

# TOPOLOGIES ON ALGEBRAIC SPACES

03Y4

## Contents

1. Introduction	1
2. The general procedure	1
3. Zariski topology	2
4. Étale topology	2
5. Smooth topology	7
6. Syntomic topology	7
7. Fppf topology	8
8. The ph topology	9
9. Fpqc topology	12
10. Other chapters	14
References	15

## 1. Introduction

03Y5 In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [Gro71], [BLR90], [LMB00] and [Knu71]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

## 2. The general procedure

03Y6 In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 2.

Let  $S$  be a base scheme. Take any category  $Sch_\alpha$  constructed as in Sets, Lemma 9.2 starting with  $S$  and any set of schemes over  $S$  you want to be included. Choose any set of coverings  $Cov_{fppf}$  on  $Sch_\alpha$  as in Sets, Lemma 11.1 starting with the category  $Sch_\alpha$  and the class of fppf coverings. Let  $Sch_{fppf}$  denote the big fppf site so obtained, and let  $(Sch/S)_{fppf}$  denote the corresponding big fppf site of  $S$ . (The above is entirely as prescribed in Topologies, Section 7.)

Given choices as above the category of algebraic spaces over  $S$  has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over  $S$  is of the form  $U/R$  for some étale equivalence relation  $j : R \rightarrow U \times_S U$  with  $U, R \in \text{Ob}((Sch/S)_{fppf})$ , see Spaces, Lemma 9.1. Hence we can find a full subcategory  $Spaces/S$  of the category of algebraic spaces over  $S$  which has a set of

---

This is a chapter of the Stacks Project, version 41964eac, compiled on Dec 16, 2020.

objects such that each algebraic space is isomorphic to an object of  $\mathcal{S}paces/S$ . We fix a choice of such a category.

In the sections below, given a topology  $\tau$ , the big site  $(\mathcal{S}paces/S)_\tau$  (resp. the big site  $(\mathcal{S}paces/X)_\tau$  of an algebraic space  $X$  over  $S$ ) has as underlying category the category  $\mathcal{S}paces/S$  (resp. the subcategory  $\mathcal{S}paces/X$  of  $\mathcal{S}paces/S$ , see Categories, Example 2.13). The procedure for turning this into a site is as usual by defining a class of  $\tau$ -coverings and using Sets, Lemma 11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site*  $X_{\text{étale}}$  of an algebraic space  $X$  has already been defined in Properties of Spaces, Definition 18.1. Its objects are schemes étale over  $X$ , of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 4.8) is the site  $X_{\text{spaces,étale}}$  of Properties of Spaces, Definition 18.2. These two sites define the same topos, see Properties of Spaces, Lemma 18.3. We will not redefine these in this chapter; instead we will simply use them.

### 3. Zariski topology

03YD In Spaces, Section 12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaced by open immersions.

041G **Definition 3.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . A *Zariski covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that each  $f_i$  is an open immersion and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

041H **Lemma 3.2.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is a Zariski covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a Zariski covering and for each  $i$  we have a Zariski covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a Zariski covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a Zariski covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a Zariski covering.*

**Proof.** Omitted. □

### 4. Étale topology

03YC In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 4.

041E **Definition 4.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . An *étale covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that each  $f_i$  is étale and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 4.1. In particular, if  $X$  and all the  $X_i$  are schemes, then we recover the usual notion of a étale covering of schemes.

0DF1 **Lemma 4.2.** *Any Zariski covering is an étale covering.*

**Proof.** This is clear from the definitions and the fact that an open immersion is an étale morphism (this follows from Morphisms, Lemma 36.9 via Spaces, Lemma 5.8 as immersions are representable).  $\square$

041F **Lemma 4.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is a étale covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a étale covering and for each  $i$  we have a étale covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a étale covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a étale covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a étale covering.*

**Proof.** Omitted.  $\square$

The following lemma tells us that the sites  $(Spaces/X)_{\acute{e}tale}$  and  $(Spaces/X)_{smooth}$  have the same categories of sheaves.

0CFV **Lemma 4.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\{X_i \rightarrow X\}_{i \in I}$  be a smooth covering of  $X$ . Then there exists an étale covering  $\{U_j \rightarrow X\}_{j \in J}$  of  $X$  which refines  $\{X_i \rightarrow X\}_{i \in I}$ .*

**Proof.** First choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow X$ . For each  $i$  choose a scheme  $W_i$  and a surjective étale morphism  $W_i \rightarrow X_i$ . Then  $\{W_i \rightarrow X\}_{i \in I}$  is a smooth covering which refines  $\{X_i \rightarrow X\}_{i \in I}$ . Hence  $\{W_i \times_X U \rightarrow U\}_{i \in I}$  is a smooth covering of schemes. By More on Morphisms, Lemma 35.7 we can choose an étale covering  $\{U_j \rightarrow U\}$  which refines  $\{W_i \times_X U \rightarrow U\}$ . Then  $\{U_j \rightarrow X\}_{j \in J}$  is an étale covering refining  $\{X_i \rightarrow X\}_{i \in I}$ .  $\square$

0DBX **Definition 4.5.** Let  $S$  be a scheme. A big étale site  $(Spaces/S)_{\acute{e}tale}$  is any site constructed as follows:

- (1) Choose a big étale site  $(Sch/S)_{\acute{e}tale}$  as in Topologies, Section 4.
- (2) As underlying category take the category  $Spaces/S$  of algebraic spaces over  $S$  (see discussion in Section 2 why this is a set).
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category  $Spaces/S$  and the class of étale coverings of Definition 4.1.

Having defined this, we can localize to get the étale site of an algebraic space.

0DBY **Definition 4.6.** Let  $S$  be a scheme. Let  $(Spaces/S)_{\acute{e}tale}$  be as in Definition 4.5. Let  $X$  be an algebraic space over  $S$ , i.e., an object of  $(Spaces/S)_{\acute{e}tale}$ . Then the big étale site  $(Spaces/X)_{\acute{e}tale}$  of  $X$  is the localization of the site  $(Spaces/S)_{\acute{e}tale}$  at  $X$  introduced in Sites, Section 25.

