and  $H^1(\mathcal{C}, \mathcal{H})$ .*

**Proof.** Let  $\mathcal{F}$  be a  $\mathcal{H}$ -torsor. Consider the free abelian sheaf  $\mathbf{Z}[\mathcal{F}]$  on  $\mathcal{F}$ . It is the sheafification of the rule which associates to  $U \in \text{Ob}(\mathcal{C})$  the collection of finite formal sums  $\sum n_i[s_i]$  with  $n_i \in \mathbf{Z}$  and  $s_i \in \mathcal{F}(U)$ . There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section  $\sum n_i[s_i]$  associates  $\sum n_i$ . The kernel of  $\sigma$  is generated by sections of the form  $[s] - [s']$ . There is a canonical map  $a : \text{Ker}(\sigma) \rightarrow \mathcal{H}$  which maps  $[s] - [s'] \mapsto h$  where  $h$  is the local section of  $\mathcal{H}$  such that  $h \cdot s = s'$ . Consider the pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} \longrightarrow 0 \\ & & \downarrow a & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} \longrightarrow 0 \end{array}$$

Here  $\mathcal{E}$  is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element  $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$  by applying the boundary operator to  $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$ .

Conversely, given  $\xi \in H^1(\mathcal{C}, \mathcal{H})$  we can associate to  $\xi$  a torsor as follows. Choose an embedding  $\mathcal{H} \rightarrow \mathcal{I}$  of  $\mathcal{H}$  into an injective abelian sheaf  $\mathcal{I}$ . We set  $\mathcal{Q} = \mathcal{I}/\mathcal{H}$  so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element  $\xi$  is the image of a global section  $q \in H^0(\mathcal{C}, \mathcal{Q})$  because  $H^1(\mathcal{C}, \mathcal{I}) = 0$  (see Derived Categories, Lemma 20.4). Let  $\mathcal{F} \subset \mathcal{I}$  be the subsheaf (of sets) of sections that map to  $q$  in the sheaf  $\mathcal{Q}$ . It is easy to verify that  $\mathcal{F}$  is a  $\mathcal{H}$ -torsor.

We omit the verification that the two constructions given above are mutually inverse.  $\square$

## 6. First cohomology and extensions

03F0

03F1 **Lemma 6.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules on  $\mathcal{C}$ . There is a canonical bijection*

$$\text{Ext}_{\text{Mod}(\mathcal{O})}^1(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$$

the image of  $1 \in \Gamma(\mathcal{C}, \mathcal{O})$  in  $H^1(\mathcal{C}, \mathcal{F})$ .

**Proof.** Let us construct the inverse of the map given in the lemma. Let  $\xi \in H^1(\mathcal{C}, \mathcal{F})$ . Choose an injection  $\mathcal{F} \subset \mathcal{I}$  with  $\mathcal{I}$  injective in  $\text{Mod}(\mathcal{O})$ . Set  $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ .

By the long exact sequence of cohomology, we see that  $\xi$  is the image of a section  $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \text{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$ . Now, we just form the pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \tilde{\xi} \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} \longrightarrow 0 \end{array}$$

see Homology, Section 6. □

The following lemma will be superseded by the more general Lemma 13.4.

03F2 **Lemma 6.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules on  $\mathcal{C}$ . Let  $\mathcal{F}_{ab}$  denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in  $\text{Ab}(\mathcal{C})$  and the right hand side is cohomology computed in  $\text{Mod}(\mathcal{O})$ .

**Proof.** Let  $\underline{\mathbf{Z}}$  denote the constant sheaf  $\mathbf{Z}$ . As  $\text{Ab}(\mathcal{C}) = \text{Mod}(\underline{\mathbf{Z}})$  we may apply Lemma 6.1 twice, and it follows that we have to show

$$\text{Ext}_{\text{Mod}(\mathcal{O})}^1(\mathcal{O}, \mathcal{F}) = \text{Ext}_{\text{Mod}(\underline{\mathbf{Z}})}^1(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$  is an extension in  $\text{Mod}(\mathcal{O})$ . Then we can use the obvious map of abelian sheaves  $1 : \underline{\mathbf{Z}} \rightarrow \mathcal{O}$  and pullback to obtain an extension  $\mathcal{E}_{ab}$ , like so:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} & \longrightarrow & \mathcal{E}_{ab} & \longrightarrow & \underline{\mathbf{Z}} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow 1 \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} \longrightarrow 0 \end{array}$$

The converse is a little more fun. Suppose that  $0 \rightarrow \mathcal{F}_{ab} \rightarrow \mathcal{E}_{ab} \rightarrow \underline{\mathbf{Z}} \rightarrow 0$  is an extension in  $\text{Mod}(\underline{\mathbf{Z}})$ . Since  $\underline{\mathbf{Z}}$  is a flat  $\underline{\mathbf{Z}}$ -module we see that the sequence

$$0 \rightarrow \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \rightarrow \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \rightarrow \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \rightarrow 0$$

is exact, see Modules on Sites, Lemma 28.8. Of course  $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$ . Hence we can form the pushout via the ( $\mathcal{O}$ -linear) multiplication map  $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \rightarrow \mathcal{F}$  to get an extension of  $\mathcal{O}$  by  $\mathcal{F}$ , like this

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{O} \longrightarrow 0 \\ & & \downarrow \mu & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} \longrightarrow 0 \end{array}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. □

## 7. First cohomology and invertible sheaves

040D The Picard group of a ringed site is defined in Modules on Sites, Section 31.

040E **Lemma 7.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a locally ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = \text{Pic}(\mathcal{O}).$$

*of abelian groups.*

**Proof.** Let  $\mathcal{L}$  be an invertible  $\mathcal{O}$ -module. Consider the presheaf  $\mathcal{L}^*$  defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if  $f \in \mathcal{O}^*(U)$  and  $s \in \mathcal{L}^*(U)$ , then clearly  $fs \in \mathcal{L}^*(U)$ . By the same token, if  $s, s' \in \mathcal{L}^*(U)$  then there exists a unique  $f \in \mathcal{O}^*(U)$  such that  $fs = s'$ . Moreover, the sheaf  $\mathcal{L}^*$  has sections locally by Modules on Sites, Lemma 39.7. In other words we see that  $\mathcal{L}^*$  is a  $\mathcal{O}^*$ -torsor. Thus we get a map

$$\begin{array}{ccc} \text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) & \longrightarrow & \text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism} & & \text{up to isomorphism} \end{array}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 5.3 the right hand side is canonically bijective to  $H^1(\mathcal{C}, \mathcal{O}^*)$ . Thus we have to show this map is injective and surjective.

**Injective.** If the torsor  $\mathcal{L}^*$  is trivial, this means by Lemma 5.2 that  $\mathcal{L}^*$  has a global section. Hence this means exactly that  $\mathcal{L} \cong \mathcal{O}$  is the neutral element in  $\text{Pic}(\mathcal{O})$ .

**Surjective.** Let  $\mathcal{F}$  be an  $\mathcal{O}^*$ -torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U)) / \mathcal{O}^*(U)$$

where the action of  $f \in \mathcal{O}^*(U)$  on  $(s, g)$  is  $(fs, f^{-1}g)$ . Then  $\mathcal{L}_1$  is a presheaf of  $\mathcal{O}$ -modules by setting  $(s, g) + (s', g') = (s, g + (s'/s)g')$  where  $s'/s$  is the local section  $f$  of  $\mathcal{O}^*$  such that  $fs = s'$ , and  $h(s, g) = (s, hg)$  for  $h$  a local section of  $\mathcal{O}$ . We omit the verification that the sheafification  $\mathcal{L} = \mathcal{L}_1^\#$  is an invertible  $\mathcal{O}$ -module whose associated  $\mathcal{O}^*$ -torsor  $\mathcal{L}^*$  is isomorphic to  $\mathcal{F}$ .  $\square$

## 8. Locality of cohomology

01FU The following lemma says there is no ambiguity in defining the cohomology of a sheaf  $\mathcal{F}$  over an object of the site.

03F3 **Lemma 8.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ .*

- (1) *If  $\mathcal{I}$  is an injective  $\mathcal{O}$ -module then  $\mathcal{I}|_U$  is an injective  $\mathcal{O}_U$ -module.*
- (2) *For any sheaf of  $\mathcal{O}$ -modules  $\mathcal{F}$  we have  $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$ .*

**Proof.** Recall that the functor  $j_U^{-1}$  of restriction to  $U$  is a right adjoint to the functor  $j_{U!}$  of extension by 0, see Modules on Sites, Section 19. Moreover,  $j_{U!}$  is exact. Hence (1) follows from Homology, Lemma 26.1.

By definition  $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$  where  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  is an injective resolution in  $\text{Mod}(\mathcal{O})$ . By the above we see that  $\mathcal{F}|_U \rightarrow \mathcal{I}^\bullet|_U$  is an injective resolution in  $\text{Mod}(\mathcal{O}_U)$ . Hence  $H^p(U, \mathcal{F}|_U)$  is equal to  $H^p(\mathcal{I}^\bullet|_U(U))$ . Of course  $\mathcal{F}(U) = \mathcal{F}|_U(U)$  for any sheaf  $\mathcal{F}$  on  $\mathcal{C}$ . Hence the equality in (2).  $\square$

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

03YU **Lemma 8.2.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be sites. Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Assume  $u$  satisfies the hypotheses of Sites, Lemma 20.8. Let  $g : \text{Sh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{D})$  be the associated morphism of topoi. For any abelian sheaf  $\mathcal{F}$  on  $\mathcal{D}$  we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular  $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$  and for any  $U \in \text{Ob}(\mathcal{C})$  we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular  $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$ . All of these isomorphisms are functorial in  $\mathcal{F}$ .

**Proof.** Since it is clear that  $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$  by hypothesis (e), it suffices to show that  $g^{-1}$  transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 26.1 to see this. The left adjoint to  $g^{-1}$  is  $g_! = f^{-1}$  with the notation of Sites, Lemma 20.8 which is an exact functor. Hence the lemma does indeed apply.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. Let  $\varphi : U \rightarrow V$  be a morphism of  $\mathcal{O}$ . Then there is a canonical *restriction mapping*

$$01\text{FV} \quad (8.2.1) \quad H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in  $\mathcal{F}$ . Namely, choose any injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ . The restriction mappings of the sheaves  $\mathcal{I}^p$  give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing  $R\Gamma(V, \mathcal{F})$  and the RHS is a complex representing  $R\Gamma(U, \mathcal{F})$ . We get the map on cohomology groups by applying the functor  $H^n$ . As indicated we will use the notation  $\xi \mapsto \xi|_U$  to denote this map. Thus the rule  $U \mapsto H^n(U, \mathcal{F})$  is a presheaf of  $\mathcal{O}$ -modules. This presheaf is customarily denoted  $\underline{H}^n(\mathcal{F})$ . We will give another interpretation of this presheaf in Lemma 11.5.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

01FW **Lemma 8.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}$ -modules. Let  $U$  be an object of  $\mathcal{C}$ . Let  $n > 0$  and let  $\xi \in H^n(U, \mathcal{F})$ . Then there exists a covering  $\{U_i \rightarrow U\}$  of  $\mathcal{C}$  such that  $\xi|_{U_i} = 0$  for all  $i \in I$ .*

**Proof.** Let  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\text{Ker}(\mathcal{I}^n(U) \rightarrow \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \rightarrow \mathcal{I}^n(U))}.$$

Pick an element  $\tilde{\xi} \in \mathcal{I}^n(U)$  representing the cohomology class in the presentation above. Since  $\mathcal{I}^\bullet$  is an injective resolution of  $\mathcal{F}$  and  $n > 0$  we see that the complex  $\mathcal{I}^\bullet$  is exact in degree  $n$ . Hence  $\text{Im}(\mathcal{I}^{n-1} \rightarrow \mathcal{I}^n) = \text{Ker}(\mathcal{I}^n \rightarrow \mathcal{I}^{n+1})$  as sheaves. Since  $\tilde{\xi}$  is a section of the kernel sheaf over  $U$  we conclude there exists a covering  $\{U_i \rightarrow U\}$  of the site such that  $\tilde{\xi}|_{U_i}$  is the image under  $d$  of a section  $\xi_i \in \mathcal{I}^{n-1}(U_i)$ . By our definition of the restriction  $\xi|_{U_i}$  as corresponding to the class of  $\tilde{\xi}|_{U_i}$  we conclude.  $\square$

072W **Lemma 8.4.** *Let  $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites corresponding to the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . For any  $\mathcal{F} \in \text{Ob}(\text{Mod}(\mathcal{O}_{\mathcal{C}}))$  the sheaf  $R^i f_* \mathcal{F}$  is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$



**Proof.** Let  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  be an injective resolution. Then  $R^i f_* \mathcal{F}$  is by definition the  $i$ th cohomology sheaf of the complex

$$f_* \mathcal{I}^0 \rightarrow f_* \mathcal{I}^1 \rightarrow f_* \mathcal{I}^2 \rightarrow \dots$$

By definition of the abelian category structure on  $\mathcal{O}_{\mathcal{D}}$ -modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \mapsto \frac{\text{Ker}(f_* \mathcal{I}^i(V) \rightarrow f_* \mathcal{I}^{i+1}(V))}{\text{Im}(f_* \mathcal{I}^{i-1}(V) \rightarrow f_* \mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\text{Ker}(\mathcal{I}^i(u(V)) \rightarrow \mathcal{I}^{i+1}(u(V)))}{\text{Im}(\mathcal{I}^{i-1}(u(V)) \rightarrow \mathcal{I}^i(u(V)))}$$

which is equal to  $H^i(u(V), \mathcal{F})$  and we win.  $\square$

### 9. The Čech complex and Čech cohomology

03AK Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target, see Sites, Definition 6.1. Assume that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{C}$ . Set

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \dots \times_U U_{i_p}).$$

This is an abelian group. For  $s \in \check{C}^p(\mathcal{U}, \mathcal{F})$  we denote  $s_{i_0 \dots i_p}$  its value in the factor  $\mathcal{F}(U_{i_0} \times_U \dots \times_U U_{i_p})$ . We define

$$d : \check{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$03AL \quad (9.0.1) \quad d(s)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_{p+1}}|_{U_{i_0} \times_U \dots \times_U U_{i_{p+1}}}$$

where the restriction is via the projection map

$$U_{i_0} \times_U \dots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \dots \times_U \widehat{U_{i_j}} \times_U \dots \times_U U_{i_{p+1}}.$$

It is straightforward to see that  $d \circ d = 0$ . In other words  $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$  is a complex.

03AM **Definition 9.1.** Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{C}$ . The complex  $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$  is the Čech complex associated to  $\mathcal{F}$  and the family  $\mathcal{U}$ . Its cohomology groups  $H^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F}))$  are called the Čech cohomology groups of  $\mathcal{F}$  with respect to  $\mathcal{U}$ . They are denoted  $\check{H}^i(\mathcal{U}, \mathcal{F})$ .

We observe that any covering  $\{U_i \rightarrow U\}$  of a site  $\mathcal{C}$  is a family of morphisms with fixed target to which the definition applies.

03AN **Lemma 9.2.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{C}$ . The following are equivalent

- (1)  $\mathcal{F}$  is an abelian sheaf on  $\mathcal{C}$  and
- (2) for every covering  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  of the site  $\mathcal{C}$  the natural map

$$\mathcal{F}(U) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{F})$$

(see Sites, Section 10) is bijective.

**Proof.** This is true since the sheaf condition is exactly that  $\mathcal{F}(U) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{F})$  is bijective for every covering of  $\mathcal{C}$ .  $\square$

Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms of  $\mathcal{C}$  with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Let  $\mathcal{V} = \{V_j \rightarrow V\}_{j \in J}$  be another. Let  $f : U \rightarrow V$ ,  $\alpha : I \rightarrow J$  and  $f_i : U_i \rightarrow V_{\alpha(i)}$  be a morphism of families of morphisms with fixed target, see Sites, Section 8. In this case we get a map of Čech complexes

$$03F4 \quad (9.2.1) \quad \varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

which in degree  $p$  is given by

$$\varphi(s)_{i_0 \dots i_p} = (f_{i_0} \times \dots \times f_{i_p})^* s_{\alpha(i_0) \dots \alpha(i_p)}$$

## 10. Čech cohomology as a functor on presheaves

03AO **Warning:** In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{C}$ . The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in  $\mathcal{F}$ . In fact, it is a functor

$$03AP \quad (10.0.1) \quad \check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \text{Comp}^+(Ab)$$

see Derived Categories, Definition 8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.9.

03AQ **Lemma 10.1.** *The functor given by Equation (10.0.1) is an exact functor (see Homology, Lemma 7.2).*

**Proof.** For any object  $W$  of  $\mathcal{C}$  the functor  $\mathcal{F} \mapsto \mathcal{F}(W)$  is an additive exact functor from  $PAb(\mathcal{C})$  to  $Ab$ . The terms  $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$  of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows.  $\square$

03AR **Lemma 10.2.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . The functors  $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$  form a  $\delta$ -functor from the abelian category  $PAb(\mathcal{C})$  to the category of  $\mathbf{Z}$ -modules (see Homology, Definition 11.1).*

**Proof.** By Lemma 10.1 a short exact sequence of abelian presheaves  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  is turned into a short exact sequence of complexes of  $\mathbf{Z}$ -modules. Hence we can use Homology, Lemma 12.12 to get the boundary maps  $\delta_{\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \rightarrow \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$  and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves.  $\square$

03AS **Lemma 10.3.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Consider the chain complex  $\mathbf{Z}_{\mathcal{U}, \bullet}$  of abelian presheaves*

$$\dots \rightarrow \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \rightarrow \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \rightarrow \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \rightarrow 0 \rightarrow \dots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0} \times_U \dots \times_U U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0} \times_U \dots \widehat{U_{i_j}} \dots \times_U U_{i_{p+1}}}$$

is given by  $(-1)^j$  times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}, \bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in  $\mathcal{F} \in \mathrm{Ob}(\mathrm{PAb}(\mathcal{C}))$ .

**Proof.** This is a tautology based on the fact that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\bigoplus_{i_0 \dots i_p} \mathbf{Z}_{U_{i_0} \times_U \dots \times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0 \dots i_p} \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U \dots \times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0 \dots i_p} \mathcal{F}(U_{i_0} \times_U \dots \times_U U_{i_p}) \end{aligned}$$

see Modules on Sites, Lemma 4.2. □

03AT **Lemma 10.4.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . The chain complex  $\mathbf{Z}_{\mathcal{U}, \bullet}$  of presheaves of Lemma 10.3 above is exact in positive degrees, i.e., the homology presheaves  $H_i(\mathbf{Z}_{\mathcal{U}, \bullet})$  are zero for  $i > 0$ .*

**Proof.** Let  $V$  be an object of  $\mathcal{C}$ . We have to show that the chain complex of abelian groups  $\mathbf{Z}_{\mathcal{U}, \bullet}(V)$  is exact in degrees  $> 0$ . This is the complex

$$\begin{array}{c} \dots \\ \downarrow \\ \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

For any morphism  $\varphi : V \rightarrow U$  denote  $\mathrm{Mor}_{\varphi}(V, U_i) = \{\varphi_i : V \rightarrow U_i \mid f_i \circ \varphi_i = \varphi\}$ . We will use a similar notation for  $\mathrm{Mor}_{\varphi}(V, U_{i_0} \times_U \dots \times_U U_{i_p})$ . Note that composing with the various projection maps between the fibred products  $U_{i_0} \times_U \dots \times_U U_{i_p}$

preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c}
\cdots \\
\downarrow \\
\bigoplus_{\varphi} \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[Mor_{\varphi}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\
\downarrow \\
\bigoplus_{\varphi} \bigoplus_{i_0 i_1} \mathbf{Z}[Mor_{\varphi}(V, U_{i_0} \times_U U_{i_1})] \\
\downarrow \\
\bigoplus_{\varphi} \bigoplus_{i_0} \mathbf{Z}[Mor_{\varphi}(V, U_{i_0})] \\
\downarrow \\
0
\end{array}$$

Next, we make the remark that we have

$$Mor_{\varphi}(V, U_{i_0} \times_U \dots \times_U U_{i_p}) = Mor_{\varphi}(V, U_{i_0}) \times \dots \times Mor_{\varphi}(V, U_{i_p})$$

Using this and the fact that  $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \amalg B]$  we see that the complex becomes

$$\begin{array}{c}
\cdots \\
\downarrow \\
\bigoplus_{\varphi} \mathbf{Z} [\amalg_{i_0 i_1 i_2} Mor_{\varphi}(V, U_{i_0}) \times Mor_{\varphi}(V, U_{i_2})] \\
\downarrow \\
\bigoplus_{\varphi} \mathbf{Z} [\amalg_{i_0 i_1} Mor_{\varphi}(V, U_{i_0}) \times Mor_{\varphi}(V, U_{i_1})] \\
\downarrow \\
\bigoplus_{\varphi} \mathbf{Z} [\amalg_{i_0} Mor_{\varphi}(V, U_{i_0})] \\
\downarrow \\
0
\end{array}$$

Finally, on setting  $S_{\varphi} = \amalg_{i \in I} Mor_{\varphi}(V, U_i)$  we see that we get

$$\bigoplus_{\varphi} (\dots \rightarrow \mathbf{Z}[S_{\varphi} \times S_{\varphi} \times S_{\varphi}] \rightarrow \mathbf{Z}[S_{\varphi} \times S_{\varphi}] \rightarrow \mathbf{Z}[S_{\varphi}] \rightarrow 0 \rightarrow \dots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set  $S$  the (extended) complex of free abelian groups

$$\dots \rightarrow \mathbf{Z}[S \times S \times S] \rightarrow \mathbf{Z}[S \times S] \rightarrow \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \rightarrow 0 \rightarrow \dots$$

is exact in all degrees. To see this fix an element  $s \in S$ , and use the homotopy

$$n_{(s_0, \dots, s_p)} \longmapsto n_{(s, s_0, \dots, s_p)}$$

with obvious notations. □

03F5 **Lemma 10.5.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . Let  $\mathcal{O}$  be a presheaf of rings on  $\mathcal{C}$ . The chain complex*

$$\mathbf{Z}_{\mathcal{U}, \bullet} \otimes_{p, \mathbf{Z}} \mathcal{O}$$

*is exact in positive degrees. Here  $\mathbf{Z}_{\mathcal{U}, \bullet}$  is the cochain complex of Lemma 10.3, and the tensor product is over the constant presheaf of rings with value  $\mathbf{Z}$ .*

**Proof.** Let  $V$  be an object of  $\mathcal{C}$ . In the proof of Lemma 10.4 we saw that  $\mathbf{Z}_{\mathcal{U}, \bullet}(V)$  is isomorphic as a complex to a direct sum of complexes which are homotopic to  $\mathbf{Z}$  placed in degree zero. Hence also  $\mathbf{Z}_{\mathcal{U}, \bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$  is isomorphic as a complex to a direct sum of complexes which are homotopic to  $\mathcal{O}(V)$  placed in degree zero. Or you can use Modules on Sites, Lemma 28.10, which applies since the presheaves  $\mathbf{Z}_{\mathcal{U}, i}$  are flat, and the proof of Lemma 10.4 shows that  $H_0(\mathbf{Z}_{\mathcal{U}, \bullet})$  is a flat presheaf also.  $\square$

03AU **Lemma 10.6.** *Let  $\mathcal{C}$  be a category. Let  $\mathcal{U} = \{f_i : U_i \rightarrow U\}_{i \in I}$  be a family of morphisms with fixed target such that all fibre products  $U_{i_0} \times_U \dots \times_U U_{i_p}$  exist in  $\mathcal{C}$ . The Čech cohomology functors  $\check{H}^p(\mathcal{U}, -)$  are canonically isomorphic as a  $\delta$ -functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

*Moreover, there is a functorial quasi-isomorphism*

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

*where the right hand side indicates the derived functor*

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

*of the left exact functor  $\check{H}^0(\mathcal{U}, -)$ .*

**Proof.** Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 6.1. Note that  $\check{H}^0(\mathcal{U}, -)$  is a left exact functor from the category of abelian presheaves to the category of  $\mathbf{Z}$ -modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 20.

Let  $\mathcal{I}$  be an injective abelian presheaf. In this case the functor  $\text{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$  is exact on  $PAb(\mathcal{C})$ . By Lemma 10.3 we have

$$\text{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}, \bullet}, \mathcal{I}) = \check{C}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 10.4 we have that  $\mathbf{Z}_{\mathcal{U}, \bullet}$  is exact in positive degrees. Hence by the exactness of  $\text{Hom}$  into  $\mathcal{I}$  mentioned above we see that  $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$  for all  $i > 0$ . Thus the  $\delta$ -functor  $(\check{H}^n, \delta)$  (see Lemma 10.2) satisfies the assumptions of Homology, Lemma 11.4, and hence is a universal  $\delta$ -functor.

By Derived Categories, Lemma 20.4 also the sequence  $R^i \check{H}^0(\mathcal{U}, -)$  forms a universal  $\delta$ -functor. By the uniqueness of universal  $\delta$ -functors, see Homology, Lemma 11.5 we conclude that  $R^i \check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$ . This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let  $\mathcal{F}$  be any abelian presheaf on  $\mathcal{C}$ . Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  in the category  $PAb(\mathcal{C})$ . Consider the double complex  $A^{\bullet, \bullet}$  with terms

$$A^{p, q} = \check{C}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex  $sA^\bullet$  associated to this double complex. There is a map of complexes

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps  $\check{C}^p(\mathcal{U}, \mathcal{F}) \rightarrow A^{p,0} = \check{C}^\bullet(\mathcal{U}, \mathcal{I}^0)$  and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps  $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \rightarrow A^{0,q} = \check{C}^0(\mathcal{U}, \mathcal{I}^q)$ . Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 22.7. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves  $\mathcal{I}^q$  are zero. Since quasi-isomorphisms become invertible in  $D^+(\mathbf{Z})$  this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial.  $\square$

## 11. Čech cohomology and cohomology

03AV The relationship between cohomology and Čech cohomology comes from the fact that the Čech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Čech cohomology in the preceding section.

03F6 **Lemma 11.1.** *Let  $\mathcal{C}$  be a site. An injective abelian sheaf is also injective as an object in the category  $PAb(\mathcal{C})$ .*

**Proof.** Apply Homology, Lemma 26.1 to the categories  $\mathcal{A} = Ab(\mathcal{C})$ ,  $\mathcal{B} = PAb(\mathcal{C})$ , the inclusion functor and sheafification. (See Modules on Sites, Section 3 to see that all assumptions of the lemma are satisfied.)  $\square$

03AW **Lemma 11.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering of  $\mathcal{C}$ . Let  $\mathcal{I}$  be an injective abelian sheaf, i.e., an injective object of  $Ab(\mathcal{C})$ . Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

**Proof.** By Lemma 11.1 we see that  $\mathcal{I}$  is an injective object in  $PAb(\mathcal{C})$ . Hence we can apply Lemma 10.6 (or its proof) to see the vanishing of higher Čech cohomology group. For the zeroth see Lemma 9.2.  $\square$

03AX **Lemma 11.3.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering of  $\mathcal{C}$ . There is a transformation*

$$\check{C}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

*of functors  $Ab(\mathcal{C}) \rightarrow D^+(\mathbf{Z})$ . In particular this gives a transformation of functors  $\check{H}^p(U, \mathcal{F}) \rightarrow H^p(U, \mathcal{F})$  for  $\mathcal{F}$  ranging over  $Ab(\mathcal{C})$ .*

**Proof.** Let  $\mathcal{F}$  be an abelian sheaf. Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ . Consider the double complex  $A^{\bullet, \bullet}$  with terms  $A^{p,q} = \check{C}^p(\mathcal{U}, \mathcal{I}^q)$ . Moreover, consider the associated simple complex  $sA^\bullet$ , see Homology, Definition 22.3. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps  $\mathcal{I}^q(U) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{I}^q)$  and a map of complexes

$$\beta : \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the map  $\mathcal{F} \rightarrow \mathcal{I}^0$ . We can apply Homology, Lemma 22.7 to see that  $\alpha$  is a quasi-isomorphism. Namely, Lemma 11.2 implies that the  $q$ th row of the double complex  $A^{\bullet, \bullet}$  is a resolution of  $\Gamma(U, \mathcal{I}^q)$ . Hence  $\alpha$  becomes invertible in  $D^+(\mathbf{Z})$  and the transformation of the lemma is the composition of  $\beta$  followed by the inverse of  $\alpha$ . We omit the verification that this is functorial.  $\square$

0A6G **Lemma 11.4.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{G}$  be an abelian sheaf on  $\mathcal{C}$ . Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering of  $\mathcal{C}$ . The map*

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(U, \mathcal{G})$$

*is injective and identifies  $\check{H}^1(\mathcal{U}, \mathcal{G})$  via the bijection of Lemma 5.3 with the set of isomorphism classes of  $\mathcal{G}|_U$ -torsors which restrict to trivial torsors over each  $U_i$ .*

**Proof.** To see this we construct an inverse map. Namely, let  $\mathcal{F}$  be a  $\mathcal{G}|_U$ -torsor on  $\mathcal{C}/U$  whose restriction to  $\mathcal{C}/U_i$  is trivial. By Lemma 5.2 this means there exists a section  $s_i \in \mathcal{F}(U_i)$ . On  $U_{i_0} \times_U U_{i_1}$  there is a unique section  $s_{i_0 i_1}$  of  $\mathcal{G}$  such that  $s_{i_0 i_1} \cdot s_{i_0}|_{U_{i_0} \times_U U_{i_1}} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}}$ . An easy computation shows that  $s_{i_0 i_1}$  is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections  $s_i$ ). The inverse maps the isomorphism class of  $\mathcal{F}$  to the cohomology class of the cocycle  $(s_{i_0 i_1})$ . We omit the verification that this map is indeed an inverse.  $\square$

03AY **Lemma 11.5.** *Let  $\mathcal{C}$  be a site. Consider the functor  $i : Ab(\mathcal{C}) \rightarrow PAb(\mathcal{C})$ . It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

*see discussion in Section 8.*

**Proof.** It is clear that  $i$  is left exact. Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ . By definition  $R^p i$  is the  $p$ th cohomology presheaf of the complex  $\mathcal{I}^\bullet$ . In other words, the sections of  $R^p i(\mathcal{F})$  over an object  $U$  of  $\mathcal{C}$  are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \rightarrow \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \rightarrow \mathcal{I}^n(U))}$$

which is the definition of  $H^p(U, \mathcal{F})$ .  $\square$

03AZ **Lemma 11.6.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering of  $\mathcal{C}$ . For any abelian sheaf  $\mathcal{F}$  there is a spectral sequence  $(E_r, d_r)_{r \geq 0}$  with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

*converging to  $H^{p+q}(U, \mathcal{F})$ . This spectral sequence is functorial in  $\mathcal{F}$ .*

**Proof.** This is a Grothendieck spectral sequence (see Derived Categories, Lemma 22.2) for the functors

$$i : Ab(\mathcal{C}) \rightarrow PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \rightarrow Ab.$$

Namely, we have  $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$  by Lemma 9.2. We have that  $i(\mathcal{I})$  is Čech acyclic by Lemma 11.2. And we have that  $\check{H}^p(\mathcal{U}, -) = R^p \check{H}^0(\mathcal{U}, -)$  as functors on  $PAb(\mathcal{C})$  by Lemma 10.6. Putting everything together gives the lemma.  $\square$

03F7 **Lemma 11.7.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering. Let  $\mathcal{F} \in \text{Ob}(Ab(\mathcal{C}))$ . Assume that  $H^i(U_{i_0} \times_U \dots \times_U U_{i_p}, \mathcal{F}) = 0$  for all  $i > 0$ , all  $p \geq 0$  and all  $i_0, \dots, i_p \in I$ . Then  $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ .*

**Proof.** We will use the spectral sequence of Lemma 11.6. The assumptions mean that  $E_2^{p,q} = 0$  for all  $(p, q)$  with  $q \neq 0$ . Hence the spectral sequence degenerates at  $E_2$  and the result follows.  $\square$

03F8 **Lemma 11.8.** *Let  $\mathcal{C}$  be a site. Let*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

*be a short exact sequence of abelian sheaves on  $\mathcal{C}$ . Let  $U$  be an object of  $\mathcal{C}$ . If there exists a cofinal system of coverings  $\mathcal{U}$  of  $U$  such that  $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ , then the map  $\mathcal{G}(U) \rightarrow \mathcal{H}(U)$  is surjective.*

**Proof.** Take an element  $s \in \mathcal{H}(U)$ . Choose a covering  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  such that (a)  $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$  and (b)  $s|_{U_i}$  is the image of a section  $s_i \in \mathcal{G}(U_i)$ . Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since  $s_i$  lifts  $s$  we see that  $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$ . By the vanishing of  $\check{H}^1(\mathcal{U}, \mathcal{F})$  we can find sections  $t_i \in \mathcal{F}(U_i)$  such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections  $s_i - t_i$  satisfy the sheaf condition and glue to a section of  $\mathcal{G}$  over  $U$  which maps to  $s$ . Hence we win.  $\square$

03F9 **Lemma 11.9.** *(Variant of Cohomology, Lemma 12.8.) Let  $\mathcal{C}$  be a site. Let  $\text{Cov}_{\mathcal{C}}$  be the set of coverings of  $\mathcal{C}$  (see Sites, Definition 6.2). Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ , and  $\text{Cov} \subset \text{Cov}_{\mathcal{C}}$  be subsets. Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ . Assume that*

- (1) *For every  $\mathcal{U} \in \text{Cov}$ ,  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  we have  $U, U_i \in \mathcal{B}$  and every  $U_{i_0} \times_U \dots \times_U U_{i_p} \in \mathcal{B}$ .*
- (2) *For every  $U \in \mathcal{B}$  the coverings of  $U$  occurring in  $\text{Cov}$  is a cofinal system of coverings of  $U$ .*
- (3) *For every  $\mathcal{U} \in \text{Cov}$  we have  $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$  for all  $p > 0$ .*

*Then  $H^p(U, \mathcal{F}) = 0$  for all  $p > 0$  and any  $U \in \mathcal{B}$ .*

**Proof.** Let  $\mathcal{F}$  and  $\text{Cov}$  be as in the lemma. We will indicate this by saying “ $\mathcal{F}$  has vanishing higher Čech cohomology for any  $\mathcal{U} \in \text{Cov}$ ”. Choose an embedding  $\mathcal{F} \rightarrow \mathcal{I}$  into an injective abelian sheaf. By Lemma 11.2  $\mathcal{I}$  has vanishing higher Čech cohomology for any  $\mathcal{U} \in \text{Cov}$ . Let  $\mathcal{Q} = \mathcal{I}/\mathcal{F}$  so that we have a short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0.$$

By Lemma 11.8 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{I}(U) \rightarrow \mathcal{Q}(U) \rightarrow 0.$$

for every  $U \in \mathcal{B}$ . Hence for any  $\mathcal{U} \in \text{Cov}$  we get a short exact sequence of Čech complexes

$$0 \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \rightarrow 0$$

since each term in the Čech complex is made up out of a product of values over elements of  $\mathcal{B}$  by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any covering  $\mathcal{U} \in \text{Cov}$ . This implies that  $\mathcal{Q}$  is also an abelian sheaf with vanishing higher Čech cohomology for all  $\mathcal{U} \in \text{Cov}$ .



Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\
& & & & & \searrow & \\
& & & & & & H^1(U, \mathcal{Q}) \\
& & & & & \swarrow & \\
& & & & & & H^1(U, \mathcal{I}) \\
& & & & & \swarrow & \\
& & & & & & H^1(U, \mathcal{F}) \\
& & & & & \swarrow & \\
& & & & & & \dots
\end{array}$$

for any  $U \in \mathcal{B}$ . Since  $\mathcal{I}$  is injective we have  $H^n(U, \mathcal{I}) = 0$  for  $n > 0$  (see Derived Categories, Lemma 20.4). By the above we see that  $H^0(U, \mathcal{I}) \rightarrow H^0(U, \mathcal{Q})$  is surjective and hence  $H^1(U, \mathcal{F}) = 0$ . Since  $\mathcal{F}$  was an arbitrary abelian sheaf with vanishing higher Čech cohomology for all  $U \in \text{Cov}$  we conclude that also  $H^1(U, \mathcal{Q}) = 0$  since  $\mathcal{Q}$  is another of these sheaves (see above). By the long exact sequence this in turn implies that  $H^2(U, \mathcal{F}) = 0$ . And so on and so forth.  $\square$

## 12. Second cohomology and gerbes

0CJZ Let  $p : \mathcal{S} \rightarrow \mathcal{C}$  be a gerbe over a site all of whose automorphism groups are commutative. In this situation the first and second cohomology groups of the sheaf of automorphisms (Stacks, Lemma 11.8) controls the existence of objects.

The following lemma will be made obsolete by a more complete discussion of this relationship we will add in the future.

0CK0 **Lemma 12.1.** *Let  $\mathcal{C}$  be a site. Let  $p : \mathcal{S} \rightarrow \mathcal{C}$  be a gerbe over a site whose automorphism sheaves are abelian. Let  $\mathcal{G}$  be the sheaf of abelian groups constructed in Stacks, Lemma 11.8. Let  $U$  be an object of  $\mathcal{C}$  such that*

- (1) *there exists a cofinal system of coverings  $\{U_i \rightarrow U\}$  of  $U$  in  $\mathcal{C}$  such that  $H^1(U_i, \mathcal{G}) = 0$  and  $H^1(U_i \times_U U_j, \mathcal{G}) = 0$  for all  $i, j$ , and*
- (2)  $H^2(U, \mathcal{G}) = 0$ .

*Then there exists an object of  $\mathcal{S}$  lying over  $U$ .*

**Proof.** By Stacks, Definition 11.1 there exists a covering  $\mathcal{U} = \{U_i \rightarrow U\}$  and  $x_i$  in  $\mathcal{S}$  lying over  $U_i$ . Write  $U_{ij} = U_i \times_U U_j$ . By (1) after refining the covering we may assume that  $H^1(U_i, \mathcal{G}) = 0$  and  $H^1(U_{ij}, \mathcal{G}) = 0$ . Consider the sheaf

$$\mathcal{F}_{ij} = \text{Isom}(x_i|_{U_{ij}}, x_j|_{U_{ij}})$$

on  $\mathcal{C}/U_{ij}$ . Since  $\mathcal{G}|_{U_{ij}} = \text{Aut}(x_i|_{U_{ij}})$  we see that there is an action

$$\mathcal{G}|_{U_{ij}} \times \mathcal{F}_{ij} \rightarrow \mathcal{F}_{ij}$$

by precomposition. It is clear that  $\mathcal{F}_{ij}$  is a pseudo  $\mathcal{G}|_{U_{ij}}$ -torsor and in fact a torsor because any two objects of a gerbe are locally isomorphic. By our choice of the covering and by Lemma 5.3 these torsors are trivial (and hence have global sections by Lemma 5.2). In other words, we can choose isomorphisms

$$\varphi_{ij} : x_i|_{U_{ij}} \longrightarrow x_j|_{U_{ij}}$$

To find an object  $x$  over  $U$  we are going to massage our choice of these  $\varphi_{ij}$  to get a descent datum (which is necessarily effective as  $p : \mathcal{S} \rightarrow \mathcal{C}$  is a stack). Namely, the

obstruction to being a descent datum is that the cocycle condition may not hold. Namely, set  $U_{ijk} = U_i \times_U U_j \times_U U_k$ . Then we can consider

$$g_{ijk} = \varphi_{ik}^{-1}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

which is an automorphism of  $x_i$  over  $U_{ijk}$ . Thus we may and do consider  $g_{ijk}$  as a section of  $\mathcal{G}$  over  $U_{ijk}$ . A computation (omitted) shows that  $(g_{i_0 i_1 i_2})$  is a 2-cocycle in the Čech complex  $\check{C}^\bullet(\mathcal{U}, \mathcal{G})$  of  $\mathcal{G}$  with respect to the covering  $\mathcal{U}$ . By the spectral sequence of Lemma 11.6 and since  $H^1(U_i, \mathcal{G}) = 0$  for all  $i$  we see that  $\check{H}^2(\mathcal{U}, \mathcal{G}) \rightarrow H^2(U, \mathcal{G})$  is injective. Hence  $(g_{i_0 i_1 i_2})$  is a coboundary by our assumption that  $H^2(U, \mathcal{G}) = 0$ . Thus we can find sections  $g_{ij} \in \mathcal{G}(U_{ij})$  such that  $g_{ik}^{-1}|_{U_{ijk}} g_{jk}|_{U_{ijk}} g_{ij}|_{U_{ijk}} = g_{ijk}$  for all  $i, j, k$ . After replacing  $\varphi_{ij}$  by  $\varphi_{ij} g_{ij}^{-1}$  we see that  $\varphi_{ij}$  gives a descent datum on the objects  $x_i$  over  $U_i$  and the proof is complete.  $\square$

### 13. Cohomology of modules

03FA Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

03FB **Lemma 13.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. An injective sheaf of modules is also injective as an object in the category  $PMod(\mathcal{O})$ .*

**Proof.** Apply Homology, Lemma 26.1 to the categories  $\mathcal{A} = Mod(\mathcal{O})$ ,  $\mathcal{B} = PMod(\mathcal{O})$ , the inclusion functor and sheafification. (See Modules on Sites, Section 11 to see that all assumptions of the lemma are satisfied.)  $\square$

06YK **Lemma 13.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Consider the functor  $i : Mod(\mathcal{C}) \rightarrow PMod(\mathcal{C})$ . It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \mapsto H^p(U, \mathcal{F})$$

*see discussion in Section 8.*

**Proof.** It is clear that  $i$  is left exact. Choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  in  $Mod(\mathcal{O})$ . By definition  $R^p i$  is the  $p$ th cohomology presheaf of the complex  $\mathcal{I}^\bullet$ . In other words, the sections of  $R^p i(\mathcal{F})$  over an object  $U$  of  $\mathcal{C}$  are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \rightarrow \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \rightarrow \mathcal{I}^n(U))}$$

which is the definition of  $H^p(U, \mathcal{F})$ .  $\square$

03FC **Lemma 13.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  be a covering of  $\mathcal{C}$ . Let  $\mathcal{I}$  be an injective  $\mathcal{O}$ -module, i.e., an injective object of  $Mod(\mathcal{O})$ . Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

**Proof.** Lemma 10.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{C}^\bullet(\mathcal{U}, \mathcal{I}) &= Mor_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U}, \bullet}, \mathcal{I}) \\ &= Mor_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U}, \bullet}, \mathcal{I}) \\ &= Mor_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U}, \bullet} \otimes_{p, \mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 9.2. By Lemma 13.1 we see that  $\mathcal{I}$  is an injective object in  $PMod(\mathcal{O})$ . Hence  $\text{Hom}_{PMod(\mathcal{O})}(-, \mathcal{I})$  is an exact functor. By Lemma 10.5 we see the vanishing of higher Čech cohomology groups. For the zeroth see Lemma 9.2.  $\square$

03FD **Lemma 13.4.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module, and denote  $\mathcal{F}_{ab}$  the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object  $U$  of  $\mathcal{C}$  we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in  $Ab(\mathcal{C})$  and the right hand side is cohomology computed in  $Mod(\mathcal{O})$ .

