

# DUALITY FOR SPACES

0E4V

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## 1. Introduction

0E4W This chapter is the analogue of the corresponding chapter for schemes, see Duality for Schemes, Section 1. The development is similar to the development in the papers [Nee96], [LN07], [Lip09], and [Nee14].

## 2. Dualizing complexes on algebraic spaces

0E4X Let  $U$  be a locally Noetherian scheme. Let  $\mathcal{O}_{\acute{e}tale}$  be the structure sheaf of  $U$  on the small étale site of  $U$ . We will say an object  $K \in D_{QCoh}(\mathcal{O}_{\acute{e}tale})$  is a dualizing complex on  $U$  if  $K = \epsilon^*(\omega_U^\bullet)$  for some dualizing complex  $\omega_U^\bullet$  in the sense of Duality for Schemes, Section 2. Here  $\epsilon^* : D_{QCoh}(\mathcal{O}_U) \rightarrow D_{QCoh}(\mathcal{O}_{\acute{e}tale})$  is the equivalence of Derived Categories of Spaces, Lemma 4.2. Most of the properties of  $\omega_U^\bullet$  studied in Duality for Schemes, Section 2 are inherited by  $K$  via the discussion in Derived Categories of Spaces, Sections 4 and 13.

We define a dualizing complex on a locally Noetherian algebraic space to be a complex which étale locally comes from a dualizing complex on the corresponding scheme.

0E4Y **Lemma 2.1.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . Let  $K$  be an object of  $D_{QCoh}(\mathcal{O}_X)$ . The following are equivalent*

- (1) *For every étale morphism  $U \rightarrow X$  where  $U$  is a scheme the restriction  $K|_U$  is a dualizing complex for  $U$  (as discussed above).*

- (2) *There exists a surjective étale morphism  $U \rightarrow X$  where  $U$  is a scheme such that  $K|_U$  is a dualizing complex for  $U$ .*

**Proof.** Assume  $U \rightarrow X$  is surjective étale where  $U$  is a scheme. Let  $V \rightarrow X$  be an étale morphism where  $V$  is a scheme. Then

$$U \leftarrow U \times_X V \rightarrow V$$

are étale morphisms of schemes with the arrow to  $V$  surjective. Hence we can use Duality for Schemes, Lemma 27.1 to see that if  $K|_U$  is a dualizing complex for  $U$ , then  $K|_V$  is a dualizing complex for  $V$ .  $\square$

0E4Z **Definition 2.2.** Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . An object  $K$  of  $D_{QCoh}(\mathcal{O}_X)$  is called a *dualizing complex* if  $K$  satisfies the equivalent conditions of Lemma 2.1.

0E50 **Lemma 2.3.** *Let  $A$  be a Noetherian ring and let  $X = \text{Spec}(A)$ . Let  $\mathcal{O}_{\acute{e}tale}$  be the structure sheaf of  $X$  on the small étale site of  $X$ . Let  $K, L$  be objects of  $D(A)$ . If  $K \in D_{Coh}(A)$  and  $L$  has finite injective dimension, then*

$$\epsilon^* R\widetilde{\text{Hom}}_A(K, L) = R\mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^* \widetilde{K}, \epsilon^* \widetilde{L})$$

in  $D(\mathcal{O}_{\acute{e}tale})$  where  $\epsilon : (X_{\acute{e}tale}, \mathcal{O}_{\acute{e}tale}) \rightarrow (X, \mathcal{O}_X)$  is as in *Derived Categories of Spaces, Section 4*.

**Proof.** By Duality for Schemes, Lemma 2.3 we have a canonical isomorphism

$$R\widetilde{\text{Hom}}_A(K, L) = R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L})$$

in  $D(\mathcal{O}_X)$ . There is a canonical map

$$\epsilon^* R\mathcal{H}om_{\mathcal{O}_X}(\widetilde{K}, \widetilde{L}) \longrightarrow R\mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^* \widetilde{K}, \epsilon^* \widetilde{L})$$

in  $D(\mathcal{O}_{\acute{e}tale})$ , see *Cohomology on Sites, Remark 28.11*. We will show the left and right hand side of this arrow have isomorphic cohomology sheaves, but we will omit the verification that the isomorphism is given by this arrow.

We may assume that  $L$  is given by a finite complex  $I^\bullet$  of injective  $A$ -modules. By induction on the length of  $I^\bullet$  and compatibility of the constructions with distinguished triangles, we reduce to the case that  $L = I[0]$  where  $I$  is an injective  $A$ -module. Recall that the cohomology sheaves of  $R\mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(\epsilon^* \widetilde{K}, \epsilon^* \widetilde{L})$  are the sheafifications of the presheaf sending  $U$  étale over  $X$  to the  $i$ th ext group between the restrictions of  $\epsilon^* \widetilde{K}$  and  $\epsilon^* \widetilde{L}$  to  $U_{\acute{e}tale}$ . See *Cohomology on Sites, Lemma 28.1*. If  $U = \text{Spec}(B)$  is affine, then this ext group is equal to  $\text{Ext}_B^i(K \otimes_A B, L \otimes_A B)$  by the equivalence of *Derived Categories of Spaces, Lemma 4.2* and *Derived Categories of Schemes, Lemma 3.5* (this also uses the compatibilities detailed in *Derived Categories of Spaces, Remark 6.3*). Since  $A \rightarrow B$  is étale, we see that  $I \otimes_A B$  is an injective  $B$ -module by *Dualizing Complexes, Lemma 26.4*. Hence we see that

$$\begin{aligned} \text{Ext}_B^n(K \otimes_A B, I \otimes_A B) &= \text{Hom}_B(H^{-n}(K \otimes_A B), I \otimes_A B) \\ &= \text{Hom}_{A_f}(H^{-n}(K) \otimes_A B, I \otimes_A B) \\ &= \text{Hom}_A(H^{-n}(K), I) \otimes_A B \\ &= \text{Ext}_A^n(K, I) \otimes_A B \end{aligned}$$

The penultimate equality because  $H^{-n}(K)$  is a finite  $A$ -module, see More on Algebra, Remark 62.20. Therefore the cohomology sheaves of the left and right hand side of the equality in the lemma are the same.  $\square$

0E51 **Lemma 2.4.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . Let  $K$  be a dualizing complex on  $X$ . Then  $K$  is an object of  $D_{Coh}(\mathcal{O}_X)$  and  $D = R\mathcal{H}om_{\mathcal{O}_X}(-, K)$  induces an anti-equivalence*

$$D : D_{Coh}(\mathcal{O}_X) \longrightarrow D_{Coh}(\mathcal{O}_X)$$

which comes equipped with a canonical isomorphism  $id \rightarrow D \circ D$ . If  $X$  is quasi-compact, then  $D$  exchanges  $D_{Coh}^+(\mathcal{O}_X)$  and  $D_{Coh}^-(\mathcal{O}_X)$  and induces an equivalence  $D_{Coh}^b(\mathcal{O}_X) \rightarrow D_{Coh}^b(\mathcal{O}_X)$ .

**Proof.** Let  $U \rightarrow X$  be an étale morphism with  $U$  affine. Say  $U = \text{Spec}(A)$  and let  $\omega_A^\bullet$  be a dualizing complex for  $A$  corresponding to  $K|_U$  as in Lemma 2.1 and Duality for Schemes, Lemma 2.1. By Lemma 2.3 the diagram

$$\begin{array}{ccc} D_{Coh}(A) & \longrightarrow & D_{Coh}(\mathcal{O}_{\acute{e}tale}) \\ R\mathcal{H}om_A(-, \omega_A^\bullet) \downarrow & & \downarrow R\mathcal{H}om_{\mathcal{O}_{\acute{e}tale}}(-, K|_U) \\ D_{Coh}(A) & \longrightarrow & D(\mathcal{O}_{\acute{e}tale}) \end{array}$$

commutes where  $\mathcal{O}_{\acute{e}tale}$  is the structure sheaf of the small étale site of  $U$ . Since formation of  $R\mathcal{H}om$  commutes with restriction, we conclude that  $D$  sends  $D_{Coh}(\mathcal{O}_X)$  into  $D_{Coh}(\mathcal{O}_X)$ . Moreover, the canonical map

$$L \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(R\mathcal{H}om_{\mathcal{O}_X}(L, K), K)$$

(Cohomology on Sites, Lemma 28.5) is an isomorphism for all  $L$  in  $D_{Coh}(\mathcal{O}_X)$  because this is true over all  $U$  as above by Dualizing Complexes, Lemma 15.2. The statement on boundedness properties of the functor  $D$  in the quasi-compact case also follows from the corresponding statements of Dualizing Complexes, Lemma 15.2.  $\square$

Let  $(\mathcal{C}, \mathcal{O})$  be a ringed site. We will say that an object  $L$  of  $D(\mathcal{O})$  is *invertible* if for every object  $U$  of  $\mathcal{C}$  there is a covering  $\{U_i \rightarrow U\}$  of  $U$  in  $\mathcal{C}$  such that  $L|_{U_i} \cong \mathcal{O}_{U_i}[-n_i]$  for some integers  $n_i$ .

Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . If  $L$  in  $D(\mathcal{O}_X)$  is invertible, then there is a disjoint union decomposition  $X = \coprod_{n \in \mathbf{Z}} X_n$  such that  $L|_{X_n}$  is an invertible module sitting in degree  $n$ . In particular, it follows that  $L = \bigoplus H^n(L)[-n]$  which gives a well defined complex of  $\mathcal{O}_X$ -modules (with zero differentials) representing  $L$ . Moreover, we see that  $L$  is a perfect object of  $D(\mathcal{O}_X)$ .

