

COHOMOLOGY OF ALGEBRAIC STACKS

073P

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1. Introduction

073Q In this chapter we write about cohomology of algebraic stacks. This means in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled “Cohomology of Schemes” and “Cohomology of Algebraic Spaces”. The results in this chapter are different from those in [LMB00] mainly because we consistently use the “big sites”. Before reading this chapter please take a quick look at the chapter “Sheaves on Algebraic Stacks” in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 1.

2. Conventions and abuse of language

073R We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Notation

073S Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{\acute{e}tale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 4.1. (Think “big site”.) Correspondingly the structure sheaf of \mathcal{X} is a sheaf on \mathcal{X}_{fppf} . On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z , and in this case $X_{\acute{e}tale}$ indicates the small étale site of X (as defined in Topologies, Definition

4.8 or Properties of Spaces, Definition 18.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say “sheaf on \mathcal{X} ” or “sheaf of $\mathcal{O}_{\mathcal{X}}$ ” modules when we mean sheaf on \mathcal{X}_{fppf} or object of $Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$.

If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don’t have to mention which topology we are working with. The same isn’t true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \rightarrow \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack \mathcal{Y} . Let \mathcal{G} be a sheaf on \mathcal{Y}_τ for some topology τ . In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on \mathcal{S}_X (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \rightarrow \mathcal{Y}$. If $\tau = \text{étale}$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{\text{étale}}}$ to denote the restriction to the étale site of X , see Sheaves on Stacks, Section 21. If \mathcal{G} is an $\mathcal{O}_{\mathcal{Y}}$ -module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{\text{étale}}}$ instead.

4. Pullback of quasi-coherent modules

076W Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_{\mathcal{Y}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{X}}),$$

see Sheaves on Stacks, Lemma 11.2. In general this functor isn’t exact, but if f is flat then it is.

076X **Lemma 4.1.** *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_{\mathcal{Y}}) \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ is an exact functor.*

Proof. Choose a scheme V and a surjective smooth morphism $V \rightarrow \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \rightarrow V \times_{\mathcal{Y}} \mathcal{X}$. Then $U \rightarrow \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad} & V \\ \downarrow & \searrow f' & \downarrow \\ \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_U) & \longleftarrow & QCoh(\mathcal{O}_V) \\ \uparrow & & \uparrow \\ QCoh(\mathcal{O}_{\mathcal{X}}) & \longleftarrow & QCoh(\mathcal{O}_{\mathcal{Y}}) \end{array}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 14.1). Since f' is a flat morphism of schemes (by

our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V . Thus we win. \square

5. The key lemma

076Y The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two versions: one for the étale topology and one for the fppf topology.

076Z **Lemma 5.1.** *Let \mathcal{M} be a rule which associates to every algebraic stack \mathcal{X} a subcategory $\mathcal{M}_{\mathcal{X}}$ of $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ such that*

- (1) $\mathcal{M}_{\mathcal{X}}$ is a weak Serre subcategory of $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ (see Homology, Definition 9.1) for all algebraic stacks \mathcal{X} ,
- (2) for a smooth morphism of algebraic stacks $f : \mathcal{Y} \rightarrow \mathcal{X}$ the functor f^* maps $\mathcal{M}_{\mathcal{X}}$ into $\mathcal{M}_{\mathcal{Y}}$,
- (3) if $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object \mathcal{F} of $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ is in $\mathcal{M}_{\mathcal{X}}$ if and only if $f_i^* \mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i , and
- (4) if $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a morphism of algebraic stacks such that \mathcal{X} and \mathcal{Y} are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$.

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of algebraic stacks $R^i f_$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$. (Higher direct images computed in étale topology.)*

Proof. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let \mathcal{F} be an object of $\mathcal{M}_{\mathcal{Y}}$. Choose a surjective smooth morphism $\mathcal{U} \rightarrow \mathcal{X}$ where \mathcal{U} is representable by a scheme. By Sheaves on Stacks, Lemma 20.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of \mathcal{F} to $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U} \times_{\mathcal{X}} \mathcal{Y}}$ because the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \rightarrow \mathcal{X}$ by the projection $\mathcal{U} \times_{\mathcal{X}} \mathcal{Y} \rightarrow \mathcal{U}$. In other words, we may assume that \mathcal{X} is representable by a scheme. Using (3) once more, we see that the question is Zariski local on \mathcal{X} , hence we may assume that \mathcal{X} is representable by an affine scheme. Since f is quasi-compact this implies that also \mathcal{Y} is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \rightarrow \mathcal{Y}$ where \mathcal{V} is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 20.1. Recall that this is a first quadrant spectral sequence hence we may use the last part of Homology, Lemma 22.6. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_{\mathcal{Y}} \dots \times_{\mathcal{Y}} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g . Thus the sheaves $g_p^* \mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_{\mathcal{Y}} \dots \times_{\mathcal{Y}} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_{\mathcal{X}}$. The algebraic stacks \mathcal{V}_p are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 7.8. Of course each \mathcal{V}_p is representable by an algebraic space (the diagonal of the algebraic stack \mathcal{Y} is representable by algebraic spaces).

This reduces us to the case where \mathcal{Y} is representable by an algebraic space and \mathcal{X} is representable by an affine scheme.

In the situation where \mathcal{Y} is representable by an algebraic space and \mathcal{X} is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \rightarrow \mathcal{Y}$ where \mathcal{V} is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \rightarrow \mathcal{X}$. But in the current situation the algebraic stacks \mathcal{V}_p are representable by quasi-compact and quasi-separated schemes (because the diagonal of an algebraic space is representable by schemes).

Thus we may assume \mathcal{Y} is representable by a scheme and \mathcal{X} is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \rightarrow \mathcal{Y}$ where \mathcal{V} is representable by an affine scheme. In this case all the algebraic stacks \mathcal{V}_p are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume \mathcal{Y} is representable by a separated scheme and \mathcal{X} is representable by an affine scheme. Choose (yet again) a surjective smooth morphism $\mathcal{V} \rightarrow \mathcal{Y}$ where \mathcal{V} is representable by an affine scheme. In this case all the algebraic stacks \mathcal{V}_p are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. \square

Here is the version for the fppf topology.

0770 **Lemma 5.2.** *Let \mathcal{M} be a rule which associates to every algebraic stack \mathcal{X} a subcategory $\mathcal{M}_{\mathcal{X}}$ of $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ such that*

- (1) $\mathcal{M}_{\mathcal{X}}$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ for all algebraic stacks \mathcal{X} ,
- (2) for a smooth morphism of algebraic stacks $f : \mathcal{Y} \rightarrow \mathcal{X}$ the functor f^* maps $\mathcal{M}_{\mathcal{X}}$ into $\mathcal{M}_{\mathcal{Y}}$,
- (3) if $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object \mathcal{F} of $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ is in $\mathcal{M}_{\mathcal{X}}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i , and
- (4) if $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a morphism of algebraic stacks and \mathcal{X} and \mathcal{Y} are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$.

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of algebraic stacks $R^i f_$ maps $\mathcal{M}_{\mathcal{Y}}$ into $\mathcal{M}_{\mathcal{X}}$. (Higher direct images computed in fppf topology.)*

Proof. Identical to the proof of Lemma 5.1. \square

6. Locally quasi-coherent modules

075X Let \mathcal{X} be an algebraic stack. Let \mathcal{F} be a presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules. We can ask whether \mathcal{F} is *locally quasi-coherent*, see Sheaves on Stacks, Definition 11.5. Briefly, this means \mathcal{F} is an $\mathcal{O}_{\mathcal{X}}$ -module for the étale topology such that for any morphism $f : U \rightarrow \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{\text{étale}}}$ is quasi-coherent on $U_{\text{étale}}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$LQCoh(\mathcal{O}_{\mathcal{X}}) \subset \text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 11.8.