**Proof.** By Derived Categories, Lemma 20.4 the  $\delta$ -functor  $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$  is universal. The functor  $Mod(\mathcal{O}) \rightarrow Ab(\mathcal{C}), \mathcal{F} \mapsto \mathcal{F}_{ab}$  is exact. Hence  $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$  is a  $\delta$ -functor also. Suppose we show that  $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$  is also universal. This will imply the second statement of the lemma by uniqueness of universal  $\delta$ -functors, see Homology, Lemma 11.5. Since  $Mod(\mathcal{O})$  has enough injectives, it suffices to show that  $H^i(U, \mathcal{I}_{ab}) = 0$  for any injective object  $\mathcal{I}$  in  $Mod(\mathcal{O})$ , see Homology, Lemma 11.4.

Let  $\mathcal{I}$  be an injective object of  $Mod(\mathcal{O})$ . Apply Lemma 11.9 with  $\mathcal{F} = \mathcal{I}, \mathcal{B} = \mathcal{C}$  and  $Cov = Cov_{\mathcal{C}}$ . Assumption (3) of that lemma holds by Lemma 13.3. Hence we see that  $H^i(U, \mathcal{I}_{ab}) = 0$  for every object  $U$  of  $\mathcal{C}$ .

If  $\mathcal{C}$  has a final object then this also implies the first equality. If not, then according to Sites, Lemma 28.5 we see that the ringed topos  $(Sh(\mathcal{C}), \mathcal{O})$  is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows.  $\square$

060L **Lemma 13.5.** *Let  $\mathcal{C}$  be a site. Let  $I$  be a set. For  $i \in I$  let  $\mathcal{F}_i$  be an abelian sheaf on  $\mathcal{C}$ . Let  $U \in Ob(\mathcal{C})$ . The canonical map*

$$H^p(U, \prod_{i \in I} \mathcal{F}_i) \longrightarrow \prod_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for  $p = 0$  and injective for  $p = 1$ .

**Proof.** The statement for  $p = 0$  is true because the product of sheaves is equal to the product of the underlying presheaves, see Sites, Lemma 10.1. Proof for  $p = 1$ . Set  $\mathcal{F} = \prod \mathcal{F}_i$ . Let  $\xi \in H^1(U, \mathcal{F})$  map to zero in  $\prod H^1(U, \mathcal{F}_i)$ . By locality of cohomology, see Lemma 8.3, there exists a covering  $\mathcal{U} = \{U_j \rightarrow U\}$  such that  $\xi|_{U_j} = 0$  for all  $j$ . By Lemma 11.4 this means  $\xi$  comes from an element  $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$ . Since the maps  $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \rightarrow H^1(U, \mathcal{F}_i)$  are injective for all  $i$  (by Lemma 11.4), and since the image of  $\xi$  is zero in  $\prod H^1(U, \mathcal{F}_i)$  we see that the image  $\check{\xi}_i = 0$  in  $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$ . However, since  $\mathcal{F} = \prod \mathcal{F}_i$  we see that  $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$  is the product of the complexes  $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$ , hence by Homology, Lemma 29.1 we conclude that  $\check{\xi} = 0$  as desired.  $\square$

093X **Lemma 13.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $a : U' \rightarrow U$  be a monomorphism in  $\mathcal{C}$ . Then for any injective  $\mathcal{O}$ -module  $\mathcal{I}$  the restriction mapping  $\mathcal{I}(U) \rightarrow \mathcal{I}(U')$  is surjective.*

**Proof.** Let  $j : \mathcal{C}/U \rightarrow \mathcal{C}$  and  $j' : \mathcal{C}/U' \rightarrow \mathcal{C}$  be the localization morphisms (Modules on Sites, Section 19). Since  $j_!$  is a left adjoint to restriction we see that for any sheaf  $\mathcal{F}$  of  $\mathcal{O}$ -modules

$$\mathrm{Hom}_{\mathcal{O}}(j_! \mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf  $j'_! \mathcal{O}_{U'}$  represents the functor  $\mathcal{F} \mapsto \mathcal{F}(U')$ . Moreover below we describe a canonical map of  $\mathcal{O}$ -modules

$$j'_! \mathcal{O}_{U'} \longrightarrow j_! \mathcal{O}_U$$

which corresponds to the restriction mapping  $\mathcal{F}(U) \rightarrow \mathcal{F}(U')$  via Yoneda's lemma (Categories, Lemma 3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 24.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object  $V$  of  $\mathcal{C}$  the  $\mathcal{O}(V)$ -module

$$\bigoplus_{\varphi' \in \text{Mor}_{\mathcal{C}}(V, U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus_{\varphi \in \text{Mor}_{\mathcal{C}}(V, U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 19.2. We take the map which maps the summand corresponding to  $\varphi'$  to the summand corresponding to  $\varphi = a \circ \varphi'$  by the identity map on  $\mathcal{O}(V)$ . As  $a$  is a monomorphism, this map is injective. As sheafification is exact, the result follows.  $\square$

#### 14. Limp sheaves

079X Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be a sheaf of sets on  $\mathcal{C}$  (we intentionally use a roman capital here to distinguish from abelian sheaves). Given an abelian sheaf  $\mathcal{F}$  we denote  $\mathcal{F}(K) = \text{Mor}_{\text{Sh}(\mathcal{C})}(K, \mathcal{F})$ . The functor  $\mathcal{F} \mapsto \mathcal{F}(K)$  is a left exact functor  $\text{Mod}(\mathcal{O}) \rightarrow \text{Ab}$  hence we have its right derived functors. We will denote these  $H^p(K, \mathcal{F})$  so that  $H^0(K, \mathcal{F}) = \mathcal{F}(K)$ .

We mention two special cases. The first is the case where  $K = h_U^\#$  for some object  $U$  of  $\mathcal{C}$ . In this case  $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$ , because  $\text{Mor}_{\text{Sh}(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ , see Sites, Section 12. The second is the case  $\mathcal{O} = \mathbf{Z}$  (the constant sheaf). In this case the cohomology groups are functors  $H^p(K, -) : \text{Ab}(\mathcal{C}) \rightarrow \text{Ab}$ . Here is the analogue of Lemma 13.4.

079Y **Lemma 14.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be a sheaf of sets on  $\mathcal{C}$ . Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module and denote  $\mathcal{F}_{ab}$  the underlying sheaf of abelian groups. Then  $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$ .*

**Proof.** Note that both  $H^p(K, \mathcal{F})$  and  $H^p(K, \mathcal{F}_{ab})$  depend only on the topos, not on the underlying site. Hence by Sites, Lemma 28.5 we may replace  $\mathcal{C}$  by a “larger” site such that  $K = h_U$  for some object  $U$  of  $\mathcal{C}$ . In this case the result follows from Lemma 13.4.  $\square$

079Z **Lemma 14.2.** *Let  $\mathcal{C}$  be a site. Let  $K' \rightarrow K$  be a surjective map of sheaves of sets on  $\mathcal{C}$ . Set  $K'_p = K' \times_K \dots \times_K K'$  ( $p+1$ -factors). For every abelian sheaf  $\mathcal{F}$  there is a spectral sequence with  $E_1^{p,q} = H^q(K'_p, \mathcal{F})$  converging to  $H^{p+q}(K, \mathcal{F})$ .*

**Proof.** After replacing  $\mathcal{C}$  by a “larger” site as in Sites, Lemma 28.5 we may assume that  $K, K'$  are objects of  $\mathcal{C}$  and that  $\mathcal{U} = \{K' \rightarrow K\}$  is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 11.6 whose  $E_1$  page is as indicated in the statement of the lemma.  $\square$

07A0 **Lemma 14.3.** *Let  $\mathcal{C}$  be a site. Let  $K$  be a sheaf of sets on  $\mathcal{C}$ . Consider the morphism of topoi  $j : \text{Sh}(\mathcal{C}/K) \rightarrow \text{Sh}(\mathcal{C})$ , see Sites, Lemma 29.3. Then  $j^{-1}$  preserves injectives and  $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$  for any abelian sheaf  $\mathcal{F}$  on  $\mathcal{C}$ .*

**Proof.** By Sites, Lemmas 29.1 and 29.3 the morphism of topoi  $j$  is equivalent to a localization. Hence this follows from Lemma 8.1.  $\square$

Keeping in mind Lemma 14.1 we see that the following definition is the “correct one” also for sheaves of modules on ringed sites.

072Y **Definition 14.4.** Let  $\mathcal{C}$  be a site. We say an abelian sheaf  $\mathcal{F}$  is *limp*<sup>1</sup> if for every sheaf of sets  $K$  we have  $H^p(K, \mathcal{F}) = 0$  for all  $p \geq 1$ .

It is clear that being limp is an intrinsic property, i.e., preserved under equivalences of topoi. A limp sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being limp is strictly stronger. Here is a characterization of limp sheaves which is sometimes useful.

07A1 **Lemma 14.5.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{F}$  be an abelian sheaf. If*

- (1)  $H^p(U, \mathcal{F}) = 0$  for  $p > 0$  and  $U \in \text{Ob}(\mathcal{C})$ , and
- (2) for every surjection  $K' \rightarrow K$  of sheaves of sets the extended Čech complex

$$0 \rightarrow H^0(K, \mathcal{F}) \rightarrow H^0(K', \mathcal{F}) \rightarrow H^0(K' \times_K K', \mathcal{F}) \rightarrow \dots$$

*is exact,*

*then  $\mathcal{F}$  is limp (and the converse holds too).*

**Proof.** By assumption (1) we have  $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$  for all  $p > 0$  and all objects  $U$  of  $\mathcal{C}$ . Note that if  $K = \coprod K_i$  is a coproduct of sheaves of sets on  $\mathcal{C}$  then  $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$ . For any sheaf of sets  $K$  there exists a surjection

$$K' = \coprod h_{U_i}^\# \longrightarrow K$$

see Sites, Lemma 12.5. Thus we conclude that: (\*) for every sheaf of sets  $K$  there exists a surjection  $K' \rightarrow K$  of sheaves of sets such that  $H^p(K', \mathcal{F}) = 0$  for  $p > 0$ . We claim that (\*) and condition (2) imply that  $\mathcal{F}$  is limp. Note that conditions (\*) and (2) only depend on  $\mathcal{F}$  as an object of the topos  $Sh(\mathcal{C})$  and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on  $n \geq 0$  that (\*) and (2) imply the following induction hypothesis  $IH_n$ :  $H^p(K, \mathcal{F}) = 0$  for all  $0 < p \leq n$  and all sheaves of sets  $K$ . Note that  $IH_0$  holds. Assume  $IH_n$ . Pick a sheaf of sets  $K$ . Pick a surjection  $K' \rightarrow K$  such that  $H^p(K', \mathcal{F}) = 0$  for all  $p > 0$ . We have a spectral sequence with

$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$

covering to  $H^{p+q}(K, \mathcal{F})$ , see Lemma 14.2. By  $IH_n$  we see that  $E_1^{p,q} = 0$  for  $0 < q \leq n$  and by assumption (2) we see that  $E_2^{p,0} = 0$  for  $p > 0$ . Finally, we have  $E_1^{0,q} = 0$  for  $q > 0$  because  $H^q(K', \mathcal{F}) = 0$  by choice of  $K'$ . Hence we conclude that  $H^{n+1}(K, \mathcal{F}) = 0$  because all the terms  $E_2^{p,q}$  with  $p + q = n + 1$  are zero.  $\square$

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<sup>1</sup>This is probably nonstandard notation. In [AGV71, V, Definition 4.1] this property is dubbed “flasque”, but we cannot use this because it would clash with our definition of flasque sheaves on topological spaces. Please email [stacks.project@gmail.com](mailto:stacks.project@gmail.com) if you have a better suggestion.

### 15. The Leray spectral sequence

072X The key to proving the existence of the Leray spectral sequence is the following lemma.

072Z **Lemma 15.1.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Then for any injective object  $\mathcal{I}$  in  $Mod(\mathcal{O}_{\mathcal{C}})$  the pushforward  $f_*\mathcal{I}$  is limp.*

**Proof.** Let  $K$  be a sheaf of sets on  $\mathcal{D}$ . By Modules on Sites, Lemma 7.2 we may replace  $\mathcal{C}$ ,  $\mathcal{D}$  by “larger” sites such that  $f$  comes from a morphism of ringed sites induced by a continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$  such that  $K = h_V$  for some object  $V$  of  $\mathcal{D}$ .

Thus we have to show that  $H^q(V, f_*\mathcal{I})$  is zero for  $q > 0$  and all objects  $V$  of  $\mathcal{D}$  when  $f$  is given by a morphism of ringed sites. Let  $\mathcal{V} = \{V_j \rightarrow V\}$  be any covering of  $\mathcal{D}$ . Since  $u$  is continuous we see that  $\mathcal{U} = \{u(V_j) \rightarrow u(V)\}$  is a covering of  $\mathcal{C}$ . Then we have an equality of Čech complexes

$$\check{C}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{C}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of  $f_*$ . By Lemma 13.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 11.9.  $\square$

For flat morphisms the functor  $f_*$  preserves injective modules. In particular the functor  $f_* : Ab(\mathcal{C}) \rightarrow Ab(\mathcal{D})$  always transforms injective abelian sheaves into injective abelian sheaves.

0730 **Lemma 15.2.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. If  $f$  is flat, then  $f_*\mathcal{I}$  is an injective  $\mathcal{O}_{\mathcal{D}}$ -module for any injective  $\mathcal{O}_{\mathcal{C}}$ -module  $\mathcal{I}$ .*

**Proof.** In this case the functor  $f^*$  is exact, see Modules on Sites, Lemma 30.2. Hence the result follows from Homology, Lemma 26.1.  $\square$

0731 **Lemma 15.3.** *Let  $(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  be a ringed topos. A limp sheaf is right acyclic for the following functors:*

- (1) *the functor  $H^0(U, -)$  for any object  $U$  of  $\mathcal{C}$ ,*
- (2) *the functor  $\mathcal{F} \mapsto \mathcal{F}(K)$  for any presheaf of sets  $K$ ,*
- (3) *the functor  $\Gamma(\mathcal{C}, -)$  of global sections,*
- (4) *the functor  $f_*$  for any morphism  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  of ringed topoi.*

**Proof.** Part (2) is the definition of a limp sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of limp sheaves. Part (3) is a special case of (2) where  $K = e$  is the final object of  $Sh(\mathcal{C})$ .

To prove (4) we may assume, by Modules on Sites, Lemma 7.2 that  $f$  is given by a morphism of sites. In this case we see that  $R^i f_*$ ,  $i > 0$  of a limp sheaf are zero by the description of higher direct images in Lemma 8.4.  $\square$

08J6 **Remark 15.4.** As a consequence of the results above we find that Derived Categories, Lemma 22.1 applies to a number of situations. For example, given a morphism  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of  $\mathcal{O}_C$ -modules  $\mathcal{F}$ . Namely, for an injective  $\mathcal{O}_X$ -module  $\mathcal{I}$  the  $\mathcal{O}_D$ -module  $f_*\mathcal{I}$  is limp by Lemma 15.1 and a limp sheaf is acyclic for  $\Gamma(\mathcal{D}, -)$  by Lemma 15.3.

0732 **Lemma 15.5** (Leray spectral sequence). *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_C) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_D)$  be a morphism of ringed topoi. Let  $\mathcal{F}^\bullet$  be a bounded below complex of  $\mathcal{O}_C$ -modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^q f_*(\mathcal{F}^\bullet))$$

*converging to  $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$ .*

**Proof.** This is just the Grothendieck spectral sequence Derived Categories, Lemma 22.2 coming from the composition of functors  $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$ . To see that the assumptions of Derived Categories, Lemma 22.2 are satisfied, see Lemmas 15.1 and 15.3.  $\square$

0733 **Lemma 15.6.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_C) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_D)$  be a morphism of ringed topoi. Let  $\mathcal{F}$  be an  $\mathcal{O}_C$ -module.*

- (1) *If  $R^q f_*\mathcal{F} = 0$  for  $q > 0$ , then  $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$  for all  $p$ .*
- (2) *If  $H^p(\mathcal{D}, R^q f_*\mathcal{F}) = 0$  for all  $q$  and  $p > 0$ , then  $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^q f_*\mathcal{F})$  for all  $q$ .*

**Proof.** These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves.  $\square$

0734 **Lemma 15.7** (Relative Leray spectral sequence). *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_C) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_D)$  and  $g : (Sh(\mathcal{D}), \mathcal{O}_D) \rightarrow (Sh(\mathcal{E}), \mathcal{O}_E)$  be morphisms of ringed topoi. Let  $\mathcal{F}$  be an  $\mathcal{O}_C$ -module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

*converging to  $R^{p+q}(g \circ f)_*\mathcal{F}$ . This spectral sequence is functorial in  $\mathcal{F}$ , and there is a version for bounded below complexes of  $\mathcal{O}_C$ -modules.*

**Proof.** This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 22.2 and Lemmas 15.1 and 15.3.  $\square$

## 16. The base change map

0735 In this section we construct the base change map in some cases; the general case is treated in Remark 20.3. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose  $g : (Sh(\mathcal{C}), \mathcal{O}_C) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_D)$  is a flat morphism of ringed topoi. By Modules on Sites, Lemma 30.2 the functor  $g^* : Mod(\mathcal{O}_D) \rightarrow Mod(\mathcal{O}_C)$  is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_D) \rightarrow D(\mathcal{O}_C)$$

which is computed by simply pulling back an representative of a given object in  $D(\mathcal{O}_D)$ , see Derived Categories, Lemma 17.9. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by  $g^*$  rather than  $Lg^*$ .

0736 **Lemma 16.1.** *Let*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ f' \downarrow & & \downarrow f \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

be a commutative diagram of ringed topoi. Let  $\mathcal{F}^\bullet$  be a bounded below complex of  $\mathcal{O}_{\mathcal{C}}$ -modules. Assume both  $g$  and  $g'$  are flat. Then there exists a canonical base change map

$$g^* Rf_* \mathcal{F}^\bullet \longrightarrow R(f')_*(g')^* \mathcal{F}^\bullet$$

in  $D^+(\mathcal{O}_{\mathcal{D}'})$ .

**Proof.** Choose injective resolutions  $\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$  and  $(g')^* \mathcal{F}^\bullet \rightarrow \mathcal{J}^\bullet$ . By Lemma 15.2 we see that  $(g')_* \mathcal{J}^\bullet$  is a complex of injectives representing  $R(g')_*(g')^* \mathcal{F}^\bullet$ . Hence by Derived Categories, Lemmas 18.6 and 18.7 the arrow  $\beta$  in the diagram

$$\begin{array}{ccc} (g')_*(g')^* \mathcal{F}^\bullet & \longrightarrow & (g')_* \mathcal{J}^\bullet \\ \text{adjunction} \uparrow & & \uparrow \beta \\ \mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

exists and is unique up to homotopy. Pushing down to  $\mathcal{D}$  we get

$$f_* \beta : f_* \mathcal{I}^\bullet \longrightarrow f_*(g')_* \mathcal{J}^\bullet = g_*(f')_* \mathcal{J}^\bullet$$

By adjunction of  $g^*$  and  $g_*$  we get a map of complexes  $g^* f_* \mathcal{I}^\bullet \rightarrow (f')_* \mathcal{J}^\bullet$ . Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map  $\beta$  and everything was done on the level of complexes.  $\square$

### 17. Cohomology and colimits

0737 Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{I} \rightarrow \text{Mod}(\mathcal{O})$ ,  $i \mapsto \mathcal{F}_i$  be a diagram over the index category  $\mathcal{I}$ , see Categories, Section 14. For each  $i$  there is a canonical map  $\mathcal{F}_i \rightarrow \text{colim}_i \mathcal{F}_i$  which induces a map on cohomology. Hence we get a canonical map

$$\text{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \text{colim}_i \mathcal{F}_i)$$

for every  $p \geq 0$  and every object  $U$  of  $\mathcal{C}$ . These maps are in general not isomorphisms, even for  $p = 0$ .

The following lemma is the analogue of Sites, Lemma 17.5 for cohomology.

0739 **Lemma 17.1.** *Let  $\mathcal{C}$  be a site. Let  $\text{Cov}_{\mathcal{C}}$  be the set of coverings of  $\mathcal{C}$  (see Sites, Definition 6.2). Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ , and  $\text{Cov} \subset \text{Cov}_{\mathcal{C}}$  be subsets. Assume that*

- (1) *For every  $\mathcal{U} \in \text{Cov}$  we have  $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$  with  $I$  finite,  $U, U_i \in \mathcal{B}$  and every  $U_{i_0} \times_U \dots \times_U U_{i_p} \in \mathcal{B}$ .*
- (2) *For every  $U \in \mathcal{B}$  the coverings of  $U$  occurring in  $\text{Cov}$  is a cofinal system of coverings of  $U$ .*

Then the map

$$\text{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \text{colim}_i \mathcal{F}_i)$$

is an isomorphism for every  $p \geq 0$ , every  $U \in \mathcal{B}$ , and every filtered diagram  $\mathcal{I} \rightarrow \text{Ab}(\mathcal{C})$ .



**Proof.** To prove the lemma we will argue by induction on  $p$ . Note that we require in (1) the coverings  $\mathcal{U} \in \text{Cov}$  to be finite, so that all the elements of  $\mathcal{B}$  are quasi-compact. Hence (2) and (1) imply that any  $U \in \mathcal{B}$  satisfies the hypothesis of Sites, Lemma 17.5 (4). Thus we see that the result holds for  $p = 0$ . Now we assume the lemma holds for  $p$  and prove it for  $p + 1$ .

Choose a filtered diagram  $\mathcal{F} : \mathcal{I} \rightarrow \text{Ab}(\mathcal{C})$ ,  $i \mapsto \mathcal{F}_i$ . Since  $\text{Ab}(\mathcal{C})$  has functorial injective embeddings, see Injectives, Theorem 7.4, we can find a morphism of filtered diagrams  $\mathcal{F} \rightarrow \mathcal{I}$  such that each  $\mathcal{F}_i \rightarrow \mathcal{I}_i$  is an injective map of abelian sheaves into an injective abelian sheaf. Denote  $\mathcal{Q}_i$  the cokernel so that we have short exact sequences

$$0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{I}_i \rightarrow \mathcal{Q}_i \rightarrow 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 8.8), we see the sequence

$$0 \rightarrow \text{colim}_i \mathcal{F}_i \rightarrow \text{colim}_i \mathcal{I}_i \rightarrow \text{colim}_i \mathcal{Q}_i \rightarrow 0.$$

is also a short exact sequence. We claim that  $H^q(U, \text{colim}_i \mathcal{I}_i) = 0$  for all  $U \in \mathcal{B}$  and all  $q \geq 1$ . Accepting this claim for the moment consider the diagram

$$\begin{array}{ccccccc} \text{colim}_i H^p(U, \mathcal{I}_i) & \longrightarrow & \text{colim}_i H^p(U, \mathcal{Q}_i) & \longrightarrow & \text{colim}_i H^{p+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^p(U, \text{colim}_i \mathcal{I}_i) & \longrightarrow & H^p(U, \text{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{p+1}(U, \text{colim}_i \mathcal{F}_i) & \longrightarrow & 0 \end{array}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves  $\mathcal{I}_i$  are injective. The top row is exact by an application of Algebra, Lemma 8.8. Hence by the snake lemma we deduce the result for  $p + 1$ .

It remains to show that the claim is true. We will use Lemma 11.9. By the result for  $p = 0$  we see that for  $\mathcal{U} \in \text{Cov}$  we have

$$\check{C}^\bullet(\mathcal{U}, \text{colim}_i \mathcal{I}_i) = \text{colim}_i \check{C}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the  $U_{j_0} \times_U \dots \times_U U_{j_p}$  are in  $\mathcal{B}$ . By Lemma 11.2 each of the complexes in the colimit of Čech complexes is acyclic in degree  $\geq 1$ . Hence by Algebra, Lemma 8.8 we see that also the Čech complex  $\check{C}^\bullet(\mathcal{U}, \text{colim}_i \mathcal{I}_i)$  is acyclic in degrees  $\geq 1$ . In other words we see that  $\check{H}^p(\mathcal{U}, \text{colim}_i \mathcal{I}_i) = 0$  for all  $p \geq 1$ . Thus the assumptions of Lemma 11.9. are satisfied and the claim follows.  $\square$

Let  $\mathcal{C}$  be a limit of sites  $\mathcal{C}_i$  as in Sites, Situation 17.6 and Lemmas 17.7, 17.8, and 17.9. In particular, all coverings in  $\mathcal{C}$  and  $\mathcal{C}_i$  have finite index sets. Moreover, assume given

- (1) an abelian sheaf  $\mathcal{F}_i$  on  $\mathcal{C}_i$  for all  $i \in \text{Ob}(\mathcal{I})$ ,
- (2) for  $a : j \rightarrow i$  a map  $\varphi_a : f_a^{-1} \mathcal{F}_i \rightarrow \mathcal{F}_j$  of abelian sheaves on  $\mathcal{C}_j$

such that  $\varphi_c = \varphi_b \circ f_b^{-1} \varphi_a$  whenever  $c = a \circ b$ .

09YP **Lemma 17.2.** *In the situation discussed above set  $\mathcal{F} = \text{colim } f_i^{-1} \mathcal{F}_i$ . Let  $i \in \text{Ob}(\mathcal{I})$ ,  $X_i \in \text{Ob}(\mathcal{C}_i)$ . Then*

$$\text{colim}_{a:j \rightarrow i} H^p(u_a(X_i), \mathcal{F}_j) = H^p(u_i(X_i), \mathcal{F})$$

for all  $p \geq 0$ .

**Proof.** The case  $p = 0$  is Sites, Lemma 17.9.

In this paragraph we show that we can find a map of systems  $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \rightarrow (\mathcal{G}_i, \psi_a)$  with  $\mathcal{G}_i$  an injective abelian sheaf and  $\gamma_i$  injective. For each  $i$  we pick an injection  $\mathcal{F}_i \rightarrow \mathcal{I}_i$  where  $\mathcal{I}_i$  is an injective abelian sheaf on  $\mathcal{C}_i$ . Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod_{b:k \rightarrow i} f_{b,*} \mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps  $f_b^{-1} \mathcal{F}_i \rightarrow \mathcal{F}_k \rightarrow \mathcal{I}_k$ . For  $a : j \rightarrow i$  in  $\mathcal{I}$  there is a canonical map

$$\psi_a : f_a^{-1} \mathcal{G}_i \rightarrow \mathcal{G}_j$$

whose components are the canonical maps  $f_b^{-1} f_{a \circ b,*} \mathcal{I}_k \rightarrow f_{b,*} \mathcal{I}_k$  for  $b : k \rightarrow j$ . Thus we find an injection  $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a\} \rightarrow (\mathcal{G}_i, \psi_a)$  of systems of abelian sheaves. Note that  $\mathcal{G}_i$  is an injective sheaf of abelian groups on  $\mathcal{C}_i$ , see Lemma 15.2 and Homology, Lemma 24.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 17.1 we see that it suffices to prove that  $H^p(X, \text{colim } f_i^{-1} \mathcal{G}_i) = 0$  for  $p > 0$ .

Set  $\mathcal{G} = \text{colim } f_i^{-1} \mathcal{G}_i$ . To show vanishing of cohomology of  $\mathcal{G}$  on every object of  $\mathcal{C}$  we show that the Čech cohomology of  $\mathcal{G}$  for any covering  $\mathcal{U}$  of  $\mathcal{C}$  is zero (Lemma 11.9). The covering  $\mathcal{U}$  comes from a covering  $\mathcal{U}_i$  of  $\mathcal{C}_i$  for some  $i$ . We have

$$\check{C}^\bullet(\mathcal{U}, \mathcal{G}) = \text{colim}_{a:j \rightarrow i} \check{C}^\bullet(u_a(\mathcal{U}_i), \mathcal{G}_j)$$

by the case  $p = 0$ . The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 11.2. See Algebra, Lemma 8.8.  $\square$

## 18. Flat resolutions

06YL In this section we redo the arguments of Cohomology, Section 27 in the setting of ringed sites and ringed topoi.

06YM **Lemma 18.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{G}^\bullet$  be a complex of  $\mathcal{O}$ -modules. The functor*

$$K(\text{Mod}(\mathcal{O})) \longrightarrow K(\text{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$$

*is an exact functor of triangulated categories.*

**Proof.** Omitted. Hint: See More on Algebra, Lemmas 55.1 and 55.2.  $\square$

06YN **Definition 18.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. A complex  $\mathcal{K}^\bullet$  of  $\mathcal{O}$ -modules is called *K-flat* if for every acyclic complex  $\mathcal{F}^\bullet$  of  $\mathcal{O}$ -modules the complex

$$\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is acyclic.

06YP **Lemma 18.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{K}^\bullet$  be a K-flat complex. Then the functor*

$$K(\text{Mod}(\mathcal{O})) \longrightarrow K(\text{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

*transforms quasi-isomorphisms into quasi-isomorphisms.*

**Proof.** Follows from Lemma 18.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones.  $\square$

0E8K **Lemma 18.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . If  $\mathcal{K}^\bullet$  is a  $K$ -flat complex of  $\mathcal{O}$ -modules, then  $\mathcal{K}^\bullet|_U$  is a  $K$ -flat complex of  $\mathcal{O}_U$ -modules.*

**Proof.** Let  $\mathcal{G}^\bullet$  be an exact complex of  $\mathcal{O}_U$ -modules. Since  $j_{U!}$  is exact (Modules on Sites, Lemma 19.3) and  $\mathcal{K}^\bullet$  is a  $K$ -flat complex of  $\mathcal{O}$ -modules we see that the complex

$$j_{U!}(\mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_U} \mathcal{K}^\bullet|_U)) = \mathrm{Tot}(j_{U!}\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is exact. Here the equality comes from Modules on Sites, Lemma 27.7 and the fact that  $j_{U!}$  commutes with direct sums (as a left adjoint). We conclude because  $j_{U!}$  reflects exactness by Modules on Sites, Lemma 19.4.  $\square$

07A2 **Lemma 18.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. If  $\mathcal{K}^\bullet, \mathcal{L}^\bullet$  are  $K$ -flat complexes of  $\mathcal{O}$ -modules, then  $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$  is a  $K$ -flat complex of  $\mathcal{O}$ -modules.*

**Proof.** Follows from the isomorphism

$$\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$$

and the definition.  $\square$

07A3 **Lemma 18.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$  be a distinguished triangle in  $K(\mathrm{Mod}(\mathcal{O}))$ . If two out of three of  $\mathcal{K}_i^\bullet$  are  $K$ -flat, so is the third.*

**Proof.** Follows from Lemma 18.1 and the fact that in a distinguished triangle in  $K(\mathrm{Mod}(\mathcal{O}))$  if two out of three are acyclic, so is the third.  $\square$

06YQ **Lemma 18.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. A bounded above complex of flat  $\mathcal{O}$ -modules is  $K$ -flat.*

**Proof.** Let  $\mathcal{K}^\bullet$  be a bounded above complex of flat  $\mathcal{O}$ -modules. Let  $\mathcal{L}^\bullet$  be an acyclic complex of  $\mathcal{O}$ -modules. Note that  $\mathcal{L}^\bullet = \mathrm{colim}_m \tau_{\leq m} \mathcal{L}^\bullet$  where we take termwise colimits. Hence also

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) = \mathrm{colim}_m \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \tau_{\leq m} \mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since  $\tau_{\leq m} \mathcal{L}^\bullet$  is acyclic this reduces us to the case where  $\mathcal{L}^\bullet$  is bounded above. In this case the spectral sequence of Homology, Lemma 22.6 has

$${}^I E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as  $\mathcal{K}^q$  is flat and  $\mathcal{L}^\bullet$  acyclic. Hence we win.  $\square$

06YR **Lemma 18.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{K}_1^\bullet \rightarrow \mathcal{K}_2^\bullet \rightarrow \dots$  be a system of  $K$ -flat complexes. Then  $\mathrm{colim}_i \mathcal{K}_i^\bullet$  is  $K$ -flat.*

**Proof.** Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_i^\bullet) = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact.  $\square$

077J **Lemma 18.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. For any complex  $\mathcal{G}^\bullet$  of  $\mathcal{O}$ -modules there exists a commutative diagram of complexes of  $\mathcal{O}$ -modules*

$$\begin{array}{ccccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \dots & & \\ \downarrow & & \downarrow & & & & \\ \tau_{\leq 1} \mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2} \mathcal{G}^\bullet & \longrightarrow & \dots & & \end{array}$$

*with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each  $\mathcal{K}_n^\bullet$  is a bounded above complex whose terms are direct sums of  $\mathcal{O}$ -modules of the form  $j_{U!} \mathcal{O}_U$ , and (3) the maps  $\mathcal{K}_n^\bullet \rightarrow \mathcal{K}_{n+1}^\bullet$  are termwise split injections whose cokernels are direct sums of  $\mathcal{O}$ -modules of the form  $j_{U!} \mathcal{O}_U$ . Moreover, the map  $\text{colim } \mathcal{K}_n^\bullet \rightarrow \mathcal{G}^\bullet$  is a quasi-isomorphism.*

**Proof.** The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 28.7 and Derived Categories, Lemma 28.1. The induced map  $\text{colim } \mathcal{K}_n^\bullet \rightarrow \mathcal{G}^\bullet$  is a quasi-isomorphism because filtered colimits are exact.  $\square$

06YS **Lemma 18.10.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. For any complex  $\mathcal{G}^\bullet$  of  $\mathcal{O}$ -modules there exists a K-flat complex  $\mathcal{K}^\bullet$  and a quasi-isomorphism  $\mathcal{K}^\bullet \rightarrow \mathcal{G}^\bullet$ .*

**Proof.** Choose a diagram as in Lemma 18.9. Each complex  $\mathcal{K}_n^\bullet$  is a bounded above complex of flat modules, see Modules on Sites, Lemma 28.6. Hence  $\mathcal{K}_n^\bullet$  is K-flat by Lemma 18.7. The induced map  $\text{colim } \mathcal{K}_n^\bullet \rightarrow \mathcal{G}^\bullet$  is a quasi-isomorphism by construction. Since  $\text{colim } \mathcal{K}_n^\bullet$  is K-flat by Lemma 18.8 we win.  $\square$

06YT **Lemma 18.11.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\alpha : \mathcal{P}^\bullet \rightarrow \mathcal{Q}^\bullet$  be a quasi-isomorphism of K-flat complexes of  $\mathcal{O}$ -modules. For every complex  $\mathcal{F}^\bullet$  of  $\mathcal{O}$ -modules the induced map*

$$\text{Tot}(id_{\mathcal{F}^\bullet} \otimes \alpha) : \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \longrightarrow \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet)$$

*is a quasi-isomorphism.*

**Proof.** Choose a quasi-isomorphism  $\mathcal{K}^\bullet \rightarrow \mathcal{F}^\bullet$  with  $\mathcal{K}^\bullet$  a K-flat complex, see Lemma 18.10. Consider the commutative diagram

$$\begin{array}{ccc} \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \\ \downarrow & & \downarrow \\ \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \end{array}$$

The result follows as by Lemma 18.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}^\bullet$  be an object of  $D(\mathcal{O})$ . Choose a K-flat resolution  $\mathcal{K}^\bullet \rightarrow \mathcal{F}^\bullet$ , see Lemma 18.10. By Lemma 18.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \text{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

By Lemma 18.3 this functor induces a functor  $D(\mathcal{O}) \rightarrow D(\mathcal{O})$  simply because  $D(\mathcal{O})$  is the localization of  $K(\mathcal{O})$  at quasi-isomorphisms. By Lemma 18.11 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

06YU **Definition 18.12.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}^\bullet$  be an object of  $D(\mathcal{O})$ . The *derived tensor product*

$$- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}^\bullet$$

for  $\mathcal{G}^\bullet$  and  $\mathcal{F}^\bullet$  in  $D(\mathcal{O})$ . Hence when we write  $\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet$  we will usually be agnostic about which variable we are using to define the derived tensor product with.

08FF **Definition 18.13.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}, \mathcal{G}$  be  $\mathcal{O}$ -modules. The *Tor's* of  $\mathcal{F}$  and  $\mathcal{G}$  are defined by the formula

$$\mathrm{Tor}_p^{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of  $\mathcal{O}$ -modules  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  we have a long exact cohomology sequence

$$\begin{array}{ccccccc} \mathcal{F}_1 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_2 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_3 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & & \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_1, \mathcal{G}) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_2, \mathcal{G}) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}_3, \mathcal{G}) \end{array}$$

for every  $\mathcal{O}$ -module  $\mathcal{G}$ . This will be called the long exact sequence of Tor associated to the situation.

08FG **Lemma 18.14.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module. The following are equivalent*

- (1)  $\mathcal{F}$  is a flat  $\mathcal{O}$ -module, and
- (2)  $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$  for every  $\mathcal{O}$ -module  $\mathcal{G}$ .

