1. Introduction

The goal of this chapter is to discuss pushouts in the category of algebraic spaces. This can be done with varying assumptions. A fairly general pushout construction is given in [TT13]: one of the morphisms is affine and the other is a closed immersion. We discuss a particular case of this in Section 2 where we assume one of the morphisms is affine and the other is a thickening, a situation that often comes up in deformation theory.

In Sections 5 and 6 we discuss diagrams

\[
\begin{array}{c}
\begin{aligned}
f^{-1}(X \setminus Z) & \overset{f^{-1}}{\longrightarrow} Y \\
\downarrow & \\
X \setminus Z & \overset{f}{\longrightarrow} X
\end{aligned}
\end{array}
\]

where \( f \) is a quasi-compact and quasi-separated morphism of algebraic spaces, \( Z \to X \) is a closed immersion of finite presentation, the map \( f^{-1}(Z) \to Z \) is an isomorphism, and \( f \) is flat along \( f^{-1}(Z) \). In this situation we glue quasi-coherent modules on \( X \setminus Z \) and \( Y \) (in Section 5) to quasi-coherent modules on \( X \) and we glue algebraic spaces over \( X \setminus Z \) and \( Y \) (in Section 6) to algebraic spaces over \( X \).

In Section 7 we discuss how proper birational morphisms of Noetherian algebraic spaces give rise to coequalizer diagrams in algebraic spaces in some sense.

In Section 8 we use the construction of elementary distinguished squares in Section 4 to prove Nagata’s theorem on compactifications in the setting of algebraic spaces.
2. Pushouts in the category of algebraic spaces

Let $S$ be a scheme. Let $I \to \mathcal{(Sch/S)_{fppf}}$, $i \mapsto X_i$ be a diagram (see Categories, Section [14]). For each $i$ we may consider the small étale site $X_i, \text{étale}$. For each morphism $i \to j$ of $I$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_j, \text{étale} \to X_i, \text{étale}$. Hence we obtain a pseudo functor from $\mathcal{T}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_i, \text{étale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \to X_i$ over $I$ such that for each $i \to j$ in $I$ the diagram

$$
\begin{array}{ccc}
U_i & \longrightarrow & U_j \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & X_j
\end{array}
$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to T$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to T$ is equal to $f_j$. Then we get a functor $T, \text{étale} \to \lim X_i, \text{étale}$. With this notation in hand we can formulate our lemma.

Lemma 2.1. Let $S$ be a scheme. Let $I \to \mathcal{(Sch/S)_{fppf}}$, $i \mapsto X_i$ be a diagram as above. Assume that

1. $X = \text{colim} X_i$ exists in the category of schemes,
2. $\coprod X_i \to X$ is surjective,
3. if $U \to X$ is étale and $U_i = X_i \times_X U$, then $U = \text{colim} U_i$ in the category of schemes, and
4. every object $(U_i \to X_i)$ of $\lim X_i, \text{étale}$ with $U_i \to X_i$ separated is in the essential image the functor $X, \text{étale} \to \lim X_i, \text{étale}$.

Then $X = \text{colim} X_i$ in the category of algebraic spaces over $S$ also.

Proof. Let $Z$ be an algebraic space over $S$. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to $f_j$. We have to construct a morphism of algebraic spaces $f : X \to Z$ such that we can recover $f_i$ as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to $Z$. We may assume that $W$ is a disjoint union of affines and in particular we may assume that $W \to Z$ is separated. For each $i$ set $U_i = W \times_{Z, f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_i, \text{étale}$ with $U_i \to X_i$ separated. By assumption (4) we can find an étale morphism $U \to X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \to W$ such that the compositions $U_i \to U \to W$ are $h_i$. Let $g : U \to Z$ be the composition of $h$ with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_Z W$. Composing with $U_i \times_X U_i \to U \times_X U$ we obtain $(h_i, h_i)$ which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_X U_i$ by (3) we see that $(h, h)$ factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$
is surjective. As $Z$ is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. □

**Lemma 2.2.** Let $S$ be a scheme. Let $X \to X'$ be a thickening of schemes over $S$ and let $X \to Y$ be an affine morphism of schemes over $S$. Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 14.3). Then $Y'$ is also a pushout in the category of algebraic spaces over $S$.

**Proof.** This is an immediate consequence of Lemma 2.1 and More on Morphisms, Lemmas 14.3, 14.4, and 14.6. □

**Lemma 2.3.** In More on Morphisms, Situation 59.1 let $Y \amalg_Z X$ be the pushout in the category of schemes (More on Morphisms, Proposition 59.3). Then $Y \amalg_Z X$ is also a pushout in the category of algebraic spaces over $S$.

**Proof.** This is a consequence of Lemma 2.1, the proposition mentioned in the lemma and More on Morphisms, Lemmas 59.6 and 59.7. Conditions (1) and (2) of Lemma 2.1 follow immediately. To see (3) and (4) note that an étale morphism is locally quasi-finite and use that the equivalence of categories of More on Morphisms, Lemma 59.7 is constructed using the pushout construction of More on Morphisms, Lemmas 59.6. Minor details omitted. □

**Lemma 2.4.** Let $S$ be a scheme. Let $X \to X'$ be a thickening of algebraic spaces over $S$ and let $X \to Y$ be an affine morphism of algebraic spaces over $S$. Then there exists a pushout

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow f & & \downarrow f' \\
Y & \longrightarrow & Y \amalg_X X'
\end{array}
$$

in the category of algebraic spaces over $S$. Moreover $Y' = Y \amalg_X X'$ is a thickening of $Y$ and

$$
\mathcal{O}_{Y'} = \mathcal{O}_{Y} \times_f \mathcal{O}_{X} f'_* \mathcal{O}_{X'}
$$

as sheaves on $Y_{\text{étale}} = (Y')_{\text{étale}}$.

**Proof.** Choose a scheme $V$ and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. This is a scheme affine over $V$ with a surjective étale morphism $U \to X$. By More on Morphisms of Spaces, Lemma 9.6 there exists a $U' \to X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \to U'$ is a thickening too. Apply More on Morphisms, Lemma 14.3 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$
\begin{array}{ccc}
U \times_X U & \longrightarrow & U' \times_{X'} U' \\
\downarrow & & \downarrow \\
V \times_Y V & \longrightarrow & R'
\end{array}
$$

in the category of schemes. Consider the morphisms

$$
U \times_X U \to U \to V', \quad U' \times_{X'} U' \to U' \to V', \quad V \times_Y V \to V \to V'
$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \to V'$ which is étale by More on Morphisms, Lemma 14.6. Similarly, we
obtain $s' : R' \to V'$ étale. The morphism $j' = (t', s') : R' \to V' \times_Y V$ is unramified (as $t'$ is étale) and a monomorphism when restricted to the closed subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that $j'$ is a monomorphism too. Finally, $j'$ is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s', V', t'} R' \to R'$ (details omitted). At this point we set $Y' = U'/R'$, see Spaces, Theorem [10.5].

We have morphisms $X' = U'/U' \times_X X', U' \to V'/R' = Y'$ and $Y = V/V \times_Y V \to V'/R' = Y'$. By construction these fit into the commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow f & & \downarrow f' \\
Y & \longrightarrow & Y'
\end{array}
$$

Since $Y \to Y'$ is a thickening we have $Y_{\text{étale}} = (Y')_{\text{étale}}$, see More on Morphisms of Spaces, Lemma [9.6]. The commutativity of the diagram gives a map of sheaves

$$
\mathcal{O}_{Y'} \to \mathcal{O}_Y \times_{f,f'} \mathcal{O}_X,
$$
on this set. By More on Morphisms, Lemma [14.3] this map is an isomorphism when we restrict to the scheme $V'$, hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let $Z$ be an algebraic space and let $a' : X' \to Z$ and $b : Y \to Z$ be morphisms of algebraic spaces. By Lemma [2.2] we obtain a unique morphism $h : V' \to Z$ fitting into the commutative diagrams

$$
\begin{array}{ccc}
U' & \longrightarrow & V' \\
\downarrow & & \downarrow h \\
X' & \longrightarrow & Z
\end{array}
$$

and

$$
\begin{array}{ccc}
V & \longrightarrow & V' \\
\downarrow & & \downarrow h \\
Y & \longrightarrow & Z
\end{array}
$$
The uniqueness shows that $h \circ t' = h \circ s'$. Hence $h$ factors uniquely as $V' \to Y' \to Z$ and we win. \hfill \square

In the following lemma we use the fibre product of categories as defined in Categories, Example [30.3].

**Lemma 2.5.** Let $S$ be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over $S$ and let $X \to Y$ be an affine morphism of algebraic spaces over $S$. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma [2.4]). Base change gives a functor

$$
F : (\text{Spaces}/Y') \longrightarrow (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X')
$$
given by $V' \mapsto (V' \times_Y Y', V' \times_Y X', 1)$ which sends $(\text{Sch}/Y')$ into $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$. The functor $F$ has a left adjoint

$$
G : (\text{Spaces}/Y) \times_{(\text{Spaces}/Y')} (\text{Spaces}/X') \longrightarrow (\text{Spaces}/Y')
$$
which sends the triple $(V, U', \varphi)$ to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over $S$. The functor $G$ sends $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ into $(\text{Sch}/Y')$.

**Proof.** The proof is completely formal. Since the morphisms $X \to X'$ and $X \to Y$ are representable it is clear that $F$ sends $(\text{Sch}/Y')$ into $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$.}
Let us construct $G$. Let $(V,U',\varphi)$ be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies $U$ with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms of Spaces, Lemma \textit{20.5}. Hence we can apply Lemma \textit{2.4} to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that $V'$ is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on $U$ as morphisms into $Y'$. Setting $G(V,U',\varphi) = V'$ gives the functor $G$.

If $(V,U',\varphi)$ is an object of $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma \textit{14.3}. By Lemma \textit{2.2} this is also a pushout in the category of schemes, hence $G$ sends $(\text{Sch}/Y) \times_{(\text{Sch}/Y')} (\text{Sch}/X')$ into $(\text{Sch}/Y')$.

Let us prove that $G$ is a left adjoint to $F$. Let $Z$ be an algebraic space over $Y'$. We have to show that

$$\text{Mor}(V',Z) = \text{Mor}((V,U',\varphi),F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of $g'$ with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then $(g,f')$ is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g,f') : (V,U',\varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of $g$, resp. $f$ by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from $U$ to $Z$. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. 

\textbf{07VZ Lemma 2.6.} Let $S$ be a scheme. Let

\begin{equation*}
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & D \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \\
& & \downarrow \\
\end{array}
\begin{array}{ccc}
C & \longrightarrow & E \\
\downarrow & & \downarrow \\
D & \longrightarrow & F \\
\end{array}
\end{equation*}

be a commutative diagram of algebraic spaces over $S$. Assume that $A,B,C,D$ and $A,B,E,F$ form cartesian squares and that $B \to D$ is surjective étale. Then $C,D,E,F$ is a cartesian square.

\textbf{Proof.} This is formal. 

\textbf{07W0 Lemma 2.7.} In the situation of Lemma \textit{2.5} the functor $F \circ G$ is isomorphic to the identity functor.

\textbf{Proof.} We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma \textit{14.4}.

Choose a scheme $Y_1$ and a surjective étale morphism $Y_1 \to Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over $Y_1$ with a surjective étale morphism $X_1 \to X$. By More on Morphisms of Spaces, Lemma \textit{9.6} there exists a $X'_1 \to X'$ surjective étale with $X_1 = X'_1 \times_{X'} X$. In particular the morphism of schemes $X_1 \to X'_1$ is a thickening too. Apply More on Morphisms, Lemma \textit{14.3} to obtain a pushout $Y'_1 = Y_1 \amalg_{X_1} X'_1$. 

in the category of schemes. In the proof of Lemma 2.4 we constructed $Y'$ as a quotient of an étale equivalence relation on $Y_1'$ such that we get a commutative diagram

where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote $F_1, G_1$ the functors constructed in More on Morphisms, Lemma 14.4 for the front square. Then the diagram of categories

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 2.2.