075Y **Lemma 6.1.** *Let \mathcal{X} be an algebraic stack. Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\acute{e}tale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is \mathcal{F} .*

Proof. We may replace each of the algebraic stacks \mathcal{X}_j by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 32.2). The pullback of \mathcal{F} to $(Sch/U_j)_{\acute{e}tale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 11.7. Then $f = \coprod f_j : U = \coprod U_j \rightarrow \mathcal{X}$ is a surjective smooth morphism. Let x be an object of \mathcal{X} . By Sheaves on Stacks, Lemma 18.10 there exists an étale covering $\{x_i \rightarrow x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{\acute{e}tale}$. This just means that x, x_i live over schemes V, V_i , that $\{V_i \rightarrow V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \rightarrow U$. The restriction $x_i^*\mathcal{F}|_{V_i, \acute{e}tale}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_i, \acute{e}tale$, see Sheaves on Stacks, Lemma 9.3. Hence $x^*\mathcal{F}|_{V, \acute{e}tale}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_i, \acute{e}tale$ for each i . This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 29.6. \square

075Z **Lemma 6.2.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let \mathcal{F} be a locally quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module on $\mathcal{X}_{\acute{e}tale}$. Then $R^i f_*\mathcal{F}$ (computed in the étale topology) is locally quasi-coherent on $\mathcal{Y}_{\acute{e}tale}$.*

Proof. We will use Lemma 5.1 to prove this. We will check its assumptions (1) – (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 11.8. Part (3) follows from Lemma 6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of algebraic stacks such that \mathcal{X} and \mathcal{Y} are representable by affine schemes X and Y . Choose any object y of \mathcal{Y} lying over a scheme V . For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V . Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ \mathcal{V} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

Thus \mathcal{Z} is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 21.3 we have

$$R^i f_*\mathcal{F}|_{V, \acute{e}tale} = R^i f'_{small,*}(g^*\mathcal{F}|_{Z, \acute{e}tale})$$

The right hand side is a quasi-coherent sheaf on $V_{\acute{e}tale}$ by Cohomology of Spaces, Lemma 3.1. This implies the left hand side is quasi-coherent which is what we had to prove. \square

07AP **Lemma 6.3.** *Let \mathcal{X} be an algebraic stack. Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on \mathcal{X}_{fppf} . If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is \mathcal{F} .*

Proof. First, suppose there is a morphism $a : \mathcal{U} \rightarrow \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow a_*a^*\mathcal{F} \rightarrow b_*b^*\mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{X}$, see Sheaves on Stacks, Proposition 18.7 and Lemma 18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 11.7). The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent by Lemma 6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that \mathcal{F} is locally quasi-coherent, see Sheaves on Stacks, Lemma 11.8.

We are going to reduce the proof of the general case the situation in the first paragraph. Let x be an object of \mathcal{X} lying over the scheme U . We have to show that $\mathcal{F}|_{U_{\acute{e}tale}}$ is a quasi-coherent \mathcal{O}_U -module. It suffices to do this (Zariski) locally on U , hence we may assume that U is affine. By Morphisms of Stacks, Lemma 26.14 there exists an fppf covering $\{a_i : U_i \rightarrow U\}$ such that each $x \circ a_i$ factors through some f_j . Hence $a_i^*\mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering we may assume $\{U_i \rightarrow U\}_{i=1, \dots, n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \dots \amalg U_n \rightarrow U$ is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{\acute{e}tale}}$ is quasi-coherent. \square

7. Flat comparison maps

0760 Let \mathcal{X} be an algebraic stack and let \mathcal{F} be an object of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$. Given an object x of \mathcal{X} lying over the scheme U the restriction $\mathcal{F}|_{U_{\acute{e}tale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U , see Sheaves on Stacks, Definition 9.2. Next, let $\varphi : x \rightarrow x'$ be a morphism of \mathcal{X} lying over a morphism of schemes $f : U \rightarrow U'$. Thus a 2-commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ & \searrow x & \swarrow x' \\ & & \mathcal{X} \end{array}$$

Associated to φ we obtain a comparison map between restrictions

0761 (7.0.1)
$$c_{\varphi} : f_{small}^*(\mathcal{F}|_{U'_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}}$$

see Sheaves on Stacks, Equation (9.4.1). In this situation we can consider the following property of \mathcal{F} .

0762 **Definition 7.1.** Let \mathcal{X} be an algebraic stack and let \mathcal{F} in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$. We say \mathcal{F} has the *flat base change property*¹ if and only if c_{φ} is an isomorphism whenever f is flat.

Here is a lemma with some properties of this notion.

0764 **Lemma 7.2.** *Let \mathcal{X} be an algebraic stack. Let \mathcal{F} be an $\mathcal{O}_{\mathcal{X}}$ -module on $\mathcal{X}_{\acute{e}tale}$.*

- (1) *If \mathcal{F} has the flat base change property then for any morphism $g : \mathcal{Y} \rightarrow \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.*
- (2) *The full subcategory of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ consisting of modules with the flat base change property is a weak Serre subcategory.*
- (3) *Let $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does \mathcal{F} .*

¹This may be nonstandard notation.

- (4) *The category of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\acute{e}tale}$ with the flat base change property has colimits and they agree with colimits in $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$.*

Proof. Let $g : \mathcal{Y} \rightarrow \mathcal{X}$ be as in (1). Let y be an object of \mathcal{Y} lying over a scheme V . By Sheaves on Stacks, Lemma 9.3 we have $(g^*\mathcal{F})|_{V_{\acute{e}tale}} = \mathcal{F}|_{V_{\acute{e}tale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on \mathcal{Y} is a special case of a comparison map for the sheaf \mathcal{F} on \mathcal{X} , see Sheaves on Stacks, Lemma 9.3. In this way (1) is clear.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 9.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f_{small}^* is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{\acute{e}tale}}$ are exact (because we are working in the étale topology). Finally, if $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ and the outer two sheaves have the flat base change property then the middle one does as well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 32.2) we may assume that each \mathcal{X}_i is representable by a scheme. Let $\varphi : x \rightarrow x'$ be a morphism of \mathcal{X} lying over a flat morphism $a : U \rightarrow U'$ of schemes. By Sheaves on Stacks, Lemma 18.10 there exists a jointly surjective family of étale morphisms $U'_i \rightarrow U'$ such that $U' \rightarrow U' \rightarrow \mathcal{X}$ factors through \mathcal{X}_i . Thus we obtain commutative diagrams

$$\begin{array}{ccccc} U_i = U \times_{U'} U'_i & \xrightarrow{a_i} & U'_i & \xrightarrow{x'_i} & \mathcal{X}_i \\ \downarrow & & \downarrow & & \downarrow f_i \\ U & \xrightarrow{a} & U' & \xrightarrow{x'} & \mathcal{X} \end{array}$$

Note that each a_i is a flat morphism of schemes as a base change of a . Denote $\psi_i : x_i \rightarrow x'_i$ the morphism of \mathcal{X}_i lying over a_i with target x'_i . By assumption the comparison maps $c_{\psi_i} : (a_i)_{small}^*(f_i^*\mathcal{F}|_{(U'_i)_{\acute{e}tale}}) \rightarrow f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ is an isomorphism. Because the vertical arrows $U'_i \rightarrow U'$ and $U_i \rightarrow U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U'_i)_{\acute{e}tale}}$ and $f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ are the restrictions of $\mathcal{F}|_{U'_{\acute{e}tale}}$ and $\mathcal{F}|_{U_{\acute{e}tale}}$ and the map c_{ψ_i} is the restriction of c_{φ} to $(U_i)_{\acute{e}tale}$, see Sheaves on Stacks, Lemma 9.3. Since $\{U_i \rightarrow U\}$ is an étale covering, this implies that the comparison map c_{φ} is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \rightarrow Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each \mathcal{F}_i has the flat base change property. Recall that $\text{colim}_i \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(\text{colim}_i \mathcal{F}_i)|_{U_{\acute{e}tale}} = \text{colim}_i \mathcal{F}_i|_{U_{\acute{e}tale}}$$

As f_{small}^* commutes with colimits (as a left adjoint) we see that (4) holds. \square

0765 **Lemma 7.3.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let \mathcal{F} be an object of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ which is locally quasi-coherent and has the flat base change property. Then each $R^i f_* \mathcal{F}$ (computed in the étale topology) has the flat base change property.*

Proof. We will use Lemma 5.1 to prove this. For every algebraic stack \mathcal{X} let $\mathcal{M}_{\mathcal{X}}$ denote the full subcategory of $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) – (4) of Lemma 5.1 the lemma will follow. Properties (1), (2), and (3) follow from Sheaves on Stacks, Lemmas 11.7 and 11.8 and Lemmas 6.1 and 7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of algebraic stacks such that \mathcal{X} and \mathcal{Y} are representable by affine schemes X and Y . In this case, suppose that $\psi : y \rightarrow y'$ is a morphism of \mathcal{Y} lying over a flat morphism $b : V \rightarrow V'$ of schemes. For clarity denote $\mathcal{V} = (\text{Sch}/V)_{fppf}$ and $\mathcal{V}' = (\text{Sch}/V')_{fppf}$ the corresponding algebraic stacks. Consider the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{a} & \mathcal{Z}' & \xrightarrow{x'} & \mathcal{X} \\ f'' \downarrow & & f' \downarrow & & \downarrow f \\ \mathcal{V} & \xrightarrow{b} & \mathcal{V}' & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated – even affine) we see that \mathcal{Z} and \mathcal{Z}' are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since \mathcal{F} has the flat base change property we see that

$$a_{small}^*(\mathcal{F}|_{\mathcal{Z}'_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{\mathcal{Z}_{\acute{e}tale}}$$

is an isomorphism. Moreover,

$$R^i f_* \mathcal{F}|_{V'_{\acute{e}tale}} = R^i (f')_{small,*}(\mathcal{F}|_{\mathcal{Z}'_{\acute{e}tale}})$$

and

$$R^i f_* \mathcal{F}|_{V_{\acute{e}tale}} = R^i (f'')_{small,*}(\mathcal{F}|_{\mathcal{Z}_{\acute{e}tale}})$$

by Sheaves on Stacks, Lemma 21.3. Hence we see that the comparison map

$$c_{\psi} : b_{small}^*(R^i f_* \mathcal{F}|_{V'_{\acute{e}tale}}) \longrightarrow R^i f_* \mathcal{F}|_{V_{\acute{e}tale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 11.2. Thus $R^i f_* \mathcal{F}$ has the flat base change property. Since $R^i f_* \mathcal{F}$ is locally quasi-coherent by Lemma 6.2 we win. \square