**Proof.** If  $\mathcal{F}$  is flat, then  $\mathcal{F} \otimes_{\mathcal{O}} -$  is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if  $\mathcal{G} \rightarrow \mathcal{H}$  is injective with cokernel  $\mathcal{Q}$ , the long exact sequence of Tor shows that the kernel of  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}$  is a quotient of  $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{Q})$  which is zero by assumption. Hence  $\mathcal{F}$  is flat.  $\square$

## 19. Derived pullback

06YV Let  $f : (\mathrm{Sh}(\mathcal{C}), \mathcal{O}) \rightarrow (\mathrm{Sh}(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \rightarrow D(\mathcal{O})$$

06YW **Lemma 19.1.** *Let  $(\mathrm{Sh}(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  be a ringed topos. For any complex of  $\mathcal{O}_{\mathcal{C}}$ -modules  $\mathcal{G}^\bullet$  there exists a quasi-isomorphism  $\mathcal{K}^\bullet \rightarrow \mathcal{G}^\bullet$  such that  $f^*\mathcal{K}^\bullet$  is a K-flat complex of  $\mathcal{O}_{\mathcal{D}}$ -modules for any morphism  $f : (\mathrm{Sh}(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \rightarrow (\mathrm{Sh}(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  of ringed topoi.*

**Proof.** In the proof of Lemma 18.10 we find a quasi-isomorphism  $\mathcal{K}^\bullet = \mathrm{colim}_i \mathcal{K}_i^\bullet \rightarrow \mathcal{G}^\bullet$  where each  $\mathcal{K}_i^\bullet$  is a bounded above complex of flat  $\mathcal{O}_{\mathcal{C}}$ -modules. Let  $f : (\mathrm{Sh}(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \rightarrow (\mathrm{Sh}(\mathcal{C}), \mathcal{O}_{\mathcal{C}})$  be a morphism of ringed topoi. By Modules on Sites, Lemma 38.1 we see that  $f^*\mathcal{K}_i^\bullet$  is a bounded above complex of flat  $\mathcal{O}_{\mathcal{D}}$ -modules. Hence  $f^*\mathcal{K}^\bullet = \mathrm{colim}_i f^*\mathcal{K}_i^\bullet$  is K-flat by Lemmas 18.7 and 18.8.  $\square$

06YY **Lemma 19.2.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

*of triangulated categories so that  $Lf^*K^\bullet = f^*K^\bullet$  for any complex as in Lemma 19.1 and in particular for any bounded above complex of flat  $\mathcal{O}'$ -modules.*

**Proof.** To see this we use the general theory developed in Derived Categories, Section 15. Set  $\mathcal{D} = K(\mathcal{O}')$  and  $\mathcal{D}' = D(\mathcal{O})$ . Let us write  $F : \mathcal{D} \rightarrow \mathcal{D}'$  the exact functor of triangulated categories defined by the rule  $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$ . We let  $S$  be the set of quasi-isomorphisms in  $\mathcal{D} = K(\mathcal{O}')$ . This gives a situation as in Derived Categories, Situation 15.1 so that Derived Categories, Definition 15.2 applies. We claim that  $LF$  is everywhere defined. This follows from Derived Categories, Lemma 15.15 with  $\mathcal{P} \subset \text{Ob}(\mathcal{D})$  the collection of complexes  $K^\bullet$  as in Lemma 19.1. Namely, (1) follows from Lemma 19.1 and to see (2) we have to show that for a quasi-isomorphism  $K_1^\bullet \rightarrow K_2^\bullet$  between elements of  $\mathcal{P}$  the map  $f^*K_1^\bullet \rightarrow f^*K_2^\bullet$  is a quasi-isomorphism. To see this write this as

$$f^{-1}K_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}K_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor  $f^{-1}$  is exact, hence the map  $f^{-1}K_1^\bullet \rightarrow f^{-1}K_2^\bullet$  is a quasi-isomorphism. The complexes  $f^{-1}K_1^\bullet$  and  $f^{-1}K_2^\bullet$  are K-flat complexes of  $f^{-1}\mathcal{O}'$ -modules by our choice of  $\mathcal{P}$  because we can consider the morphism of ringed topoi  $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ . Hence Lemma 18.11 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (15.9.1). Finally, Derived Categories, Lemma 15.15 also guarantees that  $LF(K^\bullet) = F(K^\bullet) = f^*K^\bullet$  when  $K^\bullet$  is in  $\mathcal{P}$ . Since the proof of Lemma 19.1 shows that bounded above complexes of flat modules are in  $\mathcal{P}$  we win.  $\square$

0D6D **Lemma 19.3.** *Consider morphisms of ringed topoi  $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$  and  $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \rightarrow (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ . Then  $Lf^* \circ Lg^* = L(g \circ f)^*$  as functors  $D(\mathcal{O}_\mathcal{E}) \rightarrow D(\mathcal{O}_\mathcal{C})$ .*

**Proof.** Let  $E$  be an object of  $D(\mathcal{O}_\mathcal{E})$ . By construction  $Lg^*E$  is computed by choosing a complex  $K^\bullet$  as in Lemma 19.1 representing  $E$  and setting  $Lg^*E = g^*K^\bullet$ . By transitivity of pullback functors the complex  $g^*K^\bullet$  pulled back by any morphism of ringed topoi  $(Sh(\mathcal{C}'), \mathcal{O}') \rightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$  is K-flat. Hence  $g^*K^\bullet$  is a complex as in Lemma 19.1 representing  $Lg^*E$ . We conclude  $Lf^*Lg^*E$  is given by  $f^*g^*K^\bullet = (g \circ f)^*K^\bullet$  which also represents  $L(g \circ f)^*E$ .  $\square$

07A4 **Lemma 19.4.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} Lf^*\mathcal{G}^\bullet$$

*for  $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \text{Ob}(D(\mathcal{O}'))$ .*

**Proof.** By Lemma 19.1 we may assume that  $\mathcal{F}^\bullet$  and  $\mathcal{G}^\bullet$  are K-flat complexes of  $\mathcal{O}'$ -modules such that  $f^*\mathcal{F}^\bullet$  and  $f^*\mathcal{G}^\bullet$  are K-flat complexes of  $\mathcal{O}$ -modules. In this case  $\mathcal{F}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}^\bullet$  is just the total complex associated to the double complex  $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$ .

By Lemma 18.5  $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$  is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\text{Tot}(f^* \mathcal{F}^\bullet \otimes_{\mathcal{O}} f^* \mathcal{G}^\bullet) \longrightarrow f^* \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms  $f^* \mathcal{F}^p \otimes_{\mathcal{O}} f^* \mathcal{G}^q \rightarrow f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$  of Modules on Sites, Lemma 26.1.  $\square$

08I6 **Lemma 19.5.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes_{\mathcal{O}}^L Lf^* \mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}'}^L f^{-1} \mathcal{G}^\bullet$$

for  $\mathcal{F}^\bullet$  in  $D(\mathcal{O})$  and  $\mathcal{G}^\bullet$  in  $D(\mathcal{O}')$ .

**Proof.** Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module and let  $\mathcal{G}$  be an  $\mathcal{O}'$ -module. Then  $\mathcal{F} \otimes_{\mathcal{O}} f^* \mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1} \mathcal{G}$  because  $f^* \mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1} \mathcal{G}$ . The lemma follows from this and the definitions.  $\square$

0DEN **Lemma 19.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{K}^\bullet$  be a complex of  $\mathcal{O}$ -modules.*

- (1) *If  $\mathcal{K}^\bullet$  is K-flat, then for every point  $p$  of the site  $\mathcal{C}$  the complex of  $\mathcal{O}_p$ -modules  $\mathcal{K}_p^\bullet$  is K-flat in the sense of More on Algebra, Definition 55.3*
- (2) *If  $\mathcal{C}$  has enough points, then the converse is true.*

**Proof.** Proof of (2). If  $\mathcal{C}$  has enough points and  $\mathcal{K}_p^\bullet$  is K-flat for all points  $p$  of  $\mathcal{C}$  then we see that  $\mathcal{K}^\bullet$  is K-flat because  $\otimes$  and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules on Sites, Lemma 14.4.

Proof of (1). Assume  $\mathcal{K}^\bullet$  is K-flat. Choose a quasi-isomorphism  $a : \mathcal{L}^\bullet \rightarrow \mathcal{K}^\bullet$  such that  $\mathcal{L}^\bullet$  is K-flat and such that any pullback of  $\mathcal{L}^\bullet$  is K-flat, see Lemma 19.1. In particular the stalk  $\mathcal{L}_p^\bullet$  is a K-flat complex of  $\mathcal{O}_p$ -modules. Thus the cone  $C(a)$  on  $a$  is a K-flat (Lemma 18.6) acyclic complex of  $\mathcal{O}$ -modules and it suffices to show the stalk of  $C(a)$  is K-flat (by More on Algebra, Lemma 55.7). Thus we may assume that  $\mathcal{K}^\bullet$  is K-flat and acyclic.

Assume  $\mathcal{K}^\bullet$  is acyclic and K-flat. Before continuing we replace the site  $\mathcal{C}$  by another one as in Sites, Lemma 28.5 to insure that  $\mathcal{C}$  has all finite limits. This implies the category of neighbourhoods of  $p$  is filtered (Sites, Lemma 32.1) and the colimit defining the stalk of a sheaf is filtered. Let  $M$  be a finitely presented  $\mathcal{O}_p$ -module. It suffices to show that  $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$  is acyclic, see More on Algebra, Lemma 55.11. Since  $\mathcal{O}_p$  is the filtered colimit of  $\mathcal{O}(U)$  where  $U$  runs over the neighbourhoods of  $p$ , we can find a neighbourhood  $(U, x)$  of  $p$  and a finitely presented  $\mathcal{O}(U)$ -module  $M'$  whose base change to  $\mathcal{O}_p$  is  $M$ , see Algebra, Lemma 126.6. By Lemma 18.4 we may replace  $\mathcal{C}, \mathcal{O}, \mathcal{K}^\bullet$  by  $\mathcal{C}/U, \mathcal{O}_U, \mathcal{K}^\bullet|_U$ . We conclude that we may assume there exists an  $\mathcal{O}$ -module  $\mathcal{F}$  such that  $M \cong \mathcal{F}_p$ . Since  $\mathcal{K}^\bullet$  is K-flat and acyclic, we see that  $\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$  is acyclic (as it computes the derived tensor product by definition). Taking stalks is an exact functor, hence we get that  $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$  is acyclic as desired.  $\square$

0DEP **Lemma 19.7.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. If  $\mathcal{C}$  has enough points, then the pullback of a K-flat complex of  $\mathcal{O}'$ -modules is a K-flat complex of  $\mathcal{O}$ -modules.*

**Proof.** This follows from Lemma 19.6, Modules on Sites, Lemma 35.4, and More on Algebra, Lemma 55.5.  $\square$

06YX **Remark 19.8.** The pullback of a K-flat complex is K-flat for a morphism of ringed topoi with enough points, see Lemma 19.7. This slightly improves the result of Lemma 19.1. However, in applications it seems rather that the explicit form of the K-flat complexes constructed in Lemma 18.10 is what is useful and not the plain fact that they are K-flat. Note for example that the terms of the complex constructed are each direct sums of modules of the form  $j_{U!}\mathcal{O}_U$ , see Lemma 18.9.

## 20. Cohomology of unbounded complexes

07A5 Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. The category  $\text{Mod}(\mathcal{O})$  is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus_{U \in \text{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 14 and Lemmas 28.6 and 28.7. By Injectives, Theorem 12.6 for every complex  $\mathcal{F}^\bullet$  of  $\mathcal{O}$ -modules there exists an injective quasi-isomorphism  $\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$  to a K-injective complex of  $\mathcal{O}$ -modules. Hence we can define

$$R\Gamma(\mathcal{C}, \mathcal{F}^\bullet) = \Gamma(\mathcal{C}, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 29.7. For any morphism of ringed topoi  $f : (\mathcal{S}h(\mathcal{C}), \mathcal{O}) \rightarrow (\mathcal{S}h(\mathcal{D}), \mathcal{O}')$  we obtain

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on the unbounded derived categories.

07A6 **Lemma 20.1.** *Let  $f : (\mathcal{S}h(\mathcal{C}), \mathcal{O}) \rightarrow (\mathcal{S}h(\mathcal{D}), \mathcal{O}')$  be a morphism of ringed topoi. The functor  $Rf_*$  defined above and the functor  $Lf^*$  defined in Lemma 19.2 are adjoint:*

$$\text{Hom}_{D(\mathcal{O}')} (Lf^* \mathcal{G}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{D(\mathcal{O})} (\mathcal{G}^\bullet, Rf_* \mathcal{F}^\bullet)$$

bifunctorially in  $\mathcal{F}^\bullet \in \text{Ob}(D(\mathcal{O}))$  and  $\mathcal{G}^\bullet \in \text{Ob}(D(\mathcal{O}'))$ .

**Proof.** This follows formally from the fact that  $Rf_*$  and  $Lf^*$  exist, see Derived Categories, Lemma 28.5.  $\square$

0D6E **Lemma 20.2.** *Let  $f : (\mathcal{S}h(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{S}h(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  and  $g : (\mathcal{S}h(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \rightarrow (\mathcal{S}h(\mathcal{E}), \mathcal{O}_{\mathcal{E}})$  be morphisms of ringed topoi. Then  $Rg_* \circ Rf_* = R(g \circ f)_*$  as functors  $D(\mathcal{O}_{\mathcal{C}}) \rightarrow D(\mathcal{O}_{\mathcal{E}})$ .*

**Proof.** By Lemma 20.1 we see that  $Rg_* \circ Rf_*$  is adjoint to  $Lf^* \circ Lg^*$ . We have  $Lf^* \circ Lg^* = L(g \circ f)^*$  by Lemma 19.3 and hence by uniqueness of adjoint functors we have  $Rg_* \circ Rf_* = R(g \circ f)_*$ .  $\square$

07A7 **Remark 20.3.** The construction of unbounded derived functor  $Lf^*$  and  $Rf_*$  allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} (\mathcal{S}h(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (\mathcal{S}h(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ f' \downarrow & & \downarrow f \\ (\mathcal{S}h(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (\mathcal{S}h(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let  $K$  be an object of  $D(\mathcal{O}_{\mathcal{C}})$ . Then there exists a canonical base change map

$$Lg^* Rf_* K \longrightarrow R(f')_* L(g')^* K$$



in  $D(\mathcal{O}_{\mathcal{D}'})$ . Namely, this map is adjoint to a map  $L(f')^*Lg^*Rf_*K \rightarrow L(g')^*K$ . Since  $L(f')^* \circ Lg^* = L(g')^* \circ Lf^*$  we see this is the same as a map  $L(g')^*Lf^*Rf_*K \rightarrow L(g')^*K$  which we can take to be  $L(g')^*$  of the adjunction map  $Lf^*Rf_*K \rightarrow K$ .

0E46 **Remark 20.4.** Consider a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) & \xrightarrow{k} & (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \\ f' \downarrow & & \downarrow f \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{l} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ g' \downarrow & & \downarrow g \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{m} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Then the base change maps of Remark 20.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* &= Lm^* \circ Rg_* \circ Rf_* \\ &\rightarrow Rg'_* \circ Ll^* \circ Rf_* \\ &\rightarrow Rg'_* \circ Rf'_* \circ Lk^* \\ &= R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0E47 **Remark 20.5.** Consider a commutative diagram

$$\begin{array}{ccccc} (Sh(\mathcal{C}''), \mathcal{O}_{\mathcal{C}''}) & \xrightarrow{g'} & (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ f'' \downarrow & & f' \downarrow & & \downarrow f \\ (Sh(\mathcal{D}''), \mathcal{O}_{\mathcal{D}''}) & \xrightarrow{h'} & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{h} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

of ringed topoi. Then the base change maps of Remark 20.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} L(h \circ h')^* \circ Rf_* &= L(h')^* \circ Lh_* \circ Rf_* \\ &\rightarrow L(h')^* \circ Rf'_* \circ Lg^* \\ &\rightarrow Rf''_* \circ L(g')^* \circ Lg^* \\ &= Rf''_* \circ L(g \circ g')^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0B6C **Remark 20.6.** Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. The adjointness of  $Lf^*$  and  $Rf_*$  allows us to construct a relative cup product

$$Rf_*K \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} Rf_*L \rightarrow Rf_*(K \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} L)$$

in  $D(\mathcal{O}_{\mathcal{D}})$  for all  $K, L$  in  $D(\mathcal{O}_{\mathcal{C}})$ . Namely, this map is adjoint to a map  $Lf^*(Rf_*K \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} Rf_*L) \rightarrow K \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} L$  for which we can take the composition of the isomorphism  $Lf^*(Rf_*K \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} Rf_*L) = Lf^*Rf_*K \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^*Rf_*L$  (Lemma 19.4) with the map  $Lf^*Rf_*K \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^*Rf_*L \rightarrow K \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} L$  coming from the counit  $Lf^* \circ Rf_* \rightarrow \text{id}$ .

0DD7 **Lemma 20.7.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{A} \subset \text{Ab}(\mathcal{C})$  denote the Serre subcategory consisting of torsion abelian sheaves. Then the functor  $D(\mathcal{A}) \rightarrow D_{\mathcal{A}}(\mathcal{C})$  is an equivalence.*

**Proof.** A key observation is that an injective abelian sheaf  $\mathcal{I}$  is divisible. Namely, if  $s \in \mathcal{I}(U)$  is a local section, then we interpret  $s$  as a map  $s : j_{U!}\mathbf{Z} \rightarrow \mathcal{I}$  and we apply the defining property of an injective object to the injective map of sheaves  $n : j_{U!}\mathbf{Z} \rightarrow j_{U!}\mathbf{Z}$  to see that there exists an  $s' \in \mathcal{I}(U)$  with  $ns' = s$ .

For a sheaf  $\mathcal{F}$  denote  $\mathcal{F}_{\text{tor}}$  its torsion subsheaf. We claim that if  $\mathcal{I}^{\bullet}$  is a complex of injective abelian sheaves whose cohomology sheaves are torsion, then

$$\mathcal{I}_{\text{tor}}^{\bullet} \rightarrow \mathcal{I}^{\bullet}$$

is a quasi-isomorphism. Namely, by flatness of  $\mathbf{Q}$  over  $\mathbf{Z}$  we have

$$H^p(\mathcal{I}^{\bullet}) \otimes_{\mathbf{Z}} \mathbf{Q} = H^p(\mathcal{I}^{\bullet} \otimes_{\mathbf{Z}} \mathbf{Q})$$

which is zero because the cohomology sheaves are torsion. By divisibility (shown above) we see that  $\mathcal{I}^{\bullet} \rightarrow \mathcal{I}^{\bullet} \otimes_{\mathbf{Z}} \mathbf{Q}$  is surjective with kernel  $\mathcal{I}_{\text{tor}}^{\bullet}$ . The claim follows from the long exact sequence of cohomology sheaves associated to the short exact sequence you get.

To prove the lemma we will construct right adjoint  $T : D(\mathcal{C}) \rightarrow D(\mathcal{A})$ . Namely, given  $K$  in  $D(\mathcal{C})$  we can represent  $K$  by a  $\mathbf{K}$ -injective complex  $\mathcal{I}^{\bullet}$  whose cohomology sheaves are injective, see Injectives, Theorem 12.6. Then we set  $T(K) = \mathcal{I}_{\text{tor}}^{\bullet}$ , in other words,  $T$  is the right derived functor of taking torsion. The functor  $T$  is a right adjoint to  $i : D(\mathcal{A}) \rightarrow D_{\mathcal{A}}(\mathcal{C})$ . This readily follows from the observation that if  $\mathcal{F}^{\bullet}$  is a complex of torsion sheaves, then

$$\text{Hom}_{K(\mathcal{A})}(\mathcal{F}^{\bullet}, \mathcal{I}_{\text{tor}}^{\bullet}) = \text{Hom}_{K(\text{Ab}(\mathcal{C}))}(\mathcal{F}^{\bullet}, \mathcal{I}^{\bullet})$$

in particular  $\mathcal{I}_{\text{tor}}^{\bullet}$  is a  $\mathbf{K}$ -injective complex of  $\mathcal{A}$ . Some details omitted; in case of doubt, it also follows from the more general Derived Categories, Lemma 28.5. Our claim above gives that  $L = T(i(L))$  for  $L$  in  $D(\mathcal{A})$  and  $i(T(K)) = K$  if  $K$  is in  $D_{\mathcal{A}}(\mathcal{C})$ . Using Categories, Lemma 24.3 the result follows.  $\square$

## 21. Some properties of $\mathbf{K}$ -injective complexes

08FH Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Denote  $j : (\text{Sh}(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (\text{Sh}(\mathcal{C}), \mathcal{O})$  the corresponding localization morphism. The pullback functor  $j^*$  is exact as it is just the restriction functor. Thus derived pullback  $Lj^*$  is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \rightarrow D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero  $j_! : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\mathcal{O})$  (see Modules on Sites, Definition 19.1) is an exact functor (Modules on Sites, Lemma 19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \rightarrow D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying  $j_!$  to any complex representing the object  $F$ .

08FI **Lemma 21.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . The restriction of a  $\mathbf{K}$ -injective complex of  $\mathcal{O}$ -modules to  $\mathcal{C}/U$  is a  $\mathbf{K}$ -injective complex of  $\mathcal{O}_U$ -modules.*

**Proof.** Follows immediately from Derived Categories, Lemma 29.9 and the fact that the restriction functor has the exact left adjoint  $j_!$ . See discussion above.  $\square$

0D6F **Lemma 21.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U \in \text{Ob}(\mathcal{C})$ . For  $K$  in  $D(\mathcal{O})$  we have  $H^p(U, K) = H^p(\mathcal{C}/U, K|_{\mathcal{C}/U})$ .*

**Proof.** Let  $\mathcal{I}^\bullet$  be a K-injective complex of  $\mathcal{O}$ -modules representing  $K$ . Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(\mathcal{C}/U, \mathcal{I}^\bullet|_{\mathcal{C}/U}))$$

by construction of cohomology. By Lemma 21.1 the complex  $\mathcal{I}^\bullet|_{\mathcal{C}/U}$  is a K-injective complex representing  $K|_{\mathcal{C}/U}$  and the lemma follows.  $\square$

0BKV **Lemma 21.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be an object of  $D(\mathcal{O})$ . The sheafification of*

$$U \mapsto H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$$

*is the  $q$ th cohomology sheaf  $H^q(K)$  of  $K$ .*

**Proof.** The equality  $H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$  holds by Lemma 21.2. Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $K$ . Then

$$H^q(U, K) = \frac{\text{Ker}(\mathcal{I}^q(U) \rightarrow \mathcal{I}^{q+1}(U))}{\text{Im}(\mathcal{I}^{q-1}(U) \rightarrow \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since  $H^q(K) = \text{Ker}(\mathcal{I}^q \rightarrow \mathcal{I}^{q+1})/\text{Im}(\mathcal{I}^{q-1} \rightarrow \mathcal{I}^q)$  the result is clear.  $\square$

0D6G **Lemma 21.4.** *Let  $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites corresponding to the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Given  $V \in \mathcal{D}$ , set  $U = u(V)$  and denote  $g : (\mathcal{C}/U, \mathcal{O}_U) \rightarrow (\mathcal{D}/V, \mathcal{O}_V)$  the induced morphism of ringed sites (Modules on Sites, Lemma 20.1). Then  $(Rf_*E)|_{\mathcal{D}/V} = Rg_*(E|_{\mathcal{C}/U})$  for  $E$  in  $D(\mathcal{O}_{\mathcal{C}})$ .*

**Proof.** Represent  $E$  by a K-injective complex  $\mathcal{I}^\bullet$  of  $\mathcal{O}_{\mathcal{C}}$ -modules. Then  $Rf_*(E) = f_*\mathcal{I}^\bullet$  and  $Rg_*(E|_{\mathcal{C}/U}) = g_*(\mathcal{I}^\bullet|_{\mathcal{C}/U})$  by Lemma 21.1. Since it is clear that  $(f_*\mathcal{F})|_{\mathcal{D}/V} = g_*(\mathcal{F}|_{\mathcal{C}/U})$  for any sheaf  $\mathcal{F}$  on  $\mathcal{C}$  (see Modules on Sites, Lemma 20.1 or the more basic Sites, Lemma 27.1) the result follows.  $\square$

0D6H **Lemma 21.5.** *Let  $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites corresponding to the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Then  $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$  as functors  $D(\mathcal{O}_{\mathcal{C}}) \rightarrow D(\Gamma(\mathcal{O}_{\mathcal{D}}))$ . More generally, for  $V \in \mathcal{D}$  with  $U = u(V)$  we have  $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$ .*

**Proof.** Consider the punctual topos  $pt$  endowed with  $\mathcal{O}_{pt}$  given by the ring  $\Gamma(\mathcal{O}_{\mathcal{D}})$ . There is a canonical morphism  $(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \rightarrow (pt, \mathcal{O}_{pt})$  of ringed topoi inducing the identification on global sections of structure sheaves. Then  $D(\mathcal{O}_{pt}) = D(\Gamma(\mathcal{O}_{\mathcal{D}}))$ . The assertion  $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$  follows from Lemma 20.2 applied to

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \rightarrow (pt, \mathcal{O}_{pt})$$

The second (more general) statement follows from the first statement after applying Lemma 21.4.  $\square$

0D6I **Lemma 21.6.** *Let  $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites corresponding to the continuous functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $K$  be in  $D(\mathcal{O}_{\mathcal{C}})$ . Then  $H^i(Rf_*K)$  is the sheaf associated to the presheaf*

$$V \mapsto H^i(u(V), K) = H^i(V, Rf_*K)$$

**Proof.** The equality  $H^i(u(V), K) = H^i(V, Rf_*K)$  follows upon taking cohomology from the second statement in Lemma 21.5. Then the statement on sheafification follows from Lemma 21.3.  $\square$

0D6J **Lemma 21.7.** *Let  $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$  be a ringed site. Let  $K$  be an object of  $D(\mathcal{O}_{\mathcal{C}})$  and denote  $K_{ab}$  its image in  $D(\underline{\mathbf{Z}}_{\mathcal{C}})$ .*

- (1) *There is a canonical map  $R\Gamma(\mathcal{C}, K) \rightarrow R\Gamma(\mathcal{C}, K_{ab})$  which is an isomorphism in  $D(\text{Ab})$ .*
- (2) *For any  $U \in \mathcal{C}$  there is a canonical map  $R\Gamma(U, K) \rightarrow R\Gamma(U, K_{ab})$  which is an isomorphism in  $D(\text{Ab})$ .*
- (3) *Let  $f : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites. There is a canonical map  $Rf_*K \rightarrow Rf_*(K_{ab})$  which is an isomorphism in  $D(\underline{\mathbf{Z}}_{\mathcal{D}})$ .*

**Proof.** The map is constructed as follows. Choose a  $\mathbf{K}$ -injective complex  $\mathcal{I}^\bullet$  representing  $K$ . Choose a quasi-isomorphism  $\mathcal{I}^\bullet \rightarrow \mathcal{J}^\bullet$  where  $\mathcal{J}^\bullet$  is a  $\mathbf{K}$ -injective complex of abelian groups. Then the map in (1) is given by  $\Gamma(\mathcal{C}, \mathcal{I}^\bullet) \rightarrow \Gamma(\mathcal{C}, \mathcal{J}^\bullet)$  (2) is given by  $\Gamma(U, \mathcal{I}^\bullet) \rightarrow \Gamma(U, \mathcal{J}^\bullet)$  and the map in (3) is given by  $f_*\mathcal{I}^\bullet \rightarrow f_*\mathcal{J}^\bullet$ . To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 21.2 and 21.6 it suffices to show that the map

$$H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$$

is an isomorphism. Observe that

$$H^0(\mathcal{C}, K) = \text{Hom}_{D(\mathcal{O}_{\mathcal{C}})}(\mathcal{O}_{\mathcal{C}}, K)$$

and similarly for the other group. Choose any complex  $\mathcal{K}^\bullet$  of  $\mathcal{O}_{\mathcal{C}}$ -modules representing  $K$ . By construction of the derived category as a localization we have

$$\text{Hom}_{D(\mathcal{O}_{\mathcal{C}})}(\mathcal{O}_{\mathcal{C}}, K) = \text{colim}_{s: \mathcal{F}^\bullet \rightarrow \mathcal{O}_{\mathcal{C}}} \text{Hom}_{K(\mathcal{O}_{\mathcal{C}})}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms  $s$  of complexes of  $\mathcal{O}_{\mathcal{C}}$ -modules. Similarly, we have

$$\text{Hom}_{D(\underline{\mathbf{Z}}_{\mathcal{C}})}(\underline{\mathbf{Z}}_{\mathcal{C}}, K) = \text{colim}_{s: \mathcal{G}^\bullet \rightarrow \underline{\mathbf{Z}}_{\mathcal{C}}} \text{Hom}_{K(\underline{\mathbf{Z}}_{\mathcal{C}})}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms  $s : \mathcal{G}^\bullet \rightarrow \underline{\mathbf{Z}}_{\mathcal{C}}$  with  $\mathcal{G}^\bullet$  bounded above complex of flat  $\underline{\mathbf{Z}}_{\mathcal{C}}$ -modules is cofinal in the system. (This follows from Modules on Sites, Lemma 28.7 and Derived Categories, Lemma 16.5; see discussion in Section 18.) Hence we can construct an inverse to the map  $H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$  by representing an element  $\xi \in H^0(\mathcal{C}, K_{ab})$  by a pair

$$(s : \mathcal{G}^\bullet \rightarrow \underline{\mathbf{Z}}_{\mathcal{C}}, a : \mathcal{G}^\bullet \rightarrow \mathcal{K}^\bullet)$$

with  $\mathcal{G}^\bullet$  a bounded above complex of flat  $\underline{\mathbf{Z}}_{\mathcal{C}}$ -modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_{\mathcal{C}}} \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{O}_{\mathcal{C}}, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_{\mathcal{C}}} \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 18.11 and 18.7. We omit the detailed verification that this construction is indeed an inverse.  $\square$

08FJ **Lemma 21.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Denote  $j : (\text{Sh}(\mathcal{C}/U), \mathcal{O}_U) \rightarrow (\text{Sh}(\mathcal{C}), \mathcal{O})$  the corresponding localization morphism. The restriction functor  $D(\mathcal{O}) \rightarrow D(\mathcal{O}_U)$  is a right adjoint to extension by zero  $j_! : D(\mathcal{O}_U) \rightarrow D(\mathcal{O})$ .*

**Proof.** We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex  $\mathcal{E}^\bullet$  of  $\mathcal{O}_U$ -modules representing  $E$  and choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $F$ . By Lemma 21.1 the complex  $\mathcal{I}^\bullet|_U$  is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as  $|_U$  and  $j_!$  are adjoint functors (Modules on Sites, Lemma 19.2) and Derived Categories, Lemma 29.2.  $\square$

093Y **Lemma 21.9.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a flat morphism of ringed topoi. If  $\mathcal{I}^\bullet$  is a K-injective complex of  $\mathcal{O}_{\mathcal{C}}$ -modules, then  $f_*\mathcal{I}^\bullet$  is K-injective as a complex of  $\mathcal{O}_{\mathcal{D}}$ -modules.*

**Proof.** This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_{\mathcal{D}})}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_{\mathcal{C}})}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Modules on Sites, Lemma 13.2 and the fact that  $f^*$  is exact as  $f$  is assumed to be flat.  $\square$

093Z **Lemma 21.10.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a map of sheaves of rings. If  $\mathcal{I}^\bullet$  is a K-injective complex of  $\mathcal{O}$ -modules, then  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{I}^\bullet)$  is a K-injective complex of  $\mathcal{O}'$ -modules.*

**Proof.** This is true because  $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$  by Modules on Sites, Lemma 27.6.  $\square$

## 22. Derived and homotopy limits

0940 Let  $\mathcal{C}$  be a site. Consider the category  $\mathcal{C} \times \mathbf{N}$  with  $Mor((U, n), (V, m)) = \emptyset$  if  $n > m$  and  $Mor((U, n), (V, m)) = Mor(U, V)$  else. We endow this with the structure of a site by letting coverings be families  $\{(U_i, n) \rightarrow (U, n)\}$  such that  $\{U_i \rightarrow U\}$  is a covering of  $\mathcal{C}$ . Then the reader verifies immediately that sheaves on  $\mathcal{C} \times \mathbf{N}$  are the same thing as inverse systems of sheaves on  $\mathcal{C}$ . In particular  $Ab(\mathcal{C} \times \mathbf{N})$  is inverse systems of abelian sheaves on  $\mathcal{C}$ . Consider now the functor

$$\lim : Ab(\mathcal{C} \times \mathbf{N}) \rightarrow Ab(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but  $g_*$  where  $g : Sh(\mathcal{C} \times \mathbf{N}) \rightarrow Sh(\mathcal{C})$  is the morphism of topoi associated to the continuous and cocontinuous functor  $\mathcal{C} \times \mathbf{N} \rightarrow \mathcal{C}$ . (Observe that  $g^{-1}$  assigns to a sheaf on  $\mathcal{C}$  the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \rightarrow D(\mathcal{C}).$$

As indicated this functor is often denoted  $R\lim$ .

On the other hand, the continuous and cocontinuous functors  $\mathcal{C} \rightarrow \mathcal{C} \times \mathbf{N}$ ,  $U \mapsto (U, n)$  define morphisms of topoi  $i_n : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{C} \times \mathbf{N})$ . Of course  $i_n^{-1}$  is the functor which picks the  $n$ th term of the inverse system. Thus there are transformations of functors  $i_{n+1}^{-1} \rightarrow i_n^{-1}$ . Hence given  $K \in D(\mathcal{C} \times \mathbf{N})$  we get  $K_n = i_n^{-1}K \in D(\mathcal{C})$

and maps  $K_{n+1} \rightarrow K_n$ . In Derived Categories, Definition 32.1 we have defined the notion of a homotopy limit

$$R \lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

0941 **Lemma 22.1.** *Let  $\mathcal{C}$  be a site. Let  $K$  be an object of  $D(\mathcal{C} \times \mathbf{N})$ . Set  $K_n = i_n^{-1}K$  as above. Then*

$$R \lim K \cong R \lim K_n$$

in  $D(\mathcal{C})$ .

**Proof.** To calculate  $R \lim$  on an object  $K$  of  $D(\mathcal{C} \times \mathbf{N})$  we choose a K-injective representative  $\mathcal{I}^\bullet$  whose terms are injective objects of  $Ab(\mathcal{C} \times \mathbf{N})$ , see Injectives, Theorem 12.6. We may and do think of  $\mathcal{I}^\bullet$  as an inverse system of complexes  $(\mathcal{I}_n^\bullet)$  and then we see that

$$R \lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let  $\mathcal{J} = (\mathcal{J}_n)$  be an injective object of  $Ab(\mathcal{C} \times \mathbf{N})$ . The morphisms  $(U, n) \rightarrow (U, n+1)$  are monomorphisms of  $\mathcal{C} \times \mathbf{N}$ , hence  $\mathcal{J}(U, n+1) \rightarrow \mathcal{J}(U, n)$  is surjective (Lemma 13.6). It follows that  $\mathcal{J}_{n+1} \rightarrow \mathcal{J}_n$  is surjective as a map of presheaves.

Note that the functor  $i_n^{-1}$  has an exact left adjoint  $i_{n,!}$ . Namely,  $i_{n,!}\mathcal{F}$  is the inverse system  $\dots 0 \rightarrow 0 \rightarrow \mathcal{F} \rightarrow \dots \rightarrow \mathcal{F}$ . Thus the complexes  $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$  are K-injective by Derived Categories, Lemma 29.9.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \rightarrow \lim \mathcal{I}_n^\bullet \rightarrow \prod \mathcal{I}_n^\bullet \rightarrow \prod \mathcal{I}_n^\bullet \rightarrow 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in  $D(\mathcal{C})$ , see Injectives, Lemma 13.4 and its proof (this is where we use that  $\mathcal{I}_n^\bullet$  is K-injective). Thus  $R \lim K$  is a homotopy limit of the inverse system  $(K_n)$  by definition of homotopy limits in triangulated categories.  $\square$

0D6K **Lemma 22.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. The functors  $R\Gamma(\mathcal{C}, -)$  and  $R\Gamma(U, -)$  for  $U \in \text{Ob}(\mathcal{C})$  commute with  $R \lim$ . Moreover, there are short exact sequences*

$$0 \rightarrow R^1 \lim H^{m-1}(U, K_n) \rightarrow H^m(U, R \lim K_n) \rightarrow \lim H^m(U, K_n) \rightarrow 0$$

for any inverse system  $(K_n)$  in  $D(\mathcal{O})$  and  $m \in \mathbf{Z}$ . Similar for  $H^m(\mathcal{C}, R \lim K_n)$ .

**Proof.** The first statement follows from Injectives, Lemma 13.6. Then we may apply More on Algebra, Remark 75.9 to  $R \lim R\Gamma(U, K_n) = R\Gamma(U, R \lim K_n)$  to get the short exact sequences.  $\square$

0A07 **Lemma 22.3.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. Then  $Rf_*$  commutes with  $R \lim$ , i.e.,  $Rf_*$  commutes with derived limits.*

**Proof.** Let  $(K_n)$  be an inverse system of objects of  $D(\mathcal{O})$ . By induction on  $n$  we may choose actual complexes  $\mathcal{K}_n^\bullet$  of  $\mathcal{O}$ -modules and maps of complexes  $\mathcal{K}_{n+1}^\bullet \rightarrow \mathcal{K}_n^\bullet$  representing the maps  $K_{n+1} \rightarrow K_n$  in  $D(\mathcal{O})$ . In other words, there exists an object

$K$  in  $D(\mathcal{C} \times \mathbf{N})$  whose associated inverse system is the given one. Next, consider the commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C} \times \mathbf{N}) & \xrightarrow{g} & Sh(\mathcal{C}) \\ f \times 1 \downarrow & & \downarrow f \\ Sh(\mathcal{C}' \times \mathbf{N}) & \xrightarrow{g'} & Sh(\mathcal{C}') \end{array}$$

of morphisms of topoi. It follows that  $R\lim R(f \times 1)_* K = Rf_* R\lim K$ . Working through the definitions and using Lemma 22.1 we obtain that  $R\lim(Rf_* K_n) = Rf_*(R\lim K_n)$ .

Alternate proof in case  $\mathcal{C}$  has enough points. Consider the defining distinguished triangle

$$R\lim K_n \rightarrow \prod K_n \rightarrow \prod K_n$$

in  $D(\mathcal{O})$ . Applying the exact functor  $Rf_*$  we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \rightarrow Rf_*\left(\prod K_n\right) \rightarrow Rf_*\left(\prod K_n\right)$$

in  $D(\mathcal{O}')$ . Thus we see that it suffices to prove that  $Rf_*$  commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 13.4). However, since  $Rf_*$  is a right adjoint by Lemma 20.1 this follows formally (see Categories, Lemma 24.4). Caution: Note that we cannot apply Categories, Lemma 24.4 directly as  $R\lim K_n$  is not a limit in  $D(\mathcal{O})$ .  $\square$

0BKW **Remark 22.4.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K_n)$  be an inverse system in  $D(\mathcal{O})$ . Set  $K = R\lim K_n$ . For each  $n$  and  $m$  let  $\mathcal{H}_n^m = H^m(K_n)$  be the  $m$ th cohomology sheaf of  $K_n$  and similarly set  $\mathcal{H}^m = H^m(K)$ . Let us denote  $\underline{\mathcal{H}}_n^m$  the presheaf

$$U \mapsto \underline{\mathcal{H}}_n^m(U) = H^m(U, K_n)$$

Similarly we set  $\underline{\mathcal{H}}^m(U) = H^m(U, K)$ . By Lemma 21.3 we see that  $\mathcal{H}_n^m$  is the sheafification of  $\underline{\mathcal{H}}_n^m$  and  $\mathcal{H}^m$  is the sheafification of  $\underline{\mathcal{H}}^m$ . Here is a diagram

$$\begin{array}{ccc} K & & \underline{\mathcal{H}}^m \longrightarrow \mathcal{H}^m \\ \parallel & & \downarrow \qquad \qquad \downarrow \\ R\lim K_n & & \lim \underline{\mathcal{H}}_n^m \longrightarrow \lim \mathcal{H}_n^m \end{array}$$

In general it may not be the case that  $\lim \mathcal{H}_n^m$  is the sheafification of  $\lim \underline{\mathcal{H}}_n^m$ . If  $U \in \mathcal{C}$ , then we have short exact sequences

$$0 \rightarrow R^1 \lim \underline{\mathcal{H}}_n^{m-1}(U) \rightarrow \underline{\mathcal{H}}^m(U) \rightarrow \lim \underline{\mathcal{H}}_n^m(U) \rightarrow 0$$

by Lemma 22.2.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on an algebraic space or an algebraic stack.

0BKY **Lemma 22.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{F}_n)$  be an inverse system of  $\mathcal{O}$ -modules. Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*
- (2)  *$H^p(U, \mathcal{F}_n) = 0$  for  $p > 0$  and  $U \in \mathcal{B}$ ,*
- (3) *the inverse system  $\mathcal{F}_n(U)$  has vanishing  $R^1 \lim$  for  $U \in \mathcal{B}$ .*

Then  $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$  and we have  $H^p(U, \lim \mathcal{F}_n) = 0$  for  $p > 0$  and  $U \in \mathcal{B}$ .

**Proof.** Set  $K_n = \mathcal{F}_n$  and  $K = R\lim \mathcal{F}_n$ . Using the notation of Remark 22.4 and assumption (2) we see that for  $U \in \mathcal{B}$  we have  $\underline{H}_n^m(U) = 0$  when  $m \neq 0$  and  $\underline{H}_n^0(U) = \mathcal{F}_n(U)$ . From Equation (22.4.1) and assumption (3) we see that  $\underline{H}^m(U) = 0$  when  $m \neq 0$  and equal to  $\lim \mathcal{F}_n(U)$  when  $m = 0$ . Sheafifying using (1) we find that  $\mathcal{H}^m = 0$  when  $m \neq 0$  and equal to  $\lim \mathcal{F}_n$  when  $m = 0$ . Hence  $K = \lim \mathcal{F}_n$ . Since  $H^m(U, K) = \underline{H}^m(U) = 0$  for  $m > 0$  (see above) we see that the second assertion holds.  $\square$

0D6L **Lemma 22.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K_n)$  be an inverse system in  $D(\mathcal{O})$ . Let  $V \in \text{Ob}(\mathcal{C})$  and  $m \in \mathbf{Z}$ . Assume there exist an integer  $n(V)$  and a cofinal system  $\text{Cov}_V$  of coverings of  $V$  such that for  $\{V_i \rightarrow V\} \in \text{Cov}_V$*

- (1)  $R^1 \lim H^{m-1}(V_i, K_n) = 0$ , and
- (2)  $H^m(V_i, K_n) \rightarrow H^m(V_i, K_{n(V)})$  is injective for  $n \geq n(V)$ .