Recall that given an algebraic space  $X$  over  $S$  as in the definition, we already have defined the small étale sites  $X_{spaces, \acute{e}tale}$  and  $X_{\acute{e}tale}$ , see Properties of Spaces, Section 18. We will silently identify the corresponding topoi using the inclusion functor  $X_{\acute{e}tale} \subset X_{spaces, \acute{e}tale}$  (Properties of Spaces, Lemma 18.3) and we will call it the small étale topoi of  $X$ . Next, we establish some relationships between the topoi associated to these sites.

0DF2 **Lemma 4.7.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of  $(\text{Spaces}/S)_{\acute{e}tale}$ . The inclusion functor  $Y_{\text{spaces},\acute{e}tale} \rightarrow (\text{Spaces}/X)_{\acute{e}tale}$  is cocontinuous and induces a morphism of topoi*

$$i_f : \text{Sh}(Y_{\acute{e}tale}) \longrightarrow \text{Sh}((\text{Spaces}/X)_{\acute{e}tale})$$

For a sheaf  $\mathcal{G}$  on  $(\text{Spaces}/X)_{\acute{e}tale}$  we have the formula  $(i_f^{-1}\mathcal{G})(U/Y) = \mathcal{G}(U/X)$ . The functor  $i_f^{-1}$  also has a left adjoint  $i_{f,!}$  which commutes with fibre products and equalizers.

**Proof.** Denote the functor  $u : Y_{\text{spaces},\acute{e}tale} \rightarrow (\text{Spaces}/X)_{\acute{e}tale}$ . In other words, given an étale morphism  $j : U \rightarrow Y$  corresponding to an object of  $Y_{\text{spaces},\acute{e}tale}$  we set  $u(U \rightarrow T) = (f \circ j : U \rightarrow S)$ . The category  $Y_{\text{spaces},\acute{e}tale}$  has fibre products and equalizers and  $u$  commutes with them. It is immediate that  $u$  is cocontinuous. The functor  $u$  is also continuous as  $u$  transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 21.5 and 21.6.  $\square$

0DF3 **Lemma 4.8.** *Let  $S$  be a scheme. Let  $X$  be an object of  $(\text{Spaces}/S)_{\acute{e}tale}$ . The inclusion functor  $X_{\text{spaces},\acute{e}tale} \rightarrow (\text{Spaces}/X)_{\acute{e}tale}$  satisfies the hypotheses of Sites, Lemma 21.8 and hence induces a morphism of sites*

$$\pi_X : (\text{Spaces}/X)_{\acute{e}tale} \longrightarrow X_{\text{spaces},\acute{e}tale}$$

and a morphism of topoi

$$i_X : \text{Sh}(X_{\acute{e}tale}) \longrightarrow \text{Sh}((\text{Spaces}/X)_{\acute{e}tale})$$

such that  $\pi_X \circ i_X = \text{id}$ . Moreover,  $i_X = i_{\text{id}_X}$  with  $i_{\text{id}_X}$  as in Lemma 4.7. In particular the functor  $i_X^{-1} = \pi_{X,*}$  is described by the rule  $i_X^{-1}(\mathcal{G})(U/X) = \mathcal{G}(U/X)$ .

**Proof.** In this case the functor  $u : X_{\text{spaces},\acute{e}tale} \rightarrow (\text{Spaces}/X)_{\acute{e}tale}$ , in addition to the properties seen in the proof of Lemma 4.7 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 21.8.  $\square$

0DF4 **Definition 4.9.** In the situation of Lemma 4.8 the functor  $i_X^{-1} = \pi_{X,*}$  is often called the *restriction to the small étale site*, and for a sheaf  $\mathcal{F}$  on the big étale site we often denote  $\mathcal{F}|_{X_{\acute{e}tale}}$  this restriction.

With this notation in place we have for a sheaf  $\mathcal{F}$  on the big site and a sheaf  $\mathcal{G}$  on the small site that

$$\begin{aligned} \text{Mor}_{\text{Sh}(X_{\acute{e}tale})}(\mathcal{F}|_{X_{\acute{e}tale}}, \mathcal{G}) &= \text{Mor}_{\text{Sh}((\text{Spaces}/X)_{\acute{e}tale})}(\mathcal{F}, i_{X,*}\mathcal{G}) \\ \text{Mor}_{\text{Sh}(X_{\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{X_{\acute{e}tale}}) &= \text{Mor}_{\text{Sh}((\text{Spaces}/X)_{\acute{e}tale})}(\pi_X^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have  $(i_{X,*}\mathcal{G})|_{X_{\acute{e}tale}} = \mathcal{G}$  and we have  $(\pi_X^{-1}\mathcal{G})|_{X_{\acute{e}tale}} = \mathcal{G}$ .

0DF5 **Lemma 4.10.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism in  $(\text{Spaces}/S)_{\acute{e}tale}$ . The functor*

$$u : (\text{Spaces}/Y)_{\acute{e}tale} \longrightarrow (\text{Spaces}/X)_{\acute{e}tale}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Spaces}/X)_{\acute{e}tale} \longrightarrow (\text{Spaces}/Y)_{\acute{e}tale}, \quad (U \rightarrow X) \longmapsto (U \times_X Y \rightarrow Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{\acute{e}tale}) \longrightarrow Sh((Spaces/X)_{\acute{e}tale})$$

We have  $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$ . We have  $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$ . Also,  $f_{big}^{-1}$  has a left adjoint  $f_{big}!$  which commutes with fibre products and equalizers.