0E52 **Lemma 2.5.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian algebraic space over  $S$ . If  $K$  and  $K'$  are dualizing complexes on  $X$ , then  $K'$  is isomorphic to  $K \otimes_{\mathcal{O}_X}^L L$  for some invertible object  $L$  of  $D(\mathcal{O}_X)$ .*

**Proof.** Set

$$L = R\mathcal{H}om_{\mathcal{O}_X}(K, K')$$

This is an invertible object of  $D(\mathcal{O}_X)$ , because affine locally this is true. Use Lemma 2.3 and Dualizing Complexes, Lemma 15.5 and its proof. The evaluation map  $L \otimes_{\mathcal{O}_X}^L K \rightarrow K'$  is an isomorphism for the same reason.  $\square$

0E53 **Lemma 2.6.** *Let  $S$  be a scheme. Let  $X$  be a locally Noetherian quasi-separated algebraic space over  $S$ . Let  $\omega_X^\bullet$  be a dualizing complex on  $X$ . Then  $X$  the function  $|X| \rightarrow \mathbf{Z}$  defined by*

*$x \mapsto \delta(x)$  such that  $\omega_{X,\bar{x}}^\bullet[-\delta(x)]$  is a normalized dualizing complex over  $\mathcal{O}_{X,\bar{x}}$  is a dimension function on  $|X|$ .*

**Proof.** Let  $U$  be a scheme and let  $U \rightarrow X$  be a surjective étale morphism. Let  $\omega_U^\bullet$  be the dualizing complex on  $U$  associated to  $\omega_X^\bullet|_U$ . If  $u \in U$  maps to  $x \in |X|$ , then  $\mathcal{O}_{X,\bar{x}}$  is the strict henselization of  $\mathcal{O}_{U,u}$ . By Dualizing Complexes, Lemma 22.1 we see that if  $\omega^\bullet$  is a normalized dualizing complex for  $\mathcal{O}_{U,u}$ , then  $\omega^\bullet \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\bar{x}}$  is a normalized dualizing complex for  $\mathcal{O}_{X,\bar{x}}$ . Hence we see that the dimension function  $U \rightarrow \mathbf{Z}$  of Duality for Schemes, Lemma 2.6 for the scheme  $U$  and the complex  $\omega_U^\bullet$  is equal to the composition of  $U \rightarrow |X|$  with  $\delta$ . Using the specializations in  $|X|$  lift to specializations in  $U$  and that nontrivial specializations in  $U$  map to nontrivial specializations in  $X$  (Decent Spaces, Lemmas 12.2 and 12.1) an easy topological argument shows that  $\delta$  is a dimension function on  $|X|$ .  $\square$

### 3. Right adjoint of pushforward

0E54 This is the analogue of Duality for Schemes, Section 3.

0E55 **Lemma 3.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism between quasi-separated and quasi-compact algebraic spaces over  $S$ . The functor  $Rf_* : D_{QCoh}(X) \rightarrow D_{QCoh}(Y)$  has a right adjoint.*

This is almost the same as [Nee96, Example 4.2].

**Proof.** We will prove a right adjoint exists by verifying the hypotheses of Derived Categories, Proposition 35.2. First off, the category  $D_{QCoh}(\mathcal{O}_X)$  has direct sums, see Derived Categories of Spaces, Lemma 5.3. The category  $D_{QCoh}(\mathcal{O}_X)$  is compactly generated by Derived Categories of Spaces, Theorem 15.4. Since  $X$  and  $Y$  are quasi-compact and quasi-separated, so is  $f$ , see Morphisms of Spaces, Lemmas 4.10 and 8.9. Hence the functor  $Rf_*$  commutes with direct sums, see Derived Categories of Spaces, Lemma 6.2. This finishes the proof.  $\square$

0E56 **Lemma 3.2.** *Notation and assumptions as in Lemma 3.1. Let  $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  be the right adjoint to  $Rf_*$ . Then  $a$  maps  $D_{QCoh}^+(\mathcal{O}_Y)$  into  $D_{QCoh}^+(\mathcal{O}_X)$ . In fact, there exists an integer  $N$  such that  $H^i(K) = 0$  for  $i \leq c$  implies  $H^i(a(K)) = 0$  for  $i \leq c - N$ .*

**Proof.** By Derived Categories of Spaces, Lemma 6.1 the functor  $Rf_*$  has finite cohomological dimension. In other words, there exist an integer  $N$  such that  $H^i(Rf_*L) = 0$  for  $i \geq N + c$  if  $H^i(L) = 0$  for  $i \geq c$ . Say  $K \in D_{QCoh}^+(\mathcal{O}_Y)$  has  $H^i(K) = 0$  for  $i \leq c$ . Then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(\tau_{\leq c-N}a(K), a(K)) = \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*\tau_{\leq c-N}a(K), K) = 0$$

by what we said above. Clearly, this implies that  $H^i(a(K)) = 0$  for  $i \leq c - N$ .  $\square$

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-separated and quasi-compact algebraic spaces over  $S$ . Let  $a$  denote the right adjoint to  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ . For every  $K \in D_{QCoh}(\mathcal{O}_Y)$  and  $L \in D_{QCoh}(\mathcal{O}_X)$  we obtain a canonical map

0E57 (3.2.1) 
$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

Namely, this map is constructed as the composition

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, Rf_*a(K)) \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

where the first arrow is Cohomology on Sites, Remark 28.10 and the second arrow is the counit  $Rf_*a(K) \rightarrow K$  of the adjunction.

0E58 **Lemma 3.3.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Let  $a$  be the right adjoint to  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$ . Then (3.2.1)*

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

is an isomorphism for all  $L \in D_{QCoh}(\mathcal{O}_X)$  and  $K \in D_{QCoh}(\mathcal{O}_Y)$ .

**Proof.** Let  $M \in D_{QCoh}(\mathcal{O}_Y)$ . Then we have the following

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) &= \mathrm{Hom}_X(Lf^*M, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) \\ &= \mathrm{Hom}_X(Lf^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} L, a(K)) \\ &= \mathrm{Hom}_Y(Rf_*(Lf^*M \otimes_{\mathcal{O}_X}^{\mathbf{L}} L), K) \\ &= \mathrm{Hom}_Y(M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L, K) \\ &= \mathrm{Hom}_Y(M, R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)) \end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 20.1. The second equality by Cohomology on Sites, Lemma 28.2. The third equality by construction of  $a$ . The fourth equality by Derived Categories of Spaces, Lemma 20.1 (this is the important step). The fifth by Cohomology on Sites, Lemma 28.2. Thus the result holds by the Yoneda lemma.  $\square$

0E59 **Lemma 3.4.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-separated and quasi-compact algebraic spaces over  $S$ . For all  $L \in D_{QCoh}(\mathcal{O}_X)$  and  $K \in D_{QCoh}(\mathcal{O}_Y)$  (3.2.1) induces an isomorphism  $R\mathrm{Hom}_X(L, a(K)) \rightarrow R\mathrm{Hom}_Y(Rf_*L, K)$  of global derived homs.*

**Proof.** By construction (Cohomology on Sites, Section 29) the complexes

$$R\mathrm{Hom}_X(L, a(K)) = R\Gamma(X, R\mathcal{H}om_{\mathcal{O}_X}(L, a(K))) = R\Gamma(Y, Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)))$$

and

$$R\mathrm{Hom}_Y(Rf_*L, K) = R\Gamma(Y, R\mathcal{H}om_{\mathcal{O}_X}(Rf_*L, a(K)))$$

Thus the lemma is a consequence of Lemma 3.3.  $\square$

#### 4. Right adjoint of pushforward and base change, I

0E5A Let us define the base change map between right adjoints of pushforward. Let  $S$  be a scheme. Consider a cartesian diagram

$$0E5B \quad (4.0.1) \quad \begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

where  $Y'$  and  $X$  are **Tor independent** over  $Y$ . Denote

$$a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X) \quad \text{and} \quad a' : D_{QCoh}(\mathcal{O}_{Y'}) \rightarrow D_{QCoh}(\mathcal{O}_{X'})$$

the right adjoints to  $Rf_*$  and  $Rf'_*$  (Lemma 3.1). The base change map of Cohomology on Sites, Remark 20.3 gives a transformation of functors

$$Lg^* \circ Rf_* \longrightarrow Rf'_* \circ L(g')^*$$

on derived categories of sheaves with quasi-coherent cohomology. Hence a transformation between the right adjoints in the opposite direction

$$a \circ Rg_* \longleftarrow Rg'_* \circ a'$$

0E5C **Lemma 4.1.** *In diagram (4.0.1) the map  $a \circ Rg_* \longleftarrow Rg'_* \circ a'$  is an isomorphism.*

**Proof.** The base change map  $Lg^* \circ Rf_* K \rightarrow Rf'_* \circ L(g')^* K$  is an isomorphism for every  $K$  in  $D_{QCoh}(\mathcal{O}_X)$  by Derived Categories of Spaces, Lemma 20.4 (this uses the assumption of Tor independence). Thus the corresponding transformation between adjoint functors is an isomorphism as well.  $\square$

Then we can consider the morphism of functors  $D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_{X'})$  given by the composition

0E5D (4.1.1)  $L(g')^* \circ a \rightarrow L(g')^* \circ a \circ Rg_* \circ Lg^* \longleftarrow L(g')^* \circ Rg'_* \circ a' \circ Lg^* \rightarrow a' \circ Lg^*$

The first arrow comes from the adjunction map  $\text{id} \rightarrow Rg_* Lg^*$  and the last arrow from the adjunction map  $L(g')^* Rg'_* \rightarrow \text{id}$ . We need the assumption on Tor independence to invert the arrow in the middle, see Lemma 4.1. Alternatively, we can think of (4.1.1) by adjointness of  $L(g')^*$  and  $R(g')_*$  as a natural transformation

$$a \rightarrow a \circ Rg_* \circ Lg^* \longleftarrow Rg'_* \circ a' \circ Lg^*$$

were again the second arrow is invertible. If  $M \in D_{QCoh}(\mathcal{O}_X)$  and  $K \in D_{QCoh}(\mathcal{O}_Y)$  then on Yoneda functors this map is given by

$$\begin{aligned} \text{Hom}_X(M, a(K)) &= \text{Hom}_Y(Rf_* M, K) \\ &\rightarrow \text{Hom}_Y(Rf_* M, Rg_* Lg^* K) \\ &= \text{Hom}_{Y'}(Lg^* Rf_* M, Lg^* K) \\ &\longleftarrow \text{Hom}_{Y'}(Rf'_* L(g')^* M, Lg^* K) \\ &= \text{Hom}_{X'}(L(g')^* M, a'(Lg^* K)) \\ &= \text{Hom}_X(M, Rg'_* a'(Lg^* K)) \end{aligned}$$

(were the arrow pointing left is invertible by the base change theorem given in Derived Categories of Spaces, Lemma 20.4) which makes things a little bit more explicit.

In this section we first prove that the base change map satisfies some natural compatibilities with regards to stacking squares as in Cohomology on Sites, Remarks 20.4 and 20.5 for the usual base change map. We suggest the reader skip the rest of this section on a first reading.