0771 **Proposition 7.4.** *Summary of results on locally quasi-coherent modules having the flat base change property.*

- (1) *Let \mathcal{X} be an algebraic stack. If \mathcal{F} is an object of $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ which is locally quasi-coherent and has the flat base change property, then \mathcal{F} is a sheaf for the fppf topology, i.e., it is an object of $\text{Mod}(\mathcal{O}_{\mathcal{X}})$.*
- (2) *The category of modules which are locally quasi-coherent and have the flat base change property is a weak Serre subcategory $\mathcal{M}_{\mathcal{X}}$ of both $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ and $\text{Mod}(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$.*
- (3) *Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ induces a functor $f^* : \mathcal{M}_{\mathcal{Y}} \rightarrow \mathcal{M}_{\mathcal{X}}$.*
- (4) *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and \mathcal{F} is an object of $\mathcal{M}_{\mathcal{X}}$, then*
 - (a) *the derived direct image $Rf_* \mathcal{F}$ and the higher direct images $R^i f_* \mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and*

- (b) each $R^i f_* \mathcal{F}$ is an object of $\mathcal{M}_{\mathcal{Y}}$.
- (5) The category $\mathcal{M}_{\mathcal{X}}$ has colimits and they agree with colimits in $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ as well as in $\text{Mod}(\mathcal{O}_{\mathcal{X}})$.

Proof. Part (1) is Sheaves on Stacks, Lemma 22.1.

Part (2) for the embedding $\mathcal{M}_{\mathcal{X}} \subset \text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ we have seen in the proof of Lemma 7.3. Let us prove (2) for the embedding $\mathcal{M}_{\mathcal{X}} \subset \text{Mod}(\mathcal{O}_{\mathcal{X}})$. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism between objects of $\mathcal{M}_{\mathcal{X}}$. Since $\text{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\text{Ker}(\varphi)$ is in $\mathcal{M}_{\mathcal{X}}$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in the étale topology. However, this étale cokernel is in $\mathcal{M}_{\mathcal{X}}$ hence an fppf sheaf by (1) and we see that the cokernel is in $\mathcal{M}_{\mathcal{X}}$. Finally, suppose that

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

is an exact sequence in $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ (i.e., using the fppf topology) with $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{M}_{\mathcal{X}}$. In order to show that \mathcal{F}_3 is an object of $\mathcal{M}_{\mathcal{X}}$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H_{\text{fppf}}^1(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of \mathcal{X}). This is true because $H_{\text{fppf}}^1(x, \mathcal{F}_1) = H_{\text{étale}}^1(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 22.2 and because of locality of cohomology, see Cohomology on Sites, Lemma 8.3. This proves (2).

Part (3) follows from Lemma 7.2 and Sheaves on Stacks, Lemma 11.7.

Part (4)(b) for $R^i f_* \mathcal{F}$ computed in the étale cohomology follows from Lemma 7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 22.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 11.8 and Lemma 7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1). \square

07AQ **Lemma 7.5.** *Let \mathcal{X} be an algebraic stack. With $\mathcal{M}_{\mathcal{X}}$ the category of locally quasi-coherent modules with the flat base change property.*

- (1) *Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\text{étale}}$. If each $f_j^{-1} \mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_j}$, then \mathcal{F} is in $\mathcal{M}_{\mathcal{X}}$.*
- (2) *Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\text{fppf}}$. If each $f_j^{-1} \mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_j}$, then \mathcal{F} is in $\mathcal{M}_{\mathcal{X}}$.*

Proof. Part (1) follows from a combination of Lemmas 6.1 and 7.2. The proof of (2) is analogous to the proof of Lemma 6.3. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\text{fppf}}$.

First, suppose there is a morphism $a : \mathcal{U} \rightarrow \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^* \mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow a_* a^* \mathcal{F} \rightarrow b_* b^* \mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{X}$, see Sheaves on Stacks, Proposition 18.7 and Lemma 18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 7.4. The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 7.4. We conclude that \mathcal{F} is locally quasi-coherent and has the flat base change property by Proposition 7.4.

Choose a scheme U and a surjective smooth morphism $x : U \rightarrow \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Again by part (1) it suffices to do this (Zariski) locally on U , hence we may assume that U is affine. By Morphisms of Stacks, Lemma 26.14 there exists an fppf covering $\{a_i : U_i \rightarrow U\}$ such that each $x \circ a_i$ factors through some f_j . Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \rightarrow U\}_{i=1, \dots, n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \dots \amalg U_n \rightarrow U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. \square

8. Parasitic modules

0772 The following definition is compatible with Descent, Definition 9.1.

0773 **Definition 8.1.** Let \mathcal{X} be an algebraic stack. A presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules \mathcal{F} is *parasitic* if we have $\mathcal{F}(x) = 0$ for any object x of \mathcal{X} which lies over a scheme U such that the corresponding morphism $x : U \rightarrow \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

0774 **Lemma 8.2.** *Let \mathcal{X} be an algebraic stack. Let \mathcal{F} be a presheaf of $\mathcal{O}_{\mathcal{X}}$ -modules.*

- (1) *If \mathcal{F} is parasitic and $g : \mathcal{Y} \rightarrow \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.*
- (2) *For $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$ we have*
 - (a) *the τ sheafification of a parasitic presheaf of modules is parasitic, and*
 - (b) *the full subcategory of $\text{Mod}(\mathcal{X}_{\tau}, \mathcal{O}_{\mathcal{X}})$ consisting of parasitic modules is a Serre subcategory.*
- (3) *Suppose \mathcal{F} is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is \mathcal{F} .*
- (4) *Suppose \mathcal{F} is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is \mathcal{F} .*

Proof. To see part (1) let y be an object of \mathcal{Y} which lies over a scheme V such that the corresponding morphism $y : V \rightarrow \mathcal{Y}$ is flat. Then $g(y) : V \rightarrow \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 24.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \rightarrow x\}$ is a τ -covering of \mathcal{X} , then each of the morphisms $x_i \rightarrow x$ lies over a flat morphism of schemes. Hence if x lies over a

scheme U such that $x : U \rightarrow \mathcal{X}$ is flat, so do all of the objects x_i . Hence the presheaf \mathcal{F}^+ (see Sites, Section 10) is parasitic if the presheaf \mathcal{F} is parasitic. This proves (2)(a) as the sheafification of \mathcal{F} is $(\mathcal{F}^+)^+$.

Let \mathcal{F} be a parasitic τ -module. It is immediate from the definitions that any submodule of \mathcal{F} is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient \mathcal{F}/\mathcal{F}' is a parasitic module by (2)(a). Finally, we have to show that given a short exact sequence $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ with \mathcal{F}_1 and \mathcal{F}_3 parasitic, then \mathcal{F}_2 is parasitic. This follows immediately on evaluating on x lying over a scheme flat over \mathcal{X} . This proves (2)(b), see Homology, Lemma 9.2.

Let $f_i : \mathcal{X}_i \rightarrow \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of \mathcal{X} which lies over a scheme U such that $x : U \rightarrow \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \rightarrow U \times_{x, \mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \rightarrow \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \rightarrow x\}$ is a covering for the smooth topology on \mathcal{X} . Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \rightarrow x\}_{i \in I}$ in \mathcal{X} we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as the étale topology, see More on Morphisms, Lemma 34.7. Namely, $\{x_i \rightarrow x\}_{i \in I}$ lies over a smooth covering $\{U_i \rightarrow U\}_{i \in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \rightarrow U\}_{j \in J}$ which refines $\{U_i \rightarrow U\}_{i \in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \rightarrow x\}$ is an étale covering in \mathcal{X} refining $\{x_i \rightarrow x\}_{i \in I}$. This means the map $\mathcal{F}(x) \rightarrow \prod_{j \in J} \mathcal{F}(x'_j)$, which is injective as \mathcal{F} is a sheaf in the étale topology, factors through $\mathcal{F}(x) \rightarrow \prod_{i \in I} \mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). \square

Parasitic modules are preserved under absolutely any pushforward.