**Proof.** The functor  $u$  is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.7). Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for  $f_{big}^{-1}$  and the existence of  $f_{big}!$ . Moreover, the functor  $v$  is a right adjoint because given  $U/Y$  and  $V/X$  we have  $Mor_X(u(U), V) = Mor_Y(U, V \times_X Y)$  as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for  $f_{big,*}$ .  $\square$

ODF6 **Lemma 4.11.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism in  $(Spaces/S)_{\acute{e}tale}$ .*

- (1) *We have  $i_f = f_{big} \circ i_T$  with  $i_f$  as in Lemma 4.7 and  $i_T$  as in Lemma 4.8.*
- (2) *The functor  $X_{spaces,\acute{e}tale} \rightarrow T_{spaces,\acute{e}tale}$ ,  $(U \rightarrow X) \mapsto (U \times_X Y \rightarrow Y)$  is continuous and induces a morphism of sites*

$$f_{spaces,\acute{e}tale} : Y_{spaces,\acute{e}tale} \longrightarrow X_{spaces,\acute{e}tale}$$

The corresponding morphism of small étale topoi is denoted

$$f_{small} : Sh(Y_{\acute{e}tale}) \rightarrow Sh(X_{\acute{e}tale})$$

We have  $f_{small,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$ .

- (3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} Y_{spaces,\acute{e}tale} & \xleftarrow{\pi_Y} & (Spaces/Y)_{\acute{e}tale} \\ f_{spaces,\acute{e}tale} \downarrow & & \downarrow f_{big} \\ X_{spaces,\acute{e}tale} & \xleftarrow{\pi_X} & (Spaces/X)_{\acute{e}tale} \end{array}$$

so that  $f_{small} \circ \pi_Y = \pi_X \circ f_{big}$  as morphisms of topoi.

- (4) *We have  $f_{small} = \pi_X \circ f_{big} \circ i_Y = \pi_X \circ i_f$ .*

**Proof.** The equality  $i_f = f_{big} \circ i_Y$  follows from the equality  $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$  which is clear from the descriptions of these functors above. Thus we see (1).

The functor  $u : X_{spaces,\acute{e}tale} \rightarrow Y_{spaces,\acute{e}tale}$ ,  $u(U \rightarrow X) = (U \times_X Y \rightarrow Y)$  was shown to give rise to a morphism of sites and corresponding morphism of small étale topoi in Properties of Spaces, Lemma 18.7. The description of the pushforward is clear.

Part (3) follows because  $\pi_X$  and  $\pi_Y$  are given by the inclusion functors and  $f_{spaces,\acute{e}tale}$  and  $f_{big}$  by the base change functors  $U \mapsto U \times_X Y$ .

Statement (4) follows from (3) by precomposing with  $i_Y$ .  $\square$

In the situation of the lemma, using the terminology of Definition 4.9 we have: for  $\mathcal{F}$  a sheaf on the big étale site of  $Y$

$$(f_{big,*}\mathcal{F})|_{X_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{Y_{\acute{e}tale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of  $Y$ , resp.  $X$  is given by  $\pi_{Y,*}$ , resp.  $\pi_{X,*}$ . A similar formula involving pullbacks and restrictions is false.

0DF7 **Lemma 4.12.** *Let  $S$  be a scheme. Given morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  in  $(\text{Spaces}/S)_{\acute{e}tale}$  we have  $g_{big} \circ f_{big} = (g \circ f)_{big}$  and  $g_{small} \circ f_{small} = (g \circ f)_{small}$ .*

**Proof.** This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.10. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.11.  $\square$

0DF8 **Lemma 4.13.** *Let  $S$  be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{\quad} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{\quad} & X \end{array}$$

*in  $(\text{Spaces}/S)_{\acute{e}tale}$ . Then  $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$  and  $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$ .*

**Proof.** Since the diagram is cartesian, we have for  $U'/X'$  that  $U' \times_{X'} Y' = U' \times_X Y$ . Hence both  $i_g^{-1} \circ f_{big,*}$  and  $f'_{small,*} \circ (i_{g'})^{-1}$  send a sheaf  $\mathcal{F}$  on  $(\text{Spaces}/Y)_{\acute{e}tale}$  to the sheaf  $U' \mapsto \mathcal{F}(U' \times_{X'} Y')$  on  $X'_{\acute{e}tale}$  (use Lemmas 4.7 and 4.10). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 28.1.  $\square$

0DF9 **Remark 4.14.** The sites  $(\text{Spaces}/X)_{\acute{e}tale}$  and  $X_{spaces,\acute{e}tale}$  come with structure sheaves. For the small étale site we have seen this in Properties of Spaces, Section 21. The structure sheaf  $\mathcal{O}$  on the big étale site  $(\text{Spaces}/X)_{\acute{e}tale}$  is defined by assigning to an object  $U$  the global sections of the structure sheaf of  $U$ . This makes sense because after all  $U$  is an algebraic space itself hence has a structure sheaf. Since  $\mathcal{O}_U$  is a sheaf on the étale site of  $U$ , the presheaf  $\mathcal{O}$  so defined satisfies the sheaf condition for coverings of  $U$ , i.e.,  $\mathcal{O}$  is a sheaf. We can upgrade the morphisms  $i_f$ ,  $\pi_X$ ,  $i_X$ ,  $f_{small}$ , and  $f_{big}$  defined above to morphisms of ringed sites, respectively topoi. Let us deal with these one by one.