0E5E **Lemma 4.2.** *Let  $S$  be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc}
 X' & \xrightarrow{\quad} & X \\
 f' \downarrow & & \downarrow f \\
 Y' & \xrightarrow{\quad l \quad} & Y \\
 g' \downarrow & & \downarrow g \\
 Z' & \xrightarrow{\quad m \quad} & Z
 \end{array}$$

*of quasi-compact and quasi-separated algebraic spaces over  $S$  where both diagrams are cartesian and where  $f$  and  $l$  as well as  $g$  and  $m$  are Tor independent. Then the maps (4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).*

**Proof.** It follows from the assumptions that  $g \circ f$  and  $m$  are Tor independent (details omitted), hence the statement makes sense. In this proof we write  $k^*$  in place of  $Lk^*$  and  $f_*$  instead of  $Rf_*$ . Let  $a, b$ , and  $c$  be the right adjoints of Lemma 3.1 for  $f, g$ , and  $g \circ f$  and similarly for the primed versions. The arrow corresponding to the top square is the composition

$$\gamma_{top} : k^* \circ a \rightarrow k^* \circ a \circ l_* \circ l^* \xleftarrow{\xi_{top}} k^* \circ k_* \circ a' \circ l^* \rightarrow a' \circ l^*$$

where  $\xi_{top} : k_* \circ a' \rightarrow a \circ l_*$  is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map  $l^* \circ f_* \rightarrow f'_* \circ k^*$ . The outer arrows come from the canonical maps  $1 \rightarrow l_* \circ l^*$  and  $k^* \circ k_* \rightarrow 1$ . Similarly for the second square we have

$$\gamma_{bot} : l^* \circ b \rightarrow l^* \circ b \circ m_* \circ m^* \xleftarrow{\xi_{bot}} l^* \circ l_* \circ b' \circ m^* \rightarrow b' \circ m^*$$

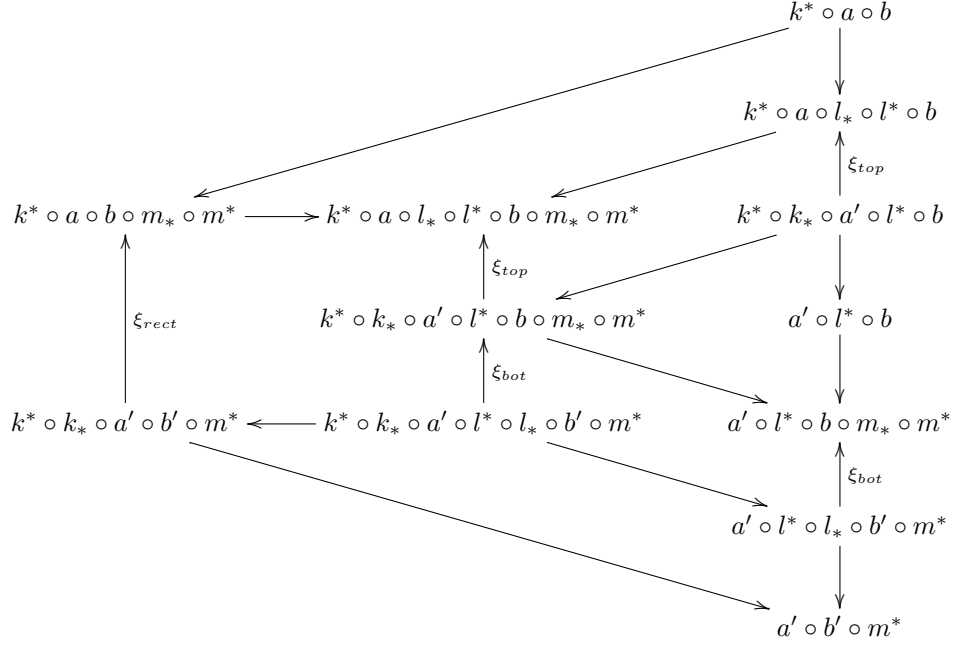
For the outer rectangle we get

$$\gamma_{rect} : k^* \circ c \rightarrow k^* \circ c \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ c' \circ m^* \rightarrow c' \circ m^*$$

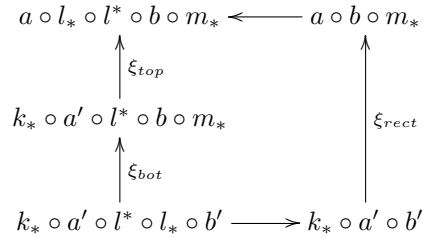
We have  $(g \circ f)_* = g_* \circ f_*$  and hence  $c = a \circ b$  and similarly  $c' = a' \circ b'$ . The statement of the lemma is that  $\gamma_{rect}$  is equal to the composition

$$k^* \circ c = k^* \circ a \circ b \xrightarrow{\gamma_{top}} a' \circ l^* \circ b \xrightarrow{\gamma_{bot}} a' \circ b' \circ m^* = c' \circ m^*$$

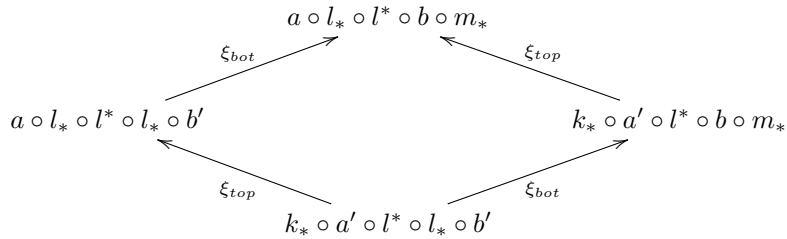
To see this we contemplate the following diagram:



Going down the right hand side we have the composition and going down the left hand side we have  $\gamma_{rect}$ . All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show the diagram



becomes commutative if we invert the arrows  $\xi_{top}$ ,  $\xi_{bot}$ , and  $\xi_{rect}$  (note that this is different from asking the diagram to be commutative). However, the diagram





commutes by Categories, Lemma 27.2. Since the diagrams

$$\begin{array}{ccc}
 a \circ l_* \circ l^* \circ b \circ m_* & \longleftarrow & a \circ b \circ m \\
 \uparrow & & \uparrow \\
 a \circ l_* \circ l^* \circ l_* \circ b' & \longleftarrow & a \circ l_* \circ b'
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 a \circ l_* \circ l^* \circ l_* \circ b' & \longrightarrow & a \circ l_* \circ b' \\
 \uparrow & & \uparrow \\
 k_* \circ a' \circ l^* \circ l_* \circ b' & \longrightarrow & k_* \circ a' \circ b'
 \end{array}$$

commute (see references cited) and since the composition of  $l_* \rightarrow l_* \circ l^* \circ l_* \rightarrow l_*$  is the identity, we find that it suffices to prove that

$$k \circ a' \circ b' \xrightarrow{\xi_{bot}} a \circ l_* \circ b \xrightarrow{\xi_{top}} a \circ b \circ m_*$$

is equal to  $\xi_{rect}$  (via the identifications  $a \circ b = c$  and  $a' \circ b' = c'$ ). This is the statement dual to Cohomology on Sites, Remark 20.4 and the proof is complete.  $\square$

0E5F **Lemma 4.3.** *Let  $S$  be a scheme. Consider a commutative diagram*

$$\begin{array}{ccccc}
 X'' & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & X \\
 f'' \downarrow & & f' \downarrow & & \downarrow f \\
 Y'' & \xrightarrow{\quad} & Y' & \xrightarrow{\quad} & Y
 \end{array}$$

of quasi-compact and quasi-separated algebraic spaces over  $S$  where both diagrams are cartesian and where  $f$  and  $h$  as well as  $f'$  and  $h'$  are Tor independent. Then the maps (4.1.1) for the two squares compose to give the base change map for the outer rectangle (see proof for a precise statement).

**Proof.** It follows from the assumptions that  $f$  and  $h \circ h'$  are Tor independent (details omitted), hence the statement makes sense. In this proof we write  $g^*$  in place of  $Lg^*$  and  $f_*$  instead of  $Rf_*$ . Let  $a$ ,  $a'$ , and  $a''$  be the right adjoints of Lemma 3.1 for  $f$ ,  $f'$ , and  $f''$ . The arrow corresponding to the right square is the composition

$$\gamma_{right} : g^* \circ a \rightarrow g^* \circ a \circ h_* \circ h^* \xleftarrow{\xi_{right}} g^* \circ g_* \circ a' \circ h^* \rightarrow a' \circ h^*$$

where  $\xi_{right} : g_* \circ a' \rightarrow a \circ h_*$  is an isomorphism (hence can be inverted) and is the arrow “dual” to the base change map  $h^* \circ f_* \rightarrow f'_* \circ g^*$ . The outer arrows come from the canonical maps  $1 \rightarrow h_* \circ h^*$  and  $g^* \circ g_* \rightarrow 1$ . Similarly for the left square we have

$$\gamma_{left} : (g')^* \circ a' \rightarrow (g')^* \circ a' \circ (h')_* \circ (h')^* \xleftarrow{\xi_{left}} (g')^* \circ (g')_* \circ a'' \circ (h')^* \rightarrow a'' \circ (h')^*$$

For the outer rectangle we get

$$\gamma_{rect} : k^* \circ a \rightarrow k^* \circ a \circ m_* \circ m^* \xleftarrow{\xi_{rect}} k^* \circ k_* \circ a'' \circ m^* \rightarrow a'' \circ m^*$$

where  $k = g \circ g'$  and  $m = h \circ h'$ . We have  $k^* = (g')^* \circ g^*$  and  $m^* = (h')^* \circ h^*$ . The statement of the lemma is that  $\gamma_{rect}$  is equal to the composition

$$k^* \circ a = (g')^* \circ g^* \circ a \xrightarrow{\gamma_{right}} (g')^* \circ a' \circ h^* \xrightarrow{\gamma_{left}} a'' \circ (h')^* \circ h^* = a'' \circ m^*$$

To see this we contemplate the following diagram

$$\begin{array}{ccc}
 & & (g')^* \circ g^* \circ a \\
 & & \downarrow \\
 & & (g')^* \circ g^* \circ a \circ h_* \circ h^* \\
 & \swarrow & \uparrow \xi_{right} \\
 (g')^* \circ g^* \circ a \circ h_* \circ (h')^* \circ (h')^* \circ h^* & & (g')^* \circ g^* \circ g_* \circ a' \circ h^* \\
 & \swarrow & \downarrow \\
 (g')^* \circ g^* \circ g_* \circ a' \circ (h')^* \circ (h')^* \circ h^* & & (g')^* \circ a' \circ h^* \\
 \uparrow \xi_{right} & \swarrow & \downarrow \\
 (g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* & & (g')^* \circ a' \circ (h')_* \circ (h')^* \circ h^* \\
 \uparrow \xi_{left} & \swarrow & \downarrow \\
 (g')^* \circ g^* \circ g_* \circ (g')_* \circ a'' \circ (h')^* \circ h^* & & (g')^* \circ (g')_* \circ a'' \circ (h')^* \circ h^* \\
 & \swarrow & \uparrow \xi_{left} \\
 & & a'' \circ (h')^* \circ h^*
 \end{array}$$