0775 **Lemma 8.3.** *Let $\tau \in \{\text{étale}, \text{fppf}\}$. Let \mathcal{X} be an algebraic stack. Let \mathcal{F} be a parasitic object of $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_{\mathcal{X}})$.*

- (1) $H_\tau^i(\mathcal{X}, \mathcal{F}) = 0$ for all i .
- (2) *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^i f_* \mathcal{F}$ (computed in τ -topology) is a parasitic object of $\text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_{\mathcal{Y}})$.*

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 20.2 we see that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$y \mapsto H_\tau^i(V \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F})$$

Here y is a typical object of \mathcal{Y} lying over the scheme V . By Lemma 8.2 it suffices to show that these cohomology groups are zero when $y : V \rightarrow \mathcal{Y}$ is flat. Note that $\text{pr} : V \times_{y, \mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$ is flat as a base change of y . Hence by Lemma 8.2 we see that $\text{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 19.1 to reduce this to the case where \mathcal{X} is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 8.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of the key Lemma

5.1) we reduce to the case where the algebraic stack \mathcal{X} is representable by a scheme X . Then $H_r^i(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 9.2. \square

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ and $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{O}_{\mathcal{X}})$ aren't exact in general.

0776 **Lemma 8.4.** *Let \mathcal{X} be an algebraic stack. Let \mathcal{F}^\bullet be an exact complex in $QCoh(\mathcal{O}_{\mathcal{X}})$. Then the cohomology sheaves of \mathcal{F}^\bullet in either the étale or the fppf topology are parasitic $\mathcal{O}_{\mathcal{X}}$ -modules.*

Proof. Let $x : U \rightarrow \mathcal{X}$ be a flat morphism where U is a scheme. Then $x^*\mathcal{F}^\bullet$ is exact by Lemma 4.1. Hence the restriction $x^*\mathcal{F}^\bullet|_{U_{\acute{e}tale}}$ is exact which is what we had to prove. \square

9. Quasi-coherent modules, I

0777 We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 14. This fact is the basis for the following.

0778 **Lemma 9.1.** *Let \mathcal{X} be an algebraic stack. Let $\mathcal{M}_{\mathcal{X}}$ be the category of locally quasi-coherent modules with the flat base change property, see Proposition 7.4. The inclusion functor $i : QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathcal{M}_{\mathcal{X}}$ has a right adjoint*

$$Q : \mathcal{M}_{\mathcal{X}} \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \rightarrow \mathcal{X}$. Set $R = U \times_{\mathcal{X}} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 16.2. We may and do replace \mathcal{X} by $[U/R]$. In the proof of Sheaves on Stacks, Proposition 13.1 we constructed a functor

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_{\mathcal{X}}).$$

The construction of the inverse functor in the proof of Sheaves on Stacks, Proposition 13.1 works for objects of $\mathcal{M}_{\mathcal{X}}$ and induces a functor

$$q_2 : \mathcal{M}_{\mathcal{X}} \longrightarrow QCoh(U, R, s, t, c).$$

Namely, if \mathcal{F} is an object of $\mathcal{M}_{\mathcal{X}}$ then we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{\acute{e}tale}}, \alpha)$$

where α is the isomorphism

$$t_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}}) \rightarrow t^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \rightarrow s^*f^*\mathcal{F}|_{R_{\acute{e}tale}} \rightarrow s_{small}^*(f^*\mathcal{F}|_{U_{\acute{e}tale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because \mathcal{F} is locally quasi-coherent (and we used the flat base change property in the construction of the descent datum α). We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 12.1)

holds. We define $Q = q_1 \circ q_2$. Let \mathcal{F} be an object of $\mathcal{M}_{\mathcal{X}}$ and let \mathcal{G} be an object of $QCoh(\mathcal{O}_{\mathcal{X}})$. We have

$$\begin{aligned} \text{Mor}_{\mathcal{M}_{\mathcal{X}}}(i(\mathcal{G}), \mathcal{F}) &= \text{Mor}_{QCoh(U, R, s, t, c)}(q_2(\mathcal{G}), q_2(\mathcal{F})) \\ &= \text{Mor}_{QCoh(\mathcal{O}_{\mathcal{X}})}(\mathcal{G}, Q(\mathcal{F})) \end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 13.2 and the second equality holds because q_1 and q_2 are inverse equivalences of categories. The assertion $Q \circ i \cong \text{id}$ is a formal consequence of the fact that i is fully faithful. \square

0779 **Lemma 9.2.** *Let \mathcal{X} be an algebraic stack. Let $Q : \mathcal{M}_{\mathcal{X}} \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ be the functor constructed in Lemma 9.1.*

- (1) *The kernel of Q is exactly the collection of parasitic objects of $\mathcal{M}_{\mathcal{X}}$.*
- (2) *For any object \mathcal{F} of $\mathcal{M}_{\mathcal{X}}$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \rightarrow \mathcal{F}$ are parasitic.*
- (3) *The functor Q is exact.*

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 9.1. Let \mathcal{F} be an object of $\mathcal{M}_{\mathcal{X}}$. It is clear from the proof of Lemma 9.1 that \mathcal{F} is in the kernel of Q if and only if $\mathcal{F}|_{U_{\acute{e}tale}} = 0$. In particular, if \mathcal{F} is parasitic then \mathcal{F} is in the kernel. Next, let $x : V \rightarrow \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_{\mathcal{X}} U$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & V \\ p \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \rightarrow U$ is flat and the projection $q : W \rightarrow V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption \mathcal{F} has the flat base change property so that we obtain $p_{small}^* \mathcal{F}|_{U_{\acute{e}tale}} \cong q_{small}^* \mathcal{F}|_{V_{\acute{e}tale}}$. Thus if \mathcal{F} is in the kernel of Q , then $\mathcal{F}|_{V_{\acute{e}tale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \rightarrow \mathcal{F}$ becomes an isomorphism after restricting to $U_{\acute{e}tale}$.

To see part (3) note that Q is left exact as a right adjoint. Suppose that $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is a short exact sequence in $\mathcal{M}_{\mathcal{X}}$. Let $\mathcal{E} = \text{Coker}(Q(\mathcal{G}) \rightarrow Q(\mathcal{H}))$ in $QCoh(\mathcal{O}_{\mathcal{X}})$. Since $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathcal{M}_{\mathcal{X}}$ is a left adjoint it is right exact. Hence we see that $Q(\mathcal{G}) \rightarrow Q(\mathcal{H}) \rightarrow \mathcal{E} \rightarrow 0$ is exact in $\mathcal{M}_{\mathcal{X}}$. Using Lemma 8.4 we find that the top row of the following commutative diagram has parasitic cohomology sheaves at $Q(\mathcal{F})$ and $Q(\mathcal{G})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\mathcal{F}) & \longrightarrow & Q(\mathcal{G}) & \longrightarrow & Q(\mathcal{H}) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & a \downarrow & & b \downarrow & & c \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 & & \end{array}$$

The bottom row is exact and the vertical arrows a, b, c have parasitic kernel and cokernels by part (2). It follows that \mathcal{E} is parasitic: in the quotient category of $Mod(\mathcal{O}_{\mathcal{X}})/\text{Parasitic}$ (see Homology, Lemma 9.6 and Lemma 8.2) we see that a, b, c are isomorphisms and that the top row becomes exact. As it is also quasi-coherent, we conclude that \mathcal{E} is zero because $\mathcal{E} = Q(\mathcal{E}) = 0$ by part (1). \square

10. Pushforward of quasi-coherent modules

070A Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : \text{Mod}(\mathcal{O}_{\mathcal{X}}) \longrightarrow \text{Mod}(\mathcal{O}_{\mathcal{Y}})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}_k^2 \setminus \{0\} \longrightarrow \mathbf{A}_k^2 = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (\text{Sch}/X)_{fppf} \rightarrow (\text{Sch}/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_{\mathcal{X}}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_{\mathcal{X}}$ is quasi-coherent on \mathcal{Y} then we would have $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$. However, consider $T = \text{Spec}(k) \rightarrow \mathbf{A}_k^2 = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_{\mathcal{X}}) = 0$ because $X \times_Y T = \emptyset$ whereas $\Gamma(T, \mathcal{O}_{\mathcal{Y}}) = k$. On the positive side, we know from Cohomology of Schemes, Lemma 5.2 that for any flat morphism $T \rightarrow Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_{\mathcal{X}}) = \Gamma(T, \mathcal{O}_{\mathcal{Y}})$ (this uses that j is quasi-compact and quasi-separated).