- (1) In Lemma 4.7 denote  $\mathcal{O}$  the structure sheaf on  $(\text{Spaces}/X)_{\acute{e}tale}$ . We have  $(i_f^{-1}\mathcal{O})(U/Y) = \mathcal{O}_U(U) = \mathcal{O}_Y(U)$  by construction. Hence an isomorphism  $i_f^\sharp : i_f^{-1}\mathcal{O} \rightarrow \mathcal{O}_Y$ .
- (2) In Lemma 4.8 it was noted that  $i_X$  is a special case of  $i_f$  with  $f = \text{id}_X$  hence we are back in case (1).
- (3) In Lemma 4.8 the morphism  $\pi_X$  satisfies  $(\pi_{X,*}\mathcal{O})(U) = \mathcal{O}(U) = \mathcal{O}_X(U)$ . Hence we can use this to define  $\pi_X^\sharp : \mathcal{O}_X \rightarrow \pi_{X,*}\mathcal{O}$ .
- (4) In Lemma 4.11 the extension of  $f_{small}$  to a morphism of ringed topoi was discussed in Properties of Spaces, Lemma 21.3.
- (5) In Lemma 4.11 the functor  $f_{big}^{-1}$  is simply the restriction via the inclusion functor  $(\text{Spaces}/Y)_{\acute{e}tale} \rightarrow (\text{Spaces}/X)_{\acute{e}tale}$ . Let  $\mathcal{O}_1$  be the structure sheaf on  $(\text{Spaces}/X)_{\acute{e}tale}$  and let  $\mathcal{O}_2$  be the structure sheaf on  $(\text{Spaces}/Y)_{\acute{e}tale}$ . We obtain a canonical isomorphism  $f_{big}^\sharp : f_{big}^{-1}\mathcal{O}_1 \rightarrow \mathcal{O}_2$ .

Moreover, with these definitions compositions work out correctly too. We omit giving a detailed statement and proof.

### 5. Smooth topology

03YB In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 5.

041C **Definition 5.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . A *smooth covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that each  $f_i$  is smooth and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 5.1. In particular, if  $X$  and all the  $X_i$  are schemes, then we recover the usual notion of a smooth covering of schemes.

0DFA **Lemma 5.2.** *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

**Proof.** This is clear from the definitions, the fact that an étale morphism is smooth (Morphisms of Spaces, Lemma 39.6), and Lemma 4.2.  $\square$

041D **Lemma 5.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is a smooth covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a smooth covering and for each  $i$  we have a smooth covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a smooth covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a smooth covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a smooth covering.*

**Proof.** Omitted.  $\square$

To be continued...

### 6. Syntomic topology

03YA In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 6.

041A **Definition 6.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . A *syntomic covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that each  $f_i$  is syntomic and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 6.1. In particular, if  $X$  and all the  $X_i$  are schemes, then we recover the usual notion of a syntomic covering of schemes.

0DFB **Lemma 6.2.** *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

**Proof.** This is clear from the definitions and the fact that a smooth morphism is syntomic (Morphisms of Spaces, Lemma 37.8), and Lemma 5.2.  $\square$

041B **Lemma 6.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is a syntomic covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a syntomic covering and for each  $i$  we have a syntomic covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a syntomic covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a syntomic covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a syntomic covering.*

**Proof.** Omitted. □

To be continued...

### 7. Fppf topology

03Y7 In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 7.

03Y8 **Definition 7.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . An *fppf covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that each  $f_i$  is flat and locally of finite presentation and such that

$$|X| = \bigcup_{i \in I} |f_i|(|X_i|),$$

i.e., the morphisms are jointly surjective.

This is exactly the same as Topologies, Definition 7.1. In particular, if  $X$  and all the  $X_i$  are schemes, then we recover the usual notion of an fppf covering of schemes.

0DFC **Lemma 7.2.** *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

**Proof.** This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation (Morphisms of Spaces, Lemmas 36.5 and 36.6) and Lemma 6.2. □

03Y9 **Lemma 7.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is an fppf covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is an fppf covering and for each  $i$  we have an fppf covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is an fppf covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is an fppf covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is an fppf covering.*

**Proof.** Omitted. □

042T **Lemma 7.4.** *Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . Suppose that  $\mathcal{U} = \{f_i : X_i \rightarrow X\}_{i \in I}$  is an fppf covering of  $X$ . Then there exists a refinement  $\mathcal{V} = \{g_i : T_i \rightarrow X\}$  of  $\mathcal{U}$  which is an fppf covering such that each  $T_i$  is a scheme.*

**Proof.** Omitted. Hint: For each  $i$  choose a scheme  $T_i$  and a surjective étale morphism  $T_i \rightarrow X_i$ . Then check that  $\{T_i \rightarrow X\}$  is an fppf covering. □

0469 **Lemma 7.5.** *Let  $S$  be a scheme. Let  $\{f_i : X_i \rightarrow X\}_{i \in I}$  be an fppf covering of algebraic spaces over  $S$ . Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$



is surjective.

**Proof.** This follows from Spaces, Lemma 5.9. See also Spaces, Remark 5.2 in case you are confused about the meaning of this lemma.  $\square$

0DBV **Definition 7.6.** Let  $S$  be a scheme. A big fppf site  $(Spaces/S)_{fppf}$  is any site constructed as follows:

- (1) Choose a big fppf site  $(Sch/S)_{fppf}$  as in Topologies, Section 7.
- (2) As underlying category take the category  $Spaces/S$  of algebraic spaces over  $S$  (see discussion in Section 2 why this is a set).
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category  $Spaces/S$  and the class of fppf coverings of Definition 7.1.

Having defined this, we can localize to get the fppf site of an algebraic space.

0DBW **Definition 7.7.** Let  $S$  be a scheme. Let  $(Spaces/S)_{fppf}$  be as in Definition 7.6. Let  $X$  be an algebraic space over  $S$ , i.e., an object of  $(Spaces/S)_{fppf}$ . Then the big fppf site  $(Spaces/X)_{fppf}$  of  $X$  is the localization of the site  $(Spaces/S)_{fppf}$  at  $X$  introduced in Sites, Section 25.

Next, we establish some relationships between the topoi associated to these sites.

0DFD **Lemma 7.8.** Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of algebraic spaces over  $S$ . The functor

$$u : (Spaces/Y)_{fppf} \longrightarrow (Spaces/X)_{fppf}, \quad V/Y \longmapsto V/X$$

is cocontinuous, and has a continuous right adjoint

$$v : (Spaces/X)_{fppf} \longrightarrow (Spaces/Y)_{fppf}, \quad (U \rightarrow Y) \longmapsto (U \times_X Y \rightarrow Y).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Spaces/Y)_{fppf}) \longrightarrow Sh((Spaces/X)_{fppf})$$

We have  $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$ . We have  $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$ . Also,  $f_{big}^{-1}$  has a left adjoint  $f_{big}!$  which commutes with fibre products and equalizers.