Let $(V, U', \varphi)$ be an object of $(Spaces/Y') \times (Spaces/Y') (Spaces/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg U'$. Choose a scheme $V_1$ and a surjective étale morphism $V_1 \to Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \to (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By More on Morphisms of Spaces, Lemma 9.6 there exists a thickening $U_1 \to U_1'$ and a surjective étale morphism $U_1' \to X_1 \times_X U'$ whose base change to $X_1 \times_X U$ is the displayed morphism. At this point $(V_1, U_1', \varphi_1)$ is an object of $(Sch/Y_1) \times (Sch/Y_1') (Sch/X_1')$. In the proof of Lemma 2.4 we constructed $G(V, U', \varphi) = V \amalg U'$ as a quotient of an étale equivalence relation on
Finally, we come to the proof of the lemma. We have to show that the adjunction is surjective étale. In particular

\[ G_1(V_1, U'_1, \varphi_1) \rightarrow G(V, U', \varphi) \]

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping \((V, U', \varphi) \rightarrow F(G(V, U', \varphi))\) is an isomorphism. We know \((V_1, U'_1, \varphi_1) \rightarrow F_1(G_1(V_1, U'_1, \varphi_1))\) is an isomorphism by More on Morphisms, Lemma 14.4. Recall that \(F\) and \(F_1\) are given by base change. Using the properties of (2.7.2) and Lemma 2.6 we see that \(V \rightarrow G(V, U', \varphi) \times_Y Y\) and \(U' \rightarrow G(V, U', \varphi) \times_Y X'\) are isomorphisms, i.e., \((V, U', \varphi) \rightarrow F(G(V, U', \varphi))\) is an isomorphism. \(\square\)

**Lemma 2.8.** Let \(S\) be a base scheme. Let \(X \rightarrow X'\) be a thickening of algebraic spaces over \(S\) and let \(X \rightarrow Y\) be an affine morphism of algebraic spaces over \(S\). Let \(Y' = Y \amalg_X X'\) be the pushout (see Lemma 2.4). Let \(V' \rightarrow Y'\) be a morphism of algebraic spaces over \(S\). Set \(V = Y \times_Y V', U' = X' \times_Y V', \) and \(U = X \times_Y V'\). There is an equivalence of categories between

1. quasi-coherent \(O_{V'}\)-modules flat over \(Y'\), and
2. the category of triples \((G, F', \varphi)\) where
   a. \(G\) is a quasi-coherent \(O_Y\)-module flat over \(Y\),
   b. \(F'\) is a quasi-coherent \(O_{V'}\)-module flat over \(X\), and
   c. \(\varphi : (U \rightarrow V)^* F' \rightarrow (U \rightarrow U')^* F'\) is an isomorphism of \(O_U\)-modules.

The equivalence maps \(G'\) to \(((V \rightarrow V')^* G', (U' \rightarrow V')^* G', \text{can})\). Suppose \(G'\) corresponds to the triple \((G, F', \varphi)\). Then

a. \(G'\) is a finite type \(O_{V'}\)-module if and only if \(G\) and \(F'\) are finite type \(O_Y\) and \(O_{V'}\)-modules.

b. if \(V' \rightarrow Y'\) is locally of finite presentation, then \(G'\) is an \(O_{V'}\)-module of finite presentation if and only if \(G\) and \(F'\) are \(O_Y\) and \(O_{V'}\)-modules of finite presentation.

**Proof.** A quasi-inverse functor assigns to the triple \((G, F', \varphi)\) the fibre product

\[ (V \rightarrow V'), G \times_{(U \rightarrow V')} F, (U' \rightarrow V'), F' \]

where \(F = (U \rightarrow U')^* F'\). This works, because on affines étale over \(V'\) and \(Y'\) we recover the equivalence of More on Algebra, Lemma 7.5. Details omitted.
Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 30) to the case where \( V' \) and \( Y' \) are affine in which case the result follows from More on Algebra, Lemmas 7.4 and 7.6.

Lemma 2.9. **In the situation of Lemma 2.7.** If \( V' = G(V, U', \varphi) \) for some triple \((V, U', \varphi)\), then

1. \( V' \to Y' \) is locally of finite type if and only if \( V \to Y \) and \( U' \to X' \) are locally of finite type,
2. \( V' \to Y' \) is flat if and only if \( V \to Y \) and \( U' \to X' \) are flat,
3. \( V' \to Y' \) is flat and locally of finite presentation if and only if \( V \to Y \) and \( U' \to X' \) are flat and locally of finite presentation,
4. \( V' \to Y' \) is smooth if and only if \( V \to Y \) and \( U' \to X' \) are smooth,
5. \( V' \to Y' \) is étale if and only if \( V \to Y \) and \( U' \to X' \) are étale, and
6. add more here as needed.

If \( W' \) is flat over \( Y' \), then the adjunction mapping \( G(F(W')) \to W' \) is an isomorphism. Hence \( F \) and \( G \) define mutually quasi-inverse functors between the category of spaces flat over \( Y' \) and the category of triples \((V, U', \varphi)\) with \( V \to Y \) and \( U' \to X' \) flat.

**Proof.** Choose a diagram (2.7.1) as in the proof of Lemma 2.7.

Proof of (1) – (5). Let \((V, U', \varphi)\) be an object of \((Spaces/Y) \times (Spaces/Y') (Spaces/X')\). Construct a diagram (2.7.2) as in the proof of Lemma 2.7. Then the base change of \( G(V, U', \varphi) \to Y' \) to \( Y_1 \) is \( G_1(V_1, U_1', \varphi_1) \to Y'_1 \). Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 14.6 for schemes.

Suppose that \( W' \to Y' \) is flat. Choose a scheme \( W'_1 \) and a surjective étale morphism \( W'_1 \to Y'_1 \times_{Y'} W' \). Observe that \( W'_1 \to W' \) is surjective étale as a composition of surjective étale morphisms. We know that \( G_1(F_1(W'_1)) \to W'_1 \) is an isomorphism by More on Morphisms, Lemma 14.6 applied to \( W'_1 \) over \( Y'_1 \) and the front of the diagram (with functors \( G_1 \) and \( F_1 \) as in the proof of Lemma 2.7). Then the construction of \( G(F(W')) \) (as a pushout, i.e., as constructed in Lemma 2.4) shows that \( G_1(F_1(W'_1)) \to G(F(W')) \) is surjective étale. Whereupon we conclude that \( G(F(W)) \to W \) is étale, see for example Properties of Spaces, Lemma 16.3. But \( G(F(W)) \to W \) is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism.

\[ \]
in the category of algebraic spaces over $S$ as in Lemma 24. Assume $i$ is a thickening. Then the essential image of the functor

$$D(O_Y) \longrightarrow D(O_Y)^{\times} \times D(O_X)$$

contains every triple $(M, K', \alpha)$ where $M \in D(O_Y)$ and $K' \in D(O_X)$ are pseudo-coherent.

**Proof.** Let $(M, K', \alpha)$ be an object of the target of the functor of the lemma. Here $\alpha : Lf^*M \to Li^*K'$ is an isomorphism which is adjoint to a map $\beta : M \to Rf_*Li^*K'$. Thus we obtain maps

$$Rj_*M \xrightarrow{Rj_*\beta} Rj_*Rf_*Li^*K' = Rf'_*Ri_*Li^*K' \leftarrow Rf'_*K'$$

where the arrow pointing left comes from $K' \to Ri_*Li^*K'$. Choose a distinguished triangle

$$M' \to Rj_*M \oplus Rf'_*K' \to Rj_*Rf_*Li^*K' \to M'[1]$$

in $D(O_Y)$. The first arrow defines canonical maps $Lj^*M' \to M$ and $L(f')^*M' \to K'$ compatible with $\alpha$. Thus it suffices to show that the maps $Lj^*M' \to M$ and $L(f')^*M' \to K$ are isomorphisms. This we may check étale locally on $Y'$, hence we may assume $Y'$ étale.

Assume $Y'$ affine and $M \in D(O_Y)$ and $K' \in D(O_X)$ are pseudo-coherent. Say our pushout corresponds to the fibre product

$$\begin{array}{ccc}
B & \to & B' \\
\downarrow & & \downarrow \\
A & \leftarrow & A'
\end{array}$$

of rings where $B' \to B$ is surjective with locally nilpotent kernel $I$ (and hence $A' \to A$ is surjective with locally nilpotent kernel $I$ as well). The assumption on $M$ and $K'$ imply that $M$ comes from a pseudo-coherent object of $D(A)$ and $K'$ comes from a pseudo-coherent object of $D(B')$, see Derived Categories of Spaces, Lemmas 13.6 and 13.2 and Derived Categories of Schemes, Lemma 3.5 and 9.2. Moreover, pushforward and derived pullback agree with the corresponding operations on derived categories of modules, see Derived Categories of Spaces, Remark 6.3 and Derived Categories of Schemes, Lemmas 3.7 and 3.8. This reduces us to the statement formulated in the next paragraph. (To be sure these references show the object $M'$ lies $D_{QCoh}(O_Y)$ as this is a triangulated subcategory of $D(O_Y)$.)

Given a diagram of rings as above and a triple $(M, K', \alpha)$ where $M \in D(A)$, $K' \in D(B')$ are pseudo-coherent and $\alpha : M \otimes^L_A B \to K' \otimes^L_{B'} B$ is an isomorphism suppose we have distinguished triangle

$$M' \to M \oplus K' \to K' \otimes^L_{B'} B \to M'[1]$$

in $D(A')$. Goal: show that the induced maps $M' \otimes^L_A A \to M$ and $M' \otimes^L_{A'} B' \to K'$ are isomorphisms. To do this, choose a bounded above complex $E^*$ of finite free $A$-modules representing $M$. Since $(B', I)$ is a henselian pair (More on Algebra, Lemma 11.2) with $B = B'/I$ we may apply More on Algebra, Lemma 70.7 to see that there exists a bounded above complex $P^*$ of free $B'$-modules such that $\alpha$ is

\footnote{All functors given by derived pullback.}
represented by an isomorphism $E^\bullet \otimes_A B \cong P^\bullet \otimes_{B'} B$. Then we can consider the short exact sequence

$$0 \to L^\bullet \to E^\bullet \oplus P^\bullet \to P^\bullet \otimes_{B'} B \to 0$$

of complexes of $B'$-modules. More on Algebra, Lemma 6.9 implies $L^\bullet$ is a bounded above complex of finite projective $A'$-modules (in fact it is rather easy to show directly that $L^n$ is finite free in our case) and that we have $L^\bullet \otimes_{A'} A = E^\bullet$ and $L^\bullet \otimes_{A'} B' = P^\bullet$. The short exact sequence gives a distinguished triangle

$$L^\bullet \to M \oplus K' \to K' \otimes_{B'} B \to (L^\bullet)[1]$$

in $D(A')$ (Derived Categories, Section 12) which is isomorphic to the given distinguished triangle by general properties of triangulated categories (Derived Categories, Section 4). In other words, $L^\bullet$ represents $M'$ compatibly with the given maps. Thus the maps $M' \otimes_{A'} L \to M$ and $M' \otimes_{A'} B' \to K'$ are isomorphisms because we just saw that the corresponding thing is true for $L^\bullet$. □

4. Constructing elementary distinguished squares

0DVH Elementary distinguished squares were defined in Derived Categories of Spaces, Section 9.

0DVI Lemma 4.1. Let $S$ be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square. Then

$$\begin{array}{ccc}
U \times_W V & \longrightarrow & V \\
\downarrow & & \downarrow f \\
U & \longrightarrow & W
\end{array}$$

is a pushout in the category of algebraic spaces over $S$.