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

- (1) f_* does preserve locally quasi-coherent modules (Lemma 6.2),
- (2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 7.3), and
- (3) locally quasi-coherent $\mathcal{O}_{\mathcal{Y}}$ -modules with the flat base change property give rise to quasi-coherent modules on a presentation of \mathcal{Y} and hence quasi-coherent modules on \mathcal{Y} , see Sheaves on Stacks, Section 14.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_{\mathcal{X}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{Y}})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_{\mathcal{Y}}) \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in \text{Ob}(\mathcal{Y})$ such that the associated 1-morphism $y : V \rightarrow \mathcal{Y}$ is flat, see (insert future reference here). Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$. However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

077A **Proposition 10.1.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_{\mathcal{Y}}) \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ has a right adjoint*

$$f_{QCoh,*} : QCoh(\mathcal{O}_{\mathcal{X}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{Y}})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathcal{M}_{\mathcal{X}} \xrightarrow{f_*} \mathcal{M}_{\mathcal{Y}} \xrightarrow{Q} QCoh(\mathcal{O}_{\mathcal{Y}})$$

where the functors f_* and Q are as in Proposition 7.4 and Lemma 9.1. Moreover, if we define $R^i f_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathcal{M}_{\mathcal{X}} \xrightarrow{R^i f_*} \mathcal{M}_{\mathcal{Y}} \xrightarrow{Q} QCoh(\mathcal{O}_{\mathcal{Y}})$$

then the sequence of functors $\{R^i f_{QCoh,*}\}_{i \geq 0}$ forms a cohomological δ -functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let \mathcal{F} be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module and let \mathcal{G} be a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$ -module. Then we have

$$\begin{aligned} \text{Mor}_{QCoh(\mathcal{O}_{\mathcal{X}})}(f^*\mathcal{G}, \mathcal{F}) &= \text{Mor}_{\mathcal{M}_{\mathcal{Y}}}(\mathcal{G}, f_*\mathcal{F}) \\ &= \text{Mor}_{QCoh(\mathcal{O}_{\mathcal{Y}})}(\mathcal{G}, Q(f_*\mathcal{F})) \\ &= \text{Mor}_{QCoh(\mathcal{O}_{\mathcal{Y}})}(\mathcal{G}, f_{QCoh,*}\mathcal{F}) \end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 7.4 we see that $f_*\mathcal{F}$ is an object of $\mathcal{M}_{\mathcal{Y}}$ (and can be computed in either the fppf or étale topology) and we obtain the second equality by Lemma 9.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^i f_{QCoh,*}\}_{i \geq 0}$ is a cohomological δ -functor as defined in Homology, Definition 11.1 let

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a short exact sequence of $QCoh(\mathcal{O}_{\mathcal{X}})$. This sequence may not be an exact sequence in $Mod(\mathcal{O}_{\mathcal{X}})$ but we know that it is up to parasitic modules, see Lemma 8.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{P}_1 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{I}_2 \rightarrow 0 \\ 0 &\rightarrow \mathcal{I}_2 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{Q}_2 \rightarrow 0 \\ 0 &\rightarrow \mathcal{P}_2 \rightarrow \mathcal{Q}_2 \rightarrow \mathcal{I}_3 \rightarrow 0 \\ 0 &\rightarrow \mathcal{I}_3 \rightarrow \mathcal{F}_3 \rightarrow \mathcal{P}_3 \rightarrow 0 \end{aligned}$$

of $Mod(\mathcal{O}_{\mathcal{X}})$ with \mathcal{P}_i parasitic. Note that each of the sheaves $\mathcal{P}_j, \mathcal{I}_j, \mathcal{Q}_j$ is an object of $\mathcal{M}_{\mathcal{X}}$, see Proposition 7.4. Applying $R^i f_*$ we obtain long exact sequences

$$\begin{aligned} 0 &\rightarrow f_*\mathcal{P}_1 \rightarrow f_*\mathcal{F}_1 \rightarrow f_*\mathcal{I}_2 \rightarrow R^1 f_*\mathcal{P}_1 \rightarrow \dots \\ 0 &\rightarrow f_*\mathcal{I}_2 \rightarrow f_*\mathcal{F}_2 \rightarrow f_*\mathcal{Q}_2 \rightarrow R^1 f_*\mathcal{I}_2 \rightarrow \dots \\ 0 &\rightarrow f_*\mathcal{P}_2 \rightarrow f_*\mathcal{Q}_2 \rightarrow f_*\mathcal{I}_3 \rightarrow R^1 f_*\mathcal{P}_2 \rightarrow \dots \\ 0 &\rightarrow f_*\mathcal{I}_3 \rightarrow f_*\mathcal{F}_3 \rightarrow f_*\mathcal{P}_3 \rightarrow R^1 f_*\mathcal{I}_3 \rightarrow \dots \end{aligned}$$

where the terms are objects of $\mathcal{M}_{\mathcal{Y}}$ by Proposition 7.4. By Lemma 8.3 the sheaves $R^i f_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q , see Lemma 9.2. Since Q is exact the maps

$$Q(R^i f_*\mathcal{F}_3) \cong Q(R^i f_*\mathcal{I}_3) \cong Q(R^i f_*\mathcal{Q}_2) \rightarrow Q(R^{i+1} f_*\mathcal{I}_2) \cong Q(R^{i+1} f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^i f_{QCoh,*}\}_{i \geq 0}$ into a cohomological δ -functor. \square

0782 **Lemma 10.2.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let \mathcal{F} be a quasi-coherent sheaf on \mathcal{X} . Then there exists a spectral sequence with E_2 -page*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 15.5 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^q f_{QCoh,*}\mathcal{F} \rightarrow R^q f_*\mathcal{F}$$

are parasitic modules on \mathcal{Y} (Lemma 9.2) hence have vanishing cohomology (Lemma 8.3). It follows formally that $H^p(\mathcal{Y}, R^q f_{QCoh,*} \mathcal{F}) = H^p(\mathcal{Y}, R^q f_* \mathcal{F})$ and we win. \square

0783 **Lemma 10.3.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let \mathcal{F} be a quasi-coherent sheaf on \mathcal{X} . Then there exists a spectral sequence with E_2 -page*

$$E_2^{p,q} = R^p g_{QCoh,*} (R^q f_{QCoh,*} \mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*} \mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 15.7 the Leray spectral sequence with

$$E_2^{p,q} = R^p g_* (R^q f_* \mathcal{F})$$

converges to $R^{p+q}(g \circ f)_* \mathcal{F}$. By the results of Proposition 7.4 all the terms of this spectral sequence are objects of $\mathcal{M}_{\mathcal{Z}}$. Applying the exact functor $Q_{\mathcal{Z}} : \mathcal{M}_{\mathcal{Z}} \rightarrow QCoh(\mathcal{O}_{\mathcal{Z}})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_{\mathcal{Z}})$ covering to $R^{p+q}(g \circ f)_{QCoh,*} \mathcal{F}$. Hence the result follows if we can show that

$$Q_{\mathcal{Z}}(R^p g_* (R^q f_* \mathcal{F})) = Q_{\mathcal{Z}}(R^p g_* (Q_{\mathcal{X}}(R^q f_* \mathcal{F})))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_{\mathcal{X}}(R^q f_* \mathcal{F}) \longrightarrow R^q f_* \mathcal{F}$$

are parasitic (Lemma 9.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 8.3). \square

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 19 and 20.

0784 **Proposition 10.4.** *Let $f : \mathcal{U} \rightarrow \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let \mathcal{F} be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module. Then there is a spectral sequence*

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^* \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{X}$ ($p+1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 19.1. \square

0785 **Proposition 10.5.** *Let $f : \mathcal{U} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \rightarrow \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that*

- (1) f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and
- (2) g is quasi-compact and quasi-separated.

If \mathcal{F} is in $QCoh(\mathcal{O}_{\mathcal{X}})$ then there is a spectral sequence

$$E_2^{p,q} = R^q (g \circ f_p)_{QCoh,*} f_p^* \mathcal{F} \Rightarrow R^{p+q} g_{QCoh,*} \mathcal{F}$$

in $QCoh(\mathcal{O}_{\mathcal{Y}})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q (g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q (g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

by Sheaves on Stacks, Proposition 20.1. Applying the exact functor $Q_{\mathcal{Y}} : \mathcal{M}_{\mathcal{Y}} \rightarrow QCoh(\mathcal{O}_{\mathcal{Y}})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_{\mathcal{Y}})$. \square

11. The lisse-étale and the flat-fppf sites

0786 In the book [LMB00] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 52 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the Stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

0787 **Definition 11.1.** Let \mathcal{X} be an algebraic stack.