*Then the map on sections  $H^m(R\lim K_n)(V) \rightarrow H^m(K_{n(V)})(V)$  is injective.*

**Proof.** Let  $\gamma \in H^m(R\lim K_n)(V)$  map to zero in  $H^m(K_{n(V)})(V)$ . Since  $H^m(R\lim K_n)$  is the sheafification of  $U \mapsto H^m(U, R\lim K_n)$  (by Lemma 21.3) we can choose  $\{V_i \rightarrow V\} \in \text{Cov}_V$  and elements  $\tilde{\gamma}_i \in H^m(V_i, R\lim K_n)$  mapping to  $\gamma|_{V_i}$ . Then  $\tilde{\gamma}_i$  maps to  $\tilde{\gamma}_{i, n(V)} \in H^m(V_i, K_{n(V)})$ . Using that  $H^m(K_{n(V)})$  is the sheafification of  $U \mapsto H^m(U, K_{n(V)})$  (by Lemma 21.3 again) we see that after replacing  $\{V_i \rightarrow V\}$  by a refinement we may assume that  $\tilde{\gamma}_{i, n(V)} = 0$  for all  $i$ . For this covering we consider the short exact sequences

$$0 \rightarrow R^1 \lim H^{m-1}(V_i, K_n) \rightarrow H^m(V_i, R\lim K_n) \rightarrow \lim H^m(V_i, K_n) \rightarrow 0$$

of Lemma 22.2. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into  $H^m(V_i, K_{n(V)})$ . We conclude  $\tilde{\gamma}_i = 0$  and hence  $\gamma = 0$  as desired.  $\square$

0D6M **Lemma 22.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E \in D(\mathcal{O})$ . Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ , and*
- (2) *for every  $V \in \mathcal{B}$  there exist a function  $p(V, -) : \mathbf{Z} \rightarrow \mathbf{Z}$  and a cofinal system  $\text{Cov}_V$  of coverings of  $V$  such that*

$$H^p(V_i, H^{m-p}(E)) = 0$$

*for all  $\{V_i \rightarrow V\} \in \text{Cov}_V$  and all integers  $p, m$  satisfying  $p > p(V, m)$ .*

*Then the canonical map  $E \rightarrow R\lim \tau_{\geq -n} E$  is an isomorphism in  $D(\mathcal{O})$ .*

**Proof.** Set  $K_n = \tau_{\geq -n} E$  and  $K = R\lim K_n$ . The canonical map  $E \rightarrow K$  comes from the canonical maps  $E \rightarrow K_n = \tau_{\geq -n} E$ . We have to show that  $E \rightarrow K$  induces an isomorphism  $H^m(E) \rightarrow H^m(K)$  of cohomology sheaves. In the rest of the proof we fix  $m$ . If  $n \geq -m$ , then the map  $E \rightarrow \tau_{\geq -n} E = K_n$  induces an isomorphism  $H^m(E) \rightarrow H^m(K_n)$ . To finish the proof it suffices to show that for every  $V \in \mathcal{B}$  there exists an integer  $n(V) \geq -m$  such that the map  $H^m(K)(V) \rightarrow H^m(K_{n(V)})(V)$  is injective. Namely, then the composition

$$H^m(E)(V) \rightarrow H^m(K)(V) \rightarrow H^m(K_{n(V)})(V)$$



is a bijection and the second arrow is injective, hence the first arrow is bijective. By property (1) this will imply  $H^m(E) \rightarrow H^m(K)$  is an isomorphism. Set

$$n(V) = 1 + \max\{-m, p(V, m-1) - m, -1 + p(V, m) - m, -2 + p(V, m+1) - m\}.$$

so that in any case  $n(V) \geq -m$ . Claim: the maps

$$H^{m-1}(V_i, K_{n+1}) \rightarrow H^{m-1}(V_i, K_n) \quad \text{and} \quad H^m(V_i, K_{n+1}) \rightarrow H^m(V_i, K_n)$$

are isomorphisms for  $n \geq n(V)$  and  $\{V_i \rightarrow V\} \in \text{Cov}_V$ . The claim implies conditions (1) and (2) of Lemma 22.6 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \rightarrow K_{n+1} \rightarrow K_n \rightarrow H^{-n-1}(E)[n+2]$$

Looking at the associated long exact cohomology sequence the claim follows if

$$H^{m+n}(V_i, H^{-n-1}(E)), \quad H^{m+n+1}(V_i, H^{-n-1}(E)), \quad H^{m+n+2}(V_i, H^{-n-1}(E))$$

are zero for  $n \geq n(V)$  and  $\{V_i \rightarrow V\} \in \text{Cov}_V$ . This follows from our choice of  $n(V)$  and the assumption in the lemma.  $\square$

0D6N **Lemma 22.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E \in D(\mathcal{O})$ . Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ , and*
- (2) *for every  $V \in \mathcal{B}$  there exist an integer  $d_V \geq 0$  and a cofinal system  $\text{Cov}_V$  of coverings of  $V$  such that*

$$H^p(V_i, H^q(E)) = 0 \text{ for } \{V_i \rightarrow V\} \in \text{Cov}_V, \quad p > d_V, \text{ and } q < 0$$

*Then the canonical map  $E \rightarrow R\lim \tau_{\geq -n} E$  is an isomorphism in  $D(\mathcal{O})$ .*

**Proof.** This follows from Lemma 22.7 with  $p(V, m) = d_V + \max(0, m)$ .  $\square$

08U3 **Lemma 22.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E \in D(\mathcal{O})$ . Assume there exists a function  $p(-) : \mathbf{Z} \rightarrow \mathbf{Z}$  and a subset  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  such that*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*
- (2)  *$H^p(V, H^{m-p}(E)) = 0$  for  $p > p(m)$  and  $V \in \mathcal{B}$ .*

*Then the canonical map  $E \rightarrow R\lim \tau_{\geq -n} E$  is an isomorphism in  $D(\mathcal{O})$ .*

**Proof.** Apply Lemma 22.7 with  $p(V, m) = p(m)$  and  $\text{Cov}_V$  equal to the set of coverings  $\{V_i \rightarrow V\}$  with  $V_i \in \mathcal{B}$  for all  $i$ .  $\square$

0D6P **Lemma 22.10.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E \in D(\mathcal{O})$ . Assume there exists an integer  $d \geq 0$  and a subset  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  such that*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*
- (2)  *$H^p(V, H^q(E)) = 0$  for  $p > d, q < 0$ , and  $V \in \mathcal{B}$ .*

*Then the canonical map  $E \rightarrow R\lim \tau_{\geq -n} E$  is an isomorphism in  $D(\mathcal{O})$ .*

**Proof.** Apply Lemma 22.8 with  $d_V = d$  and  $\text{Cov}_V$  equal to the set of coverings  $\{V_i \rightarrow V\}$  with  $V_i \in \mathcal{B}$  for all  $i$ .  $\square$

The lemmas above can be used to compute cohomology in certain situations.

0BKZ **Lemma 22.11.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be an object of  $D(\mathcal{O})$ . Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*

(2)  $H^p(U, H^q(K)) = 0$  for all  $p > 0$ ,  $q \in \mathbf{Z}$ , and  $U \in \mathcal{B}$ .

Then  $H^q(U, K) = H^0(U, H^q(K))$  for  $q \in \mathbf{Z}$  and  $U \in \mathcal{B}$ .

**Proof.** Observe that  $K = R\lim \tau_{\geq -n}K$  by Lemma 22.10 with  $d = 0$ . Let  $U \in \mathcal{B}$ . By Equation (22.4.1) we get a short exact sequence

$$0 \rightarrow R^1 \lim H^{q-1}(U, \tau_{\geq -n}K) \rightarrow H^q(U, K) \rightarrow \lim H^q(U, \tau_{\geq -n}K) \rightarrow 0$$

Condition (2) implies  $H^q(U, \tau_{\geq -n}K) = H^0(U, H^q(\tau_{\geq -n}K))$  for all  $q$  by using the spectral sequence of Derived Categories, Lemma 21.3. The spectral sequence converges because  $\tau_{\geq -n}K$  is bounded below. If  $n > -q$  then we have  $H^q(\tau_{\geq -n}K) = H^q(K)$ . Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values  $H^0(U, H^{q-1}(K))$  and  $H^0(U, H^q(K))$  and the lemma follows.  $\square$

Here is another case where we can describe the derived limit.

0A09 **Lemma 22.12.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K_n)$  be an inverse system of objects of  $D(\mathcal{O})$ . Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Assume*

- (1) every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,
- (2) for all  $U \in \mathcal{B}$  and all  $q \in \mathbf{Z}$  we have
  - (a)  $H^p(U, H^q(K_n)) = 0$  for  $p > 0$ ,
  - (b) the inverse system  $H^0(U, H^q(K_n))$  has vanishing  $R^1 \lim$ .

Then  $H^q(R\lim K_n) = \lim H^q(K_n)$  for  $q \in \mathbf{Z}$ .

**Proof.** Set  $K = R\lim K_n$ . We will use notation as in Remark 22.4. Let  $U \in \mathcal{B}$ . By Lemma 22.11 and (2)(a) we have  $H^q(U, K_n) = H^0(U, H^q(K_n))$ . Using that the functor  $R\Gamma(U, -)$  commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 75.9 and assumption (2)(b). Thus  $H^q(U, K)$  is the inverse limit the sections of the sheaves  $H^q(K_n)$  over  $U$ . Since  $\lim H^q(K_n)$  is a sheaf we find using assumption (1) that  $H^q(K)$ , which is the sheafification of the presheaf  $U \mapsto H^q(U, K)$ , is equal to  $\lim H^q(K_n)$ . This proves the lemma.  $\square$

### 23. Producing $\mathbf{K}$ -injective resolutions

070N Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{O}$ -modules. The category  $\text{Mod}(\mathcal{O})$  has enough injectives, hence we can use Derived Categories, Lemma 28.3 produce a diagram

$$\begin{array}{ccc} \dots & \longrightarrow & \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\ & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet \end{array}$$

in the category of complexes of  $\mathcal{O}$ -modules such that

- (1) the vertical arrows are quasi-isomorphisms,
- (2)  $\mathcal{I}_n^\bullet$  is a bounded below complex of injectives,
- (3) the arrows  $\mathcal{I}_{n+1}^\bullet \rightarrow \mathcal{I}_n^\bullet$  are termwise split surjections.

The category of  $\mathcal{O}$ -modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit  $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$ . By Derived Categories, Lemmas 29.4 and 29.8 this is a K-injective complex. In general the canonical map

$$070P \quad (23.0.1) \quad \mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

070Q **Lemma 23.1.** *In the situation described above. Denote  $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$  the  $m$ th cohomology sheaf. Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  be a subset. Let  $d \in \mathbf{N}$ . Assume*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*
- (2) *for every  $U \in \mathcal{B}$  we have  $H^p(U, \mathcal{H}^q) = 0$  for  $p > d$  and  $q < 0^2$ .*

*Then (23.0.1) is a quasi-isomorphism.*

**Proof.** By Derived Categories, Lemma 32.4 it suffices to show that the canonical map  $\mathcal{F}^\bullet \rightarrow R\lim_{\tau_{\geq -n}} \mathcal{F}^\bullet$  is an isomorphism. This follows from Lemma 22.10.  $\square$

Here is a technical lemma about cohomology sheaves of termwise limits of inverse systems of complexes of modules. We should avoid using this lemma as much as possible and instead use arguments with derived inverse limits.

08CT **Lemma 23.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{F}_n^\bullet)$  be an inverse system of complexes of  $\mathcal{O}$ -modules. Let  $m \in \mathbf{Z}$ . Suppose given  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  and an integer  $n_0$  such that*

- (1) *every object of  $\mathcal{C}$  has a covering whose members are elements of  $\mathcal{B}$ ,*
- (2) *for every  $U \in \mathcal{B}$* 
  - (a) *the systems of abelian groups  $\mathcal{F}_n^{m-2}(U)$  and  $\mathcal{F}_n^{m-1}(U)$  have vanishing  $R^1\lim$  (for example these have the Mittag-Leffler property),*
  - (b) *the system of abelian groups  $H^{m-1}(\mathcal{F}_n^\bullet(U))$  has vanishing  $R^1\lim$  (for example it has the Mittag-Leffler property), and*
  - (c) *we have  $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$  for all  $n \geq n_0$ .*

*Then the maps  $H^m(\mathcal{F}^\bullet) \rightarrow \lim H^m(\mathcal{F}_n^\bullet) \rightarrow H^m(\mathcal{F}_{n_0}^\bullet)$  are isomorphisms of sheaves where  $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$  is the termwise inverse limit.*

**Proof.** Let  $U \in \mathcal{B}$ . Note that  $H^m(\mathcal{F}^\bullet(U))$  is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \rightarrow \lim_n \mathcal{F}_n^{m-1}(U) \rightarrow \lim_n \mathcal{F}_n^m(U) \rightarrow \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 75.2 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all  $n \geq n_0$ . By assumption (1) we conclude that the sheafification of  $U \mapsto H^m(\mathcal{F}^\bullet(U))$  is equal to the sheafification of  $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$  for all  $n \geq n_0$ . Thus the inverse system of sheaves  $H^m(\mathcal{F}_n^\bullet)$  is constant for  $n \geq n_0$  with value  $H^m(\mathcal{F}^\bullet)$  which proves the lemma.  $\square$

<sup>2</sup>It suffices if  $\forall m, \exists p(m), H^p(U, \mathcal{H}^{m-p}) = 0$  for  $p > p(m)$ , see Lemma 22.9.

## 24. Bounded cohomological dimension

0D6Q In this section we ask when a functor  $Rf_*$  has bounded cohomological dimension. This is a rather subtle question when we consider unbounded complexes.

0D6R **Situation 24.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Let  $\mathcal{A} \subset \text{Mod}(\mathcal{O})$  be a weak Serre subcategory. We assume the following is true: there exists a subset  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  such that

- (1) every object of  $\mathcal{C}$  has a covering whose members are in  $\mathcal{B}$ , and
- (2) for every  $V \in \mathcal{B}$  there exists an integer  $d_V$  and a cofinal system  $\text{Cov}_V$  of coverings of  $V$  such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \rightarrow V\} \in \text{Cov}_V, \ p > d_V, \text{ and } \mathcal{F} \in \text{Ob}(\mathcal{A})$$

0D6S **Lemma 24.2.** *In Situation 24.1 for any  $E \in D_{\mathcal{A}}(\mathcal{O})$  the canonical map  $E \rightarrow R\lim \tau_{\geq -n} E$  is an isomorphism in  $D(\mathcal{O})$ .*

This is [LO08, Proposition 2.1.4] with slightly changed hypotheses; it is the analogue of [Spa88, Proposition 3.13] for sites.

**Proof.** Follows immediately from Lemma 22.8. □

0D6T **Lemma 24.3.** *In Situation 24.1 let  $(K_n)$  be an inverse system in  $D_{\mathcal{A}}^+(\mathcal{O})$ . Assume that for every  $j$  the inverse system  $(H^j(K_n))$  in  $\mathcal{A}$  is eventually constant with value  $\mathcal{H}^j$ . Then  $H^j(R\lim K_n) = \mathcal{H}^j$  for all  $j$ .*

**Proof.** Let  $V \in \mathcal{B}$ . Let  $\{V_i \rightarrow V\}$  be in the set  $\text{Cov}_V$  of Situation 24.1. Because  $K_n$  is bounded below there is a spectral sequence

$$E_2^{p,q} = H^p(V_i, H^q(K_n))$$

converging to  $H^{p+q}(V_i, K_n)$ . See Derived Categories, Lemma 21.3. Observe that  $E_2^{p,q} = 0$  for  $p > d_V$  by assumption. Pick  $n_0$  such that

$$\begin{aligned} \mathcal{H}^{j+1} &= H^{j+1}(K_n), \\ \mathcal{H}^j &= H^j(K_n), \\ \dots, \\ \mathcal{H}^{j-d_V-2} &= H^{j-d_V-2}(K_n) \end{aligned}$$

for all  $n \geq n_0$ . Comparing the spectral sequences above for  $K_n$  and  $K_{n_0}$ , we see that for  $n \geq n_0$  the cohomology groups  $H^{j-1}(V_i, K_n)$  and  $H^j(V_i, K_n)$  are independent of  $n$ . It follows that the map on sections  $H^j(R\lim K_n)(V) \rightarrow H^j(K_n)(V)$  is injective for  $n$  large enough (depending on  $V$ ), see Lemma 22.6. Since every object of  $\mathcal{C}$  can be covered by elements of  $\mathcal{B}$ , we conclude that the map  $H^j(R\lim K_n) \rightarrow \mathcal{H}^j$  is injective.

Surjectivity is shown in a similar manner. Namely, pick  $U \in \text{Ob}(\mathcal{C})$  and  $\gamma \in \mathcal{H}^j(U)$ . We want to lift  $\gamma$  to a section of  $H^j(R\lim K_n)$  after replacing  $U$  by the members of a covering. Hence we may assume  $U = V \in \mathcal{B}$  by property (1) of Situation 24.1. Pick  $n_0$  such that

$$\begin{aligned} \mathcal{H}^{j+1} &= H^{j+1}(K_n), \\ \mathcal{H}^j &= H^j(K_n), \\ \dots, \\ \mathcal{H}^{j-d_V-2} &= H^{j-d_V-2}(K_n) \end{aligned}$$

for all  $n \geq n_0$ . Choose an element  $\{V_i \rightarrow V\}$  of  $\text{Cov}_V$  such that  $\gamma|_{V_i} \in \mathcal{H}^j(V_i) = H^j(K_{n_0})(V_i)$  lifts to an element  $\gamma_{n_0,i} \in H^j(V_i, K_{n_0})$ . This is possible because  $H^j(K_{n_0})$  is the sheafification of  $U \mapsto H^j(U, K_{n_0})$  by Lemma 21.3. By the discussion in the first paragraph of the proof we have that  $H^{j-1}(V_i, K_n)$  and  $H^j(V_i, K_n)$  are

independent of  $n \geq n_0$ . Hence  $\gamma_{n_0, i}$  lifts to an element  $\gamma_i \in H^j(V_i, R\lim K_n)$  by Lemma 22.2. This finishes the proof.  $\square$

0D6U **Lemma 24.4.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. Let  $\mathcal{A} \subset Mod(\mathcal{O})$  and  $\mathcal{A}' \subset Mod(\mathcal{O}')$  be weak Serre subcategories. Assume there is an integer  $N$  such that*

- (1)  $\mathcal{C}, \mathcal{O}, \mathcal{A}$  satisfy the assumption of Situation 24.1,
- (2)  $\mathcal{C}', \mathcal{O}', \mathcal{A}'$  satisfy the assumption of Situation 24.1,
- (3)  $R^p f_* \mathcal{F} \in \text{Ob}(\mathcal{A}')$  for  $p \geq 0$  and  $\mathcal{F} \in \text{Ob}(\mathcal{A})$ ,
- (4)  $R^p f_* \mathcal{F} = 0$  for  $p > N$  and  $\mathcal{F} \in \text{Ob}(\mathcal{A})$ ,

Then for  $K$  in  $D_{\mathcal{A}}(\mathcal{O})$  we have

- (a)  $Rf_* K$  is in  $D_{\mathcal{A}'}(\mathcal{O}')$ ,
- (b) the map  $H^j(Rf_* K) \rightarrow H^j(Rf_*(\tau_{\geq -n} K))$  is an isomorphism for  $j \geq N - n$ .

**Proof.** By Lemma 24.2 we have  $K = R\lim \tau_{\geq -n} K$ . By Lemma 22.3 we have  $Rf_* K = R\lim Rf_* \tau_{\geq -n} K$ . The complexes  $Rf_* \tau_{\geq -n} K$  are bounded below. The spectral sequence

$$E_2^{p,q} = R^p f_* H^q(\tau_{\geq -n} K)$$

converging to  $H^{p+q}(Rf_* \tau_{\geq -n} K)$  (Derived Categories, Lemma 21.3) and assumption (3) show that  $Rf_* \tau_{\geq -n} K$  lies in  $D_{\mathcal{A}'}^+(\mathcal{O}')$ , see Homology, Lemma 21.11. Observe that for  $m \geq n$  the map

$$Rf_*(\tau_{\geq -m} K) \longrightarrow Rf_*(\tau_{\geq -n} K)$$

induces an isomorphism on cohomology sheaves in degrees  $j \geq -n + N$  by the spectral sequences above. Hence we may apply Lemma 24.3 to conclude.  $\square$

It turns out that we sometimes need a variant of the lemma above where the assumptions are slightly different.

0D6V **Situation 24.5.** *Let  $f : (\mathcal{C}, \mathcal{O}) \rightarrow (\mathcal{C}', \mathcal{O}')$  be a morphism of ringed sites. Let  $u : \mathcal{C}' \rightarrow \mathcal{C}$  be the corresponding continuous functor of sites. Let  $\mathcal{A} \subset Mod(\mathcal{O})$  be a weak Serre subcategory. We assume the following is true: there exists a subset  $\mathcal{B}' \subset \text{Ob}(\mathcal{C}')$  such that*

- (1) every object of  $\mathcal{C}'$  has a covering whose members are in  $\mathcal{B}'$ , and
- (2) for every  $V' \in \mathcal{B}'$  there exists an integer  $d_{V'}$  and a cofinal system  $\text{Cov}_{V'}$  of coverings of  $V'$  such that

$$H^p(u(V'_i), \mathcal{F}) = 0 \text{ for } \{V'_i \rightarrow V'\} \in \text{Cov}_{V'}, p > d_{V'}, \text{ and } \mathcal{F} \in \text{Ob}(\mathcal{A})$$

0D6W **Lemma 24.6.** *Let  $f : (\mathcal{C}, \mathcal{O}) \rightarrow (\mathcal{C}', \mathcal{O}')$  be a morphism of ringed sites. assume moreover there is an integer  $N$  such that*

- (1)  $\mathcal{C}, \mathcal{O}, \mathcal{A}$  satisfy the assumption of Situation 24.1,
- (2)  $f : (\mathcal{C}, \mathcal{O}) \rightarrow (\mathcal{C}', \mathcal{O}')$  and  $\mathcal{A}$  satisfy the assumption of Situation 24.5,
- (3)  $R^p f_* \mathcal{F} = 0$  for  $p > N$  and  $\mathcal{F} \in \text{Ob}(\mathcal{A})$ ,

Then for  $K$  in  $D_{\mathcal{A}}(\mathcal{O})$  the map  $H^j(Rf_* K) \rightarrow H^j(Rf_*(\tau_{\geq -n} K))$  is an isomorphism for  $j \geq N - n$ .

**Proof.** Let  $K$  be in  $D_{\mathcal{A}}(\mathcal{O})$ . By Lemma 24.2 we have  $K = R\lim \tau_{\geq -n} K$ . By Lemma 22.3 we have  $Rf_* K = R\lim Rf_*(\tau_{\geq -n} K)$ . Let  $V' \in \mathcal{B}'$  and let  $\{V'_i \rightarrow V'\}$  be an element of  $\text{Cov}_{V'}$ . Then we consider

$$H^j(V'_i, Rf_* K) = H^j(u(V'_i), K) \quad \text{and} \quad H^j(V'_i, Rf_*(\tau_{\geq -n} K)) = H^j(u(V'_i), \tau_{\geq -n} K)$$

This is a version of [LO08, Lemma 2.1.10] with slightly changed hypotheses.

This is a version of [LO08, Lemma 2.1.10] with slightly changed hypotheses.

The assumption in Situation 24.5 implies that the last group is independent of  $n$  for  $n$  large enough depending on  $j$  and  $d_{V'}$ . Some details omitted. We apply this for  $j$  and  $j - 1$  and via Lemma 22.2 this gives that

$$H^j(V'_i, Rf_*K) = \lim H^j(V'_i, Rf_*(\tau_{\geq -n}K))$$

and the system on the right is constant for  $n$  larger than a constant depending only on  $d_{V'}$  and  $j$ . Thus Lemma 22.6 implies that

$$H^j(Rf_*K)(V') \longrightarrow (\lim H^j(Rf_*(\tau_{\geq -n}K)))(V')$$

is injective. Since the elements  $V' \in \mathcal{B}'$  cover every object of  $\mathcal{C}'$  we conclude that the map  $H^j(Rf_*K) \rightarrow \lim H^j(Rf_*(\tau_{\geq -n}K))$  is injective. The spectral sequence

$$E_2^{p,q} = R^p f_* H^q(\tau_{\geq -n}K)$$

converging to  $H^{p+q}(Rf_*(\tau_{\geq -n}K))$  (Derived Categories, Lemma 21.3) and assumption (3) show that  $H^j(Rf_*(\tau_{\geq -n}K))$  is constant for  $n \geq N - j$ . Hence  $H^j(Rf_*K) \rightarrow H^j(Rf_*(\tau_{\geq -n}K))$  is injective for  $j \geq N - n$ .

Thus we proved the lemma with “isomorphism” in the last line of the lemma replaced by “injective”. However, now choose  $j$  and  $n$  with  $j \geq N - n$ . Then consider the distinguished triangle

$$\tau_{\leq -n-1}K \rightarrow K \rightarrow \tau_{\geq -n}K \rightarrow (\tau_{\leq -n-1}K)[1]$$

See Derived Categories, Remark 12.4. Since  $\tau_{\geq -n}\tau_{\leq -n-1}K = 0$ , the injectivity already proven for  $\tau_{\leq -n-1}K$  implies

$$0 = H^j(Rf_*(\tau_{\leq -n-1}K)) = H^{j+1}(Rf_*(\tau_{\leq -n-1}K)) = H^{j+2}(Rf_*(\tau_{\leq -n-1}K)) = \dots$$

By the long exact cohomology sequence associated to the distinguished triangle

$$Rf_*(\tau_{\leq -n-1}K) \rightarrow Rf_*K \rightarrow Rf_*(\tau_{\geq -n}K) \rightarrow Rf_*(\tau_{\leq -n-1}K)[1]$$

this implies that  $H^j(Rf_*K) \rightarrow H^j(Rf_*(\tau_{\geq -n}K))$  is an isomorphism.  $\square$

## 25. Cohomology on Hausdorff and locally quasi-compact spaces

09WY We continue our convention to say “Hausdorff and locally quasi-compact” instead of saying “locally compact” as is often done in the literature. Let  $LC$  denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

09WZ **Lemma 25.1.** *The category  $LC$  has fibre products and a final object and hence has arbitrary finite limits. Given morphisms  $X \rightarrow Z$  and  $Y \rightarrow Z$  in  $LC$  with  $X$  and  $Y$  quasi-compact, then  $X \times_Z Y$  is quasi-compact.*

**Proof.** The final object is the singleton space. Given morphisms  $X \rightarrow Z$  and  $Y \rightarrow Z$  of  $LC$  the fibre product  $X \times_Z Y$  is a subspace of  $X \times Y$ . Hence  $X \times_Z Y$  is Hausdorff as  $X \times Y$  is Hausdorff by Topology, Section 3.

If  $X$  and  $Y$  are quasi-compact, then  $X \times Y$  is quasi-compact by Topology, Theorem 14.4. Since  $X \times_Z Y$  is a closed subset of  $X \times Y$  (Topology, Lemma 3.4) we find that  $X \times_Z Y$  is quasi-compact by Topology, Lemma 12.3.

Finally, returning to the general case, if  $x \in X$  and  $y \in Y$  we can pick quasi-compact neighbourhoods  $x \in E \subset X$  and  $y \in F \subset Y$  and we find that  $E \times_Z F$  is a quasi-compact neighbourhood of  $(x, y)$  by the result above. Thus  $X \times_Z Y$  is an object of  $LC$  by Topology, Lemma 13.2.  $\square$

We can endow  $LC$  with a stronger topology than the usual one.

09X0 **Definition 25.2.** Let  $\{f_i : X_i \rightarrow X\}$  be a family of morphisms with fixed target in the category  $LC$ . We say this family is a *qc covering*<sup>3</sup> if for every  $x \in X$  there exist  $i_1, \dots, i_n \in I$  and quasi-compact subsets  $E_j \subset X_{i_j}$  such that  $\bigcup f_{i_j}(E_j)$  is a neighbourhood of  $x$ .

Observe that an open covering  $X = \bigcup U_i$  of an object of  $LC$  gives a qc covering  $\{U_i \rightarrow X\}$  because  $X$  is locally quasi-compact. We start with the obligatory lemma.

09X1 **Lemma 25.3.** *Let  $X$  be a Hausdorff and locally quasi-compact space, in other words, an object of  $LC$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism in  $LC$  then  $\{X' \rightarrow X\}$  is a qc covering.*
- (2) *If  $\{f_i : X_i \rightarrow X\}_{i \in I}$  is a qc covering and for each  $i$  we have a qc covering  $\{g_{ij} : X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a qc covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a qc covering and  $X' \rightarrow X$  is a morphism of  $LC$  then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a qc covering.*

**Proof.** Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let  $x \in X$ . Choose  $i_1, \dots, i_n \in I$  and  $E_a \subset X_{i_a}$  quasi-compact such that  $\bigcup f_{i_a}(E_a)$  is a neighbourhood of  $x$ . For every  $e \in E_a$  we can find a finite subset  $J_e \subset J_{i_a}$  and quasi-compact  $F_{e,j} \subset X_{ij}$ ,  $j \in J_e$  such that  $\bigcup g_{ij}(F_{e,j})$  is a neighbourhood of  $e$ . Since  $E_a$  is quasi-compact we find a finite collection  $e_1, \dots, e_{m_a}$  such that

$$E_a \subset \bigcup_{k=1, \dots, m_a} \bigcup_{j \in J_{e_k}} g_{ij}(F_{e_k, j})$$

Then we find that

$$\bigcup_{a=1, \dots, n} \bigcup_{k=1, \dots, m_a} \bigcup_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k, j}))$$

is a neighbourhood of  $x$ .

Proof of (3). Let  $x' \in X'$  be a point. Let  $x \in X$  be its image. Choose  $i_1, \dots, i_n \in I$  and quasi-compact subsets  $E_j \subset X_{i_j}$  such that  $\bigcup f_{i_j}(E_j)$  is a neighbourhood of  $x$ . Choose a quasi-compact neighbourhood  $F \subset X'$  of  $x'$  which maps into the quasi-compact neighbourhood  $\bigcup f_{i_j}(E_j)$  of  $x$ . Then  $F \times_X E_j \subset X' \times_X X_{i_j}$  is a quasi-compact subset and  $F$  is the image of the map  $\coprod F \times_X E_j \rightarrow F$ . Hence the base change is a qc covering and the proof is finished.  $\square$

09X5 **Lemma 25.4.** *Let  $f : X \rightarrow Y$  be a morphism of  $LC$ . If  $f$  is proper and surjective, then  $\{f : X \rightarrow Y\}$  is a qc covering.*

**Proof.** Let  $y \in Y$  be a point. For each  $x \in X_y$  choose a quasi-compact neighbourhood  $E_x \subset X$ . Choose  $x \in U_x \subset E_x$  open. Since  $f$  is proper the fibre  $X_y$  is quasi-compact and we find  $x_1, \dots, x_n \in X_y$  such that  $X_y \subset U_{x_1} \cup \dots \cup U_{x_n}$ . We claim that  $f(E_{x_1}) \cup \dots \cup f(E_{x_n})$  is a neighbourhood of  $y$ . Namely, as  $f$  is closed (Topology, Theorem 17.5) we see that  $Z = f(X \setminus U_{x_1} \cup \dots \cup U_{x_n})$  is a closed subset of  $Y$  not containing  $y$ . As  $f$  is surjective we see that  $Y \setminus Z$  is contained in  $f(E_{x_1}) \cup \dots \cup f(E_{x_n})$  as desired.  $\square$

<sup>3</sup>This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

Besides some set theoretic issues Lemma 25.3 shows that  $LC$  with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues)  $LC_{qc}$ .

09X2 **Remark 25.5** (Set theoretic issues). The category  $LC$  is a “big” category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the *size* of a topological space  $X$  to be the cardinality of the set of points of  $X$ . Choose a function *Bound* on cardinals, for example as in Sets, Equation (9.1.1). Finally, let  $S_0$  be an initial set of objects objects of  $LC$ , for example  $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$ . Exactly as in Sets, Lemma 9.2 we can choose a limit ordinal  $\alpha$  such that  $LC_\alpha = LC \cap V_\alpha$  contains  $S_0$  and is preserved under all countable limits and colimits which exist in  $LC$ . Moreover, if  $X \in LC_\alpha$  and if  $Y \in LC$  and  $\text{size}(Y) \leq \text{Bound}(\text{size}(X))$ , then  $Y$  is isomorphic to an object of  $LC_\alpha$ . Next, we apply Sets, Lemma 11.1 to choose set  $\text{Cov}$  of qc covering on  $LC_\alpha$  such that every qc covering in  $LC_\alpha$  is combinatorially equivalent to a covering this set. In this way we obtain a site  $(LC_\alpha, \text{Cov})$  which we will denote  $LC_{qc}$ .

There is a second topology on the site  $LC_{qc}$  of Remark 25.5. Namely, given an object  $X$  we can consider all coverings  $\{X_i \rightarrow X\}$  of  $LC_{qc}$  such that  $X_i \rightarrow X$  is an open immersion. We denote this site  $LC_{Zar}$ . The identity functor  $LC_{Zar} \rightarrow LC_{qc}$  is continuous and defines a morphism of sites

$$\epsilon : LC_{qc} \longrightarrow LC_{Zar}$$

by an application of Sites, Proposition 14.6. For a Hausdorff and locally quasi-compact topological space  $X$ , more precisely for  $X \in \text{Ob}(LC_{qc})$ , we denote the induced morphism

$$\epsilon_X : LC_{qc}/X \longrightarrow LC_{Zar}/X$$

(see Sites, Lemma 27.1). Let  $X_{Zar}$  be the site whose objects are opens of  $X$ , see Sites, Example 6.4. There is a morphism of sites

$$\pi_X : LC_{Zar}/X \longrightarrow X_{Zar}$$

given by the continuous functor  $X_{Zar} \rightarrow LC_{Zar}/X$ ,  $U \mapsto U$ . Namely,  $X_{Zar}$  has fibre products and a final object and the functor above commutes with these and Sites, Proposition 14.6 applies. We often think of  $\pi$  as a morphism of topoi

$$\pi_X : Sh(LC_{Zar}/X) \longrightarrow Sh(X)$$

using the equality  $Sh(X_{Zar}) = Sh(X)$ .

09X3 **Lemma 25.6.** *Let  $X$  be an object of  $LC_{qc}$ . Let  $\mathcal{F}$  be a sheaf on  $X$ . The rule*

$$LC_{qc}/X \longrightarrow \text{Sets}, \quad (f : Y \rightarrow X) \longmapsto \Gamma(Y, f^{-1}\mathcal{F})$$

*is a sheaf and a fortiori also a sheaf on  $LC_{Zar}/X$ . This sheaf is equal to  $\pi_X^{-1}\mathcal{F}$  on  $LC_{Zar}/X$  and  $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$  on  $LC_{qc}/X$ .*

**Proof.** Denote  $\mathcal{G}$  the presheaf given by the formula in the lemma. Of course the pullback  $f^{-1}$  in the formula denotes usual pullback of sheaves on topological spaces. It is immediate from the definitions that  $\mathcal{G}$  is a sheaf for the Zar topology.

Let  $Y \rightarrow X$  be a morphism in  $LC_{qc}$ . Let  $\mathcal{V} = \{g_i : Y_i \rightarrow Y\}_{i \in I}$  be a qc covering. To prove  $\mathcal{G}$  is a sheaf for the qc topology it suffices to show that  $\mathcal{G}(Y) \rightarrow H^0(\mathcal{V}, \mathcal{G})$  is an isomorphism, see Sites, Section 10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use



Sheaves, Lemmas 11.1 and 21.4). Thus  $\mathcal{G}$  is a separated presheaf on  $LC_{qc}$  hence it suffices to show that any element  $(s_i) \in H^0(\mathcal{V}, \mathcal{G})$  maps to an element in the image of  $\mathcal{G}(Y)$  after replacing  $\mathcal{V}$  by a refinement (Sites, Theorem 10.10).

Identifying sheaves on  $Y_{i,Zar}$  and sheaves on  $Y_i$  we find that  $\mathcal{G}|_{Y_{i,Zar}}$  is the pullback of  $f^{-1}\mathcal{F}$  under the continuous map  $g_i : Y_i \rightarrow Y$ . Thus we can choose an open covering  $Y_i = \bigcup V_{ij}$  such that for each  $j$  there is an open  $W_{ij} \subset Y$  and a section  $t_{ij} \in \mathcal{G}(W_{ij})$  such that  $V_{ij}$  maps into  $W_{ij}$  and such that  $s|_{V_{ij}}$  is the pullback of  $t_{ij}$ . In other words, after refining the covering  $\{Y_i \rightarrow Y\}$  we may assume there are opens  $W_i \subset Y$  such that  $Y_i \rightarrow Y$  factors through  $W_i$  and sections  $t_i$  of  $\mathcal{G}$  over  $W_i$  which restrict to the given sections  $s_i$ . Moreover, if  $y \in Y$  is in the image of both  $Y_i \rightarrow Y$  and  $Y_j \rightarrow Y$ , then the images  $t_{i,y}$  and  $t_{j,y}$  in the stalk  $f^{-1}\mathcal{F}_y$  agree (because  $s_i$  and  $s_j$  agree over  $Y_i \times_Y Y_j$ ). Thus for  $y \in Y$  there is a well defined element  $t_y$  of  $f^{-1}\mathcal{F}_y$  agreeing with  $t_{i,y}$  whenever  $y$  is in the image of  $Y_i \rightarrow Y$ . We will show that the element  $(t_y)$  comes from a global section of  $f^{-1}\mathcal{F}$  over  $Y$  which will finish the proof of the lemma.

It suffices to show that this is true locally on  $Y$ , see Sheaves, Section 17. Let  $y_0 \in Y$ . Pick  $i_1, \dots, i_n \in I$  and quasi-compact subsets  $E_j \subset Y_{i_j}$  such that  $\bigcup g_{i_j}(E_j)$  is a neighbourhood of  $y_0$ . Let  $V \subset Y$  be an open neighbourhood of  $y_0$  contained in  $\bigcup g_{i_j}(E_j)$  and contained in  $W_{i_1} \cap \dots \cap W_{i_n}$ . Since  $t_{i_1,y_0} = \dots = t_{i_n,y_0}$ , after shrinking  $V$  we may assume the sections  $t_{i_j}|_V$ ,  $j = 1, \dots, n$  of  $f^{-1}\mathcal{F}$  agree. As  $V \subset \bigcup g_{i_j}(E_j)$  we see that  $(t_y)_{y \in V}$  comes from this section.

We still have to show that  $\mathcal{G}$  is equal to  $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$  on  $LC_{qc}$ , resp.  $\pi_X^{-1}\mathcal{F}$  on  $LC_{Zar}$ . In both cases the pullback is defined by taking the presheaf

$$(f : Y \rightarrow X) \mapsto \operatorname{colim}_{f(Y) \subset U \subset X} \mathcal{F}(U)$$

and then sheafifying. Sheafifying in the Zar topology exactly produces our sheaf  $\mathcal{G}$  and the fact that  $\mathcal{G}$  is a qc sheaf, shows that it works as well in the qc topology.  $\square$

Let  $X \in \operatorname{Ob}(LC_{Zar})$  and let  $\mathcal{H}$  be an abelian sheaf on  $LC_{Zar}/X$ . Then we will write  $H_{Zar}^n(U, \mathcal{H})$  for the cohomology of  $\mathcal{H}$  over an object  $U$  of  $LC_{Zar}/X$ .

0DCU **Lemma 25.7.** *Let  $X$  be an object of  $LC_{Zar}$ . Then*

- (1) *for  $\mathcal{F} \in \operatorname{Ab}(X)$  we have  $H_{Zar}^n(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$ ,*
- (2)  *$\pi_{X,*} : \operatorname{Ab}(LC_{Zar}/X) \rightarrow \operatorname{Ab}(X)$  is exact,*
- (3) *the unit  $\operatorname{id} \rightarrow \pi_{X,*} \circ \pi_X^{-1}$  of the adjunction is an isomorphism, and*
- (4) *for  $K \in D(X)$  the canonical map  $K \rightarrow R\pi_{X,*}\pi_X^{-1}K$  is an isomorphism.*

*Let  $f : X \rightarrow Y$  be a morphism of  $LC_{Zar}$ . Then*

- (5) *there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{Sh}(LC_{Zar}/X) & \xrightarrow{f_{Zar}} & \operatorname{Sh}(LC_{Zar}/Y) \\ \pi_X \downarrow & & \downarrow \pi_Y \\ \operatorname{Sh}(X_{Zar}) & \xrightarrow{f} & \operatorname{Sh}(Y_{Zar}) \end{array}$$

*of topoi,*

- (6) *for  $L \in D^+(Y)$  we have  $H_{Zar}^n(X, \pi_Y^{-1}L) = H^n(X, f^{-1}L)$ ,*
- (7) *if  $f$  is proper, then we have*
  - (a)  *$\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$  as functors  $\operatorname{Sh}(X) \rightarrow \operatorname{Sh}(LC_{Zar}/Y)$ ,*

(b)  $\pi_Y^{-1} \circ Rf_* = Rf_{Zar,*} \circ \pi_X^{-1}$  as functors  $D^+(X) \rightarrow D^+(LC_{Zar}/Y)$ .

**Proof.** Proof of (1). The equality  $H_{Zar}^n(X, \pi_X^{-1}\mathcal{F}) = H^q(X, \mathcal{F})$  is a general fact coming from the trivial observation that coverings of  $X$  in  $LC_{Zar}$  are the same thing as open coverings of  $X$ . The reader who wishes to see a detailed proof should apply Lemma 8.2 to the functor  $X_{Zar} \rightarrow LC_{Zar}$ .

Proof of (2). This is true because  $\pi_{X,*} = \tau_X^{-1}$  for some morphism of topoi  $\tau_X : Sh(X_{Zar}) \rightarrow Sh(LC_{Zar})$  as follows from Sites, Lemma 20.8 applied to the functor  $X_{Zar} \rightarrow LC_{Zar}/X$  used to define  $\pi_X$ .

Proof of (3). This is true because  $\tau_X^{-1} \circ \pi_X^{-1}$  is the identity functor by Sites, Lemma 20.8. Or you can deduce it from the explicit description of  $\pi_X^{-1}$  in Lemma 25.6.

Proof of (4). Apply (3) to an complex of abelian sheaves representing  $K$ .