Going down the right hand side we have the composition and going down the left hand side we have  $\gamma_{rect}$ . All the quadrilaterals on the right hand side of this diagram commute by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. Hence we see that it suffices to show that

$$g_* \circ (g')_* \circ a'' \xrightarrow{\xi_{left}} g_* \circ a' \circ (h')_* \xrightarrow{\xi_{right}} a \circ h_* \circ (h')_*$$

is equal to  $\xi_{rect}$ . This is the statement dual to Cohomology, Remark 29.5 and the proof is complete.  $\square$

0E5G **Remark 4.4.** Let  $S$  be a scheme. Consider a commutative diagram

$$\begin{array}{ccccc}
 X'' & \longrightarrow & X' & \longrightarrow & X \\
 f'' \downarrow & & f' \downarrow & & f \downarrow \\
 Y'' & \xrightarrow{l'} & Y' & \xrightarrow{l} & Y \\
 g'' \downarrow & & g' \downarrow & & g \downarrow \\
 Z'' & \xrightarrow{m'} & Z' & \xrightarrow{m} & Z
 \end{array}$$

of quasi-compact and quasi-separated algebraic spaces over  $S$  where all squares are cartesian and where  $(f, l)$ ,  $(g, m)$ ,  $(f', l')$ ,  $(g', m')$  are Tor independent pairs of maps. Let  $a, a', a'', b, b', b''$  be the right adjoints of Lemma 3.1 for  $f, f', f'', g, g', g''$ . Let us label the squares of the diagram  $A, B, C, D$  as follows

$$\begin{array}{cc}
 A & B \\
 C & D
 \end{array}$$

Then the maps (4.1.1) for the squares are (where we use  $k^* = Lk^*$ , etc)

$$\begin{aligned} \gamma_A &: (k')^* \circ a' \rightarrow a'' \circ (l')^* & \gamma_B &: k^* \circ a \rightarrow a' \circ l^* \\ \gamma_C &: (l')^* \circ b' \rightarrow b'' \circ (m')^* & \gamma_D &: l^* \circ b \rightarrow b' \circ m^* \end{aligned}$$

For the  $2 \times 1$  and  $1 \times 2$  rectangles we have four further base change maps

$$\begin{aligned} \gamma_{A+B} &: (k \circ k')^* \circ a \rightarrow a'' \circ (l \circ l')^* \\ \gamma_{C+D} &: (l \circ l')^* \circ b \rightarrow b'' \circ (m \circ m')^* \\ \gamma_{A+C} &: (k')^* \circ (a' \circ b') \rightarrow (a'' \circ b'') \circ (m')^* \\ \gamma_{A+C} &: k^* \circ (a \circ b) \rightarrow (a' \circ b') \circ m^* \end{aligned}$$

By Lemma 4.3 we have

$$\gamma_{A+B} = \gamma_A \circ \gamma_B, \quad \gamma_{C+D} = \gamma_C \circ \gamma_D$$

and by Lemma 4.2 we have

$$\gamma_{A+C} = \gamma_C \circ \gamma_A, \quad \gamma_{B+D} = \gamma_D \circ \gamma_B$$

Here it would be more correct to write  $\gamma_{A+B} = (\gamma_A \star \text{id}_{l'}) \circ (\text{id}_{(k')^*} \star \gamma_B)$  with notation as in Categories, Section 27 and similarly for the others. However, we continue the abuse of notation used in the proofs of Lemmas 4.2 and 4.3 of dropping  $\star$  products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Having said all of this we find (a priori) two transformations

$$(k')^* \circ k^* \circ a \circ b \longrightarrow a'' \circ b'' \circ (m')^* \circ m^*$$

namely

$$\gamma_C \circ \gamma_A \circ \gamma_D \circ \gamma_B = \gamma_{A+C} \circ \gamma_{B+D}$$

and

$$\gamma_C \circ \gamma_D \circ \gamma_A \circ \gamma_B = \gamma_{C+D} \circ \gamma_{A+B}$$

The point of this remark is to point out that these transformations are equal. Namely, to see this it suffices to show that

$$\begin{array}{ccc} (k')^* \circ a' \circ l^* \circ b & \xrightarrow{\gamma_D} & (k')^* \circ a' \circ b' \circ m^* \\ \gamma_A \downarrow & & \downarrow \gamma_A \\ a'' \circ (l')^* \circ l^* \circ b & \xrightarrow{\gamma_D} & a'' \circ (l')^* \circ b' \circ m^* \end{array}$$

commutes. This is true by Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1.

## 5. Right adjoint of pushforward and base change, II

0E5H In this section we prove that the base change map of Section 4 is an isomorphism in some cases.

0E5I **Lemma 5.1.** *In diagram (4.0.1) assume in addition  $g : Y' \rightarrow Y$  is a morphism of affine schemes and  $f : X \rightarrow Y$  is proper. Then the base change map (4.1.1) induces an isomorphism*

$$L(g')^* a(K) \longrightarrow a'(Lg^* K)$$

in the following cases

- (1) for all  $K \in D_{QCoh}(\mathcal{O}_X)$  if  $f$  is flat of finite presentation,
- (2) for all  $K \in D_{QCoh}(\mathcal{O}_X)$  if  $f$  is perfect and  $Y$  Noetherian,

(3) for  $K \in D_{QCoh}^+(\mathcal{O}_X)$  if  $g$  has finite Tor dimension and  $Y$  Noetherian.

**Proof.** Write  $Y = \text{Spec}(A)$  and  $Y' = \text{Spec}(A')$ . As a base change of an affine morphism, the morphism  $g'$  is affine. Let  $M$  be a perfect generator for  $D_{QCoh}(\mathcal{O}_X)$ , see Derived Categories of Spaces, Theorem 15.4. Then  $L(g')^*M$  is a generator for  $D_{QCoh}(\mathcal{O}_{X'})$ , see Derived Categories of Spaces, Remark 15.5. Hence it suffices to show that (4.1.1) induces an isomorphism

$$0E5J \quad (5.1.1) \quad R\text{Hom}_{X'}(L(g')^*M, L(g')^*a(K)) \longrightarrow R\text{Hom}_{X'}(L(g')^*M, a'(Lg^*K))$$

of global hom complexes, see Cohomology on Sites, Section 29, as this will imply the cone of  $L(g')^*a(K) \rightarrow a'(Lg^*K)$  is zero. The structure of the proof is as follows: we will first show that these Hom complexes are isomorphic and in the last part of the proof we will show that the isomorphism is induced by (5.1.1).

The left hand side. Because  $M$  is perfect, the canonical map

$$R\text{Hom}_X(M, a(K)) \otimes_A^L A' \longrightarrow R\text{Hom}_{X'}(L(g')^*M, L(g')^*a(K))$$

is an isomorphism by Derived Categories of Spaces, Lemma 20.5. We can combine this with the isomorphism  $R\text{Hom}_Y(Rf_*M, K) = R\text{Hom}_X(M, a(K))$  of Lemma 3.4 to get that the left hand side equals  $R\text{Hom}_Y(Rf_*M, K) \otimes_A^L A'$ .

The right hand side. Here we first use the isomorphism

$$R\text{Hom}_{X'}(L(g')^*M, a'(Lg^*K)) = R\text{Hom}_{Y'}(Rf'_*L(g')^*M, Lg^*K)$$

of Lemma 3.4. Since  $f$  and  $g$  are Tor independent the base change map  $Lg^*Rf_*M \rightarrow Rf'_*L(g')^*M$  is an isomorphism by Derived Categories of Spaces, Lemma 20.4. Hence we may rewrite this as  $R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$ . Since  $Y, Y'$  are affine and  $K, Rf_*M$  are in  $D_{QCoh}(\mathcal{O}_Y)$  (Derived Categories of Spaces, Lemma 6.1) we have a canonical map

$$\beta : R\text{Hom}_Y(Rf_*M, K) \otimes_A^L A' \longrightarrow R\text{Hom}_{Y'}(Lg^*Rf_*M, Lg^*K)$$

in  $D(A')$ . This is the arrow More on Algebra, Equation (87.1.1) where we have used Derived Categories of Schemes, Lemmas 3.5 and 9.8 to translate back and forth into algebra.

- (1) If  $f$  is flat and of finite presentation, the complex  $Rf_*M$  is perfect on  $Y$  by Derived Categories of Spaces, Lemma 25.4 and  $\beta$  is an isomorphism by More on Algebra, Lemma 87.2 part (1).
- (2) If  $f$  is perfect and  $Y$  Noetherian, the complex  $Rf_*M$  is perfect on  $Y$  by More on Morphisms of Spaces, Lemma 46.5 and  $\beta$  is an isomorphism as before.
- (3) If  $g$  has finite tor dimension and  $Y$  is Noetherian, the complex  $Rf_*M$  is pseudo-coherent on  $Y$  (Derived Categories of Spaces, Lemmas 8.1 and 13.7) and  $\beta$  is an isomorphism by More on Algebra, Lemma 87.2 part (4).

We conclude that we obtain the same answer as in the previous paragraph.

In the rest of the proof we show that the identifications of the left and right hand side of (5.1.1) given in the second and third paragraph are in fact given by (5.1.1). To make our formulas manageable we will use  $(-, -)_X = R\text{Hom}_X(-, -)$ , use  $-\otimes A'$

in stead of  $-\otimes_A^{\mathbf{L}} A'$ , and we will abbreviate  $g^* = Lg^*$  and  $f_* = Rf_*$ . Consider the following commutative diagram

$$\begin{array}{ccccc}
 ((g')^*M, (g')^*a(K))_{X'} & \xleftarrow{\alpha} & (M, a(K))_X \otimes A' & \xlongequal{\quad} & (f_*M, K)_Y \otimes A' \\
 \downarrow & & \downarrow & & \downarrow \\
 ((g')^*M, (g')^*a(g_*g^*K))_{X'} & \xleftarrow{\alpha} & (M, a(g_*g^*K))_X \otimes A' & \xlongequal{\quad} & (f_*M, g_*g^*K)_Y \otimes A' \\
 \uparrow & & \uparrow & & \uparrow \\
 ((g')^*M, (g')^*g'_*a'(g^*K))_{X'} & \xleftarrow{\alpha} & (M, g'_*a'(g^*K))_X \otimes A' & & (f_*M, K)_Y \otimes A' \\
 \downarrow & \swarrow \mu & \downarrow & \searrow \mu' & \downarrow \beta \\
 ((g')^*M, a'(g^*K))_{X'} & \xlongequal{\quad} & (f'_*(g')^*M, g^*K)_{Y'} & \longrightarrow & (g^*f_*M, g^*K)_{Y'}
 \end{array}$$

The arrows labeled  $\alpha$  are the maps from Derived Categories of Spaces, Lemma 20.5 for the diagram with corners  $X', X, Y', Y$ . The upper part of the diagram is commutative as the horizontal arrows are functorial in the entries. The middle vertical arrows come from the invertible transformation  $g'_* \circ a' \rightarrow a \circ g_*$  of Lemma 4.1 and therefore the middle square is commutative. Going down the left hand side is (5.1.1). The upper horizontal arrows provide the identifications used in the second paragraph of the proof. The lower horizontal arrows including  $\beta$  provide the identifications used in the third paragraph of the proof. Given  $E \in D(A)$ ,  $E' \in D(A')$ , and  $c : E \rightarrow E'$  in  $D(A)$  we will denote  $\mu_c : E \otimes A' \rightarrow E'$  the map induced by  $c$  and the adjointness of restriction and base change; if  $c$  is clear we write  $\mu = \mu_c$ , i.e., we drop  $c$  from the notation. The map  $\mu$  in the diagram is of this form with  $c$  given by the identification  $(M, g'_*a'(g^*K))_X = ((g')^*M, a'(g^*K))_{X'}$ ; the triangle involving  $\mu$  is commutative by Derived Categories of Spaces, Remark 20.6.