- (1) The *lisse-étale site* of \mathcal{X} is the full subcategory $\mathcal{X}_{\text{lisse},\text{étale}}$ ² of \mathcal{X} whose objects are those $x \in \text{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \rightarrow \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{\text{lisse},\text{étale}}$ is a family of morphisms $\{x_i \rightarrow x\}_{i \in I}$ of $\mathcal{X}_{\text{lisse},\text{étale}}$ which forms a covering of $\mathcal{X}_{\text{étale}}$.
- (2) The *flat-fppf site* of \mathcal{X} is the full subcategory $\mathcal{X}_{\text{flat},\text{fppf}}$ of \mathcal{X} whose objects are those $x \in \text{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \rightarrow \mathcal{X}$ is flat. A covering of $\mathcal{X}_{\text{flat},\text{fppf}}$ is a family of morphisms $\{x_i \rightarrow x\}_{i \in I}$ of $\mathcal{X}_{\text{flat},\text{fppf}}$ which forms a covering of $\mathcal{X}_{\text{fppf}}$.

We denote $\mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}}$ the restriction of $\mathcal{O}_{\mathcal{X}}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to “topological” properties in this lemma).

0788 **Lemma 11.2.** Let \mathcal{X} be an algebraic stack.

- (1) The inclusion functor $\mathcal{X}_{\text{lisse},\text{étale}} \rightarrow \mathcal{X}_{\text{étale}}$ is fully faithful, continuous and cocontinuous. It follows that
 - (a) there is a morphism of topoi

$$g : \text{Sh}(\mathcal{X}_{\text{lisse},\text{étale}}) \longrightarrow \text{Sh}(\mathcal{X}_{\text{étale}})$$

with g^{-1} given by restriction,

- (b) the functor g^{-1} has a left adjoint $g_!^{\text{Sh}}$ on sheaves of sets,
- (c) the adjunction maps $g^{-1}g_* \rightarrow \text{id}$ and $\text{id} \rightarrow g^{-1}g_!^{\text{Sh}}$ are isomorphisms,
- (d) the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,
- (e) the adjunction map $\text{id} \rightarrow g^{-1}g_!$ is an isomorphism, and
- (f) we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.

- (2) The inclusion functor $\mathcal{X}_{\text{flat},\text{fppf}} \rightarrow \mathcal{X}_{\text{fppf}}$ is fully faithful, continuous and cocontinuous. It follows that

- (a) there is a morphism of topoi

$$g : \text{Sh}(\mathcal{X}_{\text{flat},\text{fppf}}) \longrightarrow \text{Sh}(\mathcal{X}_{\text{fppf}})$$

with g^{-1} given by restriction,

- (b) the functor g^{-1} has a left adjoint $g_!^{\text{Sh}}$ on sheaves of sets,
- (c) the adjunction maps $g^{-1}g_* \rightarrow \text{id}$ and $\text{id} \rightarrow g^{-1}g_!^{\text{Sh}}$ are isomorphisms,
- (d) the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,
- (e) the adjunction map $\text{id} \rightarrow g^{-1}g_!$ is an isomorphism, and
- (f) we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.

²In the literature the site is denoted $\text{Lis-ét}(\mathcal{X})$ or $\text{Lis-Et}(\mathcal{X})$ and the associated topos is denoted $\mathcal{X}_{\text{lisse-ét}}$ or $\mathcal{X}_{\text{lisse-et}}$. In the Stacks project our convention is to name the site and denote the corresponding topos by $\text{Sh}(\mathcal{C})$.

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 13.1 and 20.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 21.5 and 21.7. Parts (d), (e) follow from Modules on Sites, Lemmas 16.2 and 16.4. Part (f) is immediate. \square

0789 **Lemma 11.3.** *Let \mathcal{X} be an algebraic stack. Notation as in Lemma 11.2.*

(1) *There exists a functor*

$$g_! : \text{Mod}(\mathcal{X}_{\text{lisse}, \text{étale}}, \mathcal{O}_{\mathcal{X}_{\text{lisse}, \text{étale}}}) \longrightarrow \text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^ . Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = \text{id}$.*

(2) *There exists a functor*

$$g_! : \text{Mod}(\mathcal{X}_{\text{flat}, \text{fppf}}, \mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}) \longrightarrow \text{Mod}(\mathcal{X}_{\text{fppf}}, \mathcal{O}_{\mathcal{X}})$$

which is left adjoint to g^ . Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = \text{id}$.*

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 40.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps Modules on Sites, Equation (40.2.1) are isomorphisms.

Lisse-étale case. Let $x \in \text{Ob}(\mathcal{X}_{\text{lisse}, \text{étale}})$ lying over a scheme U with $x : U \rightarrow \mathcal{X}$ smooth. Consider the induced fully faithful functor

$$g' : \mathcal{X}_{\text{lisse}, \text{étale}}/x \longrightarrow \mathcal{X}_{\text{étale}}/x$$

The right hand side is identified with $(\text{Sch}/U)_{\text{étale}}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \rightarrow U \rightarrow \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 49.2 applies.

Flat-fppf case. Let $x \in \text{Ob}(\mathcal{X}_{\text{flat}, \text{fppf}})$ lying over a scheme U with $x : U \rightarrow \mathcal{X}$ flat. Consider the induced fully faithful functor

$$g' : \mathcal{X}_{\text{flat}, \text{fppf}}/x \longrightarrow \mathcal{X}_{\text{fppf}}/x$$

The right hand side is identified with $(\text{Sch}/U)_{\text{fppf}}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \rightarrow U \rightarrow \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 49.2 applies.

In both cases the equality $g^*g_! = \text{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 11.2. \square

078A **Lemma 11.4.** *Let \mathcal{X} be an algebraic stack. Notation as in Lemmas 11.2 and 11.3.*

(1) *We have $g_!\mathcal{O}_{\mathcal{X}_{\text{lisse}, \text{étale}}} = \mathcal{O}_{\mathcal{X}}$.*

(2) *We have $g_!\mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}} = \mathcal{O}_{\mathcal{X}}$.*

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{\text{étale}}$ (resp. $\mathcal{C} = \mathcal{X}_{\text{fppf}}$) and we denote $\mathcal{C}' = \mathcal{X}_{\text{lisse}, \text{étale}}$ (resp. $\mathcal{C}' = \mathcal{X}_{\text{flat}, \text{fppf}}$). Then \mathcal{C}' is a full subcategory of \mathcal{C} . In this proof we will think of objects V of \mathcal{C} as schemes over \mathcal{X} and objects U of \mathcal{C}' as schemes smooth (resp. flat) over \mathcal{X} . Finally, we write $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{\text{lisse}, \text{étale}}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the \mathcal{O} -module homomorphism $g_!\mathcal{O}' \rightarrow \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

Recall that $g_!\mathcal{O}'$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}'$ given by the rule

$$V \longmapsto \text{colim}_{V \rightarrow U} \mathcal{O}'(U)$$

where the colimit is taken in the category of abelian groups (Modules on Sites, Definition 16.1). Below we will use frequently that if

$$V \rightarrow U \rightarrow U'$$

are morphisms and if $f' \in \mathcal{O}'(U')$ restricts to $f \in \mathcal{O}'(U)$, then $(V \rightarrow U, f)$ and $(V \rightarrow U', f')$ define the same element of the colimit. Also, $g_! \mathcal{O}' \rightarrow \mathcal{O}$ maps the element $(V \rightarrow U, f)$ simply to the pullback of f to V .

To see that $g_! \mathcal{O}' \rightarrow \mathcal{O}$ is surjective it suffices to show that $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ is locally in the image. Choose an object U of \mathcal{C}' corresponding to a surjective smooth morphism $U \rightarrow \mathcal{X}$. Then viewing U both as an object of \mathcal{C}' and \mathcal{C} we see that $(U \rightarrow U, 1)$ is an element of the colimit above which maps to $1 \in \mathcal{O}(U)$. Since U surjects onto the final object of $Sh(\mathcal{C})$ we conclude $g_! \mathcal{O}' \rightarrow \mathcal{O}$ is surjective.