Proof. Observe that $U \amalg V \to W$ is a surjective étale morphism. The fibre product

$$(U \amalg V) \times_W (U \amalg V)$$

is the disjoint union of four pieces, namely $U = U \times_W U$, $U \times_W V$, $V \times_W U$, and $V \times_W V$. There is a surjective étale morphism

$$V \amalg (U \times_W V) \times_U (U \times_W V) \longrightarrow V \times_W V$$

because $f$ induces an isomorphism over $W \setminus U$ (part of the definition of being an elementary distinguished square). Let $B$ be an algebraic space over $S$ and let $g : V \to B$ and $h : U \to B$ be morphisms over $S$ which agree after restricting to $U \times_W V$. Then the description of $(U \amalg V) \times_W (U \amalg V)$ given above shows that $h \amalg g : U \amalg V \to B$ equalizes the two projections. Since $B$ is a sheaf for the étale topology we obtain a unique factorization of $h \amalg g$ through $W$ as desired. □

0DVJ Lemma 4.2. Let $S$ be a scheme. Let $V$, $U$ be algebraic spaces over $S$. Let $V' \subset V$ be an open subspace and let $f' : V' \to U$ be a separated étale morphism of algebraic spaces over $S$. Then there exists a pushout

$$\begin{array}{ccc}
V' & \longrightarrow & V \\
\downarrow & & \downarrow f \\
U & \longrightarrow & W
\end{array}$$
in the category of algebraic spaces over $S$ and moreover $(U \subset W, f : V \to W)$ is an elementary distinguished square.

**Proof.** We are going to construct $W$ as the quotient of an étale equivalence relation $R$ on $U \amalg V$. Such a quotient is an algebraic space for example by Bootstrap, Theorem 10.1. Moreover, the proof of Lemma 4.1 tells us to take $R = U \amalg V' \amalg V \amalg (V' \times_U V \setminus \Delta_{V'/U}(V'))$

Since we assumed $V' \to U$ is separated, the image of $\Delta_{V'/U}$ is closed and hence the complement is an open subspace. The morphism $j : R \to (U \amalg V) \times_S (U \amalg V)$ is given by $u, v', v, (v'_1, v'_2) \mapsto (u, u), (f'(v'), v'), (v', f'(v'))$, $(v, v), (v'_1, v'_2)$ with obvious notation. It is immediately verified that this is a monomorphism, an equivalence relation, and that the induced morphisms $s, t : R \to U \amalg V$ are étale. Let $W = (U \amalg V)/R$ be the quotient algebraic space. We obtain a commutative diagram as in the statement of the lemma. To finish the proof it suffices to show that this diagram is an elementary distinguished square, since then Lemma 4.1 implies that it is a pushout. Thus we have to show that $U \to W$ is open and that $f$ is étale and is an isomorphism over $W \setminus U$. This follows from the choice of $R$; we omit the details. □

5. Formal glueing of quasi-coherent modules

This section is the analogue of More on Algebra, Section 80. In the case of morphisms of schemes, the result can be found in the paper by Joyet [Joy96]; this is a good place to start reading. For a discussion of applications to descent problems for stacks, see the paper by Moret-Bailly [MB96]. In the case of an affine morphism of schemes there is a statement in the appendix of the paper [FR70] but one needs to add the hypothesis that the closed subscheme is cut out by a finitely generated ideal (as in the paper by Joyet) since otherwise the result does not hold. A generalization of this material to (higher) derived categories with potential applications to nonflat situations can be found in [Bha14, Section 5].

We start with a lemma on abelian sheaves supported on closed subsets.

**Lemma 5.1.** Let $S$ be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over $S$. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of $Y$ and $\overline{x} = f(\overline{y})$. We have

$$(Rf_* Q)_{\overline{x}} = Q_{\overline{y}}$$

in $D(\text{Ab})$ for any object $Q$ of $D(Y_{\text{étale}})$ supported on $|f^{-1}Z|$.

**Proof.** Consider the commutative diagram of algebraic spaces

$$
\begin{array}{ccc}
|f^{-1}Z| & \to & Y \\
\downarrow & & \downarrow \\
Z & \to & X
\end{array}
$$

By Cohomology of Spaces, Lemma 9.4 we can write $Q = Ri'_* K'$ for some object $K'$ of $D(f^{-1}Z_{\text{étale}})$. By Morphisms of Spaces, Lemma 53.7 we have $K' = (f)^{-1}K$ with $K = Rf'_* K'$. Then we have $Rf_* Q = Rf_* Ri'_* K' = Ri_* Rf'_* K' = Ri_* K$. Let $\overline{z}$
be the geometric point of $Z$ corresponding to $\overline{\pi}$ and let $\overline{\pi}'$ be the geometric point of $f^{-1}Z$ corresponding to $\overline{y}$. We obtain the result of the lemma as follows

$$Q_{\overline{\pi}} = (R\iota''_xK_\overline{\pi}) = K_\overline{\pi}' = (f')^{-1}K_{\overline{\pi}'} = K_{\overline{\pi}} = R\iota_xK_{\overline{\pi}} = R\iota_*Q_{\overline{\pi}}$$

The middle equality holds because of the description of the stalk of a pullback given in Properties of Spaces, Lemma [19.9] □

**Lemma 5.2.** Let $S$ be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over $S$. Let $Z \subset X$ closed subspace such that $f^{-1}Z \to Z$ is integral and universally injective. Let $\overline{y}$ be a geometric point of $Y$ and $\overline{\pi} = f(\overline{y})$. Let $\mathcal{G}$ be an abelian sheaf on $Y$. Then the map of two term complexes

$$(f_*\mathcal{G})_{\overline{\pi}} \to (f \circ j'_*)^!(\mathcal{G}|_Y)_{\overline{\pi}} \longrightarrow (\mathcal{G}_{\overline{\pi}} \to j'_!(\mathcal{G}|_Y)_{\overline{\pi}})$$

induces an isomorphism on kernels and an injection on cokernels. Here $V = Y \setminus f^{-1}Z$ and $j' : V \to Y$ is the inclusion.

**Proof.** Choose a distinguished triangle

$$\mathcal{G} \to Rj'_!\mathcal{G}|_V \to Q \to \mathcal{G}[1]$$

in $D(Y_{\text{etale}})$. The cohomology sheaves of $Q$ are supported on $|f^{-1}Z|$. We apply $Rf_*$ and we obtain

$$Rf_*\mathcal{G} \to Rf_*Rj'_!\mathcal{G}|_V \to Rf_*Q \to Rf_*\mathcal{G}[1]$$

Taking stalks at $\overline{\pi}$ we obtain an exact sequence

$$0 \to (R^{-1}f_*\mathcal{G})_{\overline{\pi}} \to f_*\mathcal{G}_{\overline{\pi}} \to (f \circ j'_*)^!(\mathcal{G}|_V)_{\overline{\pi}} \to (R^0f_*\mathcal{G})_{\overline{\pi}}$$

We can compare this with the exact sequence

$$0 \to H^{-1}(Q)_{\overline{\pi}} \to \mathcal{G}_{\overline{\pi}} \to j'_!(\mathcal{G}|_V)_{\overline{\pi}} \to H^0(Q)_{\overline{\pi}}$$

Thus we see that the lemma follows because $Q_{\overline{\pi}} = Rf_*Q_{\overline{\pi}}$ by Lemma [5.1] □

**Lemma 5.3.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\overline{\pi}$ be a geometric point of $X$ and let $\text{Spec}(\mathcal{O}_{X,\overline{\pi}}) \to X$ be the canonical morphism. For a quasi-coherent module $\mathcal{G}$ on $Y$ we have

$$f_*\mathcal{G}_{\overline{\pi}} = \Gamma(Y \times_X \text{Spec}(\mathcal{O}_{X,\overline{\pi}}), p^*\mathcal{F})$$

where $p : Y \times_X \text{Spec}(\mathcal{O}_{X,\overline{\pi}}) \to Y$ is the projection.

**Proof.** Observe that $f_*\mathcal{G}_{\overline{\pi}} = \Gamma(\text{Spec}(\mathcal{O}_{X,\overline{\pi}}), h^*f_*\mathcal{G})$ where $h : \text{Spec}(\mathcal{O}_{X,\overline{\pi}}) \to X$. Hence the result is true because $h$ is flat so that Cohomology of Spaces, Lemma [11.2] applies. □

**Lemma 5.4.** Let $S$ be a scheme. Let $X$ be an algebraic space over $S$. Let $i : Z \to X$ be a closed immersion of finite presentation. Let $Q \in D_{\text{QCoh}}(\mathcal{O}_X)$ be supported on $|Z|$. Let $\overline{\pi}$ be a geometric point of $X$ and let $I_{\overline{\pi}} \subset \mathcal{O}_{X,\overline{\pi}}$ be the stalk of the ideal sheaf of $Z$. Then the cohomology modules $H^n(Q_{\overline{\pi}})$ are $I_{\overline{\pi}}$-power torsion (see More on Algebra, Definition [79.4]).

**Proof.** Choose an affine scheme $U$ and an étale morphism $U \to X$ such that $\overline{\pi}$ lifts to a geometric point $\overline{\pi}'$ of $U$. Then we can replace $X$ by $U$, $Z$ by $U \times_Z Y$, $Q$ by the restriction $Q|_U$, and $\overline{\pi}$ by $\overline{\pi}'$. Thus we may assume that $X = \text{Spec}(A)$ is affine. Let $I \subset A$ be the ideal defining $Z$. Since $i : Z \to X$ is of finite presentation, the ideal
$I = (f_1, \ldots, f_r)$ is finitely generated. The object $Q$ comes from a complex of $A$-modules $M^\bullet$, see Derived Categories of Spaces, Lemma 3.5 and Derived Categories of Schemes, Lemma 5.5. Since the cohomology sheaves of $Q$ are supported on $Z$ we see that the localization $M^\bullet_f$ is acyclic for each $f \in I$. Take $x \in H^p(M^\bullet)$. By the above we can find $n_i$ such that $f_i^{n_i}x = 0$ in $H^p(M^\bullet)$ for each $i$. Then with $n = \sum n_i$ we see that $I^n$ annihilates $x$. Thus $H^p(M^\bullet)$ is $I$-power torsion. Since the ring map $A \to O_{X, \pi}$ is flat and since $I_\pi = IO_{X, \pi}$ we conclude. □

**Lemma 5.5.** Let $S$ be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over $S$. Let $Z \subset X$ be a closed subspace. Assume $f^{-1}Z \to Z$ is an isomorphism and that $f$ is flat in every point of $f^{-1}Z$. For any $Q$ in $D_{QCoh}(O_Y)$ supported on $|f^{-1}Z|$ we have $Lf^*Rf_*Q = Q$.

**Proof.** We show the canonical map $Lf^*Rf_*Q \to Q$ is an isomorphism by checking on stalks at $\pi$. If $\pi$ is not in $f^{-1}Z$, then both sides are zero and the result is true. Assume the image $x$ of $\pi$ is in $Z$. By Lemma 5.1 we have $Rf_*Q_\pi = Q_\pi$ and since $f$ is flat at $\pi$ we see that

$$(Lf^*Rf_*Q)_\pi = (Rf_*Q)_\pi \otimes_{O_{X, \pi}} O_{Y, \pi} = Q_\pi \otimes_{O_{X, \pi}} O_{Y, \pi}$$

Thus we have to check that the canonical map

$$Q_\pi \otimes_{O_{X, \pi}} O_{Y, \pi} \longrightarrow Q_\pi$$

is an isomorphism in the derived category. Let $I_\pi \subset O_{X, \pi}$ be the stalk of the ideal sheaf defining $Z$. Since $Z \to X$ is locally of finite presentation this ideal is finitely generated and the cohomology groups of $Q_\pi$ are $I_\pi = I_\pi O_{X, \pi}$-power torsion by Lemma 5.3 applied to $Q$ on $Y$. It follows that they are also $I_\pi$-power torsion. The ring map $O_{X, \pi} \to O_{Y, \pi}$ is flat and induces an isomorphism after dividing by $I_\pi$ and $I_\pi$ because we assumed that $f^{-1}Z \to Z$ is an isomorphism. Hence we see that the cohomology modules of $Q_\pi \otimes_{O_{X, \pi}} O_{Y, \pi}$ are equal to the cohomology modules of $Q_\pi$ by More on Algebra, Lemma 80.2 which finishes the proof. □

**Situation 5.6.** Here $S$ is a base scheme, $f : Y \to X$ is a quasi-compact and quasi-separated morphism of algebraic spaces over $S$, and $Z \to X$ is a closed immersion of finite presentation. We assume that $f^{-1}(Z) \to Z$ is an isomorphism and that $f$ is flat in every point $x \in |f^{-1}Z|$. We set $U = X \setminus Z$ and $V = Y \setminus f^{-1}(Z)$. Picture

$$
\begin{array}{ccc}
V & \longrightarrow & Y \\
\downarrow f/\psi & & \downarrow f \\
U & \longrightarrow & X
\end{array}
$$

In Situation 5.6 we define $QCoh(Y \to X, Z)$ as the category of triples $(\mathcal{H}, \mathcal{G}, \varphi)$ where $\mathcal{H}$ is a quasi-coherent sheaf of $O_U$-modules, $\mathcal{G}$ is a quasi-coherent sheaf of $O_Y$-modules, and $\varphi : f^*\mathcal{H} \to \mathcal{G}|_V$ is an isomorphism of $O_Y$-modules. There is a canonical functor

**0AEW** (5.6.1) $QCoh(O_X) \longrightarrow QCoh(Y \to X, Z)$

which maps $\mathcal{F}$ to the system $(\mathcal{F}|_U, f^*\mathcal{F}, can)$. By analogy with the proof given in the affine case, we construct a functor in the opposite direction. To an object $(\mathcal{H}, \mathcal{G}, \varphi)$ we assign the $O_X$-module

**0AEX** (5.6.2) $\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j^*), \mathcal{G}|_V)$
Observe that \( j \) and \( j' \) are quasi-compact morphisms as \( Z \to X \) is of finite presentation. Hence \( f_* \), \( j_* \), and \((f \circ j')_* \) transform quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma \[11.2\]). Thus the module \((5.6.2)\) is quasi-coherent.