Proof of (5). The morphism of topoi  $f_{Zar}$  comes from an application of Sites, Lemma 24.8 and in our case comes from the continuous functor  $Z/Y \mapsto Z \times_Y X/X$  by Sites, Lemma 26.3. The diagram commutes simply because the corresponding continuous functors compose correctly (see Sites, Lemma 14.3).

Proof of (6). We have  $H_{Zar}^n(X, \pi_Y^{-1}\mathcal{G}) = H_{Zar}^n(X, f_{Zar}^{-1}\pi_Y^{-1}\mathcal{G})$  for  $\mathcal{G}$  in  $Ab(Y)$ , see Lemma 8.1. This is equal to  $H_{Zar}^n(X, \pi_X^{-1}f^{-1}\mathcal{G})$  by the commutativity of the diagram in (5). Hence we conclude by (1) in the case  $L$  consists of a single sheaf in degree 0. The general case follows by representing  $L$  by a bounded below complex of abelian sheaves.

Proof of (7a). Let  $\mathcal{F}$  be a sheaf on  $X$ . Let  $g : Z \rightarrow Y$  be an object of  $LC_{Zar}/Y$ . Consider the fibre product

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Then we have

$$(f_{Zar,*}\pi_X^{-1}\mathcal{F})(Z/Y) = (\pi_X^{-1}\mathcal{F})(Z'/X) = \Gamma(Z', (g')^{-1}\mathcal{F}) = \Gamma(Z, f'_*(g')^{-1}\mathcal{F})$$

the second equality by Lemma 25.6. On the other hand

$$(\pi_Y^{-1}f_*\mathcal{F})(Z/Y) = \Gamma(Z, g^{-1}f_*\mathcal{F})$$

again by Lemma 25.6. Hence by proper base change for sheaves of sets (Cohomology, Lemma 19.3) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism  $\pi_Y^{-1}f_*\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$ . Thus an isomorphism of functors  $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$ .

Proof of (7b). Let  $K \in D^+(X)$ . By Lemma 21.6 the  $n$ th cohomology sheaf of  $Rf_{Zar,*}\pi_X^{-1}K$  is the sheaf associated to the presheaf

$$(g : Z \rightarrow Y) \mapsto H_{Zar}^n(Z', \pi_X^{-1}K)$$

with notation as above. Observe that

$$\begin{aligned} H_{Zar}^n(Z', \pi_X^{-1}K) &= H^n(Z', (g')^{-1}K) \\ &= H^n(Z, Rf'_*(g')^{-1}K) \\ &= H^n(Z, g^{-1}Rf_*K) \\ &= H_{Zar}^n(Z, \pi_Y^{-1}Rf_*K) \end{aligned}$$

The first equality is (6) applied to  $K$  and  $g' : Z' \rightarrow X$ . The second equality is Leray for  $f' : Z' \rightarrow Z$  (Cohomology, Lemma 14.1). The third equality is the proper base change theorem (Cohomology, Theorem 19.2). The fourth equality is (6) applied to  $g : Z \rightarrow Y$  and  $Rf_*K$ . Thus  $Rf_{Zar,*}\pi_X^{-1}K$  and  $\pi_Y^{-1}Rf_*K$  have the same cohomology sheaves. We omit the verification that the canonical base change map  $\pi_Y^{-1}Rf_*K \rightarrow Rf_{Zar,*}\pi_X^{-1}K$  induces this isomorphism.  $\square$

In the situation of Lemma 25.6 the composition of  $\epsilon$  and  $\pi$  and the equality  $Sh(X) = Sh(X_{Zar})$  determine a morphism of topoi

$$a_X : Sh(LC_{qc}/X) \longrightarrow Sh(X)$$

0D92 **Lemma 25.8.** *Let  $f : X \rightarrow Y$  be a morphism of  $LC_{qc}$ . Then there are commutative diagrams of topoi*

$$\begin{array}{ccc} Sh(LC_{qc}/X) & \xrightarrow{f_{qc}} & Sh(LC_{qc}/Y) \\ \epsilon_X \downarrow & & \downarrow \epsilon_Y \\ Sh(LC_{Zar}/X) & \xrightarrow{f_{Zar}} & Sh(LC_{Zar}/Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} Sh(LC_{qc}/X) & \xrightarrow{f_{qc}} & Sh(LC_{qc}/Y) \\ a_X \downarrow & & \downarrow a_Y \\ Sh(X) & \xrightarrow{f} & Sh(Y) \end{array}$$

with  $a_X = \pi_X \circ \epsilon_X$ ,  $a_Y = \pi_Y \circ \epsilon_Y$ . If  $f$  is proper, then  $a_Y^{-1} \circ f_* = f_{qc,*} \circ a_X^{-1}$ .

**Proof.** The morphism of topoi  $f_{qc}$  is the one from Sites, Lemma 24.8 which in our case comes from the continuous functor  $Z/Y \mapsto Z \times_Y X/X$ , see Sites, Lemma 26.3. The diagram on the left commutes because the corresponding continuous functors compose correctly (see Sites, Lemma 14.3). The diagram on the right commutes because the one on the left does and because of part (5) of Lemma 25.7.

Proof of the final assertion. The reader may repeat the proof of part (7a) of Lemma 25.7; we will instead deduce this from it. As  $\epsilon_{Y,*}$  is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 25.6 shows that  $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$  and similarly for  $X$ . To show that the canonical map  $a_Y^{-1}f_*\mathcal{F} \rightarrow f_{qc,*}a_X^{-1}\mathcal{F}$  is an isomorphism, it suffices to show that

$$\pi_Y^{-1}f_*\mathcal{F} = \epsilon_{Y,*}a_Y^{-1}f_*\mathcal{F} \rightarrow \epsilon_{Y,*}f_{qc,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$$

is an isomorphism. This is part (7a) of Lemma 25.7.  $\square$

Our next goal is to prove Lemmas 25.14 and 25.13. We will do this by an induction argument using the following two statements

( $V_n$ ) For  $X \in \text{Ob}(LC_{qc})$  and abelian sheaf  $\mathcal{F}$  on  $X$  we have  $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$  and  $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$  for  $1 \leq i \leq n$ .

( $C_n$ ) For a proper morphism  $f : X \rightarrow Y$  in  $LC_{qc}$  and abelian sheaf  $\mathcal{F}$  on  $X$  we have  $a_Y^{-1}(R^if_*\mathcal{F}) = R^if_{qc,*}(a_X^{-1}\mathcal{F})$  for  $i = 0, \dots, n$ .

Observe that  $V_0$  holds by Lemma 25.6 and that  $C_0$  holds by Lemma 25.8.

0DCV **Lemma 25.9.** *With notation as above, we have  $V_n + C_{n-1} \Rightarrow C_n$ .*

**Proof.** Let  $f : X \rightarrow Y$  and  $\mathcal{F}$  be as in  $C_n$ . We will use the commutative diagrams of Lemma 25.8 without further mention. In particular have

$$R\epsilon_{Y,*}(Rf_{qc,*}(a_X^{-1}\mathcal{F})) = Rf_{Zar,*}(R\epsilon_{X,*}(a_X^{-1}\mathcal{F}))$$

Using the spectral sequences of Lemma 15.7 on both sides, the description of  $R^i f_{qc,*} a_X^{-1}\mathcal{F}$  for  $i \leq n-1$  given to us by  $C_{n-1}$ , and the vanishing furnished to us by  $V_n$  (for both  $X$  and  $Y$ ) we obtain

$$\epsilon_{Y,*}(R^n f_{qc,*}(a_X^{-1}\mathcal{F})) = R^n f_{Zar,*}(\epsilon_{X,*}(a_X^{-1}\mathcal{F}))$$

The sheaf on the right is equal to  $R^n f_{Zar,*}(\pi_X^{-1}\mathcal{F})$  by  $V_0$ . This is equal to  $\pi_Y^{-1}R^n f_*\mathcal{F}$  by Lemma 25.7 part (7b). If we rewrite this as  $\epsilon_{Y,*}(a_Y^{-1}R^n f_*\mathcal{F})$  then we get

$$\epsilon_{Y,*}(R^n f_{qc,*}(a_X^{-1}\mathcal{F})) = \epsilon_{Y,*}(a_Y^{-1}R^n f_*\mathcal{F})$$

This finishes the proof as  $\epsilon_{Y,*}$  reflects isomorphisms (see proof of Lemma 25.8).  $\square$

If  $\mathcal{H}$  is an abelian sheaf on  $LC_{qc}/X$ , then we will write  $H_{qc}^n(U, \mathcal{H})$  for the cohomology of  $\mathcal{H}$  over an object  $U$  of  $LC_{qc}/X$ .

0DCW **Lemma 25.10.** *With notation as above, if  $V_n$  holds, then for  $X \in \text{Ob}(LC_{qc})$  and  $L \in D(X)$  with  $H^i(L) = 0$  for  $i < 0$  we have  $H^n(X, L) = H_{qc}^n(X, a_X^{-1}L)$*

**Proof.** We have  $H^n(X, L) = H_{Zar}^n(X, \pi_X^{-1}L)$  by Lemma 25.7. By Lemma 21.5 we have  $H_{qc}^n(X, a_X^{-1}L) = H_{Zar}^n(X, R\epsilon_{X,*}a_X^{-1}L)$ . Represent  $L$  by a complex  $\mathcal{F}^\bullet$  of abelian sheaves on  $X$  with  $\mathcal{F}^i = 0$  for  $i < 0$ . We have  $\epsilon_{X,*}a_X^{-1}\mathcal{F}^\bullet = \pi_X^{-1}\mathcal{F}^\bullet$  by  $V_0$ . Looking at the first spectral sequence of Derived Categories, Lemma 21.3 the vanishing of  $R^i \epsilon_{X,*}\mathcal{F}^j$  for  $i+j \leq n$  and  $i > 0$  implies that  $\tau_{\leq n}\pi_X^{-1}L \rightarrow \tau_{\leq n}R\epsilon_{X,*}a_X^{-1}L$  is an isomorphism. Combining the above gives the lemma as cohomology in degree  $n$  only depends on the truncation  $\tau_{\leq n}$  (Derived Categories, Lemma 17.1).  $\square$

0DCX **Lemma 25.11.** *With notation as above, we have  $V_n + C_n \Rightarrow V_{n+1}$ .*

**Proof.** Let  $X \in \text{Ob}(LC_{qc})$  and  $\mathcal{F} \in \text{Ob}(Ab(X))$ . Let  $\xi \in H_{qc}^{n+1}(U, a_X^{-1}\mathcal{F})$  for some  $U/X$ . We have to show that  $\xi$  restricts to zero on the members of an open covering of  $U$ . See Lemma 8.4. Pick  $u \in U$ . We have to find an open neighbourhood  $u \in V \subset U$  such that  $\xi$  maps to zero in  $H_{qc}^{n+1}(V, a_X^{-1}\mathcal{F})$ .

By locality of cohomology (Lemma 8.3) we can choose a qc covering  $\{U_i \rightarrow U\}$  such that  $\xi$  restricts to zero on  $U_i$ . Pick  $i_1, \dots, i_m \in I$  and quasi-compact subsets  $E_j \subset U_{i_j}$  such that  $\bigcup f_{i_j}(E_j)$  is a neighbourhood of  $x$ . Observe that  $Y = \coprod_{j=1, \dots, m} E_j \rightarrow U$  is proper as a continuous map from a quasi-compact space to a Hausdorff one (Topology, Lemma 17.7). Observe that  $Y = \coprod E_j$  is a disjoint union (in the sense that  $E_j \times_Y E_{j'}$  is empty if  $j \neq j'$ ) and hence  $\xi$  restricts to zero on  $Y$ . Choose an open neighbourhood  $u \in V$  contained in  $\bigcup f_{i_j}(E_j)$ . Then  $Y \times_U V \rightarrow V$  is surjective and proper and the restriction of  $\xi$  to  $Y \times_U V$  is zero. This observation, changing notation, and combining with Lemma 8.1 reduces us to the problem described in the next paragraph.

Let  $f : X \rightarrow Y$  be a surjective proper morphism. Let  $\mathcal{F}$  be an abelian sheaf on  $Y$ . Let  $\xi \in H_{qc}^{n+1}(Y, a_Y^{-1}\mathcal{F})$  which maps to zero in  $H_{qc}^{n+1}(X, a_X^{-1}\mathcal{F})$ . Problem: show that  $\xi$  restricts to zero on an open covering of  $Y$ . The assumption implies that  $\xi$  maps to zero under the map

$$a_Y^{-1}\mathcal{F} \longrightarrow Rf_{qc,*}f_{qc}^{-1}a_Y^{-1}\mathcal{F} = Rf_{qc,*}a_X^{-1}f^{-1}\mathcal{F}$$

Use Lemma 21.5. A simple argument using the distinguished triangle of truncations (Derived Categories, Remark 12.4) shows that  $\xi$  maps to zero under the map

$$a_Y^{-1}\mathcal{F} \longrightarrow \tau_{\leq n}Rf_{qc,*}a_X^{-1}f^{-1}\mathcal{F}$$

Set  $K = \tau_{\leq n}Rf_*f^{-1}\mathcal{F}$  in  $D^+(Y)$  and choose a distinguished triangle  $\mathcal{F} \rightarrow K \rightarrow L$ . Note that  $\mathcal{F} \rightarrow f_*f^{-1}\mathcal{F}$  is injective as  $f : X \rightarrow Y$  is surjective, hence  $L$  only has nonzero cohomology sheaves in degrees  $0, \dots, n$ . By  $C_n$  the canonical map  $a_Y^{-1}Rf_*f^{-1}\mathcal{F} \rightarrow Rf_{qc,*}a_X^{-1}f^{-1}\mathcal{F}$  induces an isomorphism

$$a_Y^{-1}K \longrightarrow \tau_{\leq n}Rf_{qc,*}a_X^{-1}f^{-1}\mathcal{F}$$

Pulling back the distinguished triangle  $\mathcal{F} \rightarrow K \rightarrow L$  by  $a_Y$  we find that  $\xi$  is the image of an element  $\xi' \in H_{qc}^n(Y, a_Y^{-1}L)$ . By induction, more precisely by  $V_n$ , we have

$$H^n(Y, L) = H_{qc}^n(Y, a_Y^{-1}L)$$

see Lemma 25.10. Thus we may lift  $\xi'$  to an element of  $H^n(Y, L)$  and take the boundary into  $H^{n+1}(Y, \mathcal{F})$  to see that  $\xi$  is in the image of the canonical map  $H^{n+1}(Y, \mathcal{F}) \rightarrow H_{qc}^{n+1}(Y, a_Y^{-1}\mathcal{F})$ . By locality of cohomology for  $H^{n+1}(Y, \mathcal{F})$ , see Cohomology, Lemma 8.2, we conclude.  $\square$

0DCY **Lemma 25.12.** *With notation as above,  $V_n$  and  $C_n$  are true for all  $n$ . Moreover:*

- (1) *For  $X \in \text{Ob}(LC_{qc})$  and  $K$  in  $D^+(X)$  the map  $\pi_X^{-1}K \rightarrow R\epsilon_{X,*}(a_X^{-1}K)$  is an isomorphism.*
- (2) *For a proper morphism  $f : X \rightarrow Y$  in  $LC_{qc}$  and  $K$  in  $D^+(X)$  we have  $a_Y^{-1}(Rf_*K) = Rf_{qc,*}(a_X^{-1}K)$ .*

**Proof.** Recall that  $V_0$  holds by Lemma 25.6 and that  $C_0$  holds by Lemma 25.8. Then we get  $V_n$  and  $C_n$  for all  $n$  by Lemmas 25.9 and 25.11. Finally, the reader immediately proves (1) and (2) from  $V_n$  and  $C_n$  by representing  $K$  by a bounded below complex of abelian sheaves. We omit the details.  $\square$

0D91 **Lemma 25.13.** *Let  $X$  be an object of  $LC_{qc}$ . For  $K \in D^+(X)$  the map*

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

*is an isomorphism with  $a_X : Sh(LC_{qc}/X) \rightarrow Sh(X)$  as above.*

**Proof.** This statement has an immediate reduction to the case where  $K$  is given by a single abelian sheaf. Namely, represent  $K$  by a bounded below complex  $\mathcal{F}^\bullet$ . By the case of a sheaf we see that  $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$  and that the sheaves  $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$  are zero for  $q > 0$ . By Leray's acyclicity lemma (Derived Categories, Lemma 17.7) applied to  $a_X^{-1}\mathcal{F}^\bullet$  and the functor  $a_{X,*}$  we conclude. From now on assume  $K = \mathcal{F}$ .

By Lemma 25.6 we have  $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$ . Thus it suffices to show that  $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$  for  $q > 0$ . For this we can use  $a_X = \epsilon_X \circ \pi_X$  and the Leray spectral sequence Lemma 15.7. By Lemma 25.12 we have  $R^i \epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$  for  $i > 0$ . By  $V_0$  we have  $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$  and by Lemma 25.7  $R^j \pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$  for  $j > 0$ . This concludes the proof.  $\square$

09X4 **Lemma 25.14.** *With  $X \in \text{Ob}(LC_{qc})$  and  $a_X : Sh(LC_{qc}/X) \rightarrow Sh(X)$  as above:*

- (1) *for an abelian sheaf  $\mathcal{F}$  on  $X$  we have  $H^n(X, \mathcal{F}) = H_{qc}^n(X, a_X^{-1}\mathcal{F})$ ,*
- (2) *for  $K \in D^+(X)$  we have  $H^n(X, K) = H_{qc}^n(X, a_X^{-1}K)$ .*

*For example, if  $A$  is an abelian group, then we have  $H^n(X, \underline{A}) = H_{qc}^n(X, \underline{A})$ .*

**Proof.** This follows from Lemma 25.13 by Remark 15.4.  $\square$

## 26. Spectral sequences for Ext

07A9 In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes  $\mathcal{G}^\bullet, \mathcal{F}^\bullet$  of complexes of modules on a ringed site  $(\mathcal{C}, \mathcal{O})$  we denote

$$\mathrm{Ext}_{\mathcal{O}}^n(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 27.

07AA **Example 26.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{K}^\bullet$  be a bounded above complex of  $\mathcal{O}$ -modules. Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module. Then there is a spectral sequence with  $E_2$ -page

$$E_2^{i,j} = \mathrm{Ext}_{\mathcal{O}}^i(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}_{\mathcal{O}}^{i+j}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with  $E_1$ -page

$$E_1^{i,j} = \mathrm{Ext}_{\mathcal{O}}^j(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}_{\mathcal{O}}^{i+j}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  and consider the two spectral sequences coming from the double complex  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ , see Homology, Section 22.

## 27. Hom complexes

0A8X Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{L}^\bullet$  and  $\mathcal{M}^\bullet$  be two complexes of  $\mathcal{O}$ -modules. We construct a complex of  $\mathcal{O}$ -modules  $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$ . Namely, for each  $n$  we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of  $\mathcal{H}om^n$  as the sheaf of  $\mathcal{O}$ -modules of all  $\mathcal{O}$ -linear maps from  $\mathcal{L}^\bullet$  to  $\mathcal{M}^\bullet$  (viewed as graded  $\mathcal{O}$ -modules) which are homogenous of degree  $n$ . In this terminology, we define the differential by the rule

$$d(f) = d_{\mathcal{M}} \circ f - (-1)^n f \circ d_{\mathcal{L}}$$

for  $f \in \mathcal{H}om_{\mathcal{O}}^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$ . We omit the verification that  $d^2 = 0$ . This construction is a special case of Differential Graded Algebra, Example 19.6. It follows immediately from the construction that we have

0A8Y (27.0.1)  $H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$

for all  $n \in \mathbf{Z}$  and every  $U \in \mathrm{Ob}(\mathcal{C})$ . Similarly, we have

0A8Z (27.0.2)  $H^n(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$

for the complex of global sections.

0A90 **Lemma 27.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given complexes  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules there is an isomorphism*

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of  $\mathcal{O}$ -modules functorial in  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ .

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 65.1.  $\square$

0A91 **Lemma 27.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given complexes  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules there is a canonical morphism*

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

*of complexes of  $\mathcal{O}$ -modules.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 65.2.  $\square$

0A92 **Lemma 27.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given complexes  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules there is a canonical morphism*

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

*of complexes of  $\mathcal{O}$ -modules functorial in all three complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 65.3.  $\square$

0BYT **Lemma 27.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given complexes  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules there is a canonical morphism*

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet))$$

*of complexes of  $\mathcal{O}$ -modules functorial in all three complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 65.5.  $\square$

0A93 **Lemma 27.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given complexes  $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$  of  $\mathcal{O}$ -modules there is a canonical morphism*

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet))$$

*of complexes of  $\mathcal{O}$ -modules functorial in both complexes.*

**Proof.** Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 65.6.  $\square$

0A94 **Lemma 27.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{I}^\bullet$  be a  $K$ -injective complex of  $\mathcal{O}$ -modules. Let  $\mathcal{L}^\bullet$  be a complex of  $\mathcal{O}$ -modules. Then*

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

*for all  $U \in \mathrm{Ob}(\mathcal{C})$ . Similarly,  $H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L, M)$ .*

**Proof.** We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (27.0.1). The second equality is true because  $\mathcal{I}^\bullet|_U$  is  $K$ -injective by Lemma 21.1. The proof of the last equation is similar except that it uses (27.0.2).  $\square$

0A95 **Lemma 27.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(\mathcal{I}')^\bullet \rightarrow \mathcal{I}^\bullet$  be a quasi-isomorphism of  $K$ -injective complexes of  $\mathcal{O}$ -modules. Let  $(\mathcal{L}')^\bullet \rightarrow \mathcal{L}^\bullet$  be a quasi-isomorphism of complexes of  $\mathcal{O}$ -modules. Then*

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

*is a quasi-isomorphism.*

**Proof.** Let  $M$  be the object of  $D(\mathcal{O})$  represented by  $\mathcal{I}^\bullet$  and  $(\mathcal{I}')^\bullet$ . Let  $L$  be the object of  $D(\mathcal{O})$  represented by  $\mathcal{L}^\bullet$  and  $(\mathcal{L}')^\bullet$ . By Lemma 27.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \text{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism.  $\square$

0A96 **Lemma 27.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{I}^\bullet$  be a K-injective complex of  $\mathcal{O}$ -modules. Let  $\mathcal{L}^\bullet$  be a K-flat complex of  $\mathcal{O}$ -modules. Then  $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$  is a K-injective complex of  $\mathcal{O}$ -modules.*

**Proof.** Namely, if  $\mathcal{K}^\bullet$  is an acyclic complex of  $\mathcal{O}$ -modules, then

$$\begin{aligned} \text{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\ &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\ &= \text{Hom}_{K(\mathcal{O})}(\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\ &= 0 \end{aligned}$$

The first equality by (27.0.2). The second equality by Lemma 27.1. The third equality by (27.0.2). The final equality because  $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$  is acyclic because  $\mathcal{L}^\bullet$  is K-flat (Definition 18.2) and because  $\mathcal{I}^\bullet$  is K-injective.  $\square$

## 28. Internal hom in the derived category

08J7 Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $L, M$  be objects of  $D(\mathcal{O})$ . We would like to construct an object  $R\mathcal{H}om(L, M)$  of  $D(\mathcal{O})$  such that for every third object  $K$  of  $D(\mathcal{O})$  there exists a canonical bijection

$$08J8 \quad (28.0.1) \quad \text{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \text{Hom}_{D(\mathcal{O})}(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$$

Observe that this formula defines  $R\mathcal{H}om(L, M)$  up to unique isomorphism by the Yoneda lemma (Categories, Lemma 3.5).

To construct such an object, choose a K-injective complex of  $\mathcal{O}$ -modules  $\mathcal{I}^\bullet$  representing  $M$  and any complex of  $\mathcal{O}$ -modules  $\mathcal{L}^\bullet$  representing  $L$ . Then we set Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of  $\mathcal{O}$ -modules constructed in Section 27. This is well defined by Lemma 27.7. We get a functor

$$D(\mathcal{O})^{opp} \times D(\mathcal{O}) \longrightarrow D(\mathcal{O}), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (28.0.1) we compute the cohomology groups of  $R\mathcal{H}om(K, L)$ .

08JA **Lemma 28.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L$  be objects of  $D(\mathcal{O})$ . For every object  $U$  of  $\mathcal{C}$  we have*

$$H^0(U, R\mathcal{H}om(L, M)) = \text{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and we have  $H^0(\mathcal{C}, R\mathcal{H}om(L, M)) = \text{Hom}_{D(\mathcal{O})}(L, M)$ .



**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  of  $\mathcal{O}$ -modules representing  $M$  and a K-flat complex  $\mathcal{L}^\bullet$  representing  $L$ . Then  $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$  is K-injective by Lemma 27.8. Hence we can compute cohomology over  $U$  by simply taking sections over  $U$  and the result follows from Lemma 27.6.  $\square$

08J9 **Lemma 28.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L, M$  be objects of  $D(\mathcal{O})$ . With the construction as described above there is a canonical isomorphism*

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$$

*in  $D(\mathcal{O})$  functorial in  $K, L, M$  which recovers (28.0.1) on taking  $H^0(\mathcal{C}, -)$ .*

**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $M$  and a K-flat complex of  $\mathcal{O}$ -modules  $\mathcal{L}^\bullet$  representing  $L$ . For any complex of  $\mathcal{O}$ -modules  $\mathcal{K}^\bullet$  we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 27.1. Note that the left hand side represents  $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$  (use Lemma 27.8) and that the right hand side represents  $R\mathcal{H}om(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$ . This proves the displayed formula of the lemma. Taking global sections and using Lemma 28.1 we obtain (28.0.1).  $\square$

08JB **Lemma 28.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L$  be objects of  $D(\mathcal{O})$ . The construction of  $R\mathcal{H}om(K, L)$  commutes with restrictions, i.e., for every object  $U$  of  $\mathcal{C}$  we have  $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$ .*

**Proof.** This is clear from the construction and Lemma 21.1.  $\square$

08JC **Lemma 28.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. The bifunctor  $R\mathcal{H}om(-, -)$  transforms distinguished triangles into distinguished triangles in both variables.*

**Proof.** This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted.  $\square$

0A97 **Lemma 28.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L, M$  be objects of  $D(\mathcal{O})$ . There is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

*in  $D(\mathcal{O})$  functorial in  $K, L, M$ .*

**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $M$ , a K-injective complex  $\mathcal{J}^\bullet$  representing  $L$ , and a K-flat complex  $\mathcal{K}^\bullet$  representing  $K$ . The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 27.3. By our particular choice of complexes the left hand side represents  $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} K$  and the right hand side represents  $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$ . We omit the proof that this is functorial in all three objects of  $D(\mathcal{O})$ .  $\square$

0A98 **Lemma 28.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given  $K, L, M$  in  $D(\mathcal{O})$  there is a canonical morphism*

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

*in  $D(\mathcal{O})$ .*

**Proof.** Choose a K-injective complex  $\mathcal{I}^\bullet$  representing  $M$ , a K-injective complex  $\mathcal{J}^\bullet$  representing  $L$ , and any complex of  $\mathcal{O}$ -modules  $\mathcal{K}^\bullet$  representing  $K$ . By Lemma 27.2 there is a map of complexes

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of  $\mathcal{O}$ -modules  $\mathrm{Hom}^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$ ,  $\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$ , and  $\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$  represent  $R\mathcal{H}om(L, M)$ ,  $R\mathcal{H}om(K, L)$ , and  $R\mathcal{H}om(K, M)$ . If we choose a K-flat complex  $\mathcal{H}^\bullet$  and a quasi-isomorphism  $\mathcal{H}^\bullet \rightarrow \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$ , then there is a map

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet))$$

whose source represents  $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L)$ . Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial.  $\square$

0BYU **Lemma 28.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given  $K, L, M$  in  $D(\mathcal{O})$  there is a canonical morphism*

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

*in  $D(\mathcal{O})$  functorial in  $K, L, M$ .*

**Proof.** Choose a K-flat complex  $\mathcal{K}^\bullet$  representing  $K$ , and a K-injective complex  $\mathcal{I}^\bullet$  representing  $L$ , and choose any complex of  $\mathcal{O}$ -modules  $\mathcal{M}^\bullet$  representing  $M$ . Choose a quasi-isomorphism  $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \rightarrow \mathcal{J}^\bullet$  where  $\mathcal{J}^\bullet$  is K-injective. Then we use the map

$$\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathrm{Hom}^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)) \rightarrow \mathrm{Hom}^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet)) \rightarrow \mathrm{Hom}^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 27.4.  $\square$

0A99 **Lemma 28.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Given  $K, L$  in  $D(\mathcal{O})$  there is a canonical morphism*

$$K \longrightarrow R\mathcal{H}om(L, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

*in  $D(\mathcal{O})$  functorial in both  $K$  and  $L$ .*

**Proof.** Choose a K-flat complex  $\mathcal{K}^\bullet$  representing  $K$  and any complex of  $\mathcal{O}$ -modules  $\mathcal{L}^\bullet$  representing  $L$ . Choose a K-injective complex  $\mathcal{J}^\bullet$  and a quasi-isomorphism  $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \rightarrow \mathcal{J}^\bullet$ . Then we use

$$\mathcal{K}^\bullet \rightarrow \mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) \rightarrow \mathrm{Hom}^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 27.5.  $\square$

08JD **Lemma 28.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $L$  be an object of  $D(\mathcal{O})$ . Set  $L^\vee = R\mathcal{H}om(L, \mathcal{O})$ . For  $M$  in  $D(\mathcal{O})$  there is a canonical map*

08JE (28.9.1) 
$$L^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M \longrightarrow R\mathcal{H}om(L, M)$$

*which induces a canonical map*

$$H^0(\mathcal{C}, L^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(L, M)$$

*functorial in  $M$  in  $D(\mathcal{O})$ .*

**Proof.** The map (28.9.1) is a special case of Lemma 28.6 using the identification  $M = R\mathcal{H}om(\mathcal{O}, M)$ .  $\square$

0B6D **Remark 28.10.** Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $K, L$  be objects of  $D(\mathcal{O}_{\mathcal{C}})$ . We claim there is a canonical map

$$Rf_* R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_* L, Rf_* K)$$

Namely, by (28.0.1) this is the same thing as a map  $Rf_* R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} Rf_* L \rightarrow Rf_* K$ . For this we can use the composition

$$Rf_* R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} Rf_* L \rightarrow Rf_*(R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} L) \rightarrow Rf_* K$$

where the first arrow is the relative cup product (Remark 20.6) and the second arrow is  $Rf_*$  applied to the canonical map  $R\mathcal{H}om(L, K) \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} L \rightarrow K$  coming from Lemma 28.6 (with  $\mathcal{O}_{\mathcal{C}}$  in one of the spots).

08JF **Remark 28.11.** Let  $h : (Sh(\mathcal{C}), \mathcal{O}) \rightarrow (Sh(\mathcal{C}'), \mathcal{O}')$  be a morphism of ringed topoi. Let  $K, L$  be objects of  $D(\mathcal{O}')$ . We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^* K, Lh^* L)$$

in  $D(\mathcal{O})$ . Namely, by (28.0.1) proved in Lemma 28.2 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^* K \longrightarrow Lh^* L$$

The source of this arrow is  $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$  by Lemma 19.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\text{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (28.0.1).

08JG **Remark 28.12.** Suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{h} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ f' \downarrow & & \downarrow f \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let  $K, L$  be objects of  $D(\mathcal{O}_{\mathcal{C}})$ . We claim there exists a canonical base change map

$$Lg^* Rf_* R\mathcal{H}om(K, L) \longrightarrow R(f')_* R\mathcal{H}om(Lh^* K, Lh^* L)$$

in  $D(\mathcal{O}_{\mathcal{D}'})$ . Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^* Lg^* Rf_* R\mathcal{H}om(K, L) &= Lh^* Lf^* Rf_* R\mathcal{H}om(K, L) \\ &\rightarrow Lh^* R\mathcal{H}om(K, L) \\ &\rightarrow R\mathcal{H}om(Lh^* K, Lh^* L) \end{aligned}$$

where the first arrow uses the adjunction mapping  $Lf^* Rf_* \rightarrow \text{id}$  and the second arrow is the canonical map constructed in Remark 28.11.

### 29. Global derived hom

0B6E Let  $(Sh(\mathcal{C}), \mathcal{O})$  be a ringed topos. Let  $K, L \in D(\mathcal{O})$ . Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\mathrm{Hom}_{\mathcal{O}}(K, L) = R\Gamma(X, R\mathcal{H}om(K, L))$$

in  $D(\Gamma(\mathcal{C}, \mathcal{O}))$ . By Lemma 28.1 we have

$$H^0(R\mathrm{Hom}_{\mathcal{O}}(K, L)) = \mathrm{Hom}_{D(\mathcal{O})}(K, L)$$

and

$$H^p(R\mathrm{Hom}_{\mathcal{O}}(K, L)) = \mathrm{Ext}_{D(\mathcal{O})}^p(K, L)$$

If  $f : (\mathcal{C}', \mathcal{O}') \rightarrow (\mathcal{C}, \mathcal{O})$  is a morphism of ringed topoi, then there is a canonical map

$$R\mathrm{Hom}_{\mathcal{O}}(K, L) \longrightarrow R\mathrm{Hom}_{\mathcal{O}'}(Lf^*K, Lf^*L)$$

in  $D(\Gamma(\mathcal{O}))$  by taking global sections of the map defined in Remark 28.11.

### 30. Derived lower shriek

07AB In this section we study morphisms  $g$  of ringed topoi where besides  $Lg^*$  and  $Rg_*$  there also a derived functor  $Lg_!$ .

0D6X **Lemma 30.1.** *Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a continuous and cocontinuous functor of sites. Let  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be the corresponding morphism of topoi. Let  $\mathcal{O}_{\mathcal{D}}$  be a sheaf of rings and let  $\mathcal{I}$  be an injective  $\mathcal{O}_{\mathcal{D}}$ -module. Then  $H^p(U, g^{-1}\mathcal{I}) = 0$  for all  $p > 0$  and  $U \in \mathrm{Ob}(\mathcal{C})$ .*

**Proof.** The vanishing of the lemma follows from Lemma 11.9 if we can prove vanishing of all higher Čech cohomology groups  $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$  for any covering  $\mathcal{U} = \{U_i \rightarrow U\}$  of  $\mathcal{C}$ . Since  $u$  is continuous,  $u(\mathcal{U}) = \{u(U_i) \rightarrow u(U)\}$  is a covering of  $\mathcal{D}$ , and  $u(U_{i_0} \times_U \dots \times_U U_{i_n}) = u(U_{i_0}) \times_{u(U)} \dots \times_{u(U)} u(U_{i_n})$ . Thus we have

$$\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$$

because  $g^{-1} = u^p$  by Sites, Lemma 20.5. Since  $\mathcal{I}$  is an injective  $\mathcal{O}$ -module these Čech cohomology groups vanish, see Lemma 13.3.  $\square$

07AC **Lemma 30.2.** *Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a continuous and cocontinuous functor of sites. Let  $g : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  be the corresponding morphism of topoi. Let  $\mathcal{O}_{\mathcal{D}}$  be a sheaf of rings and set  $\mathcal{O}_{\mathcal{C}} = g^{-1}\mathcal{O}_{\mathcal{D}}$ . The functor  $g_! : \mathrm{Mod}(\mathcal{O}_{\mathcal{C}}) \rightarrow \mathrm{Mod}(\mathcal{O}_{\mathcal{D}})$  (see Modules on Sites, Lemma 40.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_{\mathcal{C}}) \longrightarrow D(\mathcal{O}_{\mathcal{D}})$$

which is left adjoint to  $g^*$ . Moreover, for  $U \in \mathrm{Ob}(\mathcal{C})$  we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where  $j_{U!}$  and  $j_{u(U)!}$  are extension by zero associated to the localization morphism  $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$  and  $j_{u(U)} : \mathcal{D}/u(U) \rightarrow \mathcal{D}$ .

**Proof.** We are going to use Derived Categories, Proposition 28.2 to construct  $Lg_!$ . To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since  $g_!$  is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let  $\mathcal{P} \subset \mathrm{Ob}(\mathrm{Mod}(\mathcal{O}_{\mathcal{C}}))$  be the collection of  $\mathcal{O}_{\mathcal{C}}$ -modules which are direct sums of modules of the form

$j_{U!}\mathcal{O}_U$ . Note that  $g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$ , see proof of Modules on Sites, Lemma 40.1. Every  $\mathcal{O}_C$ -module is a quotient of an object of  $\mathcal{P}$ , see Modules on Sites, Lemma 28.7. Thus (1) holds. Finally, we have to prove (2). Let  $\mathcal{K}^\bullet$  be a bounded above acyclic complex of  $\mathcal{O}_C$ -modules with  $\mathcal{K}^n \in \mathcal{P}$  for all  $n$ . We have to show that  $g_!\mathcal{K}^\bullet$  is exact. To do this it suffices to show, for every injective  $\mathcal{O}_D$ -module  $\mathcal{I}$  that

$$\mathrm{Hom}_{D(\mathcal{O}_D)}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all  $n \in \mathbf{Z}$ . Since  $\mathcal{I}$  is injective we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_D)}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_D)}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_D}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})) \end{aligned}$$

the last equality by the adjointness of  $g_!$  and  $g^{-1}$ .

The vanishing of this group would be clear if  $g^{-1}\mathcal{I}$  were an injective  $\mathcal{O}_C$ -module. But  $g^{-1}\mathcal{I}$  isn't necessarily an injective  $\mathcal{O}_C$ -module as  $g_!$  isn't exact in general. We do know that

$$\mathrm{Ext}_{\mathcal{O}_C}^p(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Here the first equality follows from  $\mathrm{Hom}_{\mathcal{O}_C}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$  and taking derived functors and the vanishing of  $H^p(U, g^{-1}\mathcal{I})$  for  $p > 0$  and  $U \in \mathrm{Ob}(\mathcal{C})$  follows from Lemma 30.1. Since each  $\mathcal{K}^{-q}$  is a direct sum of modules of the form  $j_{U!}\mathcal{O}_U$  we see that

$$\mathrm{Ext}_{\mathcal{O}_C}^p(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 26.1)

$$E_1^{p,q} = \mathrm{Ext}_{\mathcal{O}_C}^p(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}_{\mathcal{O}_C}^{p+q}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as  $\mathcal{K}^\bullet$  is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the  $E_1$  page are the terms  $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$ . We conclude that the complex  $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$  is acyclic as desired.

Thus the left derived functor  $Lg_!$  exists. It is left adjoint to  $g^{-1} = g^* = Rg^* = Lg^*$ , i.e., we have

$$07AD \quad (30.2.1) \quad \mathrm{Hom}_{D(\mathcal{O}_C)}(K, g^*L) = \mathrm{Hom}_{D(\mathcal{O}_D)}(Lg_!K, L)$$

by Derived Categories, Lemma 28.5. This finishes the proof.  $\square$

07AE **Remark 30.3.** Warning! Let  $u : \mathcal{C} \rightarrow \mathcal{D}$ ,  $g$ ,  $\mathcal{O}_D$ , and  $\mathcal{O}_C$  be as in Lemma 30.2. In general it is **not** the case that the diagram

$$\begin{array}{ccc} D(\mathcal{O}_C) & \xrightarrow{Lg_!} & D(\mathcal{O}_D) \\ \text{forget} \downarrow & & \downarrow \text{forget} \\ D(\mathcal{C}) & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{D}) \end{array}$$

commutes where the functor  $Lg_!^{Ab}$  is the one constructed in Lemma 30.2 but using the constant sheaf  $\mathbf{Z}$  as the structure sheaf on both  $\mathcal{C}$  and  $\mathcal{D}$ . In general it isn't even the case that  $g_! = g_!^{Ab}$  (see Modules on Sites, Remark 40.2), but this phenomenon

can occur even if  $g_! = g_!^{Ab}$ . Namely, the construction of  $Lg_!$  in the proof of Lemma 30.2 shows that  $Lg_!$  agrees with  $Lg_!^{Ab}$  if and only if the canonical maps

$$Lg_!^{Ab} j_{U!} \mathcal{O}_U \longrightarrow j_{u(U)!} \mathcal{O}_{u(U)}$$

are isomorphisms in  $D(\mathcal{D})$  for all objects  $U$  in  $\mathcal{C}$ . In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ \text{forget} \longrightarrow \text{forget} \circ Lg_!$$

0D6Y **Lemma 30.4.** *Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a continuous and cocontinuous functor of sites. Let  $g : \text{Sh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{D})$  be the corresponding morphism of topoi. Let  $\mathcal{O}_{\mathcal{D}}$  be a sheaf of rings and let  $\mathcal{I}$  be an injective  $\mathcal{O}_{\mathcal{D}}$ -module. If  $g_!^{Sh} : \text{Sh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{D})$  commutes with fibre products<sup>4</sup>, then  $g^{-1}\mathcal{I}$  is limp.*

**Proof.** We will use the criterion of Lemma 14.5. Condition (1) holds by Lemma 30.1. Let  $K' \rightarrow K$  be a surjective map of sheaves of sets on  $\mathcal{C}$ . Since  $g_!^{Sh}$  is a left adjoint, we see that  $g_!^{Sh} K' \rightarrow g_!^{Sh} K$  is surjective. Observe that

$$\begin{aligned} H^0(K' \times_K \dots \times_K K', g^{-1}\mathcal{I}) &= H^0(g_!^{Sh}(K' \times_K \dots \times_K K'), \mathcal{I}) \\ &= H^0(g_!^{Sh} K' \times_{g_!^{Sh} K} \dots \times_{g_!^{Sh} K} g_!^{Sh} K', \mathcal{I}) \end{aligned}$$

by our assumption on  $g_!^{Sh}$ . Since  $\mathcal{I}$  is injective module it is limp by Lemma 15.1 (applied to the identity). Hence we can use the converse of Lemma 14.5 to see that the complex

$$0 \rightarrow H^0(K, g^{-1}\mathcal{I}) \rightarrow H^0(K', g^{-1}\mathcal{I}) \rightarrow H^0(K' \times_K K', g^{-1}\mathcal{I}) \rightarrow \dots$$

is exact as desired.  $\square$

0DD8 **Lemma 30.5.** *Let  $u : \mathcal{C} \rightarrow \mathcal{D}$  be a continuous and cocontinuous functor of sites. Let  $g : \text{Sh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{D})$  be the corresponding morphism of topoi. Let  $U \in \text{Ob}(\mathcal{C})$ .*

- (1) *For  $M$  in  $D(\mathcal{D})$  we have  $R\Gamma(U, g^{-1}M) = R\Gamma(u(U), M)$ .*
- (2) *If  $\mathcal{O}_{\mathcal{D}}$  is a sheaf of rings and  $\mathcal{O}_{\mathcal{C}} = g^{-1}\mathcal{O}_{\mathcal{D}}$ , then for  $M$  in  $D(\mathcal{O}_{\mathcal{D}})$  we have  $R\Gamma(U, g^*M) = R\Gamma(u(U), M)$ .*

**Proof.** In the bounded below case (1) and (2) can be seen by representing  $K$  by a bounded below complex of injectives and using Lemma 30.1 as well as Leray's acyclicity lemma. In the unbounded case, first note that (1) is a special case of (2). For (2) we can use

$$R\Gamma(U, g^*M) = R\text{Hom}_{\mathcal{O}_{\mathcal{C}}}(j_{U!}\mathcal{O}_U, g^*M) = R\text{Hom}_{\mathcal{O}_{\mathcal{D}}}(j_{u(U)!}\mathcal{O}_{u(U)}, M) = R\Gamma(u(U), M)$$

where the middle equality is a consequence of Lemma 30.2.  $\square$

### 31. Derived lower shriek for fibred categories

08RV In this section we work out some special cases of the situation discussed in Section 30. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

<sup>4</sup>Holds if  $\mathcal{C}$  has finite connected limits and  $u$  commutes with them, see Sites, Lemma 20.6.

