COHOMOLOGY OF ALGEBRAIC STACKS

073P

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1. Introduction

073Q In this chapter we write about cohomology of algebraic stacks. This means in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled "Cohomology of Schemes" and "Cohomology of Algebraic Spaces". The results in this chapter are different from those in [LMB00] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapter "Sheaves on Algebraic Stacks" in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 1.

2. Conventions and abuse of language

073R We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 2.

3. Notation

073S Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{étale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 4.1. (Think "big site".) Correspondingly the structure sheaf of $\mathcal{X}$ is a sheaf on $\mathcal{X}_{fppf}$. On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z, and in this case $X_{étale}$ indicates the small étale site of X (as defined in Topologies, Definition 4.8 or Properties of Spaces, Definition 18.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say "sheaf on $\mathcal{X}$" or "sheaf of $\mathcal{O}_\mathcal{X}$-modules" when we mean sheaf on $\mathcal{X}_{fppf}$ or object of $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don't have to mention which topology we are working with. The same isn't true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \to \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Let $\mathcal{G}$ be a sheaf on $\mathcal{Y}_\tau$ for some topology τ. In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on $\mathcal{S}_X$ (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \to \mathcal{Y}$. If $\tau = étale$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{étale}}$ to denote the restriction to the étale site of X, see Sheaves on Stacks, Section 22. If $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}$-module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{étale}}$ instead.

4. Pullback of quasi-coherent modules

076W Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_\mathcal{Y}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}),$$

see Sheaves on Stacks, Lemma 11.2. In general this functor isn't exact, but if f is flat then it is.

076X **Lemma 4.1.** *If $f : \mathcal{X} \to \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ is an exact functor.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_U) & \longleftarrow & QCoh(\mathcal{O}_V) \\ \uparrow & & \uparrow \\ QCoh(\mathcal{O}_\mathcal{X}) & \longleftarrow & QCoh(\mathcal{O}_\mathcal{Y}) \end{array}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 15.1). Since f' is a flat morphism of schemes (by our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V. Thus we win. □

0GQF **Lemma 4.2.** *Let $\mathcal{X}$ be an algebraic stack. Let I be a set and for $i \in I$ let $x_i : U_i \to \mathcal{X}$ be an object of $\mathcal{X}$. Assume that x_i is flat and $\coprod x_i : \coprod U_i \to \mathcal{X}$ is surjective. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an arrow of $QCoh(\mathcal{O}_\mathcal{X})$. Denote φ_i the restriction of φ to $(U_i)_{étale}$. Then φ is injective, resp. surjective, resp. an isomorphism if and only if each φ_i is so.*

Proof. Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. We may and do think of x as an object of $\mathcal{X}$. This produces a presentation $\mathcal{X} = [U/R]$ for some groupoid in spaces (U, R, s, t, c) and correspondingly an equivalence

$$QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$$

See discussion in Sheaves on Stacks, Section 15. The structure of abelian category on the right hand is such that φ is injective, resp. surjective, resp. an isomorphism if and only if the restriction $\varphi|_{U_{étale}}$ is so, see Groupoids in Spaces, Lemma 12.6.

For each i we choose an étale covering $\{W_{i,j} \to V \times_\mathcal{X} U_i\}_{j \in J_i}$ by schemes. Denote $g_{i,j} : W_{i,j} \to V$ and $h_{i,j} : W_{i,j} \to U_i$ the obvious arrows. Each of the morphisms of schemes $g_{i,j} : W_{i,j} \to U$ is flat and they are jointly surjective. Similarly, for each fixed i the morphisms of schemes $h_{i,j} : W_{i,j} \to U_i$ are flat and jointly surjective. By Sheaves on Stacks, Lemma 12.2 the pullback by $(g_{i,j})_{small}$ of the restriction $\varphi|_{U_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$ and the pullback by $(h_{i,j})_{small}$ of the restriction $\varphi|_{(U_i)_{étale}}$ is the restriction $\varphi|_{(W_{i,j})_{étale}}$. Pullback of quasi-coherent modules by a flat morphism of schemes is exact and pullback by a jointly surjective family of flat morphisms of schemes reflects injective, resp. surjective, resp. bijective maps of quasi-coherent modules (in fact this holds for all modules as we can check exactness at stalks). Thus we see

$$\varphi|_{U_{étale}} \text{ injective} \Leftrightarrow \varphi|_{(W_{i,j})_{étale}} \text{ injective for all } i, j \Leftrightarrow \varphi|_{(U_i)_{étale}} \text{ injective for all } i$$

This finishes the proof. □

5. Higher direct images of types of modules

076Y The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two versions: one for the étale topology and one for the fppf topology.

076Z **Lemma 5.1.** *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ such that*

(1) *$\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (see Homology, Definition 10.1) for all algebraic stacks $\mathcal{X}$,*

(2) *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
(3) *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
(4) *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in étale topology.)

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{Y}$. Choose a surjective smooth morphism $\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is representable by a scheme. By Sheaves on Stacks, Lemma 21.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of $\mathcal{F}$ to $\mathcal{U} \times_\mathcal{X} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U}\times_\mathcal{X}\mathcal{Y}}$ because the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \to \mathcal{X}$ by the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{U}$. In other words, we may assume that $\mathcal{X}$ is representable by a scheme. Using (3) once more, we see that the question is Zariski local on $\mathcal{X}$, hence we may assume that $\mathcal{X}$ is representable by an affine scheme. Since f is quasi-compact this implies that also $\mathcal{Y}$ is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 21.1. Recall that this is a first quadrant spectral sequence hence we may use the last part of Homology, Lemma 25.3. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g. Thus the sheaves $g_p^*\mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_\mathcal{X}$. The algebraic stacks $\mathcal{V}_p$ are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 7.8. Of course each $\mathcal{V}_p$ is representable by an algebraic space (the diagonal of the algebraic stack $\mathcal{Y}$ is representable by algebraic spaces). This reduces us to the case where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme.