**Lemma 5.7.** In Situation \[5.6\] The functor \((5.6.2)\) is right adjoint to the functor \((5.6.1)\).

**Proof.** This follows easily from the adjointness of \( f^* \) to \( f_* \) and \( j^* \) to \( j_* \). Details omitted. \( \square \)

**Lemma 5.8.** In Situation \[5.6\] Let \( X' \to X \) be a flat morphism of algebraic spaces. Set \( Z' = X' \times X Z \) and \( Y' = X' \times X Y \). The pullbacks \( \text{QCoh}(\mathcal{O}_X) \to \text{QCoh}(\mathcal{O}_{X'}) \) and \( \text{QCoh}(Y \to X, Z) \to \text{QCoh}(Y' \to X', Z') \) are compatible with the functors \((5.6.2)\) and \((5.6.1)\).

**Proof.** This is true because pullback commutes with pullback and because flat pullback commutes with pushforward along quasi-compact and quasi-separated morphisms, see Cohomology of Spaces, Lemma \[11.2\]. \( \square \)

**Proposition 5.9.** In Situation \[5.6\] the functor \((5.6.1)\) is an equivalence with quasi-inverse given by \((5.6.2)\).

**Proof.** We first treat the special case where \( X \) and \( Y \) are affine schemes and where the morphism \( f \) is flat. Say \( X = \text{Spec}(R) \) and \( Y = \text{Spec}(S) \). Then \( f \) corresponds to a flat ring map \( R \to S \). Moreover, \( Z \subset X \) is cut out by a finitely generated ideal \( I \subset R \). Choose generators \( f_1, \ldots, f_t \in I \). By the description of quasi-coherent modules in terms of modules (Schemes, Section \[7\]), we see that the category \( \text{QCoh}(Y \to X, Z) \) is canonically equivalent to the category \( \text{Glue}(R \to S, f_1, \ldots, f_t) \) of More on Algebra, Remark \[80.10\] such that the functors \((5.6.1)\) and \((5.6.2)\) correspond to the functors \( \text{Can} \) and \( H^0 \). Hence the result follows from More on Algebra, Proposition \[80.15\] in this case.

We return to the general case. Let \( \mathcal{F} \) be a quasi-coherent module on \( X \). We will show that

\[ \alpha : \mathcal{F} \longrightarrow \text{Ker}(j_*\mathcal{F}|_U \oplus f_*f^*\mathcal{F} \to (f \circ j')_*f^*\mathcal{F}|_V) \]

is an isomorphism. Let \((\mathcal{H}, \mathcal{G}, \varphi)\) be an object of \( \text{QCoh}(Y \to X, Z) \). We will show that

\[ \beta : f^*\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{G} \]

and

\[ \gamma : j^*\text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V) \longrightarrow \mathcal{H} \]

are isomorphisms. To see these statements are true it suffices to look at stalks. Let \( \overline{y} \) be a geometric point of \( Y \) mapping to the geometric point \( \overline{x} \) of \( X \).

Fix an object \((\mathcal{H}, \mathcal{G}, \varphi)\) of \( \text{QCoh}(Y \to X, Z) \). By Lemma \[5.2\] and a diagram chase (omitted) the canonical map

\[ \text{Ker}(j_*\mathcal{H} \oplus f_*\mathcal{G} \to (f \circ j')_*\mathcal{G}|_V)_{\overline{x}} \longrightarrow \text{Ker}(j_*\mathcal{H}_{\overline{x}} \oplus \mathcal{G}_{\overline{x}} \to j'_{\overline{x}}\mathcal{G}_{\overline{x}}) \]

is an isomorphism.

In particular, if \( \overline{y} \) is a geometric point of \( V \), then we see that \( j'_{\overline{x}}\mathcal{G}_{\overline{x}} = \mathcal{G}_{\overline{x}} \) and hence that this kernel is equal to \( \mathcal{H}_{\overline{x}} \). This easily implies that \( \alpha_{\overline{x}}, \beta_{\overline{x}}, \) and \( \gamma_{\overline{x}} \) are isomorphisms in this case.
Next, assume that \( \mathfrak{y} \) is a point of \( f^{-1}Z \). Let \( I_\mathfrak{y} \subset \mathcal{O}_{X, \mathfrak{y}} \), resp. \( I_\mathfrak{y} \subset \mathcal{O}_{Y, \mathfrak{y}} \) be the stalk of the ideal cutting out \( Z \), resp. \( f^{-1}Z \). Then \( I_\mathfrak{y} \) is a finitely generated ideal, \( I_\mathfrak{y} = I_\mathfrak{y} \mathcal{O}_{Y, \mathfrak{y}} \), and \( \mathcal{O}_{X, \mathfrak{y}} \to \mathcal{O}_{Y, \mathfrak{y}} \) is a flat local homomorphism inducing an isomorphism \( \mathcal{O}_{X, \mathfrak{y}} / I_\mathfrak{y} = \mathcal{O}_{Y, \mathfrak{y}} / I_\mathfrak{y} \). At this point we can bootstrap using the diagram of categories

\[
\begin{array}{ccc}
\mathcal{QCoh}(\mathcal{O}_X) & \xrightarrow{\mathfrak{y}} & \mathcal{QCoh}(Y \to X, Z) \\
\text{Mod}_{\mathcal{O}_{X, \mathfrak{y}}} \xrightarrow{\text{Can}} \text{Glue}(\mathcal{O}_{X, \mathfrak{y}} \to \mathcal{O}_{Y, \mathfrak{y}}, f_1, \ldots, f_i) \quad & \xrightarrow{H^0} & \end{array}
\]

Namely, as in the first paragraph of the proof we identify

\[
\text{Glue}(\mathcal{O}_{X, \mathfrak{y}} \to \mathcal{O}_{Y, \mathfrak{y}}, f_1, \ldots, f_i) = \mathcal{QCoh}(\text{Spec}(\mathcal{O}_{Y, \mathfrak{y}}) \to \text{Spec}(\mathcal{O}_{X, \mathfrak{y}}), V(I_\mathfrak{y}))
\]

The right vertical functor is given by pullback, and it is clear that the inner square is commutative. Our computation of the stalk of the kernel in the third paragraph of the proof combined with Lemma 5.3 implies that the outer square (using the curved arrows) commutes. Thus we conclude using the case of a flat morphism of affine schemes which we handled in the first paragraph of the proof.

**Lemma 5.10.** In Situation 5.6 the functor \( Rf_* \) induces an equivalence between \( D_{\mathcal{QCoh}, [f^{-1}Z]}(\mathcal{O}_Y) \) and \( D_{\mathcal{QCoh}, [Z]}(\mathcal{O}_X) \) with quasi-inverse given by \( Lf^* \).

**Proof.** Since \( f \) is quasi-compact and quasi-separated we see that \( Rf_* \) defines a functor from \( D_{\mathcal{QCoh}, [f^{-1}Z]}(\mathcal{O}_Y) \) to \( D_{\mathcal{QCoh}, [Z]}(\mathcal{O}_X) \), see Derived Categories of Spaces, Lemma 6.1. By Derived Categories of Spaces, Lemma 5.3 we see that \( Lf^* \) maps \( D_{\mathcal{QCoh}, [Z]}(\mathcal{O}_X) \) into \( D_{\mathcal{QCoh}, [f^{-1}Z]}(\mathcal{O}_Y) \). In Lemma 5.3 we have seen that \( Lf^* Rf_* Q = Q \) for \( Q \) in \( D_{\mathcal{QCoh}, [f^{-1}Z]}(\mathcal{O}_Y) \). By the dual of Derived Categories, Lemma 7.2 to finish the proof it suffices to show that \( Lf^* K = 0 \) implies \( K = 0 \) for \( K \) in \( D_{\mathcal{QCoh}, [Z]}(\mathcal{O}_X) \). This follows from the fact that \( f \) is flat at all points of \( f^{-1}Z \) and the fact that \( f^{-1}Z \to Z \) is surjective.

**Lemma 5.11.** In Situation 5.6 there exists an fpqc covering \( \{ X_i \to X \}_{i \in I} \) refining the family \( \{ U \to X, Y \to X \} \).

**Proof.** For the definition and general properties of fpqc coverings we refer to Topologies, Section 9. In particular, we can first choose an étale covering \( \{ X_i \to X \} \) with \( X_i \) affine and by base changing \( Y, Z, \) and \( U \) to each \( X_i \) we reduce to the case where \( X \) is affine. In this case \( U \) is quasi-compact and hence a finite union \( U = U_1 \cup \ldots \cup U_n \) of affine opens. Then \( Z \) is quasi-compact hence also \( f^{-1}Z \) is quasi-compact. Thus we can choose an affine scheme \( W \) and an étale morphism \( h : W \to Y \) such that \( h^{-1}f^{-1}Z \to f^{-1}Z \) is surjective. Say \( W = \text{Spec}(B) \) and \( h^{-1}f^{-1}Z = V(J) \) where \( J \subset B \) is an ideal of finite type. By Pro-étale Cohomology, Lemma 6.1 there exists a localization \( B \to B' \) such that points of \( \text{Spec}(B') \) correspond exactly to points of \( W = \text{Spec}(B) \) specializing to \( h^{-1}f^{-1}Z = V(J) \). It follows that the composition \( \text{Spec}(B') \to \text{Spec}(B) = W \to Y \to X \) is flat as by assumption \( f : Y \to X \) is flat at all the points of \( f^{-1}Z \). Then \( \{ \text{Spec}(B') \to X, U_1 \to X, \ldots, U_n \to X \} \) is an fpqc covering by Topologies, Lemma 9.2.
6. Formal glueing of algebraic spaces

In Situation \ref{ Situation 5.6} we consider the category \(\text{Spaces}(X \to Y, Z)\) of commutative diagrams of algebraic spaces over \(S\) of the form
\[
\begin{array}{ccc}
U' & \leftarrow & V' \\
\downarrow & & \downarrow \\
U & \leftarrow & V \\
\end{array}
\]
where both squares are cartesian. There is a canonical functor
\[
\text{Spaces}/X \rightarrow \text{Spaces}(Y \rightarrow X, Z)
\]
which maps \(X' \to X\) to the morphisms \(U \times_X X' \leftarrow V \times_X X' \to Y \times_X X'\).

\begin{lemma}
In Situation \ref{ Situation 5.6} the functor \ref{ Spaces/X \rightarrow Spaces(Y \rightarrow X, Z) restricts to an equivalence} restricts to an equivalence
\begin{enumerate}
\item from the category of algebraic spaces affine over \(X\) to the full subcategory of \(\text{Spaces}(Y \rightarrow X, Z)\) consisting of \((U' \leftarrow V' \to Y')\) with \(U' \to U, V' \to V, \) and \(Y' \to Y\) affine,
\item from the category of closed immersions \(X' \to X\) to the full subcategory of \(\text{Spaces}(Y \to X, Z)\) consisting of \((U' \leftarrow V' \to Y')\) with \(U' \to U, V' \to V, \) and \(Y' \to Y\) closed immersions, and
\item same statement as in (2) for finite morphisms.
\end{enumerate}
\end{lemma}

\textbf{Proof.} The category of algebraic spaces affine over \(X\) is equivalent to the category of quasi-coherent sheaves \(\mathcal{A}\) of \(\mathcal{O}_X\)-algebras. The full subcategory of \(\text{Spaces}(Y \to X, Z)\) consisting of \((U' \leftarrow V' \to Y')\) with \(U' \to U, V' \to V, \) and \(Y' \to Y\) affine is equivalent to the category of algebra objects of \(\text{QCoh}(Y \to X, Z)\). In both cases this follows from Morphisms of Spaces, Lemma \ref{ Lemma 20.7} with quasi-inverse given by the relative spectrum construction (Morphisms of Spaces, Definition \ref{ Definition 20.8}) which commutes with arbitrary base change. Thus part (1) of the lemma follows from Proposition \ref{ Proposition 5.9}.