08P8 **Situation 31.1.** Here  $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a ringed site and  $p : \mathcal{C} \rightarrow \mathcal{D}$  is a fibred category. We endow  $\mathcal{C}$  with the topology inherited from  $\mathcal{D}$  (Stacks, Section 10). We denote  $\pi : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$  the morphism of topoi associated to  $p$  (Stacks, Lemma 10.3). We set  $\mathcal{O}_{\mathcal{C}} = \pi^{-1}\mathcal{O}_{\mathcal{D}}$  so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$$

08P9 **Lemma 31.2.** *Assumptions and notation as in Situation 31.1. For  $U \in \text{Ob}(\mathcal{C})$  consider the induced morphism of topoi*

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

*Then there exists a morphism of topoi*

$$\sigma : Sh(\mathcal{D}/p(U)) \rightarrow Sh(\mathcal{C}/U)$$

*such that  $\pi_U \circ \sigma = \text{id}$  and  $\sigma^{-1} = \pi_{U,*}$ .*

**Proof.** Observe that  $\pi_U$  is the restriction of  $\pi$  to the localizations, see Sites, Lemma 27.4. For an object  $V \rightarrow p(U)$  of  $\mathcal{D}/p(U)$  denote  $V \times_{p(U)} U \rightarrow U$  the strongly cartesian morphism of  $\mathcal{C}$  over  $\mathcal{D}$  which exists as  $p$  is a fibred category. The functor

$$v : \mathcal{D}/p(U) \rightarrow \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on  $\mathcal{C}$ . Moreover, it is a right adjoint to  $p$  by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 21 and we see that the sheaf  $\pi_{U,*}\mathcal{F}$  is equal to  $V \mapsto \mathcal{F}(V \times_{p(U)} U)$  (see especially Sites, Lemma 21.2).

But here we have more. Namely, the functor  $v$  is also cocontinuous (as all morphisms in coverings of  $\mathcal{C}$  are strongly cartesian). Hence  $v$  defines a morphism  $\sigma$  as indicated in the lemma. The equality  $\sigma^{-1} = \pi_{U,*}$  is immediate from the definition. Since  $\pi_U^{-1}\mathcal{G}$  is given by the rule  $U'/U \mapsto \mathcal{G}(p(U')/p(U))$  it follows that  $\sigma^{-1} \circ \pi_U^{-1} = \text{id}$  which proves the equality  $\pi_U \circ \sigma = \text{id}$ .  $\square$

08PA **Situation 31.3.** Let  $(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a ringed site. Let  $u : \mathcal{C}' \rightarrow \mathcal{C}$  be a 1-morphism of fibred categories over  $\mathcal{D}$  (Categories, Definition 32.9). Endow  $\mathcal{C}$  and  $\mathcal{C}'$  with their inherited topologies (Stacks, Definition 10.2) and let  $\pi : Sh(\mathcal{C}) \rightarrow Sh(\mathcal{D})$ ,  $\pi' : Sh(\mathcal{C}') \rightarrow Sh(\mathcal{D})$ , and  $g : Sh(\mathcal{C}') \rightarrow Sh(\mathcal{C})$  be the corresponding morphisms of topoi (Stacks, Lemma 10.3). Set  $\mathcal{O}_{\mathcal{C}} = \pi^{-1}\mathcal{O}_{\mathcal{D}}$  and  $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_{\mathcal{D}}$ . Observe that  $g^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$  so that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ & \searrow \pi' & \swarrow \pi \\ & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) & \end{array}$$

is a commutative diagram of morphisms of ringed topoi.

08PB **Lemma 31.4.** *Assumptions and notation as in Situation 31.3. For  $U' \in \text{Ob}(\mathcal{C}')$  set  $U = u(U')$  and  $V = p'(U')$  and consider the induced morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_U) \\ & \searrow \pi'_{U'} & \swarrow \pi_U \\ & (Sh(\mathcal{D}/V), \mathcal{O}_V) & \end{array}$$

Then there exists a morphism of topoi

$$\sigma' : \mathit{Sh}(\mathcal{D}/V) \rightarrow \mathit{Sh}(\mathcal{C}'/U'),$$

such that setting  $\sigma = g' \circ \sigma'$  we have  $\pi'_{U'} \circ \sigma' = \text{id}$ ,  $\pi_U \circ \sigma = \text{id}$ ,  $(\sigma')^{-1} = \pi'_{U',*}$ , and  $\sigma^{-1} = \pi_{U,*}$ .

**Proof.** Let  $v' : \mathcal{D}/V \rightarrow \mathcal{C}'/U'$  be the functor constructed in the proof of Lemma 31.2 starting with  $p' : \mathcal{C}' \rightarrow \mathcal{D}'$  and the object  $U'$ . Since  $u$  is a 1-morphism of fibred categories over  $\mathcal{D}$  it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor  $v = u \circ v'$  is the functor of the proof of Lemma 31.2 relative to  $p : \mathcal{C} \rightarrow \mathcal{D}$  and  $U$ . Thus our lemma follows from that lemma.  $\square$

08PC **Lemma 31.5.** *Assumption and notation as in Situation 31.3.*

- (1) *There are left adjoints  $g_! : \text{Mod}(\mathcal{O}_{\mathcal{C}'}) \rightarrow \text{Mod}(\mathcal{O}_{\mathcal{C}})$  and  $g_!^{Ab} : \text{Ab}(\mathcal{C}') \rightarrow \text{Ab}(\mathcal{C})$  to  $g^* = g^{-1}$  on modules and on abelian sheaves.*
- (2) *The diagram*

$$\begin{array}{ccc} \text{Mod}(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g_!} & \text{Mod}(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ \text{Ab}(\mathcal{C}') & \xrightarrow{g_!^{Ab}} & \text{Ab}(\mathcal{C}) \end{array}$$

*commutes.*

- (3) *There are left adjoints  $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \rightarrow D(\mathcal{O}_{\mathcal{C}})$  and  $Lg_!^{Ab} : D(\mathcal{C}') \rightarrow D(\mathcal{C})$  to  $g^* = g^{-1}$  on derived categories of modules and abelian sheaves.*
- (4) *The diagram*

$$\begin{array}{ccc} D(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{Lg_!} & D(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ D(\mathcal{C}') & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{C}) \end{array}$$

*commutes.*

**Proof.** The functor  $u$  is continuous and cocontinuous Stacks, Lemma 10.3. Hence the existence of the functors  $g_!$ ,  $g_!^{Ab}$ ,  $Lg_!$ , and  $Lg_!^{Ab}$  can be found in Modules on Sites, Sections 16 and 40 and Section 30.

To prove (2) it suffices to show that the canonical map

$$g_!^{Ab} j_{U'}! \mathcal{O}_{U'} \rightarrow j_{u(U')}! \mathcal{O}_{u(U')}$$

is an isomorphism for all objects  $U'$  of  $\mathcal{C}'$ , see Modules on Sites, Remark 40.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{Ab} j_{U'}! \mathcal{O}_{U'} \rightarrow j_{u(U')}! \mathcal{O}_{u(U')}$$

is an isomorphism in  $D(\mathcal{C})$  for all objects  $U'$  of  $\mathcal{C}'$ , see Remark 30.3. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism  $j$  the functors  $j_!$  and  $j_!^{Ab}$  agree (see Modules on Sites, Remark 19.6) and that  $j_!$  is exact (Modules on Sites, Lemma 19.3). Let us adopt the notation of Lemma 31.4. Since  $Lg_!^{Ab} \circ j_{U'}! = j_{U'}! \circ L(g')_!^{Ab}$  (by commutativity of Sites, Lemma 27.4 and uniqueness of adjoint functors) it



suffices to prove that  $L(g')_!^{Ab} \mathcal{O}_{U'} = \mathcal{O}_U$ . Using the results of Lemma 31.4 we have for any object  $E$  of  $D(\mathcal{C}/u(U'))$  the following sequence of equalities

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{Ab} \mathcal{O}_{U'}, E) &= \mathrm{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1} E) \\
&= \mathrm{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1} \mathcal{O}_V, (g')^{-1} E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*} (g')^{-1} E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1} (g')^{-1} E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1} E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*} E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1} \mathcal{O}_V, E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E)
\end{aligned}$$

By Yoneda's lemma we conclude.  $\square$

09CY **Remark 31.6.** Assumptions and notation as in Situation 31.1. Note that setting  $\mathcal{C}' = \mathcal{D}$  and  $u$  equal to the structure functor of  $\mathcal{C}$  gives a situation as in Situation 31.3. Hence Lemma 31.5 tells us we have functors  $\pi_!$ ,  $\pi_!^{Ab}$ ,  $L\pi_!$ , and  $L\pi_!^{Ab}$  such that  $\mathit{forget} \circ \pi_! = \pi_!^{Ab} \circ \mathit{forget}$  and  $\mathit{forget} \circ L\pi_! = L\pi_!^{Ab} \circ \mathit{forget}$ .

08PD **Remark 31.7.** Assumptions and notation as in Situation 31.3. Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{C}$ , let  $\mathcal{F}'$  be an abelian sheaf on  $\mathcal{C}'$ , and let  $t : \mathcal{F}' \rightarrow g^{-1}\mathcal{F}$  be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint  $g_!\mathcal{F}' \rightarrow \mathcal{F}$  of  $t$ , the map  $Lg_!(\mathcal{F}') \rightarrow g_!\mathcal{F}'$ , and the equality  $L\pi'_! = L\pi_! \circ Lg_!$ .

08PE **Lemma 31.8.** Assumptions and notation as in Situation 31.1. For  $\mathcal{F}$  in  $Ab(\mathcal{C})$  the sheaf  $\pi_!\mathcal{F}$  is the sheaf associated to the presheaf

$$V \longmapsto \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

**Proof.** Denote  $\mathcal{H}$  be the rule of the lemma. For a morphism  $h : V' \rightarrow V$  of  $\mathcal{D}$  there is a pullback functor  $h^* : \mathcal{C}_V \rightarrow \mathcal{C}_{V'}$  of fibre categories (Categories, Definition 32.6). Moreover for  $U \in \mathrm{Ob}(\mathcal{C}_V)$  there is a strongly cartesian morphism  $h^*U \rightarrow U$  covering  $h$ . Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map  $\mathcal{H}(V) \rightarrow \mathcal{H}(V')$  between colimits, see Categories, Lemma 14.7.

To prove the lemma we show that

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf  $\mathcal{G}$  on  $\mathcal{C}$ . An element of the left hand side is a compatible system of maps  $\mathcal{F}(U) \rightarrow \mathcal{G}(p(U))$  for all  $U$  in  $\mathcal{C}$ . Since  $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$  by our choice of topology on  $\mathcal{C}$  we see the same thing is true for the right hand side and we win.  $\square$

### 32. Homology on a category

08RW In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

08PF **Example 32.1** (Category over point). Let  $\mathcal{C}$  be a category. Endow  $\mathcal{C}$  with the chaotic topology (Sites, Example 6.6). Thus presheaves and sheaves agree on  $\mathcal{C}$ . The functor  $p : \mathcal{C} \rightarrow *$  where  $*$  is the category with a single object and a single morphism is cocontinuous and continuous. Let  $\pi : Sh(\mathcal{C}) \rightarrow Sh(*)$  be the corresponding morphism of topoi. Let  $B$  be a ring. We endow  $*$  with the sheaf of rings  $B$  and  $\mathcal{C}$  with  $\mathcal{O}_{\mathcal{C}} = \pi^{-1}B$  which we will denote  $\underline{B}$ . In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \rightarrow (*, B)$$

is an example of Situation 31.1. By Remark 31.6 we do not need to distinguish between  $\pi_!$  on modules or abelian sheaves. By Lemma 31.8 we see that  $\pi_!\mathcal{F} = \text{colim}_{\mathcal{C}^{opp}} \mathcal{F}$ . Thus  $L_n\pi_!$  is the  $n$ th left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the  $n$ th homology group of  $\mathcal{F}$  on  $\mathcal{C}$ .

08PG **Example 32.2** (Computing homology). In Example 32.1 we can compute the functors  $H_n(\mathcal{C}, -)$  as follows. Let  $\mathcal{F} \in \text{Ob}(Ab(\mathcal{C}))$ . Consider the chain complex

$$K_{\bullet}(\mathcal{F}) : \dots \rightarrow \bigoplus_{U_2 \rightarrow U_1 \rightarrow U_0} \mathcal{F}(U_0) \rightarrow \bigoplus_{U_1 \rightarrow U_0} \mathcal{F}(U_0) \rightarrow \bigoplus_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \rightarrow U_1 \rightarrow U_0, s) \mapsto (U_1 \rightarrow U_0, s) - (U_2 \rightarrow U_0, s) + (U_2 \rightarrow U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \text{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_{\bullet}(\mathcal{F})),$$

see Categories, Lemma 14.11. The construction of  $K_{\bullet}(\mathcal{F})$  is functorial in  $\mathcal{F}$  and transforms short exact sequences of  $Ab(\mathcal{C})$  into short exact sequences of complexes. Thus the sequence of functors  $\mathcal{F} \mapsto H_n(K_{\bullet}(\mathcal{F}))$  forms a  $\delta$ -functor, see Homology, Definition 11.1 and Lemma 12.12. For  $\mathcal{F} = j_{U!}\mathbf{Z}_U$  the complex  $K_{\bullet}(\mathcal{F})$  is the complex associated to the free  $\mathbf{Z}$ -module on the simplicial set  $X_{\bullet}$  with terms

$$X_n = \coprod_{U_n \rightarrow \dots \rightarrow U_1 \rightarrow U_0} \text{Mor}_{\mathcal{C}}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton  $\{*\}$ . Namely, the map  $X_{\bullet} \rightarrow \{*\}$  is obvious, the map  $\{*\} \rightarrow X_n$  is given by mapping  $*$  to  $(U \rightarrow \dots \rightarrow U, \text{id}_U)$ , and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 26.2) defining the homotopy between the two maps  $X_{\bullet} \rightarrow X_{\bullet}$  are given by the rule

$$h_{n,i} : (U_n \rightarrow \dots \rightarrow U_0, f) \mapsto (U_n \rightarrow \dots \rightarrow U_i \rightarrow U \rightarrow \dots \rightarrow U, \text{id})$$

for  $i > 0$  and  $h_{n,0} = \text{id}$ . Verifications omitted. This implies that  $K_{\bullet}(j_{U!}\mathbf{Z}_U)$  has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 26.4 and the result of Simplicial, Lemma 27.1). Thus  $K_{\bullet}(\mathcal{F})$  computes the left derived functors  $H_n(\mathcal{C}, -)$  of  $H_0(\mathcal{C}, -)$  for example by (the duals of) Homology, Lemma 11.4 and Derived Categories, Lemma 17.6.

08PH **Example 32.3.** Let  $u : \mathcal{C}' \rightarrow \mathcal{C}$  be a functor. Endow  $\mathcal{C}'$  and  $\mathcal{C}$  with the chaotic topology as in Example 32.1. The functors  $u$ ,  $\mathcal{C}' \rightarrow *$ , and  $\mathcal{C} \rightarrow *$  where  $*$  is the category with a single object and a single morphism are cocontinuous and continuous. Let  $g : Sh(\mathcal{C}') \rightarrow Sh(\mathcal{C})$ ,  $\pi' : Sh(\mathcal{C}') \rightarrow Sh(*)$ , and  $\pi : Sh(\mathcal{C}) \rightarrow Sh(*)$ , be the corresponding morphisms of topoi. Let  $B$  be a ring. We endow  $*$  with the sheaf of rings  $B$  and  $\mathcal{C}'$ ,  $\mathcal{C}$  with the constant sheaf  $\underline{B}$ . In this way

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \underline{B}) & \xrightarrow{g} & (Sh(\mathcal{C}), \underline{B}) \\ & \searrow \pi' & \swarrow \pi \\ & (Sh(*), B) & \end{array}$$

is an example of Situation 31.3. Thus Lemma 31.5 applies to  $g$  so we do not need to distinguish between  $g_!$  on modules or abelian sheaves. In particular Remark 31.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have  $\mathcal{F}$  in  $Ab(\mathcal{C})$ ,  $\mathcal{F}'$  in  $Ab(\mathcal{C}')$ , and a map  $t : \mathcal{F}' \rightarrow g^{-1}\mathcal{F}$ . In terms of the computation of homology given in Example 32.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \rightarrow \dots \rightarrow U'_0, s') \mapsto (u(U'_n) \rightarrow \dots \rightarrow u(U'_0), t(s'))$$

with obvious notation.

08Q6 **Remark 32.4.** Notation and assumptions as in Example 32.1. Let  $\mathcal{F}^\bullet$  be a bounded complex of abelian sheaves on  $\mathcal{C}$ . For any object  $U$  of  $\mathcal{C}$  there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in  $D(Ab)$ . If  $\mathcal{F}^\bullet$  is a complex of  $\underline{B}$ -modules then this map is in  $D(B)$ . To prove this, note that we compute  $L\pi_!(\mathcal{F}^\bullet)$  by taking a quasi-isomorphism  $\mathcal{P}^\bullet \rightarrow \mathcal{F}^\bullet$  where  $\mathcal{P}^\bullet$  is a complex of projectives. However, since the topology is chaotic this means that  $\mathcal{P}^\bullet(U) \rightarrow \mathcal{F}^\bullet(U)$  is a quasi-isomorphism hence can be inverted in  $D(Ab)$ , resp.  $D(B)$ . Composing with the canonical map  $\mathcal{P}^\bullet(U) \rightarrow \pi_!(\mathcal{P}^\bullet)$  coming from the computation of  $\pi_!$  as a colimit we obtain the desired arrow.

08Q7 **Lemma 32.5.** *Notation and assumptions as in Example 32.1. If  $\mathcal{C}$  has either an initial or a final object, then  $L\pi_! \circ \pi^{-1} = id$  on  $D(Ab)$ , resp.  $D(B)$ .*

**Proof.** If  $\mathcal{C}$  has an initial object, then  $\pi_!$  is computed by evaluating on this object and the statement is clear. If  $\mathcal{C}$  has a final object, then  $R\pi_*$  is computed by evaluating on this object, hence  $R\pi_* \circ \pi^{-1} \cong id$  on  $D(Ab)$ , resp.  $D(B)$ . This implies that  $\pi^{-1} : D(Ab) \rightarrow D(\mathcal{C})$ , resp.  $\pi^{-1} : D(B) \rightarrow D(\underline{B})$  is fully faithful, see Categories, Lemma 24.3. Then the same lemma implies that  $L\pi_! \circ \pi^{-1} = id$  as desired.  $\square$

08Q8 **Lemma 32.6.** *Notation and assumptions as in Example 32.1. Let  $B \rightarrow B'$  be a ring map. Consider the commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}), \underline{B}) & \xleftarrow{h} & (Sh(\mathcal{C}), \underline{B}') \\ \pi \downarrow & & \downarrow \pi' \\ (*, B) & \xleftarrow{f} & (*, B') \end{array}$$

Then  $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$ .

**Proof.** Both functors are right adjoint to the obvious functor  $D(B') \rightarrow D(\underline{B})$ .  $\square$

08Q9 **Lemma 32.7.** *Notation and assumptions as in Example 32.1. Let  $U_\bullet$  be a cosimplicial object in  $\mathcal{C}$  such that for every  $U \in \text{Ob}(\mathcal{C})$  the simplicial set  $\text{Mor}_{\mathcal{C}}(U_\bullet, U)$  is homotopy equivalent to the constant simplicial set on a singleton. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in  $D(\text{Ab})$ , resp.  $D(B)$  functorially in  $\mathcal{F}$  in  $\text{Ab}(\mathcal{C})$ , resp.  $\text{Mod}(\underline{B})$ .

**Proof.** As  $L\pi_!$  agrees for modules and abelian sheaves by Lemma 31.5 it suffices to prove this when  $\mathcal{F}$  is an abelian sheaf. For  $U \in \text{Ob}(\mathcal{C})$  the abelian sheaf  $j_{U!}\mathbf{Z}_U$  is a projective object of  $\text{Ab}(\mathcal{C})$  since  $\text{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$  and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of  $j_{U!}\mathbf{Z}_U$  by Modules on Sites, Lemma 28.7. Thus we can compute  $L\pi_!(\mathcal{F})$  by choosing a resolution

$$\dots \rightarrow \mathcal{G}^{-1} \rightarrow \mathcal{G}^0 \rightarrow \mathcal{F} \rightarrow 0$$

whose terms are direct sums of sheaves of the form above and taking  $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$ . Consider the double complex  $A^{\bullet, \bullet} = \mathcal{G}^\bullet(U_\bullet)$ . The map  $\mathcal{G}^0 \rightarrow \mathcal{F}$  gives a map of complexes  $A^{0, \bullet} \rightarrow \mathcal{F}(U_\bullet)$ . Since  $\pi_!$  is computed by taking the colimit over  $\mathcal{C}^{opp}$  (Lemma 31.8) we see that the two compositions  $\mathcal{G}^m(U_1) \rightarrow \mathcal{G}^m(U_0) \rightarrow \pi_!\mathcal{G}^m$  are equal. Thus we obtain a canonical map of complexes

$$\text{Tot}(A^{\bullet, \bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\dots \rightarrow \mathcal{G}^m(U_1) \rightarrow \mathcal{G}^m(U_0) \rightarrow \pi_!\mathcal{G}^m \rightarrow 0$$

are exact, see Homology, Lemma 22.7. Since the sheaves  $\mathcal{G}^m$  are direct sums of the sheaves  $j_{U!}\mathbf{Z}_U$  we reduce to  $\mathcal{G} = j_{U!}\mathbf{Z}_U$ . The complex  $j_{U!}\mathbf{Z}_U(U_\bullet)$  is the complex of abelian groups associated to the free  $\mathbf{Z}$ -module on the simplicial set  $\text{Mor}_{\mathcal{C}}(U_\bullet, U)$  which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \rightarrow \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 26.4 and Lemma 27.1). This finishes the proof since  $\pi_!j_{U!}\mathbf{Z}_U = \mathbf{Z}$  as was shown in the proof of Lemma 31.5.  $\square$

08QA **Lemma 32.8.** *Notation and assumptions as in Example 32.3. If there exists a cosimplicial object  $U'_\bullet$  of  $\mathcal{C}'$  such that Lemma 32.7 applies to both  $U'_\bullet$  in  $\mathcal{C}'$  and  $u(U'_\bullet)$  in  $\mathcal{C}$ , then we have  $L\pi'_! \circ g^{-1} = L\pi_!$  as functors  $D(\mathcal{C}) \rightarrow D(\text{Ab})$ , resp.  $D(\mathcal{C}, \underline{B}) \rightarrow D(B)$ .*

**Proof.** Follows immediately from Lemma 32.7 and the fact that  $g^{-1}$  is given by precomposing with  $u$ .  $\square$

08QB **Lemma 32.9.** *Let  $\mathcal{C}_i$ ,  $i = 1, 2$  be categories. Let  $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_i$  be the projection functors. Let  $B$  be a ring. Let  $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \rightarrow (Sh(\mathcal{C}_i), \underline{B})$  be the corresponding morphisms of ringed topoi, see Example 32.3. For  $K_i \in D(\mathcal{C}_i, B)$  we have*

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes_{\underline{B}}^{\mathbf{L}} L\pi_{2,!}(K_2)$$

in  $D(B)$  with obvious notation.

**Proof.** As both sides commute with colimits, it suffices to prove this for  $K_1 = j_{U!}\underline{B}_U$  and  $K_2 = j_{V!}\underline{B}_V$  for  $U \in \text{Ob}(\mathcal{C}_1)$  and  $V \in \text{Ob}(\mathcal{C}_2)$ . See construction of  $L\pi_!$  in Lemma 30.2. In this case

$$g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to  $B$ , see construction of  $L\pi_!$  in Lemma 30.2.  $\square$

08QC **Lemma 32.10.** *Notation and assumptions as in Example 32.1. If there exists a cosimplicial object  $U_\bullet$  of  $\mathcal{C}$  such that Lemma 32.7 applies, then*

$$L\pi_!(K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2) = L\pi_!(K_1) \otimes_{\underline{B}}^{\mathbf{L}} L\pi_!(K_2)$$

for all  $K_i \in D(\underline{B})$ .

**Proof.** Consider the diagram of categories and functors

$$\begin{array}{ccc} & & \mathcal{C} \\ & \nearrow & \\ \mathcal{C} & \xrightarrow{u} & \mathcal{C} \times \mathcal{C} \\ & \searrow & \\ & & \mathcal{C} \end{array}$$

$u_1$  (arrow from  $\mathcal{C} \times \mathcal{C}$  to top  $\mathcal{C}$ ),  $u_2$  (arrow from  $\mathcal{C} \times \mathcal{C}$  to bottom  $\mathcal{C}$ )

where  $u$  is the diagonal functor and  $u_i$  are the projection functors. This gives morphisms of ringed topoi  $g, g_1, g_2$ . For any object  $(U_1, U_2)$  of  $\mathcal{C}$  we have

$$\text{Mor}_{\mathcal{C} \times \mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \text{Mor}_{\mathcal{C}}(U_\bullet, U_1) \times \text{Mor}_{\mathcal{C}}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 26.10. Thus Lemma 32.8 gives  $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$  for any  $K$  in  $D(\mathcal{C} \times \mathcal{C}, B)$ . Take  $K = g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2$ . Then  $g^{-1}K = K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2$  because  $g^{-1} = g^* = Lg^*$  commutes with derived tensor product (Lemma 19.4). To finish we apply Lemma 32.9.  $\square$

08QD **Remark 32.11** (Simplicial modules). Let  $\mathcal{C} = \Delta$  and let  $B$  be any ring. This is a special case of Example 32.1 where the assumptions of Lemma 32.7 hold. Namely, let  $U_\bullet$  be the cosimplicial object of  $\Delta$  given by the identity functor. To verify the condition we have to show that for  $[m] \in \text{Ob}(\Delta)$  the simplicial set  $\Delta[m] : n \mapsto \text{Mor}_{\Delta}([n], [m])$  is homotopy equivalent to a point. This is explained in Simplicial, Example 26.7.

In this situation the category  $\text{Mod}(\underline{B})$  is just the category of simplicial  $B$ -modules and the functor  $L\pi_!$  sends a simplicial  $B$ -module  $M_\bullet$  to its associated complex  $s(M_\bullet)$

of  $B$ -modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 32.10 is: if  $M_\bullet, M'_\bullet$  are flat simplicial  $B$ -modules, then the complex  $s(M_\bullet \otimes_B M'_\bullet)$  is quasi-isomorphic to the total complex associated to the double complex  $s(M_\bullet) \otimes_B s(M'_\bullet)$ . (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [EZ53].

08RX **Lemma 32.12.** *Let  $\mathcal{C}$  be a category (endowed with chaotic topology). Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a map of sheaves of rings on  $\mathcal{C}$ . Assume*

- (1) *there exists a cosimplicial object  $U_\bullet$  in  $\mathcal{C}$  as in Lemma 32.7, and*
- (2)  *$L\pi_! \mathcal{O} \rightarrow L\pi_! \mathcal{O}'$  is an isomorphism.*

*For  $K$  in  $D(\mathcal{O})$  we have*

$$L\pi_!(K) = L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}')$$

*in  $D(\text{Ab})$ .*

**Proof.** Note: in this proof  $L\pi_!$  denotes the left derived functor of  $\pi_!$  on abelian sheaves. Since  $L\pi_!$  commutes with colimits, it suffices to prove this for bounded above complexes of  $\mathcal{O}$ -modules (compare with argument of Derived Categories, Proposition 28.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of  $j_{U!} \mathcal{O}_U$  with  $U \in \text{Ob}(\mathcal{C})$ , see Modules on Sites, Lemma 28.7. Thus it suffices to prove the lemma for  $j_{U!} \mathcal{O}_U$ . By assumption

$$S_\bullet = \text{Mor}_{\mathcal{C}}(U_\bullet, U)$$

is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set  $P_n = \mathcal{O}(U_n)$  and  $P'_n = \mathcal{O}'(U_n)$ . Observe that the complex associated to the simplicial abelian group

$$X_\bullet : n \mapsto \bigoplus_{s \in S_n} P_n$$

computes  $L\pi_!(j_{U!} \mathcal{O}_U)$  by Lemma 32.7. Since  $j_{U!} \mathcal{O}_U$  is a flat  $\mathcal{O}$ -module we have  $j_{U!} \mathcal{O}_U \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}' = j_{U!} \mathcal{O}'_U$  and  $L\pi_!$  of this is computed by the complex associated to the simplicial abelian group

$$X'_\bullet : n \mapsto \bigoplus_{s \in S_n} P'_n$$

As the rule which to a simplicial set  $T_\bullet$  associated the simplicial abelian group with terms  $\bigoplus_{t \in T_n} P_n$  is a functor, we see that  $X_\bullet \rightarrow P_\bullet$  is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set  $T_\bullet$  associates the simplicial abelian group with terms  $\bigoplus_{t \in T_n} P'_n$  is a functor. Hence  $X'_\bullet \rightarrow P'_\bullet$  is a homotopy equivalence of simplicial abelian groups. By assumption  $P_\bullet \rightarrow P'_\bullet$  is a quasi-isomorphism (since  $P_\bullet$ , resp.  $P'_\bullet$  computes  $L\pi_! \mathcal{O}$ , resp.  $L\pi_! \mathcal{O}'$  by Lemma 32.7). We conclude that  $X_\bullet$  and  $X'_\bullet$  are quasi-isomorphic as desired.  $\square$

09CZ **Remark 32.13.** Let  $\mathcal{C}$  and  $B$  be as in Example 32.1. Assume there exists a cosimplicial object as in Lemma 32.7. Let  $\mathcal{O} \rightarrow \underline{B}$  be a map sheaf of rings on  $\mathcal{C}$  which induces an isomorphism  $L\pi_! \mathcal{O} \rightarrow L\pi_! \underline{B}$ . In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object  $K$  of  $D(\mathcal{O})$  we have  $L\pi_!^{Ab}(K) = L\pi_!^{Ab}(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$  by Lemma 32.12. Thus we can define the displayed functor as the composition of  $-\otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B}$  with

the functor  $L\pi_! : D(\underline{B}) \rightarrow D(B)$ . In other words, we obtain a  $B$ -module structure on  $L\pi_!(K)$  coming from the (canonical, functorial) identification of  $L\pi_!(K)$  with  $L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$  of the lemma.

### 33. Calculating derived lower shriek

08P7 In this section we apply the results from Section 32 to compute  $L\pi_!$  in Situation 31.1 and  $Lg_!$  in Situation 31.3.

08PI **Lemma 33.1.** *Assumptions and notation as in Situation 31.1. For  $\mathcal{F}$  in  $\text{PAb}(\mathcal{C})$  and  $n \geq 0$  consider the abelian sheaf  $L_n(\mathcal{F})$  on  $\mathcal{D}$  which is the sheaf associated to the presheaf*

$$V \mapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then  $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ .

**Proof.** For a morphism  $h : V' \rightarrow V$  of  $\mathcal{D}$  there is a pullback functor  $h^* : \mathcal{C}_V \rightarrow \mathcal{C}_{V'}$  of fibre categories (Categories, Definition 32.6). Moreover for  $U \in \text{Ob}(\mathcal{C}_V)$  there is a strongly cartesian morphism  $h^*U \rightarrow U$  covering  $h$ . Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \rightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 32.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \rightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote  $L_{n,p}(\mathcal{F})$  this presheaf, so that  $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^\#$ . The canonical map  $\gamma : \mathcal{F} \rightarrow \mathcal{F}^+$  (Sites, Theorem 10.10) defines a canonical map  $L_{n,p}(\mathcal{F}) \rightarrow L_{n,p}(\mathcal{F}^+)$ . We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 32.2. Denote  $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$  the complex associated to the restriction of  $\mathcal{F}$  to the fibre category  $\mathcal{C}_V$ . By the remarks above we obtain a presheaf  $K_\bullet(\mathcal{F})$  of complexes

$$V \mapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves  $L_{n,p}(\mathcal{F})$ . Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \rightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

**Injectivity.** Let  $V$  be an object of  $\mathcal{D}$  and let  $\xi \in K_n(\mathcal{F})(V)$  be an element which maps to zero in  $K_n(\mathcal{F}^+)(V)$ . We have to show there exists a covering  $\{V_j \rightarrow V\}$  such that  $\xi|_{V_j}$  is zero in  $K_n(\mathcal{F})(V_j)$ . We write

$$\xi = \sum (U_{i,n+1} \rightarrow \dots \rightarrow U_{i,0}, \sigma_i)$$

with  $\sigma_i \in \mathcal{F}(U_{i,0})$ . We arrange it so that each sequence of morphisms  $U_n \rightarrow \dots \rightarrow U_0$  of  $\mathcal{C}_V$  occurs at most once. Since the sums in the definition of the complex  $K_\bullet$  are direct sums, the only way this can map to zero in  $K_\bullet(\mathcal{F}^+)(V)$  is if all  $\sigma_i$  map to zero in  $\mathcal{F}^+(U_{i,0})$ . By construction of  $\mathcal{F}^+$  there exist coverings  $\{U_{i,0,j} \rightarrow U_{i,0}\}$  such that  $\sigma_i|_{U_{i,0,j}}$  is zero. By our construction of the topology on  $\mathcal{C}$  we can write  $U_{i,0,j} \rightarrow U_{i,0}$  as the pullback (Categories, Definition 32.6) of some morphisms  $V_{i,j} \rightarrow V$  and moreover each  $\{V_{i,j} \rightarrow V\}$  is a covering. Choose a covering  $\{V_j \rightarrow V\}$  dominating each of the coverings  $\{V_{i,j} \rightarrow V\}$ . Then it is clear that  $\xi|_{V_j} = 0$ .

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity.  $\square$

08PJ **Lemma 33.2.** *Assumptions and notation as in Situation 31.1. For  $\mathcal{F}$  in  $Ab(\mathcal{C})$  and  $n \geq 0$  the sheaf  $L_n\pi_1(\mathcal{F})$  is equal to the sheaf  $L_n(\mathcal{F})$  constructed in Lemma 33.1.*

**Proof.** Consider the sequence of functors  $\mathcal{F} \mapsto L_n(\mathcal{F})$  from  $PAb(\mathcal{C}) \rightarrow Ab(\mathcal{C})$ . Since for each  $V \in \text{Ob}(\mathcal{D})$  the sequence of functors  $H_n(\mathcal{C}_V, -)$  forms a  $\delta$ -functor so do the functors  $\mathcal{F} \mapsto L_n(\mathcal{F})$ . Our goal is to show these form a universal  $\delta$ -functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For  $U' \in \text{Ob}(\mathcal{C})$  consider the abelian presheaf  $\mathcal{F}_{U'} = j_{U'!}^{PAb} \mathbf{Z}_{U'}$  (Modules on Sites, Remark 19.7). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus_{\text{Mor}_{\mathcal{C}}(U, U')} \mathbf{Z}$$

If  $U$  lies over  $V = p(U)$  in  $\mathcal{D}$  and  $U'$  lies over  $V' = p(U')$  then any morphism  $a : U \rightarrow U'$  factors uniquely as  $U \rightarrow h^*U' \rightarrow U'$  where  $h = p(a) : V \rightarrow V'$  (see Categories, Definition 32.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus_{h \in \text{Mor}_{\mathcal{D}}(V, V')} j_{h^*U'!} \mathbf{Z}_{h^*U'}$$

where  $j_{h^*U'} : Sh(\mathcal{C}_V/h^*U') \rightarrow Sh(\mathcal{C}_V)$  is the localization morphism. The sheaves  $j_{h^*U'!} \mathbf{Z}_{h^*U'}$  have vanishing higher homology groups (see Example 32.2). We conclude that  $L_n(\mathcal{F}_{U'}) = 0$  for all  $n > 0$  and all  $U'$ . It follows that any abelian presheaf  $\mathcal{F}$  is a quotient of an abelian presheaf  $\mathcal{G}$  with  $L_n(\mathcal{G}) = 0$  for all  $n > 0$  (Modules on Sites, Lemma 28.7). Since  $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$  we see that the same thing is true for abelian sheaves. Thus the sequence of functors  $L_n(-)$  is a universal delta functor on  $Ab(\mathcal{C})$  (Homology, Lemma 11.4). Since we have agreement with  $H^{-n}(L\pi_1(-))$  for  $n = 0$  by Lemma 31.8 we conclude by uniqueness of universal  $\delta$ -functors (Homology, Lemma 11.5) and Derived Categories, Lemma 17.6.  $\square$

08PK **Lemma 33.3.** *Assumptions and notation as in Situation 31.3. For an abelian sheaf  $\mathcal{F}'$  on  $\mathcal{C}'$  the sheaf  $L_n g_1(\mathcal{F}')$  is the sheaf associated to the presheaf*

$$U \mapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

*For notation and restriction maps see proof.*

**Proof.** Say  $p(U) = V$ . The category  $\mathcal{I}_U$  is the category of pairs  $(U', \varphi)$  where  $\varphi : U \rightarrow u(U')$  is a morphism of  $\mathcal{C}$  with  $p(\varphi) = \text{id}_V$ , i.e.,  $\varphi$  is a morphism of the fibre category  $\mathcal{C}_V$ . Morphisms  $(U'_1, \varphi_1) \rightarrow (U'_2, \varphi_2)$  are given by morphisms  $a : U'_1 \rightarrow U'_2$  of the fibre category  $\mathcal{C}'_V$  such that  $\varphi_2 = u(a) \circ \varphi_1$ . The presheaf  $\mathcal{F}'_U$  sends  $(U', \varphi)$  to  $\mathcal{F}'(U')$ . We will construct the restriction mappings below.

Choose a factorization

$$\mathcal{C}' \begin{array}{c} \xrightarrow{u'} \\ \xleftarrow{w} \end{array} \mathcal{C}'' \xrightarrow{u''} \mathcal{C}$$

of  $u$  as in Categories, Lemma 32.14. Then  $g_1 = g_1'' \circ g_1'$  and similarly for derived functors. On the other hand, the functor  $g_1'$  is exact, see Modules on Sites, Lemma 16.6. Thus we get  $Lg_1(\mathcal{F}') = Lg_1''(\mathcal{F}'')$  where  $\mathcal{F}'' = g_1' \mathcal{F}'$ . Note that  $\mathcal{F}'' = h^{-1} \mathcal{F}'$  where  $h : Sh(\mathcal{C}'') \rightarrow Sh(\mathcal{C}')$  is the morphism of topoi associated to  $w$ , see Sites, Lemma 22.1. The functor  $u''$  turns  $\mathcal{C}''$  into a fibred category over  $\mathcal{C}$ , hence Lemma



33.2 applies to the computation of  $L_n g''$ . The result follows as the construction of  $\mathcal{C}''$  in the proof of Categories, Lemma 32.14 shows that the fibre category  $\mathcal{C}''_U$  is equal to  $\mathcal{I}_U$ . Moreover,  $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$  is given by the rule described above (as  $w$  is continuous and cocontinuous by Stacks, Lemma 10.3 so we may apply Sites, Lemma 20.5).  $\square$

### 34. Simplicial modules

09D0 Let  $A_\bullet$  be a simplicial ring. Recall that we may think of  $A_\bullet$  as a sheaf on  $\Delta$  (endowed with the chaotic topology), see Simplicial, Section 4. Then a simplicial module  $M_\bullet$  over  $A_\bullet$  is just a sheaf of  $A_\bullet$ -modules on  $\Delta$ . In other words, for every  $n \geq 0$  we have an  $A_n$ -module  $M_n$  and for every map  $\varphi : [n] \rightarrow [m]$  we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is  $A_\bullet(\varphi)$ -linear such that these maps compose in the usual manner.