Observe that

$$\begin{array}{ccccc}
 (M, a(g_*g^*K))_X & \xlongequal{\quad} & (f_*M, g_*g^*K)_Y & \xlongequal{\quad} & (g^*f_*M, g^*K)_{Y'} \\
 \uparrow & & & & \uparrow \\
 (M, g'_*a'(g^*K))_X & \xlongequal{\quad} & ((g')^*M, a'(g^*K))_{X'} & \xlongequal{\quad} & (f'_*(g')^*M, g^*K)_{Y'}
 \end{array}$$

is commutative by the very definition of the transformation  $g'_* \circ a' \rightarrow a \circ g_*$ . Letting  $\mu'$  be as above corresponding to the identification  $(f_*M, g_*g^*K)_Y = (g^*f_*M, g^*K)_{Y'}$ , then the hexagon commutes as well. Thus it suffices to show that  $\beta$  is equal to the composition of  $(f_*M, K)_Y \otimes A' \rightarrow (f_*M, g_*g^*K)_Y \otimes A'$  and  $\mu'$ . To do this, it suffices to prove the two induced maps  $(f_*M, K)_Y \rightarrow (g^*f_*M, g^*K)_{Y'}$  are the same. In other words, it suffices to show the diagram

$$\begin{array}{ccc}
 R\mathrm{Hom}_A(E, K) & \xrightarrow{\text{induced by } \beta} & R\mathrm{Hom}_{A'}(E \otimes_A^{\mathbf{L}} A', K \otimes_A^{\mathbf{L}} A') \\
 & \searrow & \swarrow \\
 & R\mathrm{Hom}_A(E, K \otimes_A^{\mathbf{L}} A') &
 \end{array}$$

commutes for all  $E, K \in D(A)$ . Since this is how  $\beta$  is constructed in More on Algebra, Section 87 the proof is complete.  $\square$

## 6. Right adjoint of pushforward and trace maps

0E5K Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Let  $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  be the right adjoint as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\mathrm{Tr}_f : Rf_* \circ a \longrightarrow \mathrm{id}$$

The corresponding map  $\mathrm{Tr}_{f,K} : Rf_*a(K) \rightarrow K$  for  $K \in D_{QCoh}(\mathcal{O}_Y)$  is sometimes called the *trace map*. This is the map which has the property that the bijection

$$\mathrm{Hom}_X(L, a(K)) \longrightarrow \mathrm{Hom}_Y(Rf_*L, K)$$

for  $L \in D_{QCoh}(\mathcal{O}_X)$  which characterizes the right adjoint is given by

$$\varphi \longmapsto \mathrm{Tr}_{f,K} \circ Rf_*\varphi$$

The isomorphism

$$Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) \longrightarrow R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$$

of Lemma 3.3 comes about by composition with  $\mathrm{Tr}_{f,K}$ . Every trace map we are going to consider in this section will be a special case of this trace map. Before we discuss some special cases we show that formation of the trace map commutes with base change.

0E5L **Lemma 6.1** (Trace map and base change). *Suppose we have a diagram (4.0.1). Then the maps  $1 \star \mathrm{Tr}_f : Lg^* \circ Rf_* \circ a \rightarrow Lg^*$  and  $\mathrm{Tr}_{f'} \star 1 : Rf'_* \circ a' \circ Lg^* \rightarrow Lg^*$  agree via the base change maps  $\beta : Lg^* \circ Rf_* \rightarrow Rf'_* \circ L(g')^*$  (Cohomology on Sites, Remark 20.3) and  $\alpha : L(g')^* \circ a \rightarrow a' \circ Lg^*$  (4.1.1). More precisely, the diagram*

$$\begin{array}{ccc} Lg^* \circ Rf_* \circ a & \xrightarrow{1 \star \mathrm{Tr}_f} & Lg^* \\ \beta \star 1 \downarrow & & \uparrow \mathrm{Tr}_{f'} \star 1 \\ Rf'_* \circ L(g')^* \circ a & \xrightarrow{1 \star \alpha} & Rf'_* \circ a' \circ Lg^* \end{array}$$

*of transformations of functors commutes.*

**Proof.** In this proof we write  $f_*$  for  $Rf_*$  and  $g^*$  for  $Lg^*$  and we drop  $\star$  products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that  $\beta : g^* \circ f_* \rightarrow f'_* \circ (g')^*$  is an isomorphism and that  $\alpha$  is defined using the isomorphism  $\beta^\vee : g'_* \circ a' \rightarrow a \circ g_*$  which is the adjoint of  $\beta$ , see Lemma 4.1 and its proof. First we note that the top horizontal arrow of the diagram in the lemma is equal to the composition

$$g^* \circ f_* \circ a \rightarrow g^* \circ f_* \circ a \circ g_* \circ g^* \rightarrow g^* \circ g_* \circ g^* \rightarrow g^*$$

where the first arrow is the unit for  $(g^*, g_*)$ , the second arrow is  $\mathrm{Tr}_f$ , and the third arrow is the counit for  $(g^*, g_*)$ . This is a simple consequence of the fact that the composition  $g^* \rightarrow g^* \circ g_* \circ g^* \rightarrow g^*$  of unit and counit is the identity. Consider the

diagram

$$\begin{array}{ccccc}
 & g^* \circ f_* \circ a & \xrightarrow{\text{Tr}_f} & g^* & \\
 & \beta \swarrow & & \nwarrow & \text{Tr}_{f'} \\
 f'_* \circ (g')^* \circ a & & g^* \circ f_* \circ a \circ g_* \circ g^* & \xleftarrow{\beta^\vee} & g^* \circ f_* \circ g'_* \circ a' \circ g^* & \xrightarrow{\text{Tr}_{f'}} & f'_* \circ a' \circ g^* \\
 & \beta \downarrow & & & \beta \downarrow & & \\
 & f'_* \circ (g')^* \circ a \circ g_* \circ g^* & \xleftarrow{\beta^\vee} & f'_* \circ (g')^* \circ g'_* \circ a' \circ g^* & & & 
 \end{array}$$

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the discussion above. By Categories, Lemma 24.7 the square

$$\begin{array}{ccc}
 g^* \circ f_* \circ g'_* \circ a' & \xrightarrow{\beta} & f'_* \circ (g')^* \circ g'_* \circ a' \\
 \beta^\vee \downarrow & & \downarrow \\
 g^* \circ f_* \circ a \circ g_* & \xrightarrow{\quad} & \text{id}
 \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since  $\beta$  and  $\beta^\vee$  are isomorphisms, and since going on the outside of the big diagram equals  $\text{Tr}_f \circ \alpha \circ \beta$  by definition this proves the lemma.  $\square$

Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Let  $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  be the right adjoint of  $Rf_*$  as in Lemma 3.1. By Categories, Section 24 we obtain a transformation of functors

$$\eta_f : \text{id} \rightarrow a \circ Rf_*$$

which is called the unit of the adjunction.

0E5M **Lemma 6.2.** *Suppose we have a diagram (4.0.1). Then the maps  $1 \star \eta_f : L(g')^* \rightarrow L(g')^* \circ a \circ Rf_*$  and  $\eta_{f'} \star 1 : L(g')^* \rightarrow a' \circ Rf'_* \circ L(g')^*$  agree via the base change maps  $\beta : Lg^* \circ Rf_* \rightarrow Rf'_* \circ L(g')^*$  (Cohomology on Sites, Remark 20.3) and  $\alpha : L(g')^* \circ a \rightarrow a' \circ Lg^*$  (4.1.1). More precisely, the diagram*

$$\begin{array}{ccc}
 L(g')^* & \xrightarrow{1 \star \eta_f} & L(g')^* \circ a \circ Rf_* \\
 \eta_{f'} \star 1 \downarrow & & \downarrow \alpha \\
 a' \circ Rf'_* \circ L(g')^* & \xleftarrow{\beta} & a' \circ Lg^* \circ Rf_*
 \end{array}$$

of transformations of functors commutes.