In the situation where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \to \mathcal{X}$. But in the current situation the algebraic stacks $\mathcal{V}_p$ are representable by quasi-compact and quasi-separated schemes (because the diagonal of an algebraic space is representable by schemes).

Thus we may assume $\mathcal{Y}$ is representable by a scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$

is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume $\mathcal{Y}$ is representable by a separated scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (yet again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. □

Here is the version for the fppf topology.

0770 **Lemma 5.2.** *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{O}_\mathcal{X})$ such that*

1. *$\mathcal{O}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
4. *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in fppf topology.)

Proof. Identical to the proof of Lemma 5.1. □

6. Locally quasi-coherent modules

075X Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We can ask whether $\mathcal{F}$ is *locally quasi-coherent*, see Sheaves on Stacks, Definition 12.1. Briefly, this means $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module for the étale topology such that for any morphism $f : U \to \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent on $U_{étale}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$LQCoh(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 12.4.

075Y **Lemma 6.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 12.3. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective smooth morphism. Let x be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 19.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x, x_i live over schemes V, V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. The restriction $x_i^*\mathcal{F}|_{V_{i,étale}}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_{i,étale}$, see Sheaves on Stacks, Lemma 9.3.

Hence $x^*\mathcal{F}|_{V_{étale}}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_{i,étale}$ for each i. This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 29.6. □

075Z **Lemma 6.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$. Then $R^i f_*\mathcal{F}$ (computed in the étale topology) is locally quasi-coherent on $\mathcal{Y}_{étale}$.*

Proof. We will use Lemma 5.1 to prove this. We will check its assumptions (1) – (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 12.4. Part (3) follows from Lemma 6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. Choose any object y of $\mathcal{Y}$ lying over a scheme V. For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

Thus $\mathcal{Z}$ is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 22.3 we have

$$R^i f_*\mathcal{F}|_{V_{étale}} = R^i f'_{small,*}\big(g^*\mathcal{F}|_{Z_{étale}}\big)$$

The right hand side is a quasi-coherent sheaf on $V_{étale}$ by Cohomology of Spaces, Lemma 3.1. This implies the left hand side is quasi-coherent which is what we had to prove. □

07AP **Lemma 6.3.** *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$$

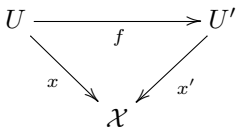
where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 19.7 and Lemma 19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 12.3). The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent by Lemma 6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Lemma 12.4.

We are going to reduce the proof of the general case the situation in the first paragraph. Let x be an object of $\mathcal{X}$ lying over the scheme U. We have to show that $\mathcal{F}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. It suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence $a_i^*\mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering

we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{étale}}$ is quasi-coherent. □

7. Flat comparison maps

0760 Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Given an object x of $\mathcal{X}$ lying over the scheme U the restriction $\mathcal{F}|_{U_{étale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U, see Sheaves on Stacks, Definition 9.2. Next, let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a morphism of schemes $f : U \to U'$. Thus a 2-commutative diagram



Associated to φ we obtain a comparison map between restrictions

0761 (7.0.1) $$c_\varphi : f^*_{small}(\mathcal{F}|_{U'_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}}$$

see Sheaves on Stacks, Equation (9.4.1). In this situation we can consider the following property of $\mathcal{F}$.

0762 **Definition 7.1.** Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. We say $\mathcal{F}$ has the *flat base change property*[1] if and only if c_φ is an isomorphism whenever f is flat.

Here is a lemma with some properties of this notion.

0764 **Lemma 7.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$.*

1. *If $\mathcal{F}$ has the flat base change property then for any morphism $g : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.*
2. *The full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of modules with the flat base change property is a weak Serre subcategory.*
3. *Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does $\mathcal{F}$.*
4. *The category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$ with the flat base change property has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}$ and $\mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ with the flat base change property then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ has the flat base change property.*
6. *Given $\mathcal{F}$ and $\mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ having the flat base change property then the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property.*

Proof. Let $g : \mathcal{Y} \to \mathcal{X}$ be as in (1). Let y be an object of $\mathcal{Y}$ lying over a scheme V. By Sheaves on Stacks, Lemma 9.3 we have $(g^*\mathcal{F})|_{V_{étale}} = \mathcal{F}|_{V_{étale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on $\mathcal{Y}$ is a special case of a comparison map for the sheaf $\mathcal{F}$ on $\mathcal{X}$, see Sheaves on Stacks, Lemma 9.3. In this way (1) is clear.

[1] This may be nonstandard notation.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 10.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f_{small}^* is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{\acute{e}tale}}$ are exact (because we are working in the étale topology). Finally, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$ and the outer two sheaves have the flat base change property then the middle one does as well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 33.2) we may assume that each $\mathcal{X}_i$ is representable by a scheme. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a flat morphism $a : U \to U'$ of schemes. By Sheaves on Stacks, Lemma 19.10 there exists a jointly surjective family of étale morphisms $U_i' \to U'$ such that $U_i' \to U' \to \mathcal{X}$ factors through $\mathcal{X}_i$. Thus we obtain