Fully faithfulness in part (2) follows from part (1). For essential surjectivity, we reduce by part (1) to proving that \(X' \to X\) is a closed immersion if and only if both \(U \times_X X' \to U\) and \(Y \times_X X' \to Y\) are closed immersions. By Lemma \ref{ Lemma 5.11} \(\{U \to X, Y \to X\}\) can be refined by an fpqc covering. Hence the result follows from Descent on Spaces, Lemma \ref{ Lemma 10.17}.

For (3) use the argument proving (2) and Descent on Spaces, Lemma \ref{ Lemma 10.23}.

\begin{lemma}
In Situation \ref{ Situation 5.6} the functor \ref{ Spaces/X \rightarrow Spaces(Y \rightarrow X, Z) restricts to an equivalence} reflects isomorphisms.
\end{lemma}

\textbf{Proof.} By a formal argument with base change, this reduces to the following question: A morphism \(a : X' \to X\) of algebraic spaces such that \(U \times_X X' \to U\) and \(Y \times_X X' \to Y\) are isomorphisms, is an isomorphism. The family \(\{U \to X, Y \to X\}\) can be refined by an fpqc covering by Lemma \ref{ Lemma 5.11} Hence the result follows from Descent on Spaces, Lemma \ref{ Lemma 10.15}.

\begin{lemma}
In Situation \ref{ Situation 5.6} the functor \ref{ Spaces/X \rightarrow Spaces(Y \rightarrow X, Z) restricts to an equivalence} is fully faithful on algebraic spaces separated over \(X\). More precisely, it induces a bijection
\[
\text{Mor}_X(X'_1, X'_2) \rightarrow \text{Mor}_{\text{Spaces}(Y \to X, Z)}(F(X'_1), F(X'_2))
\]
whenever \(X'_2 \to X\) is separated.
\end{lemma}
Proof. Since $X' \to X$ is separated, the graph $i : X'_1 \to X'_1 \times_X X'_2$ of a morphism $X'_1 \to X'_2$ over $X$ is a closed immersion, see Morphisms of Spaces, Lemma 4.6. Moreover a closed immersion $i : T \to X'_1 \times_X X'_2$ is the graph of a morphism if and only if $\text{pr}_1 \circ i$ is an isomorphism. The same is true for

(1) the graph of a morphism $U \times_X X'_1 \to U \times_X X'_2$ over $U$,
(2) the graph of a morphism $V \times_X X'_1 \to V \times_X X'_2$ over $V$, and
(3) the graph of a morphism $Y \times_X X'_1 \to Y \times_X X'_2$ over $Y$.

Moreover, if morphisms as in (1), (2), (3) fit together to form a morphism in the category $\text{Spaces}(Y \to X, Z)$, then these graphs fit together to give an object of $\text{Spaces}(Y \times_X (X'_1 \times_X X'_2) \to X'_1 \times_X X'_2, Z \times_X (X'_1 \times_X X'_2))$ whose triple of morphisms are closed immersions. The proof is finished by applying Lemmas 6.1 and 6.2. □

7. Coequalizers and glueing

Let $X$ be a Noetherian algebraic space and $Z \to X$ a closed subscheme. Let $X' \to X$ be the blowing up in $Z$. In this section we show that $X$ can be recovered from $X'$, $Z_n$ and glueing data where $Z_n$ is the $n$th infinitesimal neighbourhood of $Z$ in $X$.

Lemma 7.1. Let $S$ be a scheme. Let

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & X
\end{array}
\]

be a commutative diagram of algebraic spaces over $S$. Assume $B$ Noetherian, $g$ proper and surjective, and $X \to B$ separated of finite type. Let $R = Y \times_X Y$ with projection morphisms $t, s : R \to Y$. There exists a coequalizer $X'$ of $s, t : R \to Y$ in the category of algebraic spaces separated over $B$. The morphism $X' \to X$ is a finite universal homeomorphism.

Proof. Denote $h : R \to X$ the given morphism. The sheaves $g_* \mathcal{O}_Y$ and $h_* \mathcal{O}_R$ are coherent $\mathcal{O}_X$-algebras (Cohomology of Spaces, Lemma 20.2). The $X$-morphisms $s, t$ induce $\mathcal{O}_X$-algebra maps $s^\# , t^\#$ from the first to the second. Set $A = \text{Equalizer} (s^\#, t^\# : g_* \mathcal{O}_Y \to h_* \mathcal{O}_R)$

Then $A$ is a coherent $\mathcal{O}_X$-algebra and we can define

\[ X' = \text{Spec}_X(A) \]

as in Morphisms of Spaces, Definition 20.8. By Morphisms of Spaces, Remark 20.9 and functoriality of the $\text{Spec}$ construction there is a factorization

\[ Y \to X' \to X \]

and the morphism $g' : Y \to X'$ equalizes $s$ and $t$. Since $A$ is a coherent $\mathcal{O}_X$-module it is clear that $X' \to X$ is a finite morphism of algebraic spaces. Since the surjective morphism $g : Y \to X$ factors through $X'$ we see that $X' \to X$ is surjective.

To check that $X' \to X$ is a universal homeomorphism, it suffices to check that it is universally injective (as we’ve already seen that it is universally surjective and universally closed). To check this it suffices to check that $|X' \times_X U| \to |U|$ is
injective, for all $U 	o X$ étale, see More on Morphisms of Spaces, Lemma 36.6. It suffices to check this in all cases where $U$ is an affine scheme (minor detail omitted). Since the construction of $X'$ commutes with étale localization, we may replace $U$ by $X$. Hence it suffices to check that $|X'| \to |X|$ is injective when $X$ is moreover an affine scheme. First observe that $|Y| \to |X'|$ is surjective, because $g' : Y \to X'$ is proper by Morphisms of Spaces, Lemma 40.6 (hence the image is closed) and $\mathcal{O}_{X'} \subset g'_*\mathcal{O}_Y$ by construction. Thus if $x_1, x_2 \in |X'|$ map to the same point in $|X|$, then we can lift $x_1, x_2$ to points $y_1, y_2 \in |Y|$ mapping to the same point of $|X|$. Then we can find an $r \in |R|$ with $s(r) = y_1$ and $t(r) = y_2$, see Properties of Spaces, Lemma 4.3. Since $g'$ coequalizes $s$ and $t$ we conclude that $x_1 = x_2$ as desired.

To prove that $X'$ is the coequalizer, let $W \to B$ be a separated morphism of algebraic spaces over $S$ and let $a : Y \to W$ be a morphism over $B$ which equalizes $s$ and $t$. We will show that $a$ factors in a unique manner through the morphism $g' : Y \to X'$. We will first reduce this to the case where $W \to B$ is separated of finite type by a limit argument (we recommend the reader skip this argument). Since $Y$ is quasi-compact we can find a quasi-compact open subspace $W' \subset W$ such that $a$ factors through $W'$. After replacing $W$ by $W'$ we may assume $W$ is quasi-compact. By Limits of Spaces, Lemma 10.1 we can write $W = \lim_{i \in I} W_i$ as a cofiltered limit with affine transition morphisms with $W_i$ of finite type over $B$. After shrinking $I$ we may assume $W_i \to B$ is separated as well, see Limits of Spaces, Lemma 6.9. Since $W = \lim W_i$ we have $a = \lim a_i$ for some morphisms $a_i : Y \to W_i$. If we can prove $a_i$ factors through $g'$ for all $i$, then the same thing is true for $a$. This proves the reduction to the case of a finite type $W$.

Assume we have $a : Y \to W$ equalizing $s$ and $t$ with $W \to B$ separated and of finite type. Consider

$$\Gamma \subset X \times_B W$$

the scheme theoretic image of $(g, a) : Y \to X \times_B W$. Since $g$ is proper we conclude $Y \to \Gamma$ is surjective and the projection $p : \Gamma \to X$ is proper, see Morphisms of Spaces, Lemma 40.8. Since both $g$ and $a$ equalize $s$ and $t$, the morphism $Y \to \Gamma$ also equalizes $s$ and $t$.

We claim that $p : \Gamma \to X$ is a universal homeomorphism. As in the proof of the corresponding fact for $X' \to X$, it suffices to show that $p$ is universally injective. By More on Morphisms of Spaces, Lemma 36.6 it suffices to check $|\Gamma \times_X U| \to |U|$ is injective for every $U 
 \to X$ étale. It suffices to check this for $U$ affine (minor details omitted). Taking scheme theoretic image commutes with étale localization (Morphisms of Spaces, Lemma 16.3). Hence we may replace $X$ by $V$ and we conclude it suffices to show that $|\Gamma| \to |X|$ is injective. If $\gamma_1, \gamma_2 \in |\Gamma|$ map to the same point in $|X|$, then we can lift $\gamma_1, \gamma_2$ to points $y_1, y_2 \in |Y|$ mapping to the same point of $|X|$ (by surjectivity of $Y \to \Gamma$ we’ve seen above). Then we can find an $r \in |R|$ with $s(r) = y_1$ and $t(r) = y_2$, see Properties of Spaces, Lemma 4.3. Since $Y \to \Gamma$ coequalizes $s$ and $t$ we conclude that $\gamma_1 = \gamma_2$ as desired.

As a proper universal homeomorphism the morphism $p$ is finite (see for example More on Morphisms of Spaces, Lemma 35.1). We conclude that

$$\Gamma = \text{Spec}(p_*\mathcal{O}_{\Gamma}).$$

Since $Y \to \Gamma$ equalizes $s$ and $t$ the map $p_*\mathcal{O}_\Gamma \to g_*\mathcal{O}_Y$ factors through $A$ and we obtain a morphism $X' \to \Gamma$ by functoriality of the $\text{Spec}$ construction. We can
compose this morphism with the projection \( q : \Gamma \to W \) to get the desired morphism \( X' \to W \). We omit the proof of uniqueness of the factorization. \( \square \)

We will work in the following situation.

0AGH **Situation 7.2.** Let \( S \) be a scheme. Let \( X \to B \) be a separated finite type morphism of algebraic spaces over \( S \) with \( B \) Noetherian. Let \( Z \to X \) be a closed immersion and let \( U \subset X \) be the complementary open subspace. Finally, let \( f : X' \to X \) be a proper morphism of algebraic spaces such that \( f^{-1}(U) \to U \) is an isomorphism.

0AGI **Lemma 7.3.** In Situation 7.2 let \( Y = X' \amalg Z \) and \( R = Y \times_X Y \) with projections \( t, s : R \to Y \). There exists a coequalizer \( X_1 \) of \( s, t : R \to Y \) in the category of algebraic spaces separated over \( B \). The morphism \( X_1 \to X \) is a finite universal homeomorphism, an isomorphism over \( U \) and \( Z \to X \) lifts to \( X_1 \).