Suppose that $s \in g_! \mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$\operatorname{colim}_{V \rightarrow U} \mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \rightarrow U_i$, U_i smooth (resp. flat) over \mathcal{X} , and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_{\mathcal{X}} \dots \times_{\mathcal{X}} U_n$. Note that W is still smooth (resp. flat) over \mathcal{X} , i.e., defines an object of \mathcal{C}' . The fibre product

$$V' = V \times_{(\varphi_1, \dots, \varphi_n), U} W$$

is surjective étale over V , hence it suffices to show that s maps to zero in $g_! \mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W . In other words, we have reduced to the case of (φ, s) where $\varphi : V \rightarrow U$ is a morphism with U in \mathcal{C}' and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad} & U \times \mathbf{A}^1 \\ & \searrow_{(\varphi, 0)} & \uparrow_{(\operatorname{id}, 0)} \\ & \searrow_{\varphi} & U \end{array}$$

we see that $((\varphi, 0) : V \rightarrow U \times \mathbf{A}^1, \operatorname{pr}_2^* x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\operatorname{pr}_1^* s + \operatorname{pr}_2^* x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over \mathcal{X} . Then we see that $(V \rightarrow Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. \square

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

07AR **Lemma 11.5.** *Let \mathcal{X} be an algebraic stack.*

- (1) *Let \mathcal{F} be an $\mathcal{O}_{\mathcal{X}}$ -module with the flat base change property on $\mathcal{X}_{\text{étale}}$. The following are equivalent*
 - (a) *\mathcal{F} is parasitic, and*
 - (b) *$g^* \mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{\text{lisse, étale}}) \rightarrow Sh(\mathcal{X}_{\text{étale}})$ is as in Lemma 11.2.*
- (2) *Let \mathcal{F} be an $\mathcal{O}_{\mathcal{X}}$ -module on $\mathcal{X}_{\text{fppf}}$. The following are equivalent*
 - (a) *\mathcal{F} is parasitic, and*

(b) $g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{flat, fppf}) \rightarrow Sh(\mathcal{X}_{fppf})$ is as in Lemma 11.2.

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) \Rightarrow (1)(b) is immediate as well. To see (1)(b) \Rightarrow (1)(a) let U be a scheme and let $x : U \rightarrow \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of \mathcal{X} . Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{\acute{e}tale}} = 0$. Let $V \rightarrow \mathcal{X}$ be a flat morphism where V is a scheme. Set $W = U \times_{\mathcal{X}} V$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & V \\ p \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \rightarrow U$ is flat and the projection $q : W \rightarrow V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption \mathcal{F} has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{\acute{e}tale}} \cong q_{small}^*\mathcal{F}|_{V_{\acute{e}tale}}$. Thus if \mathcal{F} is in the kernel of g^* , then $\mathcal{F}|_{V_{\acute{e}tale}} = 0$ as desired. \square

The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks.

07AT **Lemma 11.6.** *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks.*

- (1) *If f is smooth, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{lisse, \acute{e}tale} \rightarrow \mathcal{Y}_{lisse, \acute{e}tale}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{lisse, \acute{e}tale}) & \xrightarrow{g'} & Sh(\mathcal{X}_{\acute{e}tale}) \\ f' \downarrow & & \downarrow f \\ Sh(\mathcal{Y}_{lisse, \acute{e}tale}) & \xrightarrow{g} & Sh(\mathcal{Y}_{\acute{e}tale}) \end{array}$$

We have $f'_(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.*

- (2) *If f is flat, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{flat, fppf} \rightarrow \mathcal{Y}_{flat, fppf}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{flat, fppf}) & \xrightarrow{g'} & Sh(\mathcal{X}_{fppf}) \\ f' \downarrow & & \downarrow f \\ Sh(\mathcal{Y}_{flat, fppf}) & \xrightarrow{g} & Sh(\mathcal{Y}_{fppf}) \end{array}$$

We have $f'_(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.*

Proof. The initial statement comes from the fact that if $x \in \text{Ob}(\mathcal{X})$ lies over a scheme U such that $x : U \rightarrow \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \rightarrow \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 32.2 and 24.2. The induced functor $\mathcal{X}_{lisse, \acute{e}tale} \rightarrow \mathcal{Y}_{lisse, \acute{e}tale}$ (resp. $\mathcal{X}_{flat, fppf} \rightarrow \mathcal{Y}_{flat, fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 11.2) and Sites, Lemma 21.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let \mathcal{F} be a sheaf on $\mathcal{X}_{\acute{e}tale}$ (resp. \mathcal{X}_{fppf}). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 45. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{\text{lis\`e},\acute{e}tale}$ (resp. $\mathcal{Y}_{\text{flat},fppf}$). Say y lies over the scheme V such that $y : V \rightarrow \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$(g^{-1}f_*\mathcal{F})(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \rightarrow V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \rightarrow \mathcal{X}$ is smooth (resp. flat). Denote

$$\text{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{\text{lis\`e},\acute{e}tale} \text{ (resp. } \mathcal{X}_{\text{flat},fppf}\text{)}$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (5.0.1) shows that for any sheaf \mathcal{H} on $\mathcal{X}_{\text{lis\`e},\acute{e}tale}$ (resp. $\mathcal{X}_{\text{flat},fppf}$) we have

$$07AU \quad (11.6.1) \quad f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})', (\text{pr}')^{-1}\mathcal{H})$$

Since $(g')^{-1}$ is restriction we see that

$$(f'_*(g')^{-1}\mathcal{F})(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})', \text{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'})$$

By Sheaves on Stacks, Lemma 22.3 we see that

$$\Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})', \text{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'}) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \rightarrow \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_* , the adjointness of $g_!$ and g^{-1} , and their “primed” versions. \square

12. Quasi-coherent modules, II

07AY In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

07AZ **Lemma 12.1.** *Let \mathcal{X} be an algebraic stack.*

- (1) *Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on $\mathcal{X}_{\acute{e}tale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is \mathcal{F} .*
- (2) *Let $f_j : \mathcal{X}_j \rightarrow \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules on \mathcal{X}_{fppf} . If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is \mathcal{F} .*

Proof. Proof of (1). We may replace each of the algebraic stacks \mathcal{X}_j by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 32.2). The pullback of \mathcal{F} to $(\text{Sch}/U_j)_{\acute{e}tale}$ is still quasi-coherent, see Modules on Sites, Lemma 23.4. Then $f = \coprod f_j : U = \coprod U_j \rightarrow \mathcal{X}$ is a smooth surjective morphism. Let $x : V \rightarrow \mathcal{X}$ be an object of \mathcal{X} . By Sheaves on Stacks, Lemma 18.10

there exists an étale covering $\{x_i \rightarrow x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{\acute{e}tale}$. This just means that x_i lives over a scheme V_i , that $\{V_i \rightarrow V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \rightarrow U$. Then $x_i^* \mathcal{F} = u_i^* f^* \mathcal{F}$ is quasi-coherent. This implies that $x^* \mathcal{F}$ on $(Sch/V)_{\acute{e}tale}$ is quasi-coherent, for example by Modules on Sites, Lemma 23.3. By Sheaves on Stacks, Lemma 11.4 we see that $x^* \mathcal{F}$ is an fppf sheaf and since x was arbitrary we see that \mathcal{F} is a sheaf in the fppf topology. Applying Sheaves on Stacks, Lemma 11.3 we see that \mathcal{F} is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks \mathcal{X}_j by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 24.2 and 26.2). The pullback of \mathcal{F} to $(Sch/U_j)_{\acute{e}tale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 11.2. Then $f = \coprod f_j : U = \coprod U_j \rightarrow \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \rightarrow \mathcal{X}$ be an object of \mathcal{X} . By Sheaves on Stacks, Lemma 18.10 there exists an fppf covering $\{x_i \rightarrow x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{\acute{e}tale}$. This just means that x_i lives over a scheme V_i , that $\{V_i \rightarrow V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \rightarrow U$. Then $x_i^* \mathcal{F} = u_i^* f^* \mathcal{F}$ is quasi-coherent. This implies that $x^* \mathcal{F}$ on $(Sch/V)_{\acute{e}tale}$ is quasi-coherent, for example by Modules on Sites, Lemma 23.3. By Sheaves on Stacks, Lemma 11.3 we see that \mathcal{F} is quasi-coherent. \square

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 23.