**Proof.** The functor  $u$  is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for  $f_{big}^{-1}$  and the existence of  $f_{big}!$ . Moreover, the functor  $v$  is a right adjoint because given  $U/T$  and  $V/X$  we have  $Mor_X(u(U), V) = Mor_Y(U, V \times_X Y)$  as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for  $f_{big,*}$ .  $\square$

0DFE **Lemma 7.9.** Let  $S$  be a scheme. Given morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  of algebraic spaces over  $S$  we have  $g_{big} \circ f_{big} = (g \circ f)_{big}$ .

**Proof.** This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.8.  $\square$

## 8. The ph topology

0DFF In this section we define the ph topology. This is the topology generated by étale coverings and proper surjective morphisms, see Lemma 8.7.

0DFG **Definition 8.1.** Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . A *ph covering* of  $X$  is a family of morphisms  $\{X_i \rightarrow X\}_{i \in I}$  of algebraic spaces over  $S$  such that  $f_i$  is locally of finite type and such that for every  $U \rightarrow X$  with  $U$  affine there exists a standard ph covering  $\{U_j \rightarrow U\}_{j=1, \dots, m}$  refining the family  $\{X_i \times_X U \rightarrow U\}_{i \in I}$ .

In other words, there exists indices  $i_1, \dots, i_m \in I$  and morphisms  $h_j : U_j \rightarrow X_{i_j}$  such that  $f_{i_j} \circ h_j = h \circ g_j$ . Note that if  $X$  and all  $X_i$  are representable, this is the same as a ph covering of schemes by Topologies, Definition 8.4.

0DFH **Lemma 8.2.** *Any fppf covering is a ph covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is a ph covering.*

**Proof.** We will show that an fppf covering is a ph covering, and then the rest follows from Lemma 7.2. Let  $\{X_i \rightarrow X\}_{i \in I}$  be an fppf covering of algebraic spaces over a base scheme  $S$ . Let  $U$  be an affine scheme and let  $U \rightarrow X$  be a morphism. We can refine the fppf covering  $\{X_i \times_U U \rightarrow U\}_{i \in I}$  by an fppf covering  $\{T_i \rightarrow U\}_{i \in I}$  where  $T_i$  is a scheme (Lemma 7.4). Then we can find a standard ph covering  $\{U_j \rightarrow U\}_{j=1, \dots, m}$  refining  $\{T_i \rightarrow U\}_{i \in I}$  by More on Morphisms, Lemma 44.7 (and the definition of ph coverings for schemes). Thus  $\{X_i \rightarrow X\}_{i \in I}$  is a ph covering by definition.  $\square$

0DFI **Lemma 8.3.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a surjective proper morphism of algebraic spaces over  $S$ . Then  $\{Y \rightarrow X\}$  is a ph covering.*

**Proof.** Let  $U \rightarrow X$  be a morphism with  $U$  affine. By Chow's lemma (in the weak form given as Cohomology of Spaces, Lemma 18.1) we see that there is a surjective proper morphism of schemes  $V \rightarrow U$  which factors through  $Y \times_X U \rightarrow U$ . Taking any finite affine open cover of  $V$  we obtain a standard ph covering of  $U$  refining  $\{X \times_Y U \rightarrow U\}$  as desired.  $\square$

0DFJ **Lemma 8.4.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is a ph covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a ph covering and for each  $i$  we have a ph covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a ph covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is a ph covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a ph covering.*

**Proof.** Part (1) is clear. Consider  $g : X' \rightarrow X$  and  $\{X_i \rightarrow X\}_{i \in I}$  a ph covering as in (3). By Morphisms of Spaces, Lemma 23.3 the morphisms  $X' \times_X X_i \rightarrow X'$  are locally of finite type. If  $h' : Z \rightarrow X'$  is a morphism from an affine scheme towards  $X'$ , then set  $h = g \circ h' : Z \rightarrow X$ . The assumption on  $\{X_i \rightarrow X\}_{i \in I}$  means there exists a standard ph covering  $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$  and morphisms  $Z_j \rightarrow X_{i(j)}$  covering  $h$  for certain  $i(j) \in I$ . By the universal property of the fibre product we obtain morphisms  $Z_j \rightarrow X' \times_X X_{i(j)}$  over  $h'$  also. Hence  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is a ph covering. This proves (3).

Let  $\{X_i \rightarrow X\}_{i \in I}$  and  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$  be as in (2). Let  $h : Z \rightarrow X$  be a morphism from an affine scheme towards  $X$ . By assumption there exists a standard ph covering  $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$  and morphisms  $h_j : Z_j \rightarrow X_{i(j)}$  covering  $h$  for some indices  $i(j) \in I$ . By assumption there exist standard ph coverings  $\{Z_{j,l} \rightarrow Z_j\}_{l=1, \dots, n(j)}$  and morphisms  $Z_{j,l} \rightarrow X_{i(j)j(l)}$  covering  $h_j$  for some indices  $j(l) \in$

$J_{i(j)}$ . By Topologies, Lemma 8.3 the family  $\{Z_{j,l} \rightarrow Z\}$  can be refined by a standard ph covering. Hence we conclude that  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is a ph covering.  $\square$

0DFK **Definition 8.5.** Let  $S$  be a scheme. A big ph site  $(Spaces/S)_{ph}$  is any site constructed as follows:

- (1) Choose a big ph site  $(Sch/S)_{ph}$  as in Topologies, Section 8.
- (2) As underlying category take the category  $Spaces/S$  of algebraic spaces over  $S$  (see discussion in Section 2 why this is a set).
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category  $Spaces/S$  and the class of ph coverings of Definition 8.1.

Having defined this, we can localize to get the ph site of an algebraic space.