**Proof.** This proof is dual to the proof of Lemma 6.1. In this proof we write  $f_*$  for  $Rf_*$  and  $g^*$  for  $Lg^*$  and we drop  $\star$  products with identities as one can figure out which ones to add as long as the source and target of the transformation is known. Recall that  $\beta : g^* \circ f_* \rightarrow f'_* \circ (g')^*$  is an isomorphism and that  $\alpha$  is defined using the isomorphism  $\beta^\vee : g'_* \circ a' \rightarrow a \circ g_*$  which is the adjoint of  $\beta$ , see Lemma 4.1 and its proof. First we note that the left vertical arrow of the diagram in the lemma is equal to the composition

$$(g')^* \rightarrow (g')^* \circ g'_* \circ (g')^* \rightarrow (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* \rightarrow a' \circ f'_* \circ (g')^*$$

where the first arrow is the unit for  $((g')^*, g'_*)$ , the second arrow is  $\eta_{f'}$ , and the third arrow is the counit for  $((g')^*, g'_*)$ . This is a simple consequence of the fact that the composition  $(g')^* \rightarrow (g')^* \circ (g')_* \circ (g')^* \rightarrow (g')^*$  of unit and counit is the identity. Consider the diagram

$$\begin{array}{ccccc}
 & & (g')^* \circ a \circ f_* & \longrightarrow & (g')^* \circ a \circ g_* \circ g^* \circ f_* \\
 & \nearrow \eta_f & & \searrow \beta & \uparrow \beta^\vee \\
 (g')^* & & (g')^* \circ a \circ g_* \circ f'_* \circ (g')^* & & (g')^* \circ g'_* \circ a' \circ g^* \circ f_* \\
 & \searrow & \uparrow \beta^\vee & \swarrow \beta & \downarrow \\
 & & (g')^* \circ g'_* \circ a' \circ f'_* \circ (g')^* & & a' \circ g^* \circ f_* \\
 \eta_{f'} \downarrow & & & \nearrow \beta & \\
 a' \circ f'_* \circ (g')^* & & & & 
 \end{array}$$

In this diagram the two squares commute Categories, Lemma 27.2 or more simply the discussion preceding Categories, Definition 27.1. The triangle commutes by the discussion above. By the dual of Categories, Lemma 24.7 the square

$$\begin{array}{ccc}
 \text{id} & \longrightarrow & g'_* \circ a' \circ g^* \circ f_* \\
 \downarrow & & \downarrow \beta \\
 g'_* \circ a' \circ g^* \circ f_* & \xrightarrow{\beta^\vee} & a \circ g_* \circ f'_* \circ (g')^*
 \end{array}$$

commutes which implies the pentagon in the big diagram commutes. Since  $\beta$  and  $\beta^\vee$  are isomorphisms, and since going on the outside of the big diagram equals  $\beta \circ \alpha \circ \eta_f$  by definition this proves the lemma.  $\square$

## 7. Right adjoint of pushforward and pullback

0E5N Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Let  $a$  be the right adjoint of pushforward as in Lemma 3.1. For  $K, L \in D_{QCoh}(\mathcal{O}_Y)$  there is a canonical map

$$L f^* K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L) \longrightarrow a(K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L)$$

Namely, this map is adjoint to a map

$$R f_* (L f^* K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L)) = K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} R f_* (a(L)) \longrightarrow K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L$$

(equality by Derived Categories of Spaces, Lemma 20.1) for which we use the trace map  $R f_* a(L) \rightarrow L$ . When  $L = \mathcal{O}_Y$  we obtain a map

0E5P (7.0.1) 
$$L f^* K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y) \longrightarrow a(K)$$

functorial in  $K$  and compatible with distinguished triangles.

0E5Q **Lemma 7.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . The map  $L f^* K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L) \rightarrow a(K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L)$  defined above for  $K, L \in D_{QCoh}(\mathcal{O}_Y)$  is an isomorphism if  $K$  is perfect. In particular, (7.0.1) is an isomorphism if  $K$  is perfect.*



**Proof.** Let  $K^\vee$  be the “dual” to  $K$ , see Cohomology on Sites, Lemma 39.9. For  $M \in D_{QCoh}(\mathcal{O}_X)$  we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M, K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} L) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} K^\vee, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*K^\vee, a(L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(M, Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(L)) \end{aligned}$$

Second equality by the definition of  $a$  and the projection formula (Cohomology on Sites, Lemma 40.1) or the more general Derived Categories of Spaces, Lemma 20.1. Hence the result by the Yoneda lemma.  $\square$

0E5R **Lemma 7.2.** *Suppose we have a diagram (4.0.1). Let  $K \in D_{QCoh}(\mathcal{O}_Y)$ . The diagram*

$$\begin{array}{ccc} L(g')^*(Lf^*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y)) & \longrightarrow & L(g')^*a(K) \\ \downarrow & & \downarrow \\ L(f')^*Lg^*K \otimes_{\mathcal{O}_{X'}}^{\mathbf{L}} a'(\mathcal{O}_{Y'}) & \longrightarrow & a'(Lg^*K) \end{array}$$

*commutes where the horizontal arrows are the maps (7.0.1) for  $K$  and  $Lg^*K$  and the vertical maps are constructed using Cohomology on Sites, Remark 20.3 and (4.1.1).*

**Proof.** In this proof we will write  $f_*$  for  $Rf_*$  and  $f^*$  for  $Lf^*$ , etc, and we will write  $\otimes$  for  $\otimes_{\mathcal{O}_X}^{\mathbf{L}}$ , etc. Let us write (7.0.1) as the composition

$$\begin{aligned} f^*K \otimes a(\mathcal{O}_Y) &\rightarrow a(f_*(f^*K \otimes a(\mathcal{O}_Y))) \\ &\leftarrow a(K \otimes f_*a(\mathcal{O}_K)) \\ &\rightarrow a(K \otimes \mathcal{O}_Y) \\ &\rightarrow a(K) \end{aligned}$$

Here the first arrow is the unit  $\eta_f$ , the second arrow is  $a$  applied to Cohomology on Sites, Equation (40.0.1) which is an isomorphism by Derived Categories of Spaces, Lemma 20.1, the third arrow is  $a$  applied to  $\mathrm{id}_K \otimes \mathrm{Tr}_f$ , and the fourth arrow is  $a$  applied to the isomorphism  $K \otimes \mathcal{O}_Y = K$ . The proof of the lemma consists in showing that each of these maps gives rise to a commutative square as in the statement of the lemma. For  $\eta_f$  and  $\mathrm{Tr}_f$  this is Lemmas 6.2 and 6.1. For the arrow using Cohomology on Sites, Equation (40.0.1) this is Cohomology on Sites, Remark 40.2. For the multiplication map it is clear. This finishes the proof.  $\square$

## 8. Right adjoint of pushforward for proper flat morphisms

0E5S For proper, flat, and finitely presented morphisms of quasi-compact and quasi-separated algebraic spaces the right adjoint of pushforward enjoys some remarkable properties.

0E5T **Lemma 8.1.** *Let  $S$  be a scheme. Let  $Y$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let  $a$  be the right adjoint for  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  of Lemma 3.1. Then  $a$  commutes with direct sums.*

**Proof.** Let  $P$  be a perfect object of  $D(\mathcal{O}_X)$ . By Derived Categories of Spaces, Lemma 25.4 the complex  $Rf_*P$  is perfect on  $Y$ . Let  $K_i$  be a family of objects of  $D_{QCoh}(\mathcal{O}_Y)$ . Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(\bigoplus K_i)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, \bigoplus K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_Y)}(Rf_*P, K_i) \\ &= \bigoplus \mathrm{Hom}_{D(\mathcal{O}_X)}(P, a(K_i)) \end{aligned}$$

because a perfect object is compact (Derived Categories of Spaces, Proposition 16.1). Since  $D_{QCoh}(\mathcal{O}_X)$  has a perfect generator (Derived Categories of Spaces, Theorem 15.4) we conclude that the map  $\bigoplus a(K_i) \rightarrow a(\bigoplus K_i)$  is an isomorphism, i.e.,  $a$  commutes with direct sums.  $\square$

0E5U **Lemma 8.2.** *Let  $S$  be a scheme. Let  $Y$  be a quasi-compact and quasi-separated algebraic space over  $S$ . Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which is proper, flat, and of finite presentation. The map (7.0.1) is an isomorphism for every object  $K$  of  $D_{QCoh}(\mathcal{O}_Y)$ .*

**Proof.** By Lemma 8.1 we know that  $a$  commutes with direct sums. Hence the collection of objects of  $D_{QCoh}(\mathcal{O}_Y)$  for which (7.0.1) is an isomorphism is a strictly full, saturated, triangulated subcategory of  $D_{QCoh}(\mathcal{O}_Y)$  which is moreover preserved under taking direct sums. Since  $D_{QCoh}(\mathcal{O}_Y)$  is a module category (Derived Categories of Spaces, Theorem 17.3) generated by a single perfect object (Derived Categories of Spaces, Theorem 15.4) we can argue as in More on Algebra, Remark 57.13 to see that it suffices to prove (7.0.1) is an isomorphism for a single perfect object. However, the result holds for perfect objects, see Lemma 7.1.  $\square$

0E5V **Lemma 8.3.** *Let  $Y$  be an affine scheme. Let  $f : X \rightarrow Y$  be a morphism of algebraic spaces which is proper, flat, and of finite presentation. Let  $a$  be the right adjoint for  $Rf_* : D_{QCoh}(\mathcal{O}_X) \rightarrow D_{QCoh}(\mathcal{O}_Y)$  of Lemma 3.1. Then*

- (1)  $a(\mathcal{O}_Y)$  is a  $Y$ -perfect object of  $D(\mathcal{O}_X)$ ,
- (2)  $Rf_*a(\mathcal{O}_Y)$  has vanishing cohomology sheaves in positive degrees,
- (3)  $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$  is an isomorphism.

**Proof.** We will repeatedly use that  $Rf_*R\mathcal{H}om_{\mathcal{O}_X}(L, a(K)) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*L, K)$ , see Lemma 3.3. Let  $E$  be a perfect object of  $D(\mathcal{O}_X)$  with dual  $E^\vee$ , see Cohomology on Sites, Lemma 39.9. Then

$$Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y)) = Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, a(\mathcal{O}_Y)) = R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$$

By Derived Categories of Spaces, Lemma 25.4 the complex  $Rf_*E^\vee$  is perfect. Hence the dual  $R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y)$  is perfect as well. We conclude that  $a(\mathcal{O}_Y)$  is pseudo-coherent by Derived Categories of Spaces, Lemma 25.7 and More on Morphisms of Spaces, Lemma 50.4.

Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_Y$ -module. By Lemma 8.2 we have

$$a(\mathcal{F}) = Lf^*\mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y) = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y}^{\mathbf{L}} a(\mathcal{O}_Y)$$

Second equality by Cohomology on Sites, Lemma 19.5. By Lemma 3.2 there exists an integer  $N$  such that  $H^i(a(\mathcal{F})) = 0$  for  $i \leq -N$ . Looking at stalks we conclude that  $a(\mathcal{O}_Y)$  has finite tor dimension (details omitted; hint: for  $y \in Y$  any  $\mathcal{O}_{Y,y}$ -module occurs as  $\mathcal{F}_y$  for some quasi-coherent module on the affine scheme  $Y$ ).

Combining the results of the previous two paragraphs we find that  $a(\mathcal{O}_Y)$  is  $Y$ -perfect, see More on Morphisms of Spaces, Definition 51.1. This proves (1).