**Proof.** Existence of \( X_1 \) and the fact that \( X_1 \to X \) is a finite universal homeomorphism is a special case of Lemma 7.1. The formation of \( X_1 \) commutes with étale localization on \( X \) (see proof of Lemma 7.1). Thus the morphisms \( X_n \to X \) are isomorphisms over \( U \). It is immediate from the construction that \( Z \to X \) lifts to \( X_1 \). \( \square \)

In Situation 7.2 for \( n \geq 1 \) let \( Z_n \subset X \) be the \( n \)th order infinitesimal neighbourhood of \( Z \) in \( X \), i.e., the closed subscheme defined by the \( n \)th power of the sheaf of ideals cutting out \( Z \). Consider \( Y_n = X' \amalg Z_n \) and \( R_n = Y_n \times_X Y_n \) and the coequalizer

\[
R_n \xrightarrow{\alpha_n} Y_n \xrightarrow{\beta_n} X_n \xrightarrow{\gamma_n} X
\]

as in Lemma 7.3. The maps \( Y_n \to Y_{n+1} \) and \( R_n \to R_{n+1} \) induce morphisms

0AGJ (7.3.1) \[
X_1 \to X_2 \to X_3 \to \ldots \to X
\]

Each of these morphisms is a universal homeomorphism as the morphisms \( X_n \to X \) are universal homeomorphisms.

0AGK **Lemma 7.4.** In 7.3.1 for all \( n \) large enough, there exists an \( m \) such that \( X_{n+m} \to X_n \) factors through a closed immersion \( X \to X_{n+m} \).

**Proof.** Let’s look a bit more closely at the construction of \( X_n \) and how it changes as we increase \( n \). We have \( X_n = \text{Spec}(A_n) \) where \( A_n \) is the equalizer of \( s_n^0 \) and \( t_n^0 \) going from \( g_{n,*}O_{Y_n} \) to \( h_{n,*}O_{R_n} \). Here \( g_n : Y_n = X' \amalg Z_n \to X \) and \( h_n : R_n = Y_n \times_X Y_n \to X \) are the given morphisms. Let \( \mathcal{I} \subset O_X \) be the coherent sheaf of ideals corresponding to \( Z \). Then

\[
g_{n,*}O_{Y_n} = f_*O_{X'} \times O_X/\mathcal{I}^n
\]

Similarly, we have a decomposition

\[
R_n = X' \times_X X' \amalg X' \times_X Z_n \amalg Z_n \times_X Z_n
\]

Denote \( f_n : X' \times_X Z_n \to X \) the restriction of \( f \) and denote

\[
\mathcal{A} = \text{Equalizer}(f_*O_{X'}, \mathcal{F} \times f_*O_{X' \times X'})
\]

Then we see that

\[
\mathcal{A}_n = \text{Equalizer}(\mathcal{A} \times O_X/\mathcal{I}^n \xrightarrow{\gamma_n} f_{n,*}O_{X' \times X Z_n})
\]
We have canonical maps
\[ \mathcal{O}_X \to \cdots \to \mathcal{A}_3 \to \mathcal{A}_2 \to \mathcal{A}_1 \]
of coherent \(\mathcal{O}_X\)-algebras. The statement of the lemma means that for \(n\) large enough there exists an \(m \geq 0\) such that the image of \(\mathcal{A}_{n+m} \to \mathcal{A}_n\) is isomorphic to \(\mathcal{O}_X\).

Since \(X_n \to X\) is an isomorphism over \(U\) we see that the kernel of \(\mathcal{O}_X \to \mathcal{A}_n\) is supported on \(|Z|\). Since \(X\) is Noetherian, the sequence of kernels \(J_n = \text{Ker}(\mathcal{O}_X \to \mathcal{A}_n)\) stabilizes (Cohomology of Spaces, Lemma \([13.1]\)). Say \(J_{n_0} = J_{n_0+1} = \ldots = J\). By Cohomology of Spaces, Lemma \([13.2]\) we find that \(I^t J = 0\) for some \(t \geq 0\). On the other hand, there is an \(\mathcal{O}_X\)-algebra map \(\mathcal{A}_n \to \mathcal{O}_X/I^n\) and hence \(J \subseteq I^n\) for all \(n\).

By Artin-Rees (Cohomology of Spaces, Lemma \([13.3]\)) we find that \(J \cap I^n \subseteq I^{n-c} J\) for some \(c \geq 0\) and all \(n \gg 0\). We conclude that \(J = 0\).

Pick \(n \geq n_0\) as in the previous paragraph. Then \(\mathcal{O}_X \to \mathcal{A}_n\) is injective. Hence it now suffices to find \(m \geq 0\) such that the image of \(\mathcal{A}_{n+m} \to \mathcal{A}_n\) is equal to the image of \(\mathcal{O}_X\). Observe that \(\mathcal{A}_n\) sits in a short exact sequence
\[ 0 \to \text{Ker}(A \to f_{n,*}\mathcal{O}_{X' \times X} Z_n) \to \mathcal{A}_n \to \mathcal{O}_X/I^n \to 0 \]
and similarly for \(\mathcal{A}_{n+m}\). Hence it suffices to show
\[ \text{Ker}(A \to f_{n+m,*}\mathcal{O}_{X' \times X} Z_{n+m}) \subseteq \text{Im}(I^n \to A) \]
for some \(m \geq 0\). To do this we may work étale locally on \(X\) and since \(X\) is Noetherian we may assume that \(X\) is a Noetherian affine scheme. Say \(X = \text{Spec}(R)\) and \(I\) corresponds to the ideal \(I \subseteq R\). Let \(A = \tilde{A}\) for a finite \(R\)-algebra \(A\). Let \(f_n\mathcal{O}_{X'} = \tilde{B}\) for a finite \(R\)-algebra \(B\). Then \(R \to \tilde{A} \subseteq B\) and these maps become isomorphisms on inverting any element of \(I\).

Note that \(f_n\mathcal{O}_{X' \times X} Z_n\) is equal to \(f_n(\mathcal{O}_{X'}/I^n\mathcal{O}_{X'})\) in the notation used in Cohomology of Spaces, Section \([21]\). By Cohomology of Spaces, Lemma \([21.4]\) we see that there exists a \(c \geq 0\) such that
\[ \text{Ker}(B \to \Gamma(X, f_n(\mathcal{O}_{X'}/I^{n+m+c}\mathcal{O}_{X'}))) \]
is contained in \(I^{n+m} B\). On the other hand, as \(R \to B\) is finite and an isomorphism after inverting any element of \(I\) we see that \(I^{n+m} B \subseteq \text{Im}(I^n \to B)\) for \(m\) large enough (can be chosen independent of \(n\)). This finishes the proof as \(A \subseteq B\).

0AGL Remark 7.5. The meaning of Lemma 7.4 is the system \(X_1 \to X_2 \to X_3 \to \ldots\) is essentially constant with value \(X\). See Categories, Definition \([22.1]\).

8. Compactifications

0F44 This section is the analogue of More on Flatness, Section \([33]\). The theorem in this section is the main theorem in \([CLO12]\).

Let \(B\) be a quasi-compact and quasi-separated algebraic space over some base scheme \(S\). We will say an algebraic space \(X\) over \(B\) has a compactification over \(B\) or is compactifiable over \(B\) if there exists a quasi-compact open immersion \(X \to \tilde{X}\) into an algebraic space \(\tilde{X}\) proper over \(B\). If \(X\) has a compactification over \(B\), then \(X \to B\) is separated and of finite type. The main theorem of this section is that the converse is true as well.
Lemma 8.1. Let $S$ be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over $S$. If $(U \subseteq X, f : V \to X)$ is an elementary distinguished square such that $U \to Y$ and $V \to Y$ are separated and $U \times_X V \to U \times_Y V$ is closed, then $X \to Y$ is separated.

Proof. We have to check that $\Delta : X \to X \times_Y X$ is a closed immersion. There is an étale covering of $X \times_Y X$ given by the four parts $U \times Y U$, $U \times Y V$, $V \times Y U$, and $V \times Y V$. Observe that $(U \times_Y U) \times_{(X \times_Y X)} \Delta X = U$, $(U \times_Y V) \times_{(X \times_Y X)} \Delta X = U \times_X V$, $(V \times_Y U) \times_{(X \times_Y X)} \Delta X = V \times_X U$, and $(V \times_Y V) \times_{(X \times_Y X)} \Delta X = V$. Thus the assumptions of the lemma exactly tell us that $\Delta$ is a closed immersion.

Lemma 8.2. Let $S$ be a scheme. Let $X$ be a quasi-compact and quasi-separated algebraic space over $S$. Let $U \subseteq X$ be a quasi-compact open.

1. If $Z_1, Z_2 \subseteq X$ are closed subspaces of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a $U$-admissible blowing up $X' \to X$ such that the strict transforms of $Z_1$ and $Z_2$ are disjoint.

2. If $T_1, T_2 \subseteq [U]$ are disjoint constructible closed subsets, then there is a $U$-admissible blowing up $X' \to X$ such that the closures of $T_1$ and $T_2$ are disjoint.

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $I_i$ of $Z_i$ is of finite type, see Morphisms of Spaces, Lemma 28.12 Denote $Z \subseteq X$ the closed subspace cut out by the product $I_1 I_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors on Spaces, Lemma 19.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of $Z_1$ and $Z_2$ are disjoint. Denote $Y \subseteq Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors on Spaces, Definition 19.1 and Morphisms of Spaces, Lemma 28.2). Thus the blowing up $X' \to X$ of $Y$ is a $U$-admissible blowing up. By general properties of strict transforms, the strict transform of $Z_1$, $Z_2$ with respect to $X' \to X$ is the same as the strict transform of $Z_1$, $Z_2$ with respect to $Z' \to Z$, see Divisors on Spaces, Lemma 18.3. Thus (1) is proved.

Proof of (2). By Limits of Spaces, Lemma 14.1 there exists a finite type quasi-coherent sheaf of ideals $J_i \subseteq O_X$ such that $I_i = V(J_i)$ (set theoretically). By Limits of Spaces, Lemma 9.8 there exists a finite type quasi-coherent sheaf of ideals $I_i \subseteq O_X$ whose restriction to $U$ is $J_i$. Apply the result of part (1) to the closed subspaces $Z_i = V(I_i)$ to conclude.

Lemma 8.3. Let $S$ be a scheme. Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated algebraic spaces over $S$. Let $V \subseteq Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subseteq [V]$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of $T$ in $V$. Then there exists a $V$-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of $f$ is an isomorphism over an open neighbourhood of the closure of $T$ in $|Y'|$.

Proof. Let $T' \subseteq [V]$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then $T', T$ are closed in $[V]$ and $T \cap T' = \emptyset$. Since $[V]$ is a spectral topological space (Properties of Spaces, Lemma 15.2) we can find constructible closed subsets $T_c, T'_c$ of $[V]$ with $T \subseteq T_c$, $T' \subseteq T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open $W$ of $[V]$ containing $T'$ not meeting $T$ and set $T_c = [V] \setminus W$, then choose a quasi-compact open $W'$ of $[V]$ containing $T_c$ not meeting $T'$ and set
Let $0F48$ we may, after replacing $Y$ by a $V$-admissible blowing up, assume that $T_c$ and $T'_c$ have disjoint closures in $|Y|$. Let $Y_0$ be the open subspace of $Y$ corresponding to the open $|Y| \setminus T'_c$ and set $V_0 = V \cap Y_0$, $U_0 = U \times_Y V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to Y_0$ is an isomorphism, we can find a $V_0$-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform $X'_0$ of $X_0$ maps isomorphically to $Y'_0$, see More on Morphisms of Spaces, Lemma $39.3$. By Divisors on Spaces, Lemma $19.3$ there exists a $V$-admissible blow up $Y' \to Y$ whose restriction to $Y_0$ is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of $f$, then we see what we want is true because $f'$ restricts to an isomorphism over $Y'_0$.

**Lemma 8.4.** Let $S$ be a scheme. Consider a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{f} & U & \xleftarrow{A} \\
\downarrow & & \downarrow \\
Y & \xleftarrow{V} & B
\end{array}
\]

of quasi-compact and quasi-separated algebraic spaces over $S$. Assume

1. $f$ is proper,
2. $V$ is a quasi-compact open of $Y$, $U = f^{-1}(V)$,
3. $B \subset V$ and $A \subset U$ are closed subspaces,
4. $|A|$ is a connected component of $|f(V)|$,
5. $f|A : A \to B$ is an isomorphism, and $f$ is étale at every point of $A$.