Let  $\mathcal{C}$  be a site. A *simplicial sheaf of rings*  $\mathcal{A}_\bullet$  on  $\mathcal{C}$  is a simplicial object in the category of sheaves of rings on  $\mathcal{C}$ . In this case the assignment  $U \mapsto \mathcal{A}_\bullet(U)$  is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form  $([n], U) \rightarrow ([n], V)$ . We endow the category  $\Delta \times \mathcal{C}$  with the topology inherited from  $\mathcal{C}$  (see Stacks, Section 10). The simple description of the coverings in  $\Delta \times \mathcal{C}$  (Stacks, Lemma 10.1) immediately implies that a simplicial sheaf of rings on  $\mathcal{C}$  is the same thing as a sheaf of rings on  $\Delta \times \mathcal{C}$ .

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

09D1 **Definition 34.1.** Let  $\mathcal{C}$  be a site. Let  $\mathcal{A}_\bullet$  be a simplicial sheaf of rings on  $\mathcal{C}$ . A *simplicial  $\mathcal{A}_\bullet$ -module*  $\mathcal{F}_\bullet$  (sometimes called a *simplicial sheaf of  $\mathcal{A}_\bullet$ -modules*) is a sheaf of modules over the sheaf of rings on  $\Delta \times \mathcal{C}$  associated to  $\mathcal{A}_\bullet$ .

We obtain a category  $Mod(\mathcal{A}_\bullet)$  of simplicial modules and a corresponding derived category  $D(\mathcal{A}_\bullet)$ . Given a map  $\mathcal{A}_\bullet \rightarrow \mathcal{B}_\bullet$  of simplicial sheaves of rings we obtain a functor

$$- \otimes_{\mathcal{A}_\bullet}^L \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_1 : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module  $\mathcal{F}_\bullet$  the object  $L\pi_1(\mathcal{F}_\bullet)$  is represented by the associated chain complex  $s(\mathcal{F}_\bullet)$  (Simplicial, Section 23). This follows from Lemmas 33.2 and 32.7.

09D2 **Lemma 34.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{A}_\bullet \rightarrow \mathcal{B}_\bullet$  be a homomorphism of simplicial sheaves of rings on  $\mathcal{C}$ . If  $L\pi_1\mathcal{A}_\bullet \rightarrow L\pi_1\mathcal{B}_\bullet$  is an isomorphism in  $D(\mathcal{C})$ , then we have*

$$L\pi_1(K) = L\pi_1(K \otimes_{\mathcal{A}_\bullet}^L \mathcal{B}_\bullet)$$

for all  $K$  in  $D(\mathcal{A}_\bullet)$ .

**Proof.** Let  $([n], U)$  be an object of  $\Delta \times \mathcal{C}$ . Since  $L\pi_!$  commutes with colimits, it suffices to prove this for bounded above complexes of  $\mathcal{O}$ -modules (compare with argument of Derived Categories, Proposition 28.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 28.7. Thus it suffices to prove the lemma for a flat  $\mathcal{A}_\bullet$ -module  $\mathcal{F}$ . In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 33.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 33.1. Thus it suffices to prove the result for the restriction of  $\mathcal{F}$  to the fibre categories (i.e., to  $\Delta \times U$ ). In this case the result follows from Lemma 32.12.  $\square$

09D3 **Remark 34.3.** Let  $\mathcal{C}$  be a site. Let  $\epsilon : \mathcal{A}_\bullet \rightarrow \mathcal{O}$  be an augmentation (Simplicial, Definition 20.1) in the category of sheaves of rings. Assume  $\epsilon$  induces a quasi-isomorphism  $s(\mathcal{A}_\bullet) \rightarrow \mathcal{O}$ . In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object  $K$  of  $D(\mathcal{A}_\bullet)$  we have  $L\pi_!(K) = L\pi_!(K \otimes_{\mathcal{A}_\bullet}^{\mathbf{L}} \mathcal{O})$  by Lemma 34.2. Thus we can define the displayed functor as the composition of  $-\otimes_{\mathcal{A}_\bullet}^{\mathbf{L}} \mathcal{O}$  with the functor  $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \rightarrow D(\mathcal{O})$  of Remark 31.6. In other words, we obtain a  $\mathcal{O}$ -module structure on  $L\pi_!(K)$  coming from the (canonical, functorial) identification of  $L\pi_!(K)$  with  $L\pi_!(K \otimes_{\mathcal{A}_\bullet}^{\mathbf{L}} \mathcal{O})$  of the lemma.

### 35. Cohomology on a category

08RY In the situation of Example 32.1 in addition to the derived functor  $L\pi_!$ , we also have the functor  $R\pi_*$ . For an abelian sheaf  $\mathcal{F}$  on  $\mathcal{C}$  we have  $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$  and  $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$ .

08RZ **Example 35.1** (Computing cohomology). In Example 32.1 we can compute the functors  $H^n(\mathcal{C}, -)$  as follows. Let  $\mathcal{F} \in \text{Ob}(Ab(\mathcal{C}))$ . Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod_{U_0} \mathcal{F}(U_0) \rightarrow \prod_{U_0 \rightarrow U_1} \mathcal{F}(U_0) \rightarrow \prod_{U_0 \rightarrow U_1 \rightarrow U_2} \mathcal{F}(U_0) \rightarrow \dots$$

where the transition maps are given by

$$(s_{U_0 \rightarrow U_1}) \mapsto ((U_0 \rightarrow U_1 \rightarrow U_2) \mapsto s_{U_0 \rightarrow U_1} - s_{U_0 \rightarrow U_2} + s_{U_1 \rightarrow U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 14.10. The construction of  $K^\bullet(\mathcal{F})$  is functorial in  $\mathcal{F}$  and transforms short exact sequences of  $Ab(\mathcal{C})$  into short exact sequences of complexes. Thus the sequence of functors  $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$  forms a  $\delta$ -functor, see Homology, Definition 11.1 and Lemma 12.12. For an object  $U$  of  $\mathcal{C}$  denote  $p_U : Sh(*) \rightarrow Sh(\mathcal{C})$  the corresponding point with  $p_U^{-1}$  equal to evaluation at  $U$ , see Sites, Example 32.7. Let  $A$  be an abelian group and set  $\mathcal{F} = p_{U,*}A$ . In this case the complex  $K^\bullet(\mathcal{F})$  is the complex with terms  $\text{Map}(X_n, A)$  where

$$X_n = \prod_{U_0 \rightarrow \dots \rightarrow U_{n-1} \rightarrow U_n} \text{Mor}_{\mathcal{C}}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton  $\{*\}$ . Namely, the map  $X_\bullet \rightarrow \{*\}$  is obvious, the map  $\{*\} \rightarrow X_n$  is given by mapping  $*$  to  $(U \rightarrow \dots \rightarrow U, \text{id}_U)$ , and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 26.2) defining the homotopy between the two maps  $X_\bullet \rightarrow X_\bullet$  are given by the rule

$$h_{n,i} : (U_0 \rightarrow \dots \rightarrow U_n, f) \longmapsto (U \rightarrow \dots \rightarrow U \rightarrow U_i \rightarrow \dots \rightarrow U_n, \text{id})$$

for  $i > 0$  and  $h_{n,0} = \text{id}$ . Verifications omitted. Since  $\text{Map}(-, A)$  is a contravariant functor, implies that  $K^\bullet(p_{U,*}A)$  has trivial cohomology in positive degrees (by the functoriality of *Simplicial*, Remark 26.4 and the result of *Simplicial*, Lemma 28.5). This implies that  $K^\bullet(\mathcal{F})$  is acyclic in positive degrees also if  $\mathcal{F}$  is a product of sheaves of the form  $p_{U,*}A$ . As every abelian sheaf on  $\mathcal{C}$  embeds into such a product we conclude that  $K^\bullet(\mathcal{F})$  computes the left derived functors  $H^n(\mathcal{C}, -)$  of  $H^0(\mathcal{C}, -)$  for example by *Homology*, Lemma 11.4 and *Derived Categories*, Lemma 17.6.

08S0 **Example 35.2** (Computing Exts). In Example 32.1 assume we are moreover given a sheaf of rings  $\mathcal{O}$  on  $\mathcal{C}$ . Let  $\mathcal{F}, \mathcal{G}$  be  $\mathcal{O}$ -modules. Consider the complex  $K^\bullet(\mathcal{G}, \mathcal{F})$  with degree  $n$  term

$$\prod_{U_0 \rightarrow U_1 \rightarrow \dots \rightarrow U_n} \text{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \rightarrow U_1}) \longmapsto ((U_0 \rightarrow U_1 \rightarrow U_2) \mapsto \varphi_{U_0 \rightarrow U_1} \circ \rho_{U_1}^{U_2} - \varphi_{U_0 \rightarrow U_2} + \rho_{U_0}^{U_1} \circ \varphi_{U_1 \rightarrow U_2}$$

and similarly in other degrees. Here the  $\rho$ 's indicate restriction maps. By construction

$$\text{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of  $\mathcal{O}$ -modules  $\mathcal{F}, \mathcal{G}$ . The assignment  $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$  is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\text{Ext}_{\mathcal{O}}^i(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for  $i \geq 0$  provided either

- (1)  $\mathcal{G}(U)$  is a projective  $\mathcal{O}(U)$ -module for all  $U \in \text{Ob}(\mathcal{C})$ , or
- (2)  $\mathcal{F}(U)$  is an injective  $\mathcal{O}(U)$ -module for all  $U \in \text{Ob}(\mathcal{C})$ .

Namely, case (1) the functor  $K^\bullet(\mathcal{G}, -)$  is an exact functor from the category of  $\mathcal{O}$ -modules to the category of cochain complexes of abelian groups. Thus, arguing as in Example 35.1, it suffices to show that  $K^\bullet(\mathcal{G}, \mathcal{F})$  is acyclic in positive degrees when  $\mathcal{F}$  is  $p_{U,*}A$  for an  $\mathcal{O}(U)$ -module  $A$ . Choose a short exact sequence

08S1 (35.2.1) 
$$0 \rightarrow \mathcal{G}' \rightarrow \bigoplus j_{U_i!} \mathcal{O}_{U_i} \rightarrow \mathcal{G} \rightarrow 0$$

see *Modules on Sites*, Lemma 28.7. Since (1) holds for the middle and right sheaves, it also holds for  $\mathcal{G}'$  and evaluating (35.2.1) on an object of  $\mathcal{C}$  gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \rightarrow K^\bullet(\mathcal{G}, \mathcal{F}) \rightarrow \prod K^\bullet(j_{U_i!} \mathcal{O}_{U_i}, \mathcal{F}) \rightarrow K^\bullet(\mathcal{G}', \mathcal{F}) \rightarrow 0$$

for any  $\mathcal{F}$ , in particular  $\mathcal{F} = p_{U,*}A$ . On  $H^0$  we obtain

$$0 \rightarrow \text{Hom}(\mathcal{G}, p_{U,*}A) \rightarrow \text{Hom}\left(\prod j_{U_i!} \mathcal{O}_{U_i}, p_{U,*}A\right) \rightarrow \text{Hom}(\mathcal{G}', p_{U,*}A) \rightarrow 0$$

which is exact as  $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$  and the sequence of sections of (35.2.1) over  $U$  is split exact. Thus we can use dimension shifting to see that it suffices to prove  $K^\bullet(j_{U'}\mathcal{O}_{U'}, p_{U,*}A)$  is acyclic in positive degrees for all  $U, U' \in \mathrm{Ob}(\mathcal{C})$ . In this case  $K^n(j_{U'}\mathcal{O}_{U'}, p_{U,*}A)$  is equal to

$$\prod_{U \rightarrow U_0 \rightarrow U_1 \rightarrow \dots \rightarrow U_n \rightarrow U'} A$$

In other words,  $K^\bullet(j_{U'}\mathcal{O}_{U'}, p_{U,*}A)$  is the complex with terms  $\mathrm{Map}(X_\bullet, A)$  where

$$X_n = \prod_{U_0 \rightarrow \dots \rightarrow U_{n-1} \rightarrow U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0) \times \mathrm{Mor}_{\mathcal{C}}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton  $\{*\}$  as can be proved in exactly the same way as the corresponding statement in Example 35.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

### 36. Strictly perfect complexes

08FK This section is the analogue of Cohomology, Section 40.

08FL **Definition 36.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^\bullet$  be a complex of  $\mathcal{O}$ -modules. We say  $\mathcal{E}^\bullet$  is *strictly perfect* if  $\mathcal{E}^i$  is zero for all but finitely many  $i$  and  $\mathcal{E}^i$  is a direct summand of a finite free  $\mathcal{O}$ -module for all  $i$ .

Let  $U$  be an object of  $\mathcal{C}$ . We will often say “Let  $\mathcal{E}^\bullet$  be a strictly perfect complex of  $\mathcal{O}_U$ -modules” to mean  $\mathcal{E}^\bullet$  is a strictly perfect complex of modules on the ringed site  $(\mathcal{C}/U, \mathcal{O}_U)$ , see Modules on Sites, Definition 19.1.

08FM **Lemma 36.2.** *The cone on a morphism of strictly perfect complexes is strictly perfect.*

**Proof.** This is immediate from the definitions. □

09J8 **Lemma 36.3.** *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

**Proof.** Omitted. □

08H3 **Lemma 36.4.** *Let  $(f, f^\#) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. If  $\mathcal{F}^\bullet$  is a strictly perfect complex of  $\mathcal{O}_{\mathcal{D}}$ -modules, then  $f^*\mathcal{F}^\bullet$  is a strictly perfect complex of  $\mathcal{O}_{\mathcal{C}}$ -modules.*

**Proof.** We have seen in Modules on Sites, Lemma 17.2 that the pullback of a finite free module is finite free. The functor  $f^*$  is additive functor hence preserves direct summands. The lemma follows. □

08FN **Lemma 36.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Given a solid diagram of  $\mathcal{O}_U$ -modules*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ & \searrow \text{dotted} & \uparrow p \\ & & \mathcal{G} \end{array}$$

*with  $\mathcal{E}$  a direct summand of a finite free  $\mathcal{O}_U$ -module and  $p$  surjective, then there exists a covering  $\{U_i \rightarrow U\}$  such that a dotted arrow making the diagram commute exists over each  $U_i$ .*

**Proof.** We may assume  $\mathcal{E} = \mathcal{O}_U^{\oplus n}$  for some  $n$ . In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in  $\Gamma(U, \mathcal{F})$ . This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 11).  $\square$

08FP **Lemma 36.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ .*

- (1) *Let  $\alpha : \mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet$  be a morphism of complexes of  $\mathcal{O}_U$ -modules with  $\mathcal{E}^\bullet$  strictly perfect and  $\mathcal{F}^\bullet$  acyclic. Then there exists a covering  $\{U_i \rightarrow U\}$  such that each  $\alpha|_{U_i}$  is homotopic to zero.*
- (2) *Let  $\alpha : \mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet$  be a morphism of complexes of  $\mathcal{O}_U$ -modules with  $\mathcal{E}^\bullet$  strictly perfect,  $\mathcal{E}^i = 0$  for  $i < a$ , and  $H^i(\mathcal{F}^\bullet) = 0$  for  $i \geq a$ . Then there exists a covering  $\{U_i \rightarrow U\}$  such that each  $\alpha|_{U_i}$  is homotopic to zero.*

**Proof.** The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex  $\mathcal{E}^\bullet$ . If  $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$  for some direct summand  $\mathcal{E}$  of a finite free  $\mathcal{O}$ -module and integer  $n \geq a$ , then the result follows from Lemma 36.5 and the fact that  $\mathcal{F}^{n-1} \rightarrow \text{Ker}(\mathcal{F}^n \rightarrow \mathcal{F}^{n+1})$  is surjective by the assumed vanishing of  $H^n(\mathcal{F}^\bullet)$ . If  $\mathcal{E}^i$  is zero except for  $i \in [a, b]$ , then we have a split exact sequence of complexes

$$0 \rightarrow \mathcal{E}^b[-b] \rightarrow \mathcal{E}^\bullet \rightarrow \sigma_{\leq b-1} \mathcal{E}^\bullet \rightarrow 0$$

which determines a distinguished triangle in  $K(\mathcal{O}_U)$ . Hence an exact sequence

$$\text{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1} \mathcal{E}^\bullet, \mathcal{F}^\bullet) \rightarrow \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \rightarrow \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition  $\mathcal{E}^b[-b] \rightarrow \mathcal{F}^\bullet$  is homotopic to zero on the members of a covering of  $U$  by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of  $U$  by induction hypothesis.  $\square$

08FQ **Lemma 36.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Given a solid diagram of complexes of  $\mathcal{O}_U$ -modules*

$$\begin{array}{ccc} \mathcal{E}^\bullet & \xrightarrow{\alpha} & \mathcal{F}^\bullet \\ & \searrow & \uparrow f \\ & & \mathcal{G}^\bullet \end{array}$$

*with  $\mathcal{E}^\bullet$  strictly perfect,  $\mathcal{E}^j = 0$  for  $j < a$  and  $H^j(f)$  an isomorphism for  $j > a$  and surjective for  $j = a$ , then there exists a covering  $\{U_i \rightarrow U\}$  and for each  $i$  a dotted arrow over  $U_i$  making the diagram commute up to homotopy.*

**Proof.** Our assumptions on  $f$  imply the cone  $C(f)^\bullet$  has vanishing cohomology sheaves in degrees  $\geq a$ . Hence Lemma 36.6 guarantees there is a covering  $\{U_i \rightarrow U\}$  such that the composition  $\mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow C(f)^\bullet$  is homotopic to zero over  $U_i$ . Since

$$\mathcal{G}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow C(f)^\bullet \rightarrow \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in  $K(\mathcal{O}_{U_i})$  we see that we can lift  $\alpha|_{U_i}$  up to homotopy to a map  $\alpha_i : \mathcal{E}^\bullet|_{U_i} \rightarrow \mathcal{G}^\bullet|_{U_i}$  as desired.  $\square$

08FR **Lemma 36.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . Let  $\mathcal{E}^\bullet, \mathcal{F}^\bullet$  be complexes of  $\mathcal{O}_U$ -modules with  $\mathcal{E}^\bullet$  strictly perfect.*

- (1) For any element  $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  there exists a covering  $\{U_i \rightarrow U\}$  such that  $\alpha|_{U_i}$  is given by a morphism of complexes  $\alpha_i : \mathcal{E}^\bullet|_{U_i} \rightarrow \mathcal{F}^\bullet|_{U_i}$ .
- (2) Given a morphism of complexes  $\alpha : \mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet$  whose image in the group  $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  is zero, there exists a covering  $\{U_i \rightarrow U\}$  such that  $\alpha|_{U_i}$  is homotopic to zero.

**Proof.** Proof of (1). By the construction of the derived category we can find a quasi-isomorphism  $f : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  and a map of complexes  $\beta : \mathcal{E}^\bullet \rightarrow \mathcal{G}^\bullet$  such that  $\alpha = f^{-1}\beta$ . Thus the result follows from Lemma 36.7. We omit the proof of (2).  $\square$

08JH **Lemma 36.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^\bullet, \mathcal{F}^\bullet$  be complexes of  $\mathcal{O}$ -modules with  $\mathcal{E}^\bullet$  strictly perfect. Then the internal hom  $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  is represented by the complex  $\mathcal{H}^\bullet$  with terms*

$$\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 28.

**Proof.** Choose a quasi-isomorphism  $\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$  into a K-injective complex. Let  $(\mathcal{H}')^\bullet$  be the complex with terms

$$(\mathcal{H}')^n = \prod_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents  $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  by the construction in Section 28. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object  $U$  of  $\mathcal{C}$  we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \rightarrow H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 36.8 the sheafification of  $U \mapsto H^0(\mathcal{H}^\bullet(U))$  is equal to the sheafification of  $U \mapsto H^0((\mathcal{H}')^\bullet(U))$ . A similar argument can be given for the other cohomology sheaves. Thus  $\mathcal{H}^\bullet$  is quasi-isomorphic to  $(\mathcal{H}')^\bullet$  which proves the lemma.  $\square$

08JI **Lemma 36.10.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^\bullet, \mathcal{F}^\bullet$  be complexes of  $\mathcal{O}$ -modules with*

- (1)  $\mathcal{F}^n = 0$  for  $n \ll 0$ ,
- (2)  $\mathcal{E}^n = 0$  for  $n \gg 0$ , and
- (3)  $\mathcal{E}^n$  isomorphic to a direct summand of a finite free  $\mathcal{O}$ -module.

Then the internal hom  $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  is represented by the complex  $\mathcal{H}^\bullet$  with terms

$$\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 28.

**Proof.** Choose a quasi-isomorphism  $\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$  where  $\mathcal{I}^\bullet$  is a bounded below complex of injectives. Note that  $\mathcal{I}^\bullet$  is K-injective (Derived Categories, Lemma 29.4). Hence the construction in Section 28 shows that  $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$  is represented by the complex  $(\mathcal{H}')^\bullet$  with terms

$$(\mathcal{H}')^n = \prod_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that  $\mathcal{H}^\bullet$  is the total complex associated to the double complex with terms  $\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$  and similarly for  $(\mathcal{H}')^\bullet$ . The natural map  $(\mathcal{H}')^\bullet \rightarrow \mathcal{H}^\bullet$  comes from a map of double

complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 22.6)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^{\bullet}))$$

converging to  $H^{p+q}(\mathcal{H}^{\bullet})$  and similarly for  $(\mathcal{H}')^{\bullet}$ . To finish the proof of the lemma it suffices to show that  $\mathcal{F}^{\bullet} \rightarrow \mathcal{I}^{\bullet}$  induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F}^{\bullet})) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{I}^{\bullet}))$$

on cohomology sheaves whenever  $\mathcal{E}$  is a direct summand of a finite free  $\mathcal{O}$ -module. Since this is clear when  $\mathcal{E}$  is finite free the result follows.  $\square$

### 37. Pseudo-coherent modules

08FS In this section we discuss pseudo-coherent complexes.

08FT **Definition 37.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^{\bullet}$  be a complex of  $\mathcal{O}$ -modules. Let  $m \in \mathbf{Z}$ .

- (1) We say  $\mathcal{E}^{\bullet}$  is *m-pseudo-coherent* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  and for each  $i$  a morphism of complexes  $\alpha_i : \mathcal{E}_i^{\bullet} \rightarrow \mathcal{E}^{\bullet}|_{U_i}$  where  $\mathcal{E}_i^{\bullet}$  is a strictly perfect complex of  $\mathcal{O}_{U_i}$ -modules and  $H^j(\alpha_i)$  is an isomorphism for  $j > m$  and  $H^m(\alpha_i)$  is surjective.
- (2) We say  $\mathcal{E}^{\bullet}$  is *pseudo-coherent* if it is *m-pseudo-coherent* for all  $m$ .
- (3) We say an object  $E$  of  $D(\mathcal{O})$  is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a *m-pseudo-coherent* (resp. *pseudo-coherent*) complex of  $\mathcal{O}$ -modules.

If  $\mathcal{C}$  has a final object  $X$  which is quasi-compact (for example if every covering of  $X$  can be refined by a finite covering), then an *m-pseudo-coherent* object of  $D(\mathcal{O})$  is in  $D^-(\mathcal{O})$ . But this need not be the case in general.

08FU **Lemma 37.2.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ .

- (1) If  $\mathcal{C}$  has a final object  $X$  and if there exist a covering  $\{U_i \rightarrow X\}$ , strictly perfect complexes  $\mathcal{E}_i^{\bullet}$  of  $\mathcal{O}_{U_i}$ -modules, and maps  $\alpha_i : \mathcal{E}_i^{\bullet} \rightarrow E|_{U_i}$  in  $D(\mathcal{O}_{U_i})$  with  $H^j(\alpha_i)$  an isomorphism for  $j > m$  and  $H^m(\alpha_i)$  surjective, then  $E$  is *m-pseudo-coherent*.
- (2) If  $E$  is *m-pseudo-coherent*, then any complex of  $\mathcal{O}$ -modules representing  $E$  is *m-pseudo-coherent*.
- (3) If for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  such that  $E|_{U_i}$  is *m-pseudo-coherent*, then  $E$  is *m-pseudo-coherent*.

**Proof.** Let  $\mathcal{F}^{\bullet}$  be any complex representing  $E$  and let  $X$ ,  $\{U_i \rightarrow X\}$ , and  $\alpha_i : \mathcal{E}_i^{\bullet} \rightarrow E|_{U_i}$  be as in (1). We will show that  $\mathcal{F}^{\bullet}$  is *m-pseudo-coherent* as a complex, which will prove (1) and (2) in case  $\mathcal{C}$  has a final object. By Lemma 36.8 we can after refining the covering  $\{U_i \rightarrow X\}$  represent the maps  $\alpha_i$  by maps of complexes  $\alpha_i : \mathcal{E}_i^{\bullet} \rightarrow \mathcal{F}^{\bullet}|_{U_i}$ . By assumption  $H^j(\alpha_i)$  are isomorphisms for  $j > m$ , and  $H^m(\alpha_i)$  is surjective whence  $\mathcal{F}^{\bullet}$  is *m-pseudo-coherent*.

Proof of (2). By the above we see that  $\mathcal{F}^{\bullet}|_U$  is *m-pseudo-coherent* as a complex of  $\mathcal{O}_U$ -modules for all objects  $U$  of  $\mathcal{C}$ . It is a formal consequence of the definitions that  $\mathcal{F}^{\bullet}$  is *m-pseudo-coherent*.

Proof of (3). Follows from the definitions and Sites, Definition 6.2 part (2).  $\square$

08H4 **Lemma 37.3.** *Let  $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites. Let  $E$  be an object of  $D(\mathcal{O}_{\mathcal{C}})$ . If  $E$  is  $m$ -pseudo-coherent, then  $Lf^*E$  is  $m$ -pseudo-coherent.*

**Proof.** Say  $f$  is given by the functor  $u : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $U$  be an object of  $\mathcal{C}$ . By Sites, Lemma 14.9 we can find a covering  $\{U_i \rightarrow U\}$  and for each  $i$  a morphism  $U_i \rightarrow u(V_i)$  for some object  $V_i$  of  $\mathcal{D}$ . By Lemma 37.2 it suffices to show that  $Lf^*E|_{U_i}$  is  $m$ -pseudo-coherent. To do this it is enough to show that  $Lf^*E|_{u(V_i)}$  is  $m$ -pseudo-coherent, since  $Lf^*E|_{U_i}$  is the restriction of  $Lf^*E|_{u(V_i)}$  to  $\mathcal{C}/U_i$  (via Modules on Sites, Lemma 19.5). By the commutative diagram of Modules on Sites, Lemma 20.1 it suffices to prove the lemma for the morphism of ringed sites  $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \rightarrow (\mathcal{D}/V_i, \mathcal{O}_{V_i})$ . Thus we may assume  $\mathcal{D}$  has a final object  $Y$  such that  $X = u(Y)$  is a final object of  $\mathcal{C}$ .

Let  $\{V_i \rightarrow Y\}$  be a covering such that for each  $i$  there exists a strictly perfect complex  $\mathcal{F}_i^\bullet$  of  $\mathcal{O}_{V_i}$ -modules and a morphism  $\alpha_i : \mathcal{F}_i^\bullet \rightarrow E|_{V_i}$  of  $D(\mathcal{O}_{V_i})$  such that  $H^j(\alpha_i)$  is an isomorphism for  $j > m$  and  $H^m(\alpha_i)$  is surjective. Arguing as above it suffices to prove the result for  $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \rightarrow (\mathcal{D}/V_i, \mathcal{O}_{V_i})$ . Hence we may assume that there exists a strictly perfect complex  $\mathcal{F}^\bullet$  of  $\mathcal{O}_{\mathcal{D}}$ -modules and a morphism  $\alpha : \mathcal{F}^\bullet \rightarrow E$  of  $D(\mathcal{O}_{\mathcal{D}})$  such that  $H^j(\alpha)$  is an isomorphism for  $j > m$  and  $H^m(\alpha)$  is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \rightarrow E \rightarrow C \rightarrow \mathcal{F}^\bullet[1]$$

The assumption on  $\alpha$  means exactly that the cohomology sheaves  $H^j(C)$  are zero for all  $j \geq m$ . Applying  $Lf^*$  we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \rightarrow Lf^*E \rightarrow Lf^*C \rightarrow Lf^*\mathcal{F}^\bullet[1]$$

By the construction of  $Lf^*$  as a left derived functor we see that  $H^j(Lf^*C) = 0$  for  $j \geq m$  (by the dual of Derived Categories, Lemma 17.1). Hence  $H^j(Lf^*\alpha)$  is an isomorphism for  $j > m$  and  $H^m(Lf^*\alpha)$  is surjective. On the other hand, since  $\mathcal{F}^\bullet$  is a bounded above complex of flat  $\mathcal{O}_{\mathcal{D}}$ -modules we see that  $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$ . Applying Lemma 36.4 we conclude.  $\square$

08FV **Lemma 37.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site and  $m \in \mathbf{Z}$ . Let  $(K, L, M, f, g, h)$  be a distinguished triangle in  $D(\mathcal{O})$ .*

- (1) *If  $K$  is  $(m+1)$ -pseudo-coherent and  $L$  is  $m$ -pseudo-coherent then  $M$  is  $m$ -pseudo-coherent.*
- (2) *If  $K$  and  $M$  are  $m$ -pseudo-coherent, then  $L$  is  $m$ -pseudo-coherent.*
- (3) *If  $L$  is  $(m+1)$ -pseudo-coherent and  $M$  is  $m$ -pseudo-coherent, then  $K$  is  $(m+1)$ -pseudo-coherent.*

**Proof.** Proof of (1). Let  $U$  be an object of  $\mathcal{C}$ . Choose a covering  $\{U_i \rightarrow U\}$  and maps  $\alpha_i : \mathcal{K}_i^\bullet \rightarrow K|_{U_i}$  in  $D(\mathcal{O}_{U_i})$  with  $\mathcal{K}_i^\bullet$  strictly perfect and  $H^j(\alpha_i)$  isomorphisms for  $j > m+1$  and surjective for  $j = m+1$ . We may replace  $\mathcal{K}_i^\bullet$  by  $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$  and hence we may assume that  $\mathcal{K}_i^j = 0$  for  $j < m+1$ . After refining the covering we may choose maps  $\beta_i : \mathcal{L}_i^\bullet \rightarrow L|_{U_i}$  in  $D(\mathcal{O}_{U_i})$  with  $\mathcal{L}_i^\bullet$  strictly perfect such that  $H^j(\beta)$  is an isomorphism for  $j > m$  and surjective for  $j = m$ . By Lemma 36.7 we can, after refining the covering, find maps of complexes  $\gamma_i : \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  such that the



diagrams

$$\begin{array}{ccc} K|_{U_i} & \longrightarrow & L|_{U_i} \\ \alpha_i \uparrow & & \uparrow \beta_i \\ \mathcal{K}_i^\bullet & \xrightarrow{\gamma_i} & \mathcal{L}_i^\bullet \end{array}$$

are commutative in  $D(\mathcal{O}_{U_i})$  (this requires representing the maps  $\alpha_i$ ,  $\beta_i$  and  $K|_{U_i} \rightarrow L|_{U_i}$  by actual maps of complexes; some details omitted). The cone  $C(\gamma_i)^\bullet$  is strictly perfect (Lemma 36.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 5.19 and 5.20 that  $C(\gamma_i)^\bullet \rightarrow M|_{U_i}$  induces an isomorphism on cohomology in degrees  $> m$  and a surjection in degree  $m$ . Hence  $M$  is  $m$ -pseudo-coherent by Lemma 37.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle.  $\square$

09J9 **Lemma 37.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L$  be objects of  $D(\mathcal{O})$ .*

- (1) *If  $K$  is  $n$ -pseudo-coherent and  $H^i(K) = 0$  for  $i > a$  and  $L$  is  $m$ -pseudo-coherent and  $H^j(L) = 0$  for  $j > b$ , then  $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$  is  $t$ -pseudo-coherent with  $t = \max(m + a, n + b)$ .*
- (2) *If  $K$  and  $L$  are pseudo-coherent, then  $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$  is pseudo-coherent.*

**Proof.** Proof of (1). Let  $U$  be an object of  $\mathcal{C}$ . By replacing  $U$  by the members of a covering and replacing  $\mathcal{C}$  by the localization  $\mathcal{C}/U$  we may assume there exist strictly perfect complexes  $\mathcal{K}^\bullet$  and  $\mathcal{L}^\bullet$  and maps  $\alpha : \mathcal{K}^\bullet \rightarrow K$  and  $\beta : \mathcal{L}^\bullet \rightarrow L$  with  $H^i(\alpha)$  and isomorphism for  $i > n$  and surjective for  $i = n$  and with  $H^i(\beta)$  and isomorphism for  $i > m$  and surjective for  $i = m$ . Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \rightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} L$$

induces isomorphisms on cohomology sheaves in degree  $i$  for  $i > t$  and a surjection for  $i = t$ . This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let  $U$  be an object of  $\mathcal{C}$ . We may first replace  $U$  by the members of a covering and  $\mathcal{C}$  by the localization  $\mathcal{C}/U$  to reduce to the case that  $K$  and  $L$  are bounded above. Then the statement follows immediately from case (1).  $\square$

08FW **Lemma 37.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $m \in \mathbf{Z}$ . If  $K \oplus L$  is  $m$ -pseudo-coherent (resp. pseudo-coherent) in  $D(\mathcal{O})$  so are  $K$  and  $L$ .*

**Proof.** Assume that  $K \oplus L$  is  $m$ -pseudo-coherent. Let  $U$  be an object of  $\mathcal{C}$ . After replacing  $U$  by the members of a covering we may assume  $K \oplus L \in D^-(\mathcal{O}_U)$ , hence  $L \in D^-(\mathcal{O}_U)$ . Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 4.9. By Lemma 37.4 we see that  $L \oplus L[1]$  is  $m$ -pseudo-coherent. Hence also  $L[1] \oplus L[2]$  is  $m$ -pseudo-coherent. By induction  $L[n] \oplus L[n+1]$  is  $m$ -pseudo-coherent. Since  $L$  is bounded above we see that  $L[n]$  is  $m$ -pseudo-coherent for large  $n$ . Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that  $L[n-1], L[n-2], \dots, L$  are  $m$ -pseudo-coherent as desired.  $\square$

08FX **Lemma 37.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be an object of  $D(\mathcal{O})$ . Let  $m \in \mathbf{Z}$ .*

- (1) *If  $K$  is  $m$ -pseudo-coherent and  $H^i(K) = 0$  for  $i > m$ , then  $H^m(K)$  is a finite type  $\mathcal{O}$ -module.*
- (2) *If  $K$  is  $m$ -pseudo-coherent and  $H^i(K) = 0$  for  $i > m + 1$ , then  $H^{m+1}(K)$  is a finitely presented  $\mathcal{O}$ -module.*

**Proof.** Proof of (1). Let  $U$  be an object of  $\mathcal{C}$ . We have to show that  $H^m(K)$  can be generated by finitely many sections over the members of a covering of  $U$  (see Modules on Sites, Definition 23.1). Thus during the proof we may (finitely often) choose a covering  $\{U_i \rightarrow U\}$  and replace  $\mathcal{C}$  by  $\mathcal{C}/U_i$  and  $U$  by  $U_i$ . In particular, by our definitions we may assume there exists a strictly perfect complex  $\mathcal{E}^\bullet$  and a map  $\alpha : \mathcal{E}^\bullet \rightarrow K$  which induces an isomorphism on cohomology in degrees  $> m$  and a surjection in degree  $m$ . It suffices to prove the result for  $\mathcal{E}^\bullet$ . Let  $n$  be the largest integer such that  $\mathcal{E}^n \neq 0$ . If  $n = m$ , then  $H^m(\mathcal{E}^\bullet)$  is a quotient of  $\mathcal{E}^n$  and the result is clear. If  $n > m$ , then  $\mathcal{E}^{n-1} \rightarrow \mathcal{E}^n$  is surjective as  $H^n(\mathcal{E}^\bullet) = 0$ . By Lemma 36.5 we can (after replacing  $U$  by the members of a covering) find a section of this surjection and write  $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$ . Hence it suffices to prove the result for the complex  $(\mathcal{E}')^\bullet$  which is the same as  $\mathcal{E}^\bullet$  except has  $\mathcal{E}'$  in degree  $n-1$  and 0 in degree  $n$ . We win by induction on  $n$ .

Proof of (2). Pick an object  $U$  of  $\mathcal{C}$ . As in the proof of (1) we may work locally on  $U$ . Hence we may assume there exists a strictly perfect complex  $\mathcal{E}^\bullet$  and a map  $\alpha : \mathcal{E}^\bullet \rightarrow K$  which induces an isomorphism on cohomology in degrees  $> m$  and a surjection in degree  $m$ . As in the proof of (1) we can reduce to the case that  $\mathcal{E}^i = 0$  for  $i > m + 1$ . Then we see that  $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \text{Coker}(\mathcal{E}^m \rightarrow \mathcal{E}^{m+1})$  which is of finite presentation.  $\square$

### 38. Tor dimension

08FY In this section we take a closer look at resolutions by flat modules.

08FZ **Definition 38.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ . Let  $a, b \in \mathbf{Z}$  with  $a \leq b$ .

- (1) We say  $E$  has *tor-amplitude in  $[a, b]$*  if  $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}) = 0$  for all  $\mathcal{O}$ -modules  $\mathcal{F}$  and all  $i \notin [a, b]$ .
- (2) We say  $E$  has *finite tor dimension* if it has tor-amplitude in  $[a, b]$  for some  $a, b$ .
- (3) We say  $E$  *locally has finite tor dimension* if for any object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  such that  $E|_{U_i}$  has finite tor dimension for all  $i$ .

Note that if  $E$  has finite tor dimension, then  $E$  is an object of  $D^b(\mathcal{O})$  as can be seen by taking  $\mathcal{F} = \mathcal{O}$  in the definition above.

08G0 **Lemma 38.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^\bullet$  be a bounded above complex of flat  $\mathcal{O}$ -modules with tor-amplitude in  $[a, b]$ . Then  $\text{Coker}(d_{\mathcal{E}^\bullet}^{a-1})$  is a flat  $\mathcal{O}$ -module.*

**Proof.** As  $\mathcal{E}^\bullet$  is a bounded above complex of flat modules we see that  $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$  for any  $\mathcal{O}$ -module  $\mathcal{F}$ . Hence for every  $\mathcal{O}$ -module  $\mathcal{F}$  the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F}$$

is exact in the middle. Since  $\mathcal{E}^{a-2} \rightarrow \mathcal{E}^{a-1} \rightarrow \mathcal{E}^a \rightarrow \text{Coker}(d^{a-1}) \rightarrow 0$  is a flat resolution this implies that  $\text{Tor}_1^{\mathcal{O}}(\text{Coker}(d^{a-1}), \mathcal{F}) = 0$  for all  $\mathcal{O}$ -modules  $\mathcal{F}$ . This means that  $\text{Coker}(d^{a-1})$  is flat, see Lemma 18.14.  $\square$

08G1 **Lemma 38.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ . Let  $a, b \in \mathbf{Z}$  with  $a \leq b$ . The following are equivalent*

- (1)  *$E$  has tor-amplitude in  $[a, b]$ .*
- (2)  *$E$  is represented by a complex  $\mathcal{E}^\bullet$  of flat  $\mathcal{O}$ -modules with  $\mathcal{E}^i = 0$  for  $i \notin [a, b]$ .*

**Proof.** If (2) holds, then we may compute  $E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$  and it is clear that (1) holds.

Assume that (1) holds. We may represent  $E$  by a bounded above complex of flat  $\mathcal{O}$ -modules  $\mathcal{K}^\bullet$ , see Section 18. Let  $n$  be the largest integer such that  $\mathcal{K}^n \neq 0$ . If  $n > b$ , then  $\mathcal{K}^{n-1} \rightarrow \mathcal{K}^n$  is surjective as  $H^n(\mathcal{K}^\bullet) = 0$ . As  $\mathcal{K}^n$  is flat we see that  $\text{Ker}(\mathcal{K}^{n-1} \rightarrow \mathcal{K}^n)$  is flat (Modules on Sites, Lemma 28.9). Hence we may replace  $\mathcal{K}^\bullet$  by  $\tau_{\leq n-1}\mathcal{K}^\bullet$ . Thus, by induction on  $n$ , we reduce to the case that  $\mathcal{K}^\bullet$  is a complex of flat  $\mathcal{O}$ -modules with  $\mathcal{K}^i = 0$  for  $i > b$ .