Let  $M$  be an object of  $D_{QCoh}(\mathcal{O}_Y)$ . Then

$$\begin{aligned} \mathrm{Hom}_Y(M, Rf_*a(\mathcal{O}_Y)) &= \mathrm{Hom}_X(Lf^*M, a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*Lf^*M, \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(M \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*\mathcal{O}_Y, \mathcal{O}_Y) \end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 20.1. The second equality by construction of  $a$ . The third equality by Derived Categories of Spaces, Lemma 20.1. Recall  $Rf_*\mathcal{O}_X$  is perfect of tor amplitude in  $[0, N]$  for some  $N$ , see Derived Categories of Spaces, Lemma 25.4. Thus we can represent  $Rf_*\mathcal{O}_X$  by a complex of finite projective modules sitting in degrees  $[0, N]$  (using More on Algebra, Lemma 69.2 and the fact that  $Y$  is affine). Hence if  $M = \mathcal{O}_Y[-i]$  for some  $i > 0$ , then the last group is zero. Since  $Y$  is affine we conclude that  $H^i(Rf_*a(\mathcal{O}_Y)) = 0$  for  $i > 0$ . This proves (2).

Let  $E$  be a perfect object of  $D_{QCoh}(\mathcal{O}_X)$ . Then we have

$$\begin{aligned} \mathrm{Hom}_X(E, R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))) &= \mathrm{Hom}_X(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y), a(\mathcal{O}_Y)) \\ &= \mathrm{Hom}_Y(Rf_*(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} a(\mathcal{O}_Y)), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(E^\vee, a(\mathcal{O}_Y))), \mathcal{O}_Y) \\ &= \mathrm{Hom}_Y(R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E^\vee, \mathcal{O}_Y), \mathcal{O}_Y) \\ &= R\Gamma(Y, Rf_*E^\vee) \\ &= \mathrm{Hom}_X(E, \mathcal{O}_X) \end{aligned}$$

The first equality holds by Cohomology on Sites, Lemma 28.2. The second equality is the definition of  $a$ . The third equality comes from the construction of the dual perfect complex  $E^\vee$ , see Cohomology on Sites, Lemma 39.9. The fourth equality is given in the first line of the proof. The fifth equality holds by double duality for perfect complexes (Cohomology on Sites, Lemma 39.9) and the fact that  $Rf_*E$  is perfect by Derived Categories of Spaces, Lemma 25.4. The last equality is Leray for  $f$ . This string of equalities essentially shows (3) holds by the Yoneda lemma. Namely, the object  $R\mathcal{H}om(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$  is in  $D_{QCoh}(\mathcal{O}_X)$  by Derived Categories of Spaces, Lemma 13.10. Taking  $E = \mathcal{O}_X$  in the above we get a map  $\alpha : \mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(a(\mathcal{O}_Y), a(\mathcal{O}_Y))$  corresponding to  $\mathrm{id}_{\mathcal{O}_X} \in \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$ . Since all the isomorphisms above are functorial in  $E$  we see that the cone on  $\alpha$  is an object  $C$  of  $D_{QCoh}(\mathcal{O}_X)$  such that  $\mathrm{Hom}(E, C) = 0$  for all perfect  $E$ . Since the perfect objects generate (Derived Categories of Spaces, Theorem 15.4) we conclude that  $\alpha$  is an isomorphism.  $\square$

### 9. Relative dualizing complexes for proper flat morphisms

0E5W Motivated by Duality for Schemes, Sections 12 and 28 and the material in Section 8 we make the following definition.

0E5X **Definition 9.1.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a proper, flat morphism of algebraic spaces over  $S$  which is of finite presentation. A *relative dualizing complex*

for  $X/Y$  is a pair  $(\omega_{X/Y}^\bullet, \tau)$  consisting of a  $Y$ -perfect object  $\omega_{X/Y}^\bullet$  of  $D(\mathcal{O}_X)$  and a map

$$\tau : Rf_*\omega_{X/Y}^\bullet \longrightarrow \mathcal{O}_Y$$

such that for any cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

where  $Y'$  is an affine scheme the pair  $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau)$  is isomorphic to the pair  $(a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$  studied in Sections 3, 4, 5, 6, 7, and 8.

There are several remarks we should make here.

- (1) In Definition 9.1 one may drop the assumption that  $\omega_{X/Y}^\bullet$  is  $Y$ -perfect. Namely, running  $Y'$  through the members of an étale covering of  $Y$  by affines, we see from Lemma 8.3 that the restrictions of  $\omega_{X/Y}^\bullet$  to the members of an étale covering of  $X$  are  $Y$ -perfect, which implies  $\omega_{X/Y}^\bullet$  is  $Y$ -perfect, see More on Morphisms of Spaces, Section 51.
- (2) Consider a relative dualizing complex  $(\omega_{X/Y}^\bullet, \tau)$  and a cartesian square as in Definition 9.1. We are going to think of the existence of the isomorphism  $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau) \cong (a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$  as follows: it says that for any  $M' \in D_{QCoh}(\mathcal{O}_{X'})$  the map

$$\text{Hom}_X(M', L(g')^*\omega_{X/Y}^\bullet) \longrightarrow \text{Hom}_Y(Rf'_*M', \mathcal{O}_{Y'}), \quad \varphi' \longmapsto Lg^*\tau \circ Rf'_*\varphi'$$

is an isomorphism. This follows from the definition of  $a'$  and the discussion in Section 6. In particular, the Yoneda lemma guarantees that the isomorphism is unique.

- (3) If  $Y$  is affine itself, then a relative dualizing complex  $(\omega_{X/Y}^\bullet, \tau)$  exists and is canonically isomorphic to  $(a(\mathcal{O}_Y), \text{Tr}_{f, \mathcal{O}_Y})$  where  $a$  is the right adjoint for  $Rf_*$  as in Lemma 3.1 and  $\text{Tr}_f$  is as in Section 6. Namely, given a diagram as in the definition we get an isomorphism  $L(g')^*a(\mathcal{O}_Y) \rightarrow a'(\mathcal{O}_{Y'})$  by Lemma 5.1 which is compatible with trace maps by Lemma 6.1.

This produces exactly enough information to glue the locally given relative dualizing complexes to global ones. We suggest the reader skip the proofs of the following lemmas.

**0E5Y Lemma 9.2.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a proper, flat morphism of algebraic spaces which is of finite presentation. If  $(\omega_{X/Y}^\bullet, \tau)$  is a relative dualizing complex, then  $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/Y}^\bullet, \omega_{X/Y}^\bullet)$  is an isomorphism and  $Rf_*\omega_{X/Y}^\bullet$  has vanishing cohomology sheaves in positive degrees.*

**Proof.** It suffices to prove this after base change to an affine scheme étale over  $Y$  in which case it follows from Lemma 8.3.  $\square$

**0E5Z Lemma 9.3.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a proper, flat morphism of algebraic spaces which is of finite presentation. If  $(\omega_j^\bullet, \tau_j)$ ,  $j = 1, 2$  are two relative dualizing complexes on  $X/Y$ , then there is a unique isomorphism  $(\omega_1^\bullet, \tau_1) \rightarrow (\omega_2^\bullet, \tau_2)$ .*

**Proof.** Consider  $g : Y' \rightarrow Y$  étale with  $Y'$  an affine scheme and denote  $X' = Y' \times_Y X$  the base change. By Definition 9.1 and the discussion following, there is a unique isomorphism  $\iota : (\omega_1^\bullet|_{X'}, \tau_1|_{Y'}) \rightarrow (\omega_2^\bullet|_{X'}, \tau_2|_{Y'})$ . If  $Y'' \rightarrow Y'$  is a further étale morphism of affines and  $X'' = Y'' \times_Y X$ , then  $\iota|_{X''}$  is the unique isomorphism  $(\omega_1^\bullet|_{X''}, \tau_1|_{Y''}) \rightarrow (\omega_2^\bullet|_{X''}, \tau_2|_{Y''})$  (by uniqueness). Also we have

$$\mathrm{Ext}_{X'}^p(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'}) = 0, \quad p < 0$$

because  $\mathcal{O}_{X'} \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_1^\bullet|_{X'}) \cong R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_1^\bullet|_{X'}, \omega_2^\bullet|_{X'})$  by Lemma 9.2.

Choose a étale hypercovering  $b : V \rightarrow Y$  such that each  $V_n = \coprod_{i \in I_n} Y_{n,i}$  with  $Y_{n,i}$  affine. This is possible by Hypercoverings, Lemma 12.2 and Remark 12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote  $X_{n,i} = Y_{n,i} \times_Y X$  and  $U_n = V_n \times_Y X$  so that we obtain an étale hypercovering  $a : U \rightarrow X$  (Hypercoverings, Lemma 12.4) with  $U_n = \coprod X_{n,i}$ . The assumptions of Simplicial Spaces, Lemma 36.1 are satisfied for  $a : U \rightarrow X$  and the complexes  $\omega_1^\bullet$  and  $\omega_2^\bullet$ . Hence we obtain a unique morphism  $\iota : \omega_1^\bullet \rightarrow \omega_2^\bullet$  whose restriction to  $X_{0,i}$  is the unique isomorphism  $(\omega_1^\bullet|_{X_{0,i}}, \tau_1|_{Y_{0,i}}) \rightarrow (\omega_2^\bullet|_{X_{0,i}}, \tau_2|_{Y_{0,i}})$ . We still have to see that the diagram

$$\begin{array}{ccc} Rf_*\omega_1^\bullet & \xrightarrow{Rf_*\iota} & Rf_*\omega_2^\bullet \\ & \searrow \tau_1 & \swarrow \tau_2 \\ & \mathcal{O}_Y & \end{array}$$

is commutative. However, we know that  $Rf_*\omega_1^\bullet$  and  $Rf_*\omega_2^\bullet$  have vanishing cohomology sheaves in positive degrees (Lemma 9.2) thus this commutativity may be proved after restricting to the affines  $Y_{0,i}$  where it holds by construction.  $\square$

0E60 **Lemma 9.4.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a proper, flat morphism of algebraic spaces which is of finite presentation. Let  $(\omega^\bullet, \tau)$  be a pair consisting of a  $Y$ -perfect object of  $D(\mathcal{O}_X)$  and a map  $\tau : Rf_*\omega^\bullet \rightarrow \mathcal{O}_Y$ . Assume we have cartesian diagrams*

$$\begin{array}{ccc} X_i & \xrightarrow{g'_i} & X \\ f_i \downarrow & & \downarrow f \\ Y_i & \xrightarrow{g_i} & Y \end{array}$$

with  $Y_i$  affine such that  $\{g_i : Y_i \rightarrow Y\}$  is an étale covering and isomorphisms of pairs  $(\omega^\bullet|_{X_i}, \tau|_{Y_i}) \rightarrow (a_i(\mathcal{O}_{Y_i}), \mathrm{Tr}_{f_i, \mathcal{O}_{Y_i}})$  as in Definition 9.1. Then  $(\omega^\bullet, \tau)$  is a relative dualizing complex for  $X$  over  $Y$ .