Then there exists a $V$-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ satisfies: for every geometric point $\overline{\alpha}$ of the closure of $|A|$ in $|X'|$ there exists a quotient $\mathcal{O}_{X', \overline{\alpha}} \to \mathcal{O}$ such that $\mathcal{O}_{Y', f'(\overline{\alpha})} \to \mathcal{O}$ is finite flat.

As you can see from the proof, more is true, but the statement is already long enough and this will be sufficient later on.

**Proof.** Let $T' \subseteq |U|$ be the complement of the maximal open on which $f|U$ is étale. Then $T'$ is closed in $|U|$ and disjoint from $|A|$. Since $|U|$ is a spectral topological space (Properties of Spaces, Lemma $15.2$) we can find constructible closed subsets $T_c, T'_c$ of $|U|$ with $|A| \subseteq T_c, T' \subseteq T'_c$ such that $T_c \cap T'_c = \emptyset$ (see proof of Lemma $8.3$). By Lemma $8.2$ there is a $U$-admissible blowing up $X_1 \to X$ such that $T_c$ and $T'_c$ have disjoint closures in $|X_1|$. Let $X_{1,0}$ be the open subspace of $X_1$ corresponding to the open $|X_1| \setminus T'_c$, and set $U_0 = U \cap X_{1,0}$. Observe that the scheme theoretic image $\overline{A}_1 \subseteq X_1$ of $A$ is contained in $X_{1,0}$ by construction.

After replacing $Y$ by a $V$-admissible blowing up and taking strict transforms, we may assume $X_{1,0} \to Y$ is flat, quasi-finite, and of finite presentation, see More on Morphisms of Spaces, Lemmas $39.1$ and $37.3$ Consider the commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{V} & B
\end{array}
\]

and the diagram

\[
\begin{array}{ccc}
\overline{A}_1 & \xrightarrow{f} & \overline{A} \\
\downarrow & & \downarrow \\
\overline{B}
\end{array}
\]

of scheme theoretic images. The morphism $\overline{A}_1 \to \overline{A}$ is surjective because it is proper and hence the scheme theoretic image of $\overline{A}_1 \to \overline{A}$ must be equal to $\overline{A}$ and then we can use Morphisms of Spaces, Lemma $40.8$. The statement on étale local rings follows by choosing a lift of the geometric point $\overline{\alpha}$ to a geometric point $\overline{\alpha}_1$.
of \( \overline{\mathcal{X}}_1 \) and setting \( \mathcal{O} = \mathcal{O}_{\overline{\mathcal{X}}, \pi_1} \). Namely, since \( X_1 \to Y \) is flat and quasi-finite on \( X_1 \to \overline{\mathcal{X}}_1 \), the map \( \mathcal{O}_{Y, f_{\pi_1}} \to \mathcal{O}_{\overline{\mathcal{X}}, \pi_1} \) is finite flat, see Algebra, Lemmas \([150.15, 148.3]\).

**Lemma 8.5.** Let \( S \) be a scheme. Let \( X \to B \) and \( Y \to B \) be morphisms of algebraic spaces over \( S \). Let \( U \subset X \) be an open subspace. Let \( V \to X \times_B Y \) be a quasi-compact morphism whose composition with the first projection maps into \( U \). Let \( Z \subset X \times_B Y \) be the scheme theoretic image of \( V \to X \times_B Y \). Let \( X' \to X \) be a \( U \)-admissible blowup. Then the scheme theoretic image of \( V \to X' \times_B Y \) is the strict transform of \( Z \) with respect to the blowing up.

**Proof.** Denote \( Z' \to Z \) the strict transform. The morphism \( Z' \to X' \) induces a morphism \( Z' \to X' \times_B Y \) which is a closed immersion (as \( Z' \) is a closed subspace of \( X' \times_X Z \) by definition). Thus to finish the proof it suffices to show that the scheme theoretic image \( Z'' \) of \( V \to Z' \) is \( Z' \). Observe that \( Z'' \subset Z' \) is a closed subspace such that \( V \to Z' \) factors through \( Z'' \). Since both \( V \to X \times_B Y \) and \( V \to X' \times_B Y \) are quasi-compact (for the latter this follows from Morphisms of Spaces, Lemma \( \text{[8.9]} \) and the fact that \( X' \times_B Y \to X \times_B Y \) is separated as a base change of a proper morphism), by Morphisms of Spaces, Lemma \( \text{[16.3]} \) we see that \( Z \cap (U \times_B Y) = Z'' \cap (U \times_B Y) \). Thus the inclusion morphism \( Z'' \to Z' \) is an isomorphism away from the exceptional divisor \( E \) of \( Z' \to Z \). However, the structure sheaf of \( Z' \) does not have any nonzero sections supported on \( E \) (by definition of strict transforms) and we conclude that the surjection \( \mathcal{O}_{Z'} \to \mathcal{O}_{Z''} \) must be an isomorphism.

**Lemma 8.6.** Let \( S \) be a scheme. Let \( B \) be a quasi-compact and quasi-separated algebraic space over \( S \). Let \( U \) be an algebraic space of finite type and separated over \( B \). Let \( V \to U \) be an étale morphism. If \( V \) has a compactification \( V \subset Y \) over \( B \), then there exists a \( V \)-admissible blowing up \( Y' \to Y \) and an open \( V \subset V' \subset Y' \) such that \( V \to U \) extends to a proper morphism \( V' \to U \).

**Proof.** Consider the scheme theoretic image \( Z \subset Y \times_B U \) of the “diagonal” morphism \( V \to Y \times_B U \). If we replace \( Y \) by a \( V \)-admissible blowing up, then \( Z \) is replaced by the strict transform with respect to this blowing up, see Lemma \( \text{[8.5]} \). Hence by More on Morphisms of Spaces, Lemma \( \text{[39.3]} \) we may assume \( Z \to Y \) is an open immersion. If \( V' \subset Y' \) denotes the image, then we see that the induced morphism \( V' \to U \) is proper because the projection \( Y \times_B U \to U \) is proper and \( V' \cong Z \) is a closed subspace of \( Y \times_B U \).

The following lemma is formulated for finite type separated algebraic spaces over a finite type algebraic space over \( Z \). The version for quasi-compact and quasi-separated algebraic spaces is true as well (with essentially the same proof), but will be trivially implied by the main theorem in this section. We strongly urge the reader to read the proof of this lemma in the case of schemes first.

**Lemma 8.7.** Let \( B \) be an algebraic space of finite type over \( Z \). Let \( U \) be an algebraic space of finite type and separated over \( B \). Let \( U_2 \subset U \) be an elementary distinguished square. Assume \( U_1 \) and \( U_2 \) have compactifications over \( B \) and \( U_2 \times_U U_2 \to U \) has dense image. Then \( U \) has a compactification over \( B \).

**Proof.** Choose a compactification \( U_i \subset X_i \) over \( B \) for \( i = 1, 2 \). We may assume \( U_i \) is scheme theoretically dense in \( X_i \). We may assume there is an open \( V_i \subset X_i \) and
a proper morphism \( \psi_i : V_i \to U \) extending \( U_i \to U \), see Lemma 8.6

\[
\begin{array}{c}
U_i \\
\downarrow \psi_i \\
V_i \\
\downarrow \\
X_i
\end{array}
\]

Denote \( Z_1 \subset U \) the reduced closed subspace corresponding to the closed subset \(|U| \setminus |U_2|\). Recall that \( f^{-1}Z_1 \) is a closed subspace of \( U_1 \) mapping isomorphically to \( Z_1 \). Denote \( Z_2 \subset U \) the reduced closed subspace corresponding to the closed subset \(|U| \setminus \text{Im}(f)| = |U_2| \setminus \text{Im}(U_1 \times_U U_2 \to |U_2|)\). Thus we have

\[
U = U_2 \amalg Z_1 = Z_2 \amalg \text{Im}(f) = Z_2 \amalg \text{Im}(U_1 \times_U U_2 \to U_2) \amalg Z_1
\]

set theoretically. Denote \( Z_{i,j} \subset V_i \) the inverse image of \( Z_i \) under \( \psi_i \). Observe that \( \psi_2 \) is an isomorphism over an open neighbourhood of \( Z_2 \). Observe that \( Z_{1,1} = \psi_1^{-1}Z_1 = f^{-1}Z_1 \amalg T \) for some closed subspace \( T \subset V_1 \) disjoint from \( f^{-1}Z_1 \) and furthermore \( \psi_1 \) is étale along \( f^{-1}Z_1 \). Denote \( Z_{i,j} \subset V_i \) the inverse image of \( Z_j \) under \( \psi_i \). Observe that \( \psi_i : Z_{i,j} \to Z_j \) is a proper morphism. Since \( Z_i \) and \( Z_j \) are disjoint closed subspaces of \( U \), we see that \( Z_{i,i} \) and \( Z_{i,j} \) are disjoint closed subspaces of \( V_i \).

Denote \( \overline{Z}_{i,i} \) and \( \overline{Z}_{i,j} \) the scheme theoretic images of \( Z_{i,i} \) and \( Z_{i,j} \) in \( X_i \). We recall that \( |Z_{i,j}| \) is dense in \( |Z_{i,j}| \), see Morphisms of Spaces, Lemma 17.7 After replacing \( X_i \) by a \( V_i \)-admissible blowup we may assume that \( \overline{Z}_{i,i} \) and \( \overline{Z}_{i,j} \) are disjoint, see Lemma 8.2 We assume this holds for both \( X_1 \) and \( X_2 \). Observe that this property is preserved if we replace \( X_i \) by a further \( V_i \)-admissible blowup. Hence we may replace \( X_1 \) by another \( V_1 \)-admissible blowup and assume \( |\overline{Z}_{1,1}| \) is the disjoint union of the closures of \( |T| \) and \( |f^{-1}Z_1| \) in \( |X_1| \).

Consider the scheme theoretic image \( X_{12} \subset X_1 \times_B X_2 \) of the immersion \( (U_1 \times_U U_2) \to X_1 \times_B X_2 \) of \( (U_1 \times_U U_2) \to U_1 \to X_1 \) and \( (U_1 \times_U U_2) \to U_2 \to X_2 \). The projection morphisms

\[
p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2
\]

are proper as \( X_1 \) and \( X_2 \) are proper over \( B \). If we replace \( X_1 \) by a \( V_1 \)-admissible blowing up, then \( X_{12} \) is replaced by the strict transform with respect to this blowing up, see Lemma 8.5

Denote \( V_{12} \subset X_{12} \) the open subspace \( V_{12} = p_1^{-1}(V_1) = p_2^{-1}(V_2) \) and denote \( \psi : V_{12} \to U \) the compositions \( \psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}} \). Consider the closed subspace

\[
Z_{12,2} = p_1^{-1}Z_{1,2} = p_2^{-1}Z_{2,2} = \psi^{-1}Z_2 \subset V_{12}
\]

The morphism \( p_1|_{V_{12}} : V_{12} \to V_1 \) is an isomorphism over an open neighbourhood of \( Z_{1,2} \) because \( \psi_2 : V_2 \to U \) is an isomorphism over an open neighbourhood of \( Z_2 \), see Morphisms of Spaces, Lemma 16.7. By Lemma 8.3 there exists a \( V_1 \)-admissible blowing up \( X'_1 \to X_1 \) such that the strict transform \( p'_1 : X'_{12} \to X'_1 \) of \( p_1 \) is an isomorphism over an open neighbourhood of the closure of \( |Z_{1,2}| \) in \( |X'_1| \). After replacing \( X_1 \) by \( X'_1 \) and \( X_12 \) by \( X'_12 \) we may assume that \( p_1 \) is an isomorphism over an open neighbourhood of \( |\overline{Z}_{1,2}| \).

The result of the previous paragraph tells us that

\[
X_{12} \cap (\overline{Z}_{1,2} \times_B \overline{Z}_{2,1}) = \emptyset
\]
where the intersection taken in $X_1 \times_B X_2$. Namely, the inverse image $p_1^{-1}Z_{1,2}$ in $X_{12}$ maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $|Z_{12,2}|$ is dense in $|p_1^{-1}Z_{1,2}|$. Thus $p_2$ maps $|p_1^{-1}Z_{1,2}|$ into $|Z_{2,2}|$. Since $|Z_{2,2}| \cap |Z_{2,1}| = \emptyset$ we conclude.