0DFL **Definition 8.6.** Let  $S$  be a scheme. Let  $(Spaces/S)_{ph}$  be as in Definition 8.5. Let  $X$  be an algebraic space over  $S$ , i.e., an object of  $(Spaces/S)_{ph}$ . Then the big ph site  $(Spaces/X)_{ph}$  of  $X$  is the localization of the site  $(Spaces/S)_{ph}$  at  $X$  introduced in Sites, Section 25.

Here is the promised characterization of ph sheaves.

0DFM **Lemma 8.7.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $\mathcal{F}$  be a presheaf on  $(Spaces/X)_{ph}$ . Then  $\mathcal{F}$  is a sheaf if and only if*

- (1)  $\mathcal{F}$  satisfies the sheaf condition for étale coverings, and
- (2) if  $f : V \rightarrow U$  is a proper surjective morphism of  $(Spaces/X)_{ph}$ , then  $\mathcal{F}(U)$  maps bijectively to the equalizer of the two maps  $\mathcal{F}(V) \rightarrow \mathcal{F}(V \times_U V)$ .

**Proof.** We will show that if (1) and (2) hold, then  $\mathcal{F}$  is sheaf. Let  $\{T_i \rightarrow T\}$  be a ph covering, i.e., a covering in  $(Spaces/X)_{ph}$ . We will verify the sheaf condition for this covering. Let  $s_i \in \mathcal{F}(T_i)$  be sections which restrict to the same section over  $T_i \times_T T_{i'}$ . We will show that there exists a unique section  $s \in \mathcal{F}$  restricting to  $s_i$  over  $T_i$ . Let  $\{U_j \rightarrow T\}$  be an étale covering with  $U_j$  affine. By property (1) it suffices to produce sections  $s_j \in \mathcal{F}(U_j)$  which agree on  $U_j \cap U_{j'}$  in order to produce  $s$ . Consider the ph coverings  $\{T_i \times_T U_j \rightarrow U_j\}$ . Then  $s_{ji} = s_i|_{T_i \times_T U_j}$  are sections agreeing over  $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$ . Choose a proper surjective morphism  $V_j \rightarrow U_j$  and a finite affine open covering  $V_j = \bigcup V_{jk}$  such that the standard ph covering  $\{V_{jk} \rightarrow U_j\}$  refines  $\{T_i \times_T U_j \rightarrow U_j\}$ . If  $s_{jk} \in \mathcal{F}(V_{jk})$  denotes the pullback of  $s_{ji}$  to  $V_{jk}$  by the implied morphisms, then we find that  $s_{jk}$  glue to a section  $s'_j \in \mathcal{F}(V_j)$ . Using the agreement on overlaps once more, we find that  $s'_j$  is in the equalizer of the two maps  $\mathcal{F}(V_j) \rightarrow \mathcal{F}(V_j \times_{U_j} V_j)$ . Hence by (2) we find that  $s'_j$  comes from a unique section  $s_j \in \mathcal{F}(U_j)$ . We omit the verification that these sections  $s_j$  have all the desired properties.  $\square$

Next, we establish some relationships between the topoi associated to these sites.

0DFN **Lemma 8.8.** *Let  $S$  be a scheme. Let  $f : Y \rightarrow X$  be a morphism of algebraic spaces over  $S$ . The functor*

$$u : (Spaces/Y)_{ph} \longrightarrow (Spaces/X)_{ph}, \quad V/Y \longmapsto V/X$$

*is cocontinuous, and has a continuous right adjoint*

$$v : (Spaces/X)_{ph} \longrightarrow (Spaces/Y)_{ph}, \quad (U \rightarrow Y) \longmapsto (U \times_X Y \rightarrow Y).$$

*They induce the same morphism of topoi*

$$f_{big} : Sh((Spaces/Y)_{ph}) \longrightarrow Sh((Spaces/X)_{ph})$$

We have  $f_{big}^{-1}(\mathcal{G})(U/Y) = \mathcal{G}(U/X)$ . We have  $f_{big,*}(\mathcal{F})(U/X) = \mathcal{F}(U \times_X Y/Y)$ . Also,  $f_{big}^{-1}$  has a left adjoint  $f_{big}!$  which commutes with fibre products and equalizers.

**Proof.** The functor  $u$  is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 21.5 and 21.6 apply and we deduce the formula for  $f_{big}^{-1}$  and the existence of  $f_{big}!$ . Moreover, the functor  $v$  is a right adjoint because given  $U/T$  and  $V/X$  we have  $Mor_X(u(U), V) = Mor_Y(U, V \times_X Y)$  as desired. Thus we may apply Sites, Lemmas 22.1 and 22.2 to get the formula for  $f_{big,*}$ .  $\square$

0DFP **Lemma 8.9.** *Let  $S$  be a scheme. Given morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  of algebraic spaces over  $S$  we have  $g_{big} \circ f_{big} = (g \circ f)_{big}$ .*

**Proof.** This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 8.8.  $\square$

0DK6 **Lemma 8.10.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ . Let  $P$  be a property of objects in  $(Spaces/X)_{fppf}$  such that whenever  $\{U_i \rightarrow U\}$  is a covering in  $(Spaces/X)_{fppf}$ , then*

$$P(U_{i_0} \times_U \dots \times_U U_{i_p}) \text{ for all } p \geq 0, i_0, \dots, i_p \in I \Rightarrow P(U)$$

*If  $P(U)$  for all  $U$  affine and flat, locally of finite presentation over  $X$ , then  $P(X)$ .*

**Proof.** Let  $U$  be a separated algebraic space locally of finite presentation over  $X$ . Then we can choose an étale covering  $\{U_i \rightarrow U\}_{i \in I}$  with  $V_i$  affine. Since  $U$  is separated, we conclude that  $U_{i_0} \times_U \dots \times_U U_{i_p}$  is always affine. Whence  $P(U_{i_0} \times_U \dots \times_U U_{i_p})$  always. Hence  $P(U)$  holds. Choose a scheme  $U$  which is a disjoint union of affines and a surjective étale morphism  $U \rightarrow X$ . Then  $U \times_X \dots \times_X U$  (with  $p+1$  factors) is a separated algebraic space étale over  $X$ . Hence  $P(U \times_X \dots \times_X U)$  by the above. We conclude that  $P(X)$  is true.  $\square$

## 9. Fpqc topology

03MP We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 9. We will show in Descent on Spaces, Proposition 4.1 that quasi-coherent sheaves descent along these.