**Proof.** Let  $g : Y' \rightarrow Y$  and  $X', f', g', a'$  be as in Definition 9.1. Set  $((\omega')^\bullet, \tau') = (L(g')^*\omega^\bullet, Lg'^*\tau)$ . We can find a finite étale covering  $\{Y'_j \rightarrow Y'\}$  by affines which refines  $\{Y_i \times_Y Y' \rightarrow Y'\}$  (Topologies, Lemma 4.4). Thus for each  $j$  there is an  $i_j$  and a morphism  $k_j : Y'_j \rightarrow Y_{i_j}$  over  $Y$ . Consider the fibre products

$$\begin{array}{ccc} X'_j & \xrightarrow{h'_j} & X' \\ f'_j \downarrow & & \downarrow f' \\ Y'_j & \xrightarrow{h_j} & Y' \end{array}$$

Denote  $k'_j : X'_j \rightarrow X_{i_j}$  the induced morphism (base change of  $k_j$  by  $f_{i_j}$ ). Restricting the given isomorphisms to  $Y'_j$  via the morphism  $k'_j$  we get isomorphisms of pairs  $((\omega')^\bullet|_{X'_j}, \tau'|_{Y'_j}) \rightarrow (a_j(\mathcal{O}_{Y'_j}), \text{Tr}_{f'_j, \mathcal{O}_{Y'_j}})$ . After replacing  $f : X \rightarrow Y$  by  $f' : X' \rightarrow Y'$  we reduce to the problem solved in the next paragraph.

Assume  $Y$  is affine. Problem: show  $(\omega^\bullet, \tau)$  is isomorphic to  $(\omega_{X/Y}^\bullet, \text{Tr}) = (a(\mathcal{O}_Y), \text{Tr}_{f, \mathcal{O}_Y})$ . We may assume our covering  $\{Y_i \rightarrow Y\}$  is given by a single surjective étale morphism  $\{g : Y' \rightarrow Y\}$  of affines. Namely, we can first replace  $\{g_i : Y_i \rightarrow Y\}$  by a finite subcovering, and then we can set  $g = \coprod g_i : Y' = \coprod Y_i \rightarrow Y$ ; some details omitted. Set  $X' = Y' \times_Y X$  with maps  $f', g'$  as in Definition 9.1. Then all we're given is that we have an isomorphism

$$(\omega^\bullet|_{X'}, \tau|_{Y'}) \rightarrow (a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$$

Since  $(\omega_{X/Y}^\bullet, \text{Tr})$  is a relative dualizing complex (see discussion following Definition 9.1) there is a unique isomorphism

$$(\omega_{X/Y}^\bullet|_{X'}, \text{Tr}|_{Y'}) \rightarrow (a'(\mathcal{O}_{Y'}), \text{Tr}_{f', \mathcal{O}_{Y'}})$$

Uniqueness by Lemma 9.3 for example. Combining the displayed isomorphisms we find an isomorphism

$$\alpha : (\omega^\bullet|_{X'}, \tau|_{Y'}) \rightarrow (\omega_{X/Y}^\bullet|_{X'}, \text{Tr}|_{Y'})$$

Set  $Y'' = Y' \times_Y Y'$  and  $X'' = Y'' \times_Y X$  the two pullbacks of  $\alpha$  to  $X''$  have to be the same by uniqueness again. Since we have vanishing negative self exts for  $\omega_{X'/Y'}^\bullet$  over  $X'$  (Lemma 9.2) and since this remains true after pulling back by any projection  $Y' \times_Y \dots \times_Y Y' \rightarrow Y'$  (small detail omitted – compare with the proof of Lemma 9.3), we find that  $\alpha$  descends to an isomorphism  $\omega^\bullet \rightarrow \omega_{X/Y}^\bullet$  over  $X$  by Simplicial Spaces, Lemma 36.1.  $\square$

0E61 **Lemma 9.5.** *Let  $S$  be a scheme. Let  $X \rightarrow Y$  be a proper, flat morphism of algebraic spaces which is of finite presentation. There exists a relative dualizing complex  $(\omega_{X/Y}^\bullet, \tau)$ .*

**Proof.** Choose a étale hypercovering  $b : V \rightarrow Y$  such that each  $V_n = \coprod_{i \in I_n} Y_{n,i}$  with  $Y_{n,i}$  affine. This is possible by Hypercoverings, Lemma 12.2 and Remark 12.9 (to replace the hypercovering produced in the lemma by the one having disjoint unions in each degree). Denote  $X_{n,i} = Y_{n,i} \times_Y X$  and  $U_n = V_n \times_Y X$  so that we obtain an étale hypercovering  $a : U \rightarrow X$  (Hypercoverings, Lemma 12.4) with  $U_n = \coprod X_{n,i}$ . For each  $n, i$  there exists a relative dualizing complex  $(\omega_{n,i}^\bullet, \tau_{n,i})$  on  $X_{n,i}/Y_{n,i}$ . See discussion following Definition 9.1. For  $\varphi : [m] \rightarrow [n]$  and  $i \in I_n$  consider the morphisms  $g_{\varphi,i} : Y_{n,i} \rightarrow Y_{m,\alpha(\varphi)}$  and  $g'_{\varphi,i} : X_{n,i} \rightarrow X_{m,\alpha(\varphi)}$  which are part of the structure of the given hypercoverings (Hypercoverings, Section 12). Then we have a unique isomorphisms

$$\iota_{n,i,\varphi} : (L(g'_{n,i})^* \omega_{n,i}^\bullet, Lg_{n,i}^* \tau_{n,i}) \longrightarrow (\omega_{m,\alpha(\varphi)(i)}^\bullet, \tau_{m,\alpha(\varphi)(i)})$$

of pairs, see discussion following Definition 9.1. Observe that  $\omega_{n,i}^\bullet$  has vanishing negative self exts on  $X_{n,i}$  by Lemma 9.2. Denote  $(\omega_n^\bullet, \tau_n)$  the pair on  $U_n/V_n$  constructed using the pairs  $(\omega_{n,i}^\bullet, \tau_{n,i})$  for  $i \in I_n$ . For  $\varphi : [m] \rightarrow [n]$  and  $i \in I_n$  consider the morphisms  $g_\varphi : V_n \rightarrow V_m$  and  $g'_\varphi : U_n \rightarrow U_m$  which are part of

the structure of the simplicial algebraic spaces  $V$  and  $U$ . Then we have unique isomorphisms

$$\iota_\varphi : (L(g'_\varphi)^*\omega_n^\bullet, Lg_\varphi^*\tau_n) \longrightarrow (\omega_m^\bullet, \tau_m)$$

of pairs constructed from the isomorphisms on the pieces. The uniqueness guarantees that these isomorphisms satisfy the transitivity condition as formulated in *Simplicial Spaces*, Definition 15.1. The assumptions of *Simplicial Spaces*, Lemma 36.2 are satisfied for  $a : U \rightarrow X$ , the complexes  $\omega_n^\bullet$  and the isomorphisms  $\iota_\varphi^1$ . Thus we obtain an object  $\omega^\bullet$  of  $D_{QCoh}(\mathcal{O}_X)$  together with an isomorphism  $\iota_0 : \omega^\bullet|_{U_0} \rightarrow \omega_0^\bullet$  compatible with the two isomorphisms  $\iota_{\delta_0^1}$  and  $\iota_{\delta_1^1}$ . Finally, we apply *Simplicial Spaces*, Lemma 36.1 to find a unique morphism

$$\tau : Rf_*\omega^\bullet \longrightarrow \mathcal{O}_Y$$

whose restriction to  $V_0$  agrees with  $\tau_0$ ; some details omitted – compare with the end of the proof of Lemma 9.3 for example to see why we have the required vanishing of negative exts. By Lemma 9.4 the pair  $(\omega^\bullet, \tau)$  is a relative dualizing complex and the proof is complete.  $\square$

0E6C **Lemma 9.6.** *Let  $S$  be a scheme. Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*of algebraic spaces over  $S$ . Assume  $X \rightarrow Y$  is proper, flat, and of finite presentation. Let  $(\omega_{X/Y}^\bullet, \tau)$  be a relative dualizing complex for  $f$ . Then  $(L(g')^*\omega_{X/Y}^\bullet, Lg^*\tau)$  is a relative dualizing complex for  $f'$ .*

**Proof.** Observe that  $L(g')^*\omega_{X/Y}^\bullet$  is  $Y'$ -perfect by More on Morphisms of Spaces, Lemma 51.6. The other condition of Definition 9.1 holds by transitivity of fibre products.  $\square$

### 10. Comparison with the case of schemes

0E6D We should add a lot more in this section.

0E6E **Lemma 10.1.** *Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of quasi-compact and quasi-separated algebraic spaces over  $S$ . Assume  $X$  and  $Y$  are representable and let  $f_0 : X_0 \rightarrow Y_0$  be a morphism of schemes representing  $f$  (awkward but temporary notation). Let  $a : D_{QCoh}(\mathcal{O}_Y) \rightarrow D_{QCoh}(\mathcal{O}_X)$  be the right adjoint of  $Rf_*$  from Lemma 3.1. Let  $a_0 : D_{QCoh}(\mathcal{O}_{Y_0}) \rightarrow D_{QCoh}(\mathcal{O}_{X_0})$  be the right adjoint of  $Rf_{0*}$  from *Duality for Schemes*, Lemma 3.1. Then*

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal[\text{Derived Categories of Spaces, Lemma 4.2}]{} & D_{QCoh}(\mathcal{O}_X) \\ a_0 \uparrow & & \uparrow a \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal[\text{Derived Categories of Spaces, Lemma 4.2}]{} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

*is commutative.*

<sup>1</sup>This lemma uses only  $\omega_0^\bullet$  and the two maps  $\delta_1^1, \delta_0^1 : [1] \rightarrow [0]$ . The reader can skip the first few lines of the proof of the referenced lemma because here we actually are already given a simplicial system of the derived category of modules.

**Proof.** Follows from uniqueness of adjoints and the compatibilities of Derived Categories of Spaces, Remark 6.3.  $\square$

## 11. Other chapters

- |                                    |                                       |
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| (2) Conventions                    | (43) Picard Schemes of Curves         |
| (3) Set Theory                     | (44) Adequate Modules                 |
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| (7) Sites and Sheaves              | (48) Local Cohomology                 |
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| (38) Groupoid Schemes              | (78) Duality for Spaces               |
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(95) Cohomology of Algebraic Stacks	
(96) Derived Categories of Stacks	

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