It turns out that we need to do one additional blowing up before we can conclude the argument. Namely, let $V_2 \subset W_2 \subset X_2$ be the open subspace with underlying topological space $|V_2| \cup (|X_2| \setminus |\overline{Z}_{2,1}|)$. Since $p_2(p_1^{-1}Z_{1,2})$ is contained in $W_2$ (see above) we see that replacing $X_2$ by a $W_2$-admissible blowup and $X_{21}$ by the corresponding strict transform will preserve the property of $p_1$ being an isomorphism over an open neighbourhood of $Z_{1,2}$. Since $Z_{2,1} \cap W_2 = Z_{2,2}$ and since

$$p_2^{-1}Z_{2,1} = p_1^{-1}Z_{1,1} = p_1^{-1}f^{-1}Z_1 \amalg p_1^{-1}T$$

and $p_2$ is étale along $p_1^{-1}f^{-1}Z_1$ as $\psi_1$ is étale along $f^{-1}Z_1$. Thus we may apply Lemma 8.1. Hence after replacing $X_2$ by a $W_2$-admissible blowup and $X_{12}$ by the corresponding strict transform, we obtain for every geometric point $\overline{\gamma}$ of the closure of $|p_1^{-1}f^{-1}Z_1|$ a local ring map $\mathcal{O}_{X_{12}, \overline{\gamma}} \to \mathcal{O}$ such that $\mathcal{O}_{X_{2, p_2(\overline{\gamma})}} \to \mathcal{O}$ is finite flat.

Consider the algebraic space

$$W_2 = U \coprod_{U_2} \left( X_2 \setminus \overline{Z}_{2,1} \right) ,$$

and with $T \subset V_1$ as in the first paragraph the algebraic space

$$W_1 = U \coprod_{U_1} \left( X_1 \setminus \overline{Z}_{1,2} \cup T \right) ,$$

obtained by pushout, see Lemma 4.2. Let us apply Lemma 8.1 to see that $W_1 \to B$ is separated. First, $U \to B$ and $X_i \to B$ are separated. Let us check the quasi-compact immersion $U_i \to U \times_B (X_i \setminus \overline{Z}_{i,j})$ is closed using the valuative criterion, see Morphisms of Spaces, Lemma 42.14. Choose a valuation ring $A$ over $B$ with fraction field $K$ and compatible morphisms $(u, x_i) : \text{Spec}(A) \to U \times_B X_i$ and $u_i : \text{Spec}(K) \to U_i$. Since $\psi_i$ is proper, we can find a unique $v_i : \text{Spec}(A) \to V_i$ compatible with $u$ and $u_i$. Since $X_i$ is proper over $B$ we see that $x_i = v_i$. If $v_i$ does not factor through $U_i \subset V_i$, then we conclude that $x_i$ maps the closed point of $\text{Spec}(A)$ into $Z_{i,j}$ or $T$ when $i = 1$. This finishes the proof because we removed $\overline{Z}_{i,j}$ and $T$ in the construction of $W_i$.

On the other hand, for any valuation ring $A$ over $B$ with fraction field $K$ and any morphism

$$\gamma : \text{Spec}(K) \to \text{Im}(U_1 \times_U U_2 \to U)$$

over $B$, we claim that after replacing $A$ by an extension of valuation rings, there is an $i$ and an extension of $\gamma$ to a morphism $h_i : \text{Spec}(A) \to W_i$. Namely, we first extend $\gamma$ to a morphism $g_2 : \text{Spec}(A) \to X_2$ using the valuative criterion of properness. If the image of $g_2$ does not meet $\overline{Z}_{2,1}$, then we obtain our morphism into $W_2$. Otherwise, denote $\overline{\gamma} \in \overline{Z}_{2,1}$ a geometric point lying over the image of the closed point under $g_2$. We may lift this to a geometric point $\overline{\gamma}$ of $X_{12}$ in the closure of $|p_1^{-1}f^{-1}Z_1|$ because the map of spaces $|p_1^{-1}f^{-1}Z_1| \to \overline{|Z}_{2,1}|$ is closed with image containing the dense open $|Z_{2,1}|$. After replacing $A$ by its strict henselization (More
on Algebra, Lemma \[103.5\) we get the following diagram

\[
\begin{array}{ccc}
A \ar[r] & A' \\
\ar[u] & \ar[u] \\
\mathcal{O}_{X_{12}} \ar[r] & \mathcal{O}_{X_{12, \mathcal{O}}} \ar[r] & \mathcal{O}
\end{array}
\]

where \(\mathcal{O}_{X_{12, \mathcal{O}}} \to \mathcal{O}\) is the map we found in the 5th paragraph of the proof. Since the horizontal composition is finite and flat we can find an extension of valuation rings \(A'/A\) and dotted arrow making the diagram commute. After replacing \(A\) by \(A'\) this means that we obtain a lift \(g_{12} : \text{Spec}(A) \to X_{12}\) whose closed point maps into the closure of \(|p_1^{-1}f^{-1}Z_1|\). Then \(g_1 = p_1 \circ g_{12} : \text{Spec}(A) \to X_1\) is a morphism whose closed point maps into the closure of \(|f^{-1}Z_1|\). Since the closure of \(|f^{-1}Z_1|\) is disjoint from the closure of \(|T|\) and contained in \([Z_{1,1}|\) which is disjoint from \([Z_{1,2}|\) we conclude that \(g_1\) defines a morphism \(h_1 : \text{Spec}(A) \to W_1\) as desired.

Consider a diagram

\[
\begin{array}{ccc}
W'_1 \ar[r] & W \ar[l] & W'_2 \\
\ar[ur] & \ar[dr] & \\
W_1 \ar[r] & U \ar[r] & W_2
\end{array}
\]

as in More on Morphisms of Spaces, Lemma \[40.1\). By the previous paragraph for every solid diagram

\[
\begin{array}{ccc}
\text{Spec}(K) \ar[r] & W \\
\ar[ur] & \ar[dr] & \\
\text{Spec}(A) \ar[r] & B
\end{array}
\]

where \(\text{Im}(\gamma) \subset \text{Im}(U_1 \times_U U_2 \to U)\) there is an \(i\) and an extension \(h_i : \text{Spec}(A) \to W_i\) of \(\gamma\) after possibly replacing \(A\) by an extension of valuation rings. Using the valuative criterion of properness for \(W'_i \to W_i\), we can then lift \(h_i\) to \(h'_i : \text{Spec}(A) \to W'_i\). Hence the dotted arrow in the diagram exists after possibly extending \(A\). Since \(W\) is separated over \(B\), we see that the choice of extension isn’t needed and the arrow is unique as well, see Morphisms of Spaces, Lemmas \[41.5\] and \[43.1\]. Then finally the existence of the dotted arrow implies that \(W \to B\) is universally closed by Morphisms of Spaces, Lemma \[42.5\]. As \(W \to B\) is already of finite type and separated, we win. \(\square\)

**Lemma 8.8.** Let \(S\) be a scheme. Let \(X\) be a Noetherian algebraic space over \(S\). Let \(U \subset X\) be a proper dense open subspace. Then there exists an affine scheme \(V\) and an étale morphism \(V \to X\) such that

1. the open subspace \(W = U \cup \text{Im}(V \to X)\) is strictly larger than \(U\),
2. \((U \subset W, V \to W)\) is a distinguished square, and
3. \(U \times_W V \to U\) has dense image.

**Proof.** Choose a stratification

\[0 = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X\]

and morphisms \(f_p : V_p \to U_p\) as in Decent Spaces, Lemma \[8.6\]. Let \(p\) be the smallest integer such that \(U_p \not\subset U\) (this is possible as \(U \neq X\)). Choose an affine
open $V \subset V_p$ such that the étale morphism $f_p|_V : V \to X$ does not factor through $U$. Consider the open $W = U \cup \text{Im}(V \to X)$ and the reduced closed subspace $Z \subset W$ with $|Z| = |W| \setminus |U|$. Then $f^{-1}Z \to Z$ is an isomorphism because we have the corresponding property for the morphism $f_p$, see the lemma cited above. Thus $(U \subset W, f : V \to W)$ is a distinguished square. It may not be true that the open $I = \text{Im}(U \times_W V \to U)$ is dense in $U$. The algebraic space $U' \subset U$ whose underlying set is $|U| \setminus |I|$ is Noetherian and hence we can find a dense open subscheme $U'' \subset U'$, see for example Properties of Spaces, Proposition 13.3. Then we can find a dense open affine $U''' \subset U''$, see Properties, Lemmas 5.7 and 29.1. After we replace $f$ by $V \amalg U''' \to X$ everything is clear. 

0F4D Theorem 8.9. Let $S$ be a scheme. Let $B$ be a quasi-compact and quasi-separated algebraic space over $S$. Let $X \to B$ be a separated, finite type morphism. Then $X$ has a compactification over $B$.

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. First, we may replace $S$ by $\text{Spec}(\mathbb{Z})$. See Spaces, Section 16 and Properties of Spaces, Definition 3.1. There exists a closed immersion $X \to \overline{X}$ with $X' \to B$ of finite presentation and separated. See Limits of Spaces, Proposition 11.7.

If we find a compactification of $X'$ over $B$, then taking the scheme theoretic closure of $X$ in this will give a compactification of $X$ over $B$. Thus we may assume $X \to B$ is separated and of finite presentation. We may write $B = \lim B_i$, as a directed limit of a system of Noetherian algebraic spaces of finite type over $\text{Spec}(\mathbb{Z})$ with affine transition morphisms. See Limits of Spaces, Proposition 8.1. We can choose an $i$ and a morphism $X_i \to B_i$ of finite presentation whose base change to $B$ is $X \to B$, see Limits of Spaces, Lemma 7.1. After increasing $i$ we may assume $X_i \to B_i$ is separated, see Limits of Spaces, Lemma 6.9. If we can find a compactification of $X_i$ over $B_i$, then the base change of this to $B$ will be a compactification of $X$ over $B$. This reduces us to the case discussed in the next paragraph.

Assume $B$ is of finite type over $\mathbb{Z}$ in addition to being quasi-compact and quasi-separated. Let $U \to X$ be an étale morphism of algebraic spaces such that $U$ has a compactification $Y$ over $\text{Spec}(\mathbb{Z})$. The morphism $U \to B \times_{\text{Spec}(\mathbb{Z})} Y$ is separated and quasi-finite by Morphisms of Spaces, Lemma 27.10 (the displayed morphism factors into an immersion hence is a monomorphism). Hence by Zariski’s main theorem (More on Morphisms of Spaces, Lemma 34.3) there is an open immersion of $U$ into an algebraic space $Y'$ finite over $B \times_{\text{Spec}(\mathbb{Z})} Y$. Then $Y' \to B$ is proper as the composition $Y' \to B \times_{\text{Spec}(\mathbb{Z})} Y \to B$ of two proper morphisms (use Morphisms of Spaces, Lemmas 45.9, 40.4 and 40.3). We conclude that $U$ has a compactification over $B$.

There is a dense open subspace $U \subset X$ which is a scheme. (Properties of Spaces, Proposition 13.3.) In fact, we may choose $U$ to be an affine scheme (Properties, Lemmas 5.7 and 29.1). Thus $U$ has a compactification over $\text{Spec}(\mathbb{Z})$; this is easily shown directly but also follows from the theorem for schemes, see More on Flatness, Theorem 33.8. By the previous paragraph $U$ has a compactification over $B$. By Noetherian induction we can find a maximal dense open subspace $U \subset X$ which has a compactification over $B$. We will show that the assumption that $U \neq X$ leads to a contradiction. Namely, by Lemma 8.8 we can find a strictly larger open
$U \subset W \subset X$ and a distinguished square $(U \subset W, f : V \to W)$ with $V$ affine and $U \times_W V$ dense image in $U$. Since $V$ is affine, as before it has a compactification over $B$. Hence Lemma 8.7 applies to show that $W$ has a compactification over $B$ which is the desired contradiction. \hfill \square

9. Other chapters
Deformation Theory

Topics in Moduli Theory

Miscellany

Algebraic Stacks

References