07B0 **Lemma 12.2.** *Let \mathcal{X} be an algebraic stack. Notation as in Lemma 11.2.*

- (1) *Let \mathcal{H} be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{isse}, \acute{e}tale}}$ -module on the lisse-étale site of \mathcal{X} . Then $g_! \mathcal{H}$ is a quasi-coherent module on \mathcal{X} .*
- (2) *Let \mathcal{H} be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}$ -module on the flat-fppf site of \mathcal{X} . Then $g_! \mathcal{H}$ is a quasi-coherent module on \mathcal{X} .*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \rightarrow \mathcal{X}$. By Modules on Sites, Definition 23.1 there exists an étale (resp. fppf) covering $\{U_i \rightarrow U\}_{i \in I}$ such that each pullback $f_i^{-1} \mathcal{H}$ has a global presentation (see Modules on Sites, Definition 17.1). Here $f_i : U_i \rightarrow \mathcal{X}$ is the composition $U_i \rightarrow U \rightarrow \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback “is” the restriction to \mathcal{X}/f_i , see Sheaves on Stacks, Definition 9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 11.6 we see that $f_i^{-1} g_! \mathcal{H} = g_{i,!} (f_i')^{-1} \mathcal{H}$. Using Lemma 12.1 we reduce the statement of the lemma to the case where \mathcal{H} has a global presentation. Say we have

$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

of \mathcal{O} -modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{isse}, \acute{e}tale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{flat}, \text{fppf}}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 11.3 and Categories, Lemma 24.5) we conclude that there exists an exact sequence

$$\bigoplus_{j \in J} g_! \mathcal{O} \longrightarrow \bigoplus_{i \in I} g_! \mathcal{O} \longrightarrow g_! \mathcal{H} \longrightarrow 0$$

Lemma 11.4 shows that $g_! \mathcal{O} = \mathcal{O}_{\mathcal{X}}$. In case (2) we are done. In case (1) we apply Sheaves on Stacks, Lemma 11.4 to conclude. \square

07B1 **Lemma 12.3.** *Let \mathcal{X} be an algebraic stack. Let $\mathcal{M}_{\mathcal{X}}$ be the category of locally quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -modules with the flat base change property.*

- (1) *With g as in Lemma 11.2 for the lisse-étale site we have*
 (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \begin{array}{c} \xrightarrow{g^{-1}} \\ \xleftarrow{g_!} \end{array} QCoh(\mathcal{X}_{\text{lisse},\text{étale}}, \mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}})$$

- (b) *if \mathcal{F} is in $\mathcal{M}_{\mathcal{X}}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{\text{lisse},\text{étale}}, \mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}})$ and*
 (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 9.1.*
 (2) *With g as in Lemma 11.2 for the flat-fppf site we have*
 (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \begin{array}{c} \xrightarrow{g^{-1}} \\ \xleftarrow{g_!} \end{array} QCoh(\mathcal{X}_{\text{flat},\text{fppf}}, \mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$$

- (b) *if \mathcal{F} is in $\mathcal{M}_{\mathcal{X}}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{\text{flat},\text{fppf}}, \mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$ and*
 (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 9.1.*

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 23.4. Hence g^{-1} preserves the categories of quasi-coherent modules; here we use that $QCoh(\mathcal{O}_{\mathcal{X}}) = QCoh(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ by Sheaves on Stacks, Lemma 11.4. The same is true for $g_!$ by Lemma 12.2. We know that $\mathcal{H} \rightarrow g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 11.2. Conversely, if \mathcal{F} is in $QCoh(\mathcal{O}_{\mathcal{X}})$ then the map $g_!g^{-1}\mathcal{F} \rightarrow \mathcal{F}$ is a map of quasi-coherent modules on \mathcal{X} whose restriction to any scheme smooth over \mathcal{X} is an isomorphism. Then the discussion in Sheaves on Stacks, Sections 13 and 14 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let \mathcal{F} be an object of $\mathcal{M}_{\mathcal{X}}$. By Lemma 9.2 the kernel and cokernel of the map $Q(\mathcal{F}) \rightarrow \mathcal{F}$ are parasitic. Hence by Lemma 11.5 and since $g^* = g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F}) \rightarrow g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F}) \rightarrow Q(\mathcal{F})$ is an isomorphism by our arguments above. \square

07B2 **Remark 12.4.** Let \mathcal{X} be an algebraic stack. The results of Lemmas 9.1 and 9.2 imply that

$$QCoh(\mathcal{O}_{\mathcal{X}}) = \mathcal{M}_{\mathcal{X}}/\text{Parasitic} \cap \mathcal{M}_{\mathcal{X}}$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 9.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathcal{M}_{\mathcal{X}}$ which is left adjoint to the quotient functor means that $\mathcal{M}_{\mathcal{X}} \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ is a *Bousfield colocalization* or a *right Bousfield localization* (insert future reference here). Our next goal is to show a similar result holds on the level of derived categories.

07B4 **Lemma 12.5.** *Let \mathcal{X} be an algebraic stack.*

- (1) *$QCoh(\mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}})$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{\text{lisse},\text{étale}}})$.*
 (2) *$QCoh(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\mathcal{X}_{\text{flat},\text{fppf}}})$.*

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 9.3. Since 0 is a quasi-coherent module on any ringed site we see that (1) holds. By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $Mod(\mathcal{O})$, so (2) holds. Let $\varphi : \mathcal{G} \rightarrow \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{\text{étale}}$ or $\mathcal{X}_{\text{fppf}}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for \mathcal{G} and φ , see Lemma 11.3. By Lemma 12.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -modules. Hence we see that $\text{Ker}(g_!\varphi)$ and $\text{Coker}(g_!\varphi)$ are quasi-coherent modules on \mathcal{X} . Since g^* is exact (see Lemma 11.2) we see that $g^*\text{Ker}(g_!\varphi) = \text{Ker}(g^*g_!\varphi) = \text{Ker}(\varphi)$ and $g^*\text{Coker}(g_!\varphi) = \text{Coker}(g^*g_!\varphi) = \text{Coker}(\varphi)$ are quasi-coherent too (see Lemma 12.3). This proves (3).

Finally, suppose that

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$$

is an extension of $\mathcal{O}_{\mathcal{X}_{\text{étale}}}$ -modules (resp. $\mathcal{O}_{\mathcal{X}_{\text{fppf}}}$ -modules) with \mathcal{F} and \mathcal{G} quasi-coherent. We have to show that \mathcal{E} is quasi-coherent on $\mathcal{X}_{\text{étale}}$ (resp. $\mathcal{X}_{\text{fppf}}$). We strongly urge the reader to write out what this means on a napkin and prove it him/herself rather than reading the somewhat labyrinthine proof that follows. By Lemma 12.3 this is true if and only if $g_!\mathcal{E}$ is quasi-coherent. By Lemmas 12.1 and Lemma 11.6 we may check this after replacing \mathcal{X} by a smooth (resp. fppf) cover. Choose a scheme U and a smooth surjective morphism $U \rightarrow \mathcal{X}$. By definition there exists an étale (resp. fppf) covering $\{U_i \rightarrow U\}_i$ such that \mathcal{G} has a global presentation over each U_i . Replacing \mathcal{X} by U_i (which is permissible by the discussion above) we may assume that the site $\mathcal{X}_{\text{étale}}$ (resp. $\mathcal{X}_{\text{fppf}}$) has a final object U (in other words \mathcal{X} is representable by the scheme U) and that \mathcal{G} has a global presentation

$$\bigoplus_{j \in J} \mathcal{O} \rightarrow \bigoplus_{i \in I} \mathcal{O} \rightarrow \mathcal{G} \rightarrow 0$$

of \mathcal{O} -modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{étale}}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{\text{fppf}}}$). Let \mathcal{E}' be the pullback of \mathcal{E} via the map $\bigoplus_{i \in I} \mathcal{O} \rightarrow \mathcal{G}$. Then we see that \mathcal{E} is the cokernel of a map $\bigoplus_{j \in J} \mathcal{O} \rightarrow \mathcal{E}'$ hence by property (3) which we proved above, it suffices to prove that \mathcal{E}' is quasi-coherent. Consider the exact sequence

$$L_1g_!\left(\bigoplus_{i \in I} \mathcal{O}\right) \rightarrow g_!\mathcal{F} \rightarrow g_!\mathcal{E}' \rightarrow g_!\left(\bigoplus_{i \in I} \mathcal{O}\right) \rightarrow 0$$

where $L_1g_!$ is the first left derived functor of $g_! : Mod(\mathcal{O}_{\mathcal{X}_{\text{étale}}}) \rightarrow Mod(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$ (resp. $g_! : Mod(\mathcal{X}_{\text{fppf}}, \mathcal{O}_{\mathcal{X}_{\text{fppf}}}) \rightarrow Mod(\mathcal{X}_{\text{fppf}}, \mathcal{O}_{\mathcal{X}})$). This derived functor exists by Cohomology on Sites, Lemma 36.2. Moreover, since $\mathcal{O} = j_{U!}\mathcal{O}_U$ we have $Lg_!\mathcal{O} = g_!\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ also by Cohomology on Sites, Lemma 36.2. Thus the left hand term vanishes and we obtain a short exact sequence

$$0 \rightarrow g_!\mathcal{F} \rightarrow g_!\mathcal{E}' \rightarrow \bigoplus_{i \in I} \mathcal{O}_{\mathcal{X}} \rightarrow 0$$

By Proposition 7.4 it follows that $g_!\mathcal{E}'$ is locally quasi-coherent with the flat base change property. Finally, Lemma 12.3 implies that $\mathcal{E}' = g^{-1}g_!\mathcal{E}'$ is quasi-coherent as desired. \square

13. Other chapters

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- (2) Conventions

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- (4) Categories
- (5) Topology
- (6) Sheaves on Spaces

- (7) Sites and Sheaves
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