Set  $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$ . Everything is clear except that  $\mathcal{E}^a$  is flat which follows immediately from Lemma 38.2 and the definitions.  $\square$

08H5 **Lemma 38.4.** *Let  $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites. Let  $E$  be an object of  $D(\mathcal{O}_{\mathcal{D}})$ . If  $E$  has tor amplitude in  $[a, b]$ , then  $Lf^*E$  has tor amplitude in  $[a, b]$ .*

**Proof.** Assume  $E$  has tor amplitude in  $[a, b]$ . By Lemma 38.3 we can represent  $E$  by a complex of  $\mathcal{E}^\bullet$  of flat  $\mathcal{O}$ -modules with  $\mathcal{E}^i = 0$  for  $i \notin [a, b]$ . Then  $Lf^*E$  is represented by  $f^*\mathcal{E}^\bullet$ . By Modules on Sites, Lemma 38.1 the module  $f^*\mathcal{E}^i$  are flat. Thus by Lemma 38.3 we conclude that  $Lf^*E$  has tor amplitude in  $[a, b]$ .  $\square$

08G2 **Lemma 38.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K, L, M, f, g, h)$  be a distinguished triangle in  $D(\mathcal{O})$ . Let  $a, b \in \mathbf{Z}$ .*

- (1) *If  $K$  has tor-amplitude in  $[a+1, b+1]$  and  $L$  has tor-amplitude in  $[a, b]$  then  $M$  has tor-amplitude in  $[a, b]$ .*
- (2) *If  $K$  and  $M$  have tor-amplitude in  $[a, b]$ , then  $L$  has tor-amplitude in  $[a, b]$ .*
- (3) *If  $L$  has tor-amplitude in  $[a+1, b+1]$  and  $M$  has tor-amplitude in  $[a, b]$ , then  $K$  has tor-amplitude in  $[a+1, b+1]$ .*

**Proof.** Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that  $-\otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$  preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation.  $\square$

09JA **Lemma 38.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K, L$  be objects of  $D(\mathcal{O})$ . If  $K$  has tor-amplitude in  $[a, b]$  and  $L$  has tor-amplitude in  $[c, d]$  then  $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$  has tor amplitude in  $[a+c, b+d]$ .*

**Proof.** Omitted. Hint: use the spectral sequence for tors.  $\square$

08G3 **Lemma 38.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $a, b \in \mathbf{Z}$ . For  $K, L$  objects of  $D(\mathcal{O})$  if  $K \oplus L$  has tor amplitude in  $[a, b]$  so do  $K$  and  $L$ .*

**Proof.** Clear from the fact that the Tor functors are additive.  $\square$

0942 **Lemma 38.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{I} \subset \mathcal{O}$  be a sheaf of ideals. Let  $K$  be an object of  $D(\mathcal{O})$ .*

- (1) *If  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$  is bounded above, then  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$  is uniformly bounded above for all  $n$ .*
- (2) *If  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$  as an object of  $D(\mathcal{O}/\mathcal{I})$  has tor amplitude in  $[a, b]$ , then  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$  as an object of  $D(\mathcal{O}/\mathcal{I}^n)$  has tor amplitude in  $[a, b]$  for all  $n$ .*

**Proof.** Proof of (1). Assume that  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$  is bounded above, say  $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}) = 0$  for  $i > b$ . Note that we have distinguished triangles

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} \rightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^{n+1} \rightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \rightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} = (K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that  $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n) = 0$  for  $i > b$  for all  $n$ .

Proof of (2). Assume  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$  as an object of  $D(\mathcal{O}/\mathcal{I})$  has tor amplitude in  $[a, b]$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}/\mathcal{I}^n$ -modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \dots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of  $\mathcal{O}/\mathcal{I}$ -modules. Thus to prove that  $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$  has tor amplitude in  $[a, b]$  it suffices to show  $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G})$  is zero for  $i \notin [a, b]$  for all  $\mathcal{O}/\mathcal{I}$ -modules  $\mathcal{G}$ . Since

$$(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n) \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G} = (K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{G}$$

for every sheaf of  $\mathcal{O}/\mathcal{I}$ -modules  $\mathcal{G}$  the result follows.  $\square$

0DJJ **Lemma 38.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ . Let  $a, b \in \mathbf{Z}$ .*

- (1) *If  $E$  has tor amplitude in  $[a, b]$ , then for every point  $p$  of the site  $\mathcal{C}$  the object  $E_p$  of  $D(\mathcal{O}_p)$  has tor amplitude in  $[a, b]$ .*
- (2) *If  $\mathcal{C}$  has enough points, then the converse is true.*

**Proof.** Proof of (1). This follows because taking stalks at  $p$  is the same as pulling back by the morphism of ringed sites  $(p, \mathcal{O}_p) \rightarrow (\mathcal{C}, \mathcal{O})$  and hence we can apply Lemma 38.4.

Proof of (2). If  $\mathcal{C}$  has enough points, then we can check vanishing of  $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})$  at stalks, see Modules on Sites, Lemma 14.4. Since  $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})_p = H^i(E_p \otimes_{\mathcal{O}_p}^{\mathbf{L}} \mathcal{F}_p)$  we conclude.  $\square$

### 39. Perfect complexes

08G4 In this section we discuss properties of perfect complexes on ringed sites.

08G5 **Definition 39.1.** Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $\mathcal{E}^\bullet$  be a complex of  $\mathcal{O}$ -modules. We say  $\mathcal{E}^\bullet$  is *perfect* if for every object  $U$  of  $\mathcal{C}$  there exists a covering  $\{U_i \rightarrow U\}$  such that for each  $i$  there exists a morphism of complexes  $\mathcal{E}_i^\bullet \rightarrow \mathcal{E}^\bullet|_{U_i}$  which is a quasi-isomorphism with  $\mathcal{E}_i^\bullet$  strictly perfect. An object  $E$  of  $D(\mathcal{O})$  is *perfect* if it can be represented by a perfect complex of  $\mathcal{O}$ -modules.

08G6 **Lemma 39.2.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ .*

- (1) If  $\mathcal{C}$  has a final object  $X$  and there exist a covering  $\{U_i \rightarrow X\}$ , strictly perfect complexes  $\mathcal{E}_i^\bullet$  of  $\mathcal{O}_{U_i}$ -modules, and isomorphisms  $\alpha_i : \mathcal{E}_i^\bullet \rightarrow E|_{U_i}$  in  $D(\mathcal{O}_{U_i})$ , then  $E$  is perfect.
- (2) If  $E$  is perfect, then any complex representing  $E$  is perfect.

**Proof.** Identical to the proof of Lemma 37.2.  $\square$

08G7 **Lemma 39.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ . Let  $a \leq b$  be integers. If  $E$  has tor amplitude in  $[a, b]$  and is  $(a-1)$ -pseudo-coherent, then  $E$  is perfect.*

**Proof.** Let  $U$  be an object of  $\mathcal{C}$ . After replacing  $U$  by the members of a covering and  $\mathcal{C}$  by the localization  $\mathcal{C}/U$  we may assume there exists a strictly perfect complex  $\mathcal{E}^\bullet$  and a map  $\alpha : \mathcal{E}^\bullet \rightarrow E$  such that  $H^i(\alpha)$  is an isomorphism for  $i \geq a$ . We may and do replace  $\mathcal{E}^\bullet$  by  $\sigma_{\geq a-1}\mathcal{E}^\bullet$ . Choose a distinguished triangle

$$\mathcal{E}^\bullet \rightarrow E \rightarrow C \rightarrow \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of  $E$  and  $\mathcal{E}^\bullet$  and the assumption on  $\alpha$  we obtain  $C \cong \mathcal{K}[a-2]$  with  $\mathcal{K} = \text{Ker}(\mathcal{E}^{a-1} \rightarrow \mathcal{E}^a)$ . Let  $\mathcal{F}$  be an  $\mathcal{O}$ -module. Applying  $-\otimes_{\mathcal{O}}^L \mathcal{F}$  the assumption that  $E$  has tor amplitude in  $[a, b]$  implies  $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$  has image  $\text{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$ . It follows that  $\text{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$  where  $\mathcal{E}' = \text{Coker}(\mathcal{E}^{a-1} \rightarrow \mathcal{E}^a)$ . Hence  $\mathcal{E}'$  is flat (Lemma 18.14). Thus there exists a covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{E}'|_{U_i}$  is a direct summand of a finite free module by Modules on Sites, Lemma 28.13. Thus the complex

$$\mathcal{E}'|_{U_i} \rightarrow \mathcal{E}^{a-1}|_{U_i} \rightarrow \dots \rightarrow \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to  $E|_{U_i}$  and  $E$  is perfect.  $\square$

08G8 **Lemma 39.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $E$  be an object of  $D(\mathcal{O})$ . The following are equivalent*

- (1)  $E$  is perfect, and  
(2)  $E$  is pseudo-coherent and locally has finite tor dimension.

**Proof.** Assume (1). Let  $U$  be an object of  $\mathcal{C}$ . By definition there exists a covering  $\{U_i \rightarrow U\}$  such that  $E|_{U_i}$  is represented by a strictly perfect complex. Thus  $E$  is pseudo-coherent (i.e.,  $m$ -pseudo-coherent for all  $m$ ) by Lemma 37.2. Moreover, a direct summand of a finite free module is flat, hence  $E|_{U_i}$  has finite Tor dimension by Lemma 38.3. Thus (2) holds.

Assume (2). Let  $U$  be an object of  $\mathcal{C}$ . After replacing  $U$  by the members of a covering we may assume there exist integers  $a \leq b$  such that  $E|_U$  has tor amplitude in  $[a, b]$ . Since  $E|_U$  is  $m$ -pseudo-coherent for all  $m$  we conclude using Lemma 39.3.  $\square$

08H6 **Lemma 39.5.** *Let  $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed sites. Let  $E$  be an object of  $D(\mathcal{O}_{\mathcal{D}})$ . If  $E$  is perfect in  $D(\mathcal{O}_{\mathcal{D}})$ , then  $Lf^*E$  is perfect in  $D(\mathcal{O}_{\mathcal{C}})$ .*

**Proof.** This follows from Lemma 39.4, 38.4, and 37.3.  $\square$

08G9 **Lemma 39.6.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K, L, M, f, g, h)$  be a distinguished triangle in  $D(\mathcal{O})$ . If two out of three of  $K, L, M$  are perfect then the third is also perfect.*

**Proof.** First proof: Combine Lemmas 39.4, 37.4, and 38.5. Second proof (sketch): Say  $K$  and  $L$  are perfect. Let  $U$  be an object of  $\mathcal{C}$ . After replacing  $U$  by the members of a covering we may assume that  $K|_U$  and  $L|_U$  are represented by strictly perfect complexes  $\mathcal{K}^\bullet$  and  $\mathcal{L}^\bullet$ . After replacing  $U$  by the members of a covering we may assume the map  $K|_U \rightarrow L|_U$  is given by a map of complexes  $\alpha : \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$ , see Lemma 36.8. Then  $M|_U$  is isomorphic to the cone of  $\alpha$  which is strictly perfect by Lemma 36.2.  $\square$

09JB **Lemma 39.7.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. If  $K, L$  are perfect objects of  $D(\mathcal{O})$ , then so is  $K \otimes_{\mathcal{O}}^{\mathbf{L}} L$ .*

**Proof.** Follows from Lemmas 39.4, 37.5, and 38.6.  $\square$

08GA **Lemma 39.8.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. If  $K \oplus L$  is a perfect object of  $D(\mathcal{O})$ , then so are  $K$  and  $L$ .*

**Proof.** Follows from Lemmas 39.4, 37.6, and 38.7.  $\square$

08JJ **Lemma 39.9.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $K$  be a perfect object of  $D(\mathcal{O})$ . Then  $K^\vee = R\mathcal{H}om(K, \mathcal{O})$  is a perfect object too and  $(K^\vee)^\vee = K$ . There are functorial isomorphisms*

$$K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M = R\mathcal{H}om_{\mathcal{O}}(K, M)$$

and

$$H^0(\mathcal{C}, K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M) = \text{Hom}_{D(\mathcal{O})}(K, M)$$

for  $M$  in  $D(\mathcal{O})$ .

**Proof.** We will use without further mention that formation of internal hom commutes with restriction (Lemma 28.3). In particular we may check the first two statements locally, i.e., given any object  $U$  of  $\mathcal{C}$  it suffices to prove there is a covering  $\{U_i \rightarrow U\}$  such that the statement is true after restricting to  $\mathcal{C}/U_i$  for each  $i$ . By Lemma 28.9 to see the final statement it suffices to check that the map (28.9.1)

$$K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is a local question as well. Hence it suffices to prove the lemma when  $K$  is represented by a strictly perfect complex.

Assume  $K$  is represented by the strictly perfect complex  $\mathcal{E}^\bullet$ . Then it follows from Lemma 36.9 that  $K^\vee$  is represented by the complex whose terms are  $(\mathcal{E}^n)^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^n, \mathcal{O})$  in degree  $-n$ . Since  $\mathcal{E}^n$  is a direct summand of a finite free  $\mathcal{O}$ -module, so is  $(\mathcal{E}^n)^\vee$ . Hence  $K^\vee$  is represented by a strictly perfect complex too. It is also clear that  $(K^\vee)^\vee = K$  as we have  $((\mathcal{E}^n)^\vee)^\vee = \mathcal{E}^n$ . To see that (28.9.1) is an isomorphism, represent  $M$  by a  $K$ -flat complex  $\mathcal{F}^\bullet$ . By Lemma 36.9 the complex  $R\mathcal{H}om(K, M)$  is represented by the complex with terms

$$\bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object  $K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} M$  is represented by the complex with terms

$$\bigoplus_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (28.9.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when  $\mathcal{E}$  is a direct summand of a finite free  $\mathcal{O}$ -module and  $\mathcal{F}$  is any  $\mathcal{O}$ -module. This follows immediately from the corresponding statement when  $\mathcal{E}$  is finite free.  $\square$

0A0A **Lemma 39.10.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $(K_n)_{n \in \mathbf{N}}$  be a system of perfect objects of  $D(\mathcal{O})$ . Let  $K = \text{hocolim} K_n$  be the derived colimit (Derived Categories, Definition 31.1). Then for any object  $E$  of  $D(\mathcal{O})$  we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes_{\mathcal{O}}^{\mathbf{L}} K_n^{\vee}$$

where  $(K_n^{\vee})$  is the inverse system of dual perfect complexes.

**Proof.** By Lemma 39.9 we have  $R\lim E \otimes_{\mathcal{O}}^{\mathbf{L}} K_n^{\vee} = R\lim R\mathcal{H}om(K_n, E)$  which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \rightarrow \prod R\mathcal{H}om(K_n, E) \rightarrow \prod R\mathcal{H}om(K_n, E)$$

Because  $K$  similarly fits into the distinguished triangle  $\bigoplus K_n \rightarrow \bigoplus K_n \rightarrow K$  it suffices to show that  $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$ . This is a formal consequence of (28.0.1) and the fact that derived tensor product commutes with direct sums.  $\square$

#### 40. Projection formula

0943 Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $E \in D(\mathcal{O}_{\mathcal{C}})$  and  $K \in D(\mathcal{O}_{\mathcal{D}})$ . Without any further assumptions there is a map

0B56 (40.0.1) 
$$Rf_* E \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} K \longrightarrow Rf_*(E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^* K)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_* E \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} K) = Lf^* Rf_* E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^* K \longrightarrow E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^* K$$

coming from the map  $Lf^* Rf_* E \rightarrow E$  and Lemmas 19.4 and 20.1. A reasonably general version of the projection formula is the following.

0944 **Lemma 40.1.** *Let  $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \rightarrow (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$  be a morphism of ringed topoi. Let  $E \in D(\mathcal{O}_{\mathcal{C}})$  and  $K \in D(\mathcal{O}_{\mathcal{D}})$ . If  $K$  is perfect, then*

$$Rf_* E \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} K = Rf_*(E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^* K)$$

in  $D(\mathcal{O}_{\mathcal{D}})$ .

**Proof.** To check (40.0.1) is an isomorphism we may work locally on  $\mathcal{D}$ , i.e., for any object  $V$  of  $\mathcal{D}$  we have to find a covering  $\{V_j \rightarrow V\}$  such that the map restricts to an isomorphism on  $V_j$ . By definition of perfect objects, this means we may assume  $K$  is represented by a strictly perfect complex of  $\mathcal{O}_{\mathcal{D}}$ -modules. Note that, completely generally, the statement is true for  $K = K_1 \oplus K_2$ , if and only if the statement is true for  $K_1$  and  $K_2$ . Hence we may assume  $K$  is a finite complex of finite free  $\mathcal{O}_{\mathcal{D}}$ -modules. In this case a simple argument involving stupid truncations reduces the statement to the case where  $K$  is represented by a finite free  $\mathcal{O}_{\mathcal{D}}$ -module. Since the statement is invariant under finite direct summands in the  $K$  variable, we conclude it suffices to prove it for  $K = \mathcal{O}_{\mathcal{D}}[n]$  in which case it is trivial.  $\square$

0E48 **Remark 40.2.** The map (40.0.1) is compatible with the base change map of Remark 20.3 in the following sense. Namely, suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ f' \downarrow & & \downarrow f \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let  $E \in D(\mathcal{O}_{\mathcal{C}})$  and  $K \in D(\mathcal{O}_{\mathcal{D}})$ . Then the diagram

$$\begin{array}{ccc} Lg^*(Rf_*E \otimes_{\mathcal{O}_{\mathcal{D}}}^{\mathbf{L}} K) & \xrightarrow{p} & Lg^*Rf_*(E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^*K) \\ \downarrow t & & \downarrow b \\ Lg^*Rf_*E \otimes_{\mathcal{O}_{\mathcal{D}'}}^{\mathbf{L}} Lg^*K & & Rf'_*L(g')^*(E \otimes_{\mathcal{O}_{\mathcal{C}}}^{\mathbf{L}} Lf^*K) \\ \downarrow b & & \downarrow t \\ Rf'_*L(g')^*E \otimes_{\mathcal{O}_{\mathcal{D}'}}^{\mathbf{L}} Lg^*K & & Rf'_*(L(g')^*E \otimes_{\mathcal{O}_{\mathcal{D}'}}^{\mathbf{L}} L(g')^*Lf^*K) \\ & \searrow p & \downarrow c \\ & & Rf'_*(L(g')^*E \otimes_{\mathcal{O}_{\mathcal{D}'}}^{\mathbf{L}} L(f')^*Lg^*K) \end{array}$$

is commutative. Here arrows labeled  $t$  are gotten by an application of Lemma 19.4, arrows labeled  $b$  by an application of Remark 20.3, arrows labeled  $p$  by an application of (40.0.1), and  $c$  comes from  $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$ . We omit the verification.

#### 41. Weakly contractible objects

0945 An object  $U$  of a site is *weakly contractible* if every surjection  $\mathcal{F} \rightarrow \mathcal{G}$  of sheaves of sets gives rise to a surjection  $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ , see Sites, Definition 39.2.

0946 **Lemma 41.1.** *Let  $\mathcal{C}$  be a site. Let  $U$  be a weakly contractible object of  $\mathcal{C}$ . Then*

- (1) *the functor  $\mathcal{F} \mapsto \mathcal{F}(U)$  is an exact functor  $Ab(\mathcal{C}) \rightarrow Ab$ ,*
- (2)  *$H^p(U, \mathcal{F}) = 0$  for every abelian sheaf  $\mathcal{F}$  and all  $p \geq 1$ , and*
- (3) *for any sheaf of groups  $\mathcal{G}$  any  $\mathcal{G}$ -torsor has a section over  $U$ .*

**Proof.** The first statement follows immediately from the definition (see also Homology, Section 7). The higher derived functors vanish by Derived Categories, Lemma 17.9. Let  $\mathcal{F}$  be a  $\mathcal{G}$ -torsor. Then  $\mathcal{F} \rightarrow *$  is a surjective map of sheaves. Hence (3) follows from the definition as well.  $\square$

It is convenient to list some consequences of having enough weakly contractible objects here.

0947 **Proposition 41.2.** *Let  $\mathcal{C}$  be a site. Let  $\mathcal{B} \subset \text{Ob}(\mathcal{C})$  such that every  $U \in \mathcal{B}$  is weakly contractible and every object of  $\mathcal{C}$  has a covering by elements of  $\mathcal{B}$ . Let  $\mathcal{O}$  be a sheaf of rings on  $\mathcal{C}$ . Then*

- (1) *A complex  $\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3$  of  $\mathcal{O}$ -modules is exact, if and only if  $\mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U) \rightarrow \mathcal{F}_3(U)$  is exact for all  $U \in \mathcal{B}$ .*



- (2) Every object  $K$  of  $D(\mathcal{O})$  is a derived limit of its canonical truncations:  $K = R\lim \tau_{\geq -n}K$ .
- (3) Given an inverse system  $\dots \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1$  with surjective transition maps, the projection  $\lim \mathcal{F}_n \rightarrow \mathcal{F}_1$  is surjective.
- (4) Products are exact on  $\text{Mod}(\mathcal{O})$ .
- (5) Products on  $D(\mathcal{O})$  can be computed by taking products of any representative complexes.
- (6) If  $(\mathcal{F}_n)$  is an inverse system of  $\mathcal{O}$ -modules, then  $R^p \lim \mathcal{F}_n = 0$  for all  $p > 1$  and

$$R^1 \lim \mathcal{F}_n = \text{Coker}(\prod \mathcal{F}_n \rightarrow \prod \mathcal{F}_n)$$

where the map is  $(x_n) \mapsto (x_n - f(x_{n+1}))$ .

- (7) If  $(K_n)$  is an inverse system of objects of  $D(\mathcal{O})$ , then there are short exact sequences

$$0 \rightarrow R^1 \lim H^{p-1}(K_n) \rightarrow H^p(R \lim K_n) \rightarrow \lim H^p(K_n) \rightarrow 0$$

**Proof.** Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of  $\mathcal{C}$  gives an exact sequence by Lemma 41.1. Conversely, assume that  $\mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U) \rightarrow \mathcal{F}_3(U)$  is exact for all  $U \in \mathcal{B}$ . Let  $V$  be an object of  $\mathcal{C}$  and let  $s \in \mathcal{F}_2(V)$  be an element of the kernel of  $\mathcal{F}_2 \rightarrow \mathcal{F}_3$ . By assumption there exists a covering  $\{U_i \rightarrow V\}$  with  $U_i \in \mathcal{B}$ . Then  $s|_{U_i}$  lifts to a section  $s_i \in \mathcal{F}_1(U_i)$ . Thus  $s$  is a section of the image sheaf  $\text{Im}(\mathcal{F}_1 \rightarrow \mathcal{F}_2)$ . In other words, the sequence  $\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3$  is exact.

Proof of (2). This holds by Lemma 22.10 with  $d = 0$ .

Proof of (3). Let  $(\mathcal{F}_n)$  be a system as in (2) and set  $\mathcal{F} = \lim \mathcal{F}_n$ . If  $U \in \mathcal{B}$ , then  $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$  surjects onto  $\mathcal{F}_1(U)$  as all the transition maps  $\mathcal{F}_{n+1}(U) \rightarrow \mathcal{F}_n(U)$  are surjective. Thus  $\mathcal{F} \rightarrow \mathcal{F}_1$  is surjective by Sites, Definition 11.1 and the assumption that every object has a covering by elements of  $\mathcal{B}$ .

Proof of (4). Let  $\mathcal{F}_{i,1} \rightarrow \mathcal{F}_{i,2} \rightarrow \mathcal{F}_{i,3}$  be a family of exact sequences of  $\mathcal{O}$ -modules. We want to show that  $\prod \mathcal{F}_{i,1} \rightarrow \prod \mathcal{F}_{i,2} \rightarrow \prod \mathcal{F}_{i,3}$  is exact. We use the criterion of (1). Let  $U \in \mathcal{B}$ . Then

$$(\prod \mathcal{F}_{i,1})(U) \rightarrow (\prod \mathcal{F}_{i,2})(U) \rightarrow (\prod \mathcal{F}_{i,3})(U)$$

is the same as

$$\prod \mathcal{F}_{i,1}(U) \rightarrow \prod \mathcal{F}_{i,2}(U) \rightarrow \prod \mathcal{F}_{i,3}(U)$$

Each of the sequences  $\mathcal{F}_{i,1}(U) \rightarrow \mathcal{F}_{i,2}(U) \rightarrow \mathcal{F}_{i,3}(U)$  are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 29.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 32.2.

Proof of (6) and (7). We refer to Section 22 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 32.1 we have a distinguished triangle

$$R \lim K_n \rightarrow \prod K_n \rightarrow \prod K_n \rightarrow R \lim K_n[1]$$

Taking the long exact sequence of cohomology sheaves we obtain

$$H^{p-1}(\prod K_n) \rightarrow H^{p-1}(\prod K_n) \rightarrow H^p(R \lim K_n) \rightarrow H^p(\prod K_n) \rightarrow H^p(\prod K_n)$$

Since products are exact by (4) this becomes

$$\prod H^{p-1}(K_n) \rightarrow \prod H^{p-1}(K_n) \rightarrow H^p(R\lim K_n) \rightarrow \prod H^p(K_n) \rightarrow \prod H^p(K_n)$$

Now we first apply this to the case  $K_n = \mathcal{F}_n[0]$  where  $(\mathcal{F}_n)$  is as in (6). We conclude that (6) holds. Next we apply it to  $(K_n)$  as in (7) and we conclude (7) holds.  $\square$

## 42. Compact objects

0948 In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 34.1.

09JC **Lemma 42.1.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Assume  $\mathcal{C}$  has the following properties*

- (1)  $\mathcal{C}$  has a quasi-compact final object  $X$ ,
- (2) every quasi-compact object of  $\mathcal{C}$  has a cofinal system of coverings which are finite and consist of quasi-compact objects,
- (3) for a finite covering  $\{U_i \rightarrow U\}_{i \in I}$  with  $U, U_i$  quasi-compact the fibre products  $U_i \times_U U_j$  are quasi-compact.

*Then any perfect object of  $D(\mathcal{O})$  is compact.*

**Proof.** Let  $K$  be a perfect object and let  $K^\vee$  be its dual, see Lemma 39.9. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

functorially in  $M$  in  $D(\mathcal{O}_X)$ . Since  $K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} -$  commutes with direct sums (by construction) and  $H^0$  does by Lemma 17.1 and the construction of direct sums in Injectives, Lemma 13.4 we obtain the result of the lemma.  $\square$

094B **Lemma 42.2.** *Let  $\mathcal{A}$  be a Grothendieck abelian category. Let  $S \subset \mathrm{Ob}(\mathcal{A})$  be a set of objects such that*

- (1) any object of  $\mathcal{A}$  is a quotient of a direct sum of elements of  $S$ , and
- (2) for any  $E \in S$  the functor  $\mathrm{Hom}_{\mathcal{A}}(E, -)$  commutes with direct sums.

*Then every compact object of  $D(\mathcal{A})$  is a direct summand in  $D(\mathcal{A})$  of a finite complex of finite direct sums of elements of  $S$ .*

**Proof.** Assume  $K \in D(\mathcal{A})$  is a compact object. Represent  $K$  by a complex  $K^\bullet$  and consider the map

$$K^\bullet \longrightarrow \bigoplus_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 14. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that  $K \rightarrow \tau_{\geq n} K$  is zero for at least one  $n$ , i.e.,  $K$  is in  $D^-(R)$ .

We may represent  $K$  by a bounded above complex  $K^\bullet$  each of whose terms is a direct sum of objects from  $S$ , see Derived Categories, Lemma 16.5. Note that we have

$$K^\bullet = \bigcup_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 14. Hence by Derived Categories, Lemmas 31.7 and 31.9 we see that  $1 : K^\bullet \rightarrow K^\bullet$  factors through  $\sigma_{\geq n} K^\bullet \rightarrow K^\bullet$  in  $D(R)$ . Thus we see that  $1 : K^\bullet \rightarrow K^\bullet$  factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in  $D(\mathcal{A})$  for some complex  $L^\bullet$  which is bounded and whose terms are direct sums of elements of  $S$ . Say  $L^i$  is zero for  $i \notin [a, b]$ . Let  $c$  be the largest integer  $\leq b + 1$  such that  $L^i$  a finite direct sum of elements of  $S$  for  $i < c$ . Claim: if  $c < b + 1$ , then we can modify  $L^\bullet$  to increase  $c$ . By induction this claim will show we have a factorization of  $1_K$  as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in  $D(\mathcal{A})$  where  $L$  can be represented by a finite complex of finite direct sums of elements of  $S$ . Note that  $e = \varphi \circ \psi \in \text{End}_{D(\mathcal{A})}(L)$  is an idempotent. By Derived Categories, Lemma 4.13 we see that  $L = \text{Ker}(e) \oplus \text{Ker}(1 - e)$ . The map  $\varphi : K \rightarrow L$  induces an isomorphism with  $\text{Ker}(1 - e)$  in  $D(\mathcal{A})$  and we conclude.

Proof of the claim. Write  $L^c = \bigoplus_{\lambda \in \Lambda} E_\lambda$ . Since  $L^{c-1}$  is a finite direct sum of elements of  $S$  we can by assumption (2) find a finite subset  $\Lambda' \subset \Lambda$  such that  $L^{c-1} \rightarrow L^c$  factors through  $\bigoplus_{\lambda \in \Lambda'} E_\lambda \subset L^c$ . Consider the map of complexes

$$\pi : L^\bullet \longrightarrow \left( \bigoplus_{\lambda \in \Lambda \setminus \Lambda'} E_\lambda \right)[-i]$$

given by the projection onto the factors corresponding to  $\Lambda \setminus \Lambda'$  in degree  $i$ . By our assumption on  $K$  we see that, after possibly replacing  $\Lambda'$  by a larger finite subset, we may assume that  $\pi \circ \varphi = 0$  in  $D(\mathcal{A})$ . Let  $(L')^\bullet \subset L^\bullet$  be the kernel of  $\pi$ . Since  $\pi$  is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in  $D(\mathcal{A})$  (see Derived Categories, Lemma 12.1). Since  $\text{Hom}_{D(\mathcal{A})}(K, -)$  is homological (see Derived Categories, Lemma 4.2) and  $\pi \circ \varphi = 0$ , we can find a morphism  $\varphi' : K^\bullet \rightarrow (L')^\bullet$  in  $D(\mathcal{A})$  whose composition with  $(L')^\bullet \rightarrow L^\bullet$  gives  $\varphi$ . Setting  $\psi'$  equal to the composition of  $\psi$  with  $(L')^\bullet \rightarrow L^\bullet$  we obtain a new factorization. Since  $(L')^\bullet$  agrees with  $L^\bullet$  except in degree  $c$  and since  $(L')^c = \bigoplus_{\lambda \in \Lambda'} E_\lambda$  the claim is proved.  $\square$

094C **Lemma 42.3.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Assume every object of  $\mathcal{C}$  has a covering by quasi-compact objects. Then every compact object of  $D(\mathcal{O})$  is a direct summand in  $D(\mathcal{O})$  of a finite complex whose terms are finite direct sums of  $\mathcal{O}$ -modules of the form  $j_! \mathcal{O}_U$  where  $U$  is a quasi-compact object of  $\mathcal{C}$ .*

**Proof.** Apply Lemma 42.2 where  $S \subset \text{Ob}(\text{Mod}(\mathcal{O}))$  is the set of modules of the form  $j_! \mathcal{O}_U$  with  $U \in \text{Ob}(\mathcal{C})$  quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 28.7 and the assumption that every  $U$  can be covered by quasi-compact objects. Assumption (2) follows as

$$\text{Hom}_{\mathcal{O}}(j_! \mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 17.5.  $\square$

In the situation of the lemma above it is not always true that the modules  $j_! \mathcal{O}_U$  are compact objects of  $D(\mathcal{O})$  (even if  $U$  is a quasi-compact object of  $\mathcal{C}$ ). Here is a criterion.

094D **Lemma 42.4.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$ . The  $\mathcal{O}$ -module  $j_! \mathcal{O}_U$  is a compact object of  $D(\mathcal{O})$  if there exists an integer  $d$  such that*

- (1)  $H^p(U, \mathcal{F}) = 0$  for all  $p > d$ , and
- (2) the functors  $\mathcal{F} \mapsto H^p(U, \mathcal{F})$  commute with direct sums.

**Proof.** Assume (1) and (2). The first means that the functor  $F = H^0(U, -)$  has finite cohomological dimension. Moreover, any direct sum of injective modules is acyclic for  $F$  by (2). Since we may compute  $RF$  by applying  $F$  to any complex of acyclics (Derived Categories, Lemma 30.2). Thus, if  $K_i$  be a family of objects of  $D(\mathcal{O})$ , then we can choose K-injective representatives  $I_i^\bullet$  and we see that  $\bigoplus K_i$  is represented by  $\bigoplus I_i^\bullet$ . Thus  $H^0(U, -)$  commutes with direct sums.  $\square$

094E **Lemma 42.5.** *Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. Let  $U$  be an object of  $\mathcal{C}$  which is quasi-compact and weakly contractible. Then  $j_! \mathcal{O}_U$  is a compact object of  $D(\mathcal{O})$ .*

**Proof.** Combine Lemmas 42.4 and 41.1 with Modules on Sites, Lemma 29.2.  $\square$

### 43. Complexes with locally constant cohomology sheaves

094F Locally constant sheaves are introduced in Modules on Sites, Section 42. Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring. We denote  $D(\mathcal{C}, \Lambda)$  the derived category of the abelian category of  $\Lambda$ -modules on  $\mathcal{C}$ .

094G **Lemma 43.1.** *Let  $\mathcal{C}$  be a site with final object  $X$ . Let  $\Lambda$  be a Noetherian ring. Let  $K \in D^b(\mathcal{C}, \Lambda)$  with  $H^i(K)$  locally constant sheaves of  $\Lambda$ -modules of finite type. Then there exists a covering  $\{U_i \rightarrow X\}$  such that each  $K|_{U_i}$  is represented by a complex of locally constant sheaves of  $\Lambda$ -modules of finite type.*

**Proof.** Let  $a \leq b$  be such that  $H^i(K) = 0$  for  $i \notin [a, b]$ . By induction on  $b - a$  we will prove there exists a covering  $\{U_i \rightarrow X\}$  such that  $K|_{U_i}$  can be represented by a complex  $\underline{M}^\bullet_{U_i}$  with  $M^p$  a finite type  $\Lambda$ -module and  $M^p = 0$  for  $p \notin [a, b]$ . If  $b = a$ , then this is clear. In general, we may replace  $X$  by the members of a covering and assume that  $H^b(K)$  is constant, say  $H^b(K) = \underline{M}$ . By Modules on Sites, Lemma 41.5 the module  $M$  is a finite  $\Lambda$ -module. Choose a surjection  $\Lambda^{\oplus r} \rightarrow M$  given by generators  $x_1, \dots, x_r$  of  $M$ .

By a slight generalization of Lemma 8.3 (details omitted) there exists a covering  $\{U_i \rightarrow X\}$  such that  $x_i \in H^0(X, H^b(K))$  lifts to an element of  $H^b(U_i, K)$ . Thus, after replacing  $X$  by the  $U_i$  we reach the situation where there is a map  $\underline{\Lambda}^{\oplus r}[-b] \rightarrow K$  inducing a surjection on cohomology sheaves in degree  $b$ . Choose a distinguished triangle

$$\underline{\Lambda}^{\oplus r}[-b] \rightarrow K \rightarrow L \rightarrow \underline{\Lambda}^{\oplus r}[-b+1]$$

Now the cohomology sheaves of  $L$  are nonzero only in the interval  $[a, b-1]$ , agree with the cohomology sheaves of  $K$  in the interval  $[a, b-2]$  and there is a short exact sequence

$$0 \rightarrow H^{b-1}(K) \rightarrow H^{b-1}(L) \rightarrow \underline{\text{Ker}}(\underline{\Lambda}^{\oplus r} \rightarrow M) \rightarrow 0$$

in degree  $b-1$ . By Modules on Sites, Lemma 42.5 we see that  $H^{b-1}(L)$  is locally constant of finite type. By induction hypothesis we obtain an isomorphism  $\underline{M}^\bullet \rightarrow L$  in  $D(\mathcal{C}, \underline{\Lambda})$  with  $M^p$  a finite  $\Lambda$ -module and  $M^p = 0$  for  $p \notin [a, b-1]$ . The map  $L \rightarrow \underline{\Lambda}^{\oplus r}[-b+1]$  gives a map  $\underline{M}^{b-1} \rightarrow \underline{\Lambda}^{\oplus r}$  which locally is constant (Modules on Sites, Lemma 42.3). Thus we may assume it is given by a map  $M^{b-1} \rightarrow \Lambda^{\oplus r}$ . The distinguished triangle shows that the composition  $M^{b-2} \rightarrow M^{b-1} \rightarrow \Lambda^{\oplus r}$  is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M}^a \rightarrow \dots \rightarrow \underline{M}^{b-1} \rightarrow \underline{\Lambda}^{\oplus r} \rightarrow K$$

in  $D(\mathcal{C}, \Lambda)$ .  $\square$

Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a ring. Using the morphism  $Sh(\mathcal{C}) \rightarrow Sh(pt)$  we see that there is a functor  $D(\Lambda) \rightarrow D(\mathcal{C}, \Lambda)$ ,  $K \mapsto \underline{K}$ .

09BD **Lemma 43.2.** *Let  $\mathcal{C}$  be a site with final object  $X$ . Let  $\Lambda$  be a ring. Let*

- (1)  $K$  a perfect object of  $D(\Lambda)$ ,
- (2) a finite complex  $K^\bullet$  of finite projective  $\Lambda$ -modules representing  $K$ ,
- (3)  $\mathcal{L}^\bullet$  a complex of sheaves of  $\Lambda$ -modules, and
- (4)  $\varphi : \underline{K} \rightarrow \mathcal{L}^\bullet$  a map in  $D(\mathcal{C}, \Lambda)$ .

*Then there exists a covering  $\{U_i \rightarrow X\}$  and maps of complexes  $\alpha_i : \underline{K}^\bullet|_{U_i} \rightarrow \mathcal{L}^\bullet|_{U_i}$  representing  $\varphi|_{U_i}$ .*

**Proof.** Follows immediately from Lemma 36.8.  $\square$

09BE **Lemma 43.3.** *Let  $\mathcal{C}$  be a site with final object  $X$ . Let  $\Lambda$  be a ring. Let  $K, L$  be objects of  $D(\Lambda)$  with  $K$  perfect. Let  $\varphi : \underline{K} \rightarrow \underline{L}$  be map in  $D(\mathcal{C}, \Lambda)$ . There exists a covering  $\{U_i \rightarrow X\}$  such that  $\varphi|_{U_i}$  is equal to  $\underline{\alpha}_i$  for some map  $\alpha_i : K \rightarrow L$  in  $D(\Lambda)$ .*

**Proof.** Follows from Lemma 43.2 and Modules on Sites, Lemma 42.3.  $\square$

094H **Lemma 43.4.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a Noetherian ring. Let  $K, L \in D^-(\mathcal{C}, \Lambda)$ . If the cohomology sheaves of  $K$  and  $L$  are locally constant sheaves of  $\Lambda$ -modules of finite type, then the cohomology sheaves of  $K \otimes_\Lambda^L L$  are locally constant sheaves of  $\Lambda$ -modules of finite type.*

**Proof.** We'll prove this as an application of Lemma 43.1. Note that  $H^i(K \otimes_\Lambda^L L)$  is the same as  $H^i(\tau_{\geq i-1} K \otimes_\Lambda^L \tau_{\geq i-1} L)$ . Thus we may assume  $K$  and  $L$  are bounded. By Lemma 43.1 we may assume that  $K$  and  $L$  are represented by complexes of locally constant sheaves of  $\Lambda$ -modules of finite type. Then we can replace these complexes by bounded above complexes of finite free  $\Lambda$ -modules. In this case the result is clear.  $\square$

094I **Lemma 43.5.** *Let  $\mathcal{C}$  be a site. Let  $\Lambda$  be a Noetherian ring. Let  $I \subset \Lambda$  be an ideal. Let  $K \in D^-(\mathcal{C}, \Lambda)$ . If the cohomology sheaves of  $K \otimes_\Lambda^L \underline{\Lambda/I}$  are locally constant sheaves of  $\Lambda/I$ -modules of finite type, then the cohomology sheaves of  $K \otimes_\Lambda^L \underline{\Lambda/I^n}$  are locally constant sheaves of  $\Lambda/I^n$ -modules of finite type for all  $n \geq 1$ .*

**Proof.** Recall that the locally constant sheaves of  $\Lambda$ -modules of finite type form a weak Serre subcategory of all  $\Lambda$ -modules, see Modules on Sites, Lemma 42.5. Thus the subcategory of  $D(\mathcal{C}, \Lambda)$  consisting of complexes whose cohomology sheaves are locally constant sheaves of  $\Lambda$ -modules of finite type forms a strictly full, saturated triangulated subcategory of  $D(\mathcal{C}, \Lambda)$ , see Derived Categories, Lemma 13.1. Next, consider the distinguished triangles

$$K \otimes_\Lambda^L \underline{I^n/I^{n+1}} \rightarrow K \otimes_\Lambda^L \underline{\Lambda/I^{n+1}} \rightarrow K \otimes_\Lambda^L \underline{\Lambda/I^n} \rightarrow K \otimes_\Lambda^L \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^L \underline{I^n/I^{n+1}} = \left( K \otimes_\Lambda^L \underline{\Lambda/I} \right) \otimes_{\Lambda/I}^L \underline{I^n/I^{n+1}}$$

Combined with Lemma 43.4 we obtain the result.  $\square$

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