03MQ **Definition 9.1.** Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . An *fpqc covering* of  $X$  is a family of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  of algebraic spaces such that each  $f_i$  is flat and such that for every affine scheme  $Z$  and morphism  $h : Z \rightarrow X$  there exists a standard fpqc covering  $\{g_j : Z_j \rightarrow Z\}_{j=1, \dots, m}$  which refines the family  $\{X_i \times_X Z \rightarrow Z\}_{i \in I}$ .

In other words, there exists indices  $i_1, \dots, i_m \in I$  and morphisms  $h_j : U_j \rightarrow X_{i_j}$  such that  $f_{i_j} \circ h_j = h \circ g_j$ . Note that if  $X$  and all  $X_i$  are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 9.11.

0DFQ **Lemma 9.2.** *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

**Proof.** We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 7.2. Let  $\{f_i : U_i \rightarrow U\}_{i \in I}$  be an fppf covering of algebraic spaces over  $S$ . By definition this means that the  $f_i$  are flat which checks the first condition of Definition 9.1. To check the second, let  $V \rightarrow U$  be a morphism with

$V$  affine. We may choose an étale covering  $\{V_{ij} \rightarrow V \times_U U_i\}$  with  $V_{ij}$  affine. Then the compositions  $f_{ij} : V_{ij} \rightarrow V \times_U U_i \rightarrow V$  are flat and locally of finite presentation as compositions of such (Morphisms of Spaces, Lemmas 28.2, 30.3, 39.7, and 39.8). Hence these morphisms are open (Morphisms of Spaces, Lemma 30.6) and we see that  $|V| = \bigcup_{i \in I} \bigcup_{j \in J_i} f_{ij}(|V_{ij}|)$  is an open covering of  $|V|$ . Since  $|V|$  is quasi-compact, this covering has a finite refinement. Say  $V_{i_1 j_1}, \dots, V_{i_N j_N}$  do the job. Then  $\{V_{i_k j_k} \rightarrow V\}_{k=1, \dots, N}$  is a standard fpqc covering of  $V$  refining the family  $\{U_i \times_U V \rightarrow V\}$ . This finishes the proof.  $\square$

03MR **Lemma 9.3.** *Let  $S$  be a scheme. Let  $X$  be an algebraic space over  $S$ .*

- (1) *If  $X' \rightarrow X$  is an isomorphism then  $\{X' \rightarrow X\}$  is an fpqc covering of  $X$ .*
- (2) *If  $\{X_i \rightarrow X\}_{i \in I}$  is an fpqc covering and for each  $i$  we have an fpqc covering  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$ , then  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is an fpqc covering.*
- (3) *If  $\{X_i \rightarrow X\}_{i \in I}$  is an fpqc covering and  $X' \rightarrow X$  is a morphism of algebraic spaces then  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is an fpqc covering.*

**Proof.** Part (1) is clear. Consider  $g : X' \rightarrow X$  and  $\{X_i \rightarrow X\}_{i \in I}$  an fpqc covering as in (3). By Morphisms of Spaces, Lemma 30.4 the morphisms  $X' \times_X X_i \rightarrow X'$  are flat. If  $h' : Z \rightarrow X'$  is a morphism from an affine scheme towards  $X'$ , then set  $h = g \circ h' : Z \rightarrow X$ . The assumption on  $\{X_i \rightarrow X\}_{i \in I}$  means there exists a standard fpqc covering  $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$  and morphisms  $Z_j \rightarrow X_{i(j)}$  covering  $h$  for certain  $i(j) \in I$ . By the universal property of the fibre product we obtain morphisms  $Z_j \rightarrow X' \times_X X_{i(j)}$  over  $h'$  also. Hence  $\{X' \times_X X_i \rightarrow X'\}_{i \in I}$  is an fpqc covering. This proves (3).

Let  $\{X_i \rightarrow X\}_{i \in I}$  and  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$  be as in (2). Let  $h : Z \rightarrow X$  be a morphism from an affine scheme towards  $X$ . By assumption there exists a standard fpqc covering  $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$  and morphisms  $h_j : Z_j \rightarrow X_{i(j)}$  covering  $h$  for some indices  $i(j) \in I$ . By assumption there exist standard fpqc coverings  $\{Z_{j,l} \rightarrow Z_j\}_{l=1, \dots, n(j)}$  and morphisms  $Z_{j,l} \rightarrow X_{i(j)j(l)}$  covering  $h_j$  for some indices  $j(l) \in J_{i(j)}$ . By Topologies, Lemma 9.10 the family  $\{Z_{j,l} \rightarrow Z\}$  is a standard fpqc covering. Hence we conclude that  $\{X_{ij} \rightarrow X\}_{i \in I, j \in J_i}$  is an fpqc covering.  $\square$

03MS **Lemma 9.4.** *Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . Suppose that  $\{f_i : X_i \rightarrow X\}_{i \in I}$  is a family of morphisms of algebraic spaces with target  $X$ . Let  $U \rightarrow X$  be a surjective étale morphism from a scheme towards  $X$ . Then  $\{f_i : X_i \rightarrow X\}_{i \in I}$  is an fpqc covering of  $X$  if and only if  $\{U \times_X X_i \rightarrow U\}_{i \in I}$  is an fpqc covering of  $U$ .*

**Proof.** If  $\{X_i \rightarrow X\}_{i \in I}$  is an fpqc covering, then so is  $\{U \times_X X_i \rightarrow U\}_{i \in I}$  by Lemma 9.3. Assume that  $\{U \times_X X_i \rightarrow U\}_{i \in I}$  is an fpqc covering. Let  $h : Z \rightarrow X$  be a morphism from an affine scheme towards  $X$ . Then we see that  $U \times_X Z \rightarrow Z$  is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens  $W_1, \dots, W_t$  of  $U \times_X Z$  whose images cover  $Z$ . For each  $j$  we may apply the condition that  $\{U \times_X X_i \rightarrow U\}_{i \in I}$  is an fpqc covering to the morphism  $W_j \rightarrow U$ , and obtain a standard fpqc covering  $\{W_{jl} \rightarrow W_j\}$  which refines  $\{W_j \times_X X_i \rightarrow W_j\}_{i \in I}$ . Hence  $\{W_{jl} \rightarrow Z\}$  is a standard fpqc covering of  $Z$  (see Topologies, Lemma 9.10) which refines  $\{Z \times_X X_i \rightarrow Z\}$  and we win.  $\square$

0419 **Lemma 9.5.** *Let  $S$  be a scheme, and let  $X$  be an algebraic space over  $S$ . Suppose that  $\mathcal{U} = \{f_i : X_i \rightarrow X\}_{i \in I}$  is an fpqc covering of  $X$ . Then there exists a refinement  $\mathcal{V} = \{g_i : T_i \rightarrow X\}$  of  $\mathcal{U}$  which is an fpqc covering such that each  $T_i$  is a scheme.*

**Proof.** Omitted. Hint: For each  $i$  choose a scheme  $T_i$  and a surjective étale morphism  $T_i \rightarrow X_i$ . Then check that  $\{T_i \rightarrow X\}$  is an fpqc covering.  $\square$

To be continued...

## 10. Other chapters

Preliminaries	(41) Étale Morphisms of Schemes
(1) Introduction	Topics in Scheme Theory
(2) Conventions	(42) Chow Homology
(3) Set Theory	(43) Intersection Theory
(4) Categories	(44) Picard Schemes of Curves
(5) Topology	(45) Weil Cohomology Theories
(6) Sheaves on Spaces	(46) Adequate Modules
(7) Sites and Sheaves	(47) Dualizing Complexes
(8) Stacks	(48) Duality for Schemes
(9) Fields	(49) Discriminants and Differents
(10) Commutative Algebra	(50) de Rham Cohomology
(11) Brauer Groups	(51) Local Cohomology
(12) Homological Algebra	(52) Algebraic and Formal Geometry
(13) Derived Categories	(53) Algebraic Curves
(14) Simplicial Methods	(54) Resolution of Surfaces
(15) More on Algebra	(55) Semistable Reduction
(16) Smoothing Ring Maps	(56) Derived Categories of Varieties
(17) Sheaves of Modules	(57) Fundamental Groups of Schemes
(18) Modules on Sites	(58) Étale Cohomology
(19) Injectives	(59) Crystalline Cohomology
(20) Cohomology of Sheaves	(60) Pro-étale Cohomology
(21) Cohomology on Sites	(61) More Étale Cohomology
(22) Differential Graded Algebra	(62) The Trace Formula
(23) Divided Power Algebra	Algebraic Spaces
(24) Differential Graded Sheaves	(63) Algebraic Spaces
(25) Hypercoverings	(64) Properties of Algebraic Spaces
Schemes	(65) Morphisms of Algebraic Spaces
(26) Schemes	(66) Decent Algebraic Spaces
(27) Constructions of Schemes	(67) Cohomology of Algebraic Spaces
(28) Properties of Schemes	(68) Limits of Algebraic Spaces
(29) Morphisms of Schemes	(69) Divisors on Algebraic Spaces
(30) Cohomology of Schemes	(70) Algebraic Spaces over Fields
(31) Divisors	(71) Topologies on Algebraic Spaces
(32) Limits of Schemes	(72) Descent and Algebraic Spaces
(33) Varieties	(73) Derived Categories of Spaces
(34) Topologies on Schemes	(74) More on Morphisms of Spaces
(35) Descent	(75) Flatness on Algebraic Spaces
(36) Derived Categories of Schemes	(76) Groupoids in Algebraic Spaces
(37) More on Morphisms	(77) More on Groupoids in Spaces
(38) More on Flatness	(78) Bootstrap
(39) Groupoid Schemes	(79) Pushouts of Algebraic Spaces
(40) More on Groupoid Schemes	

Topics in Geometry	(98) Properties of Algebraic Stacks
(80) Chow Groups of Spaces	(99) Morphisms of Algebraic Stacks
(81) Quotients of Groupoids	(100) Limits of Algebraic Stacks
(82) More on Cohomology of Spaces	(101) Cohomology of Algebraic Stacks
(83) Simplicial Spaces	(102) Derived Categories of Stacks
(84) Duality for Spaces	(103) Introducing Algebraic Stacks
(85) Formal Algebraic Spaces	(104) More on Morphisms of Stacks
(86) Restricted Power Series	(105) The Geometry of Stacks
(87) Resolution of Surfaces Revisited	Topics in Moduli Theory
Deformation Theory	(106) Moduli Stacks
(88) Formal Deformation Theory	(107) Moduli of Curves
(89) Deformation Theory	Miscellany
(90) The Cotangent Complex	(108) Examples
(91) Deformation Problems	(109) Exercises
Algebraic Stacks	(110) Guide to Literature
(92) Algebraic Stacks	(111) Desirables
(93) Examples of Stacks	(112) Coding Style
(94) Sheaves on Algebraic Stacks	(113) Obsolete
(95) Criteria for Representability	(114) GNU Free Documentation License
(96) Artin's Axioms	(115) Auto Generated Index
(97) Quot and Hilbert Spaces	

### References

- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 21, Springer-Verlag, 1990.
- [Gro71] Alexander Grothendieck, *Revêtements étales et groupe fondamental (sga 1)*, Lecture notes in mathematics, vol. 224, Springer-Verlag, 1971.
- [Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, vol. 203, Springer-Verlag, 1971.
- [LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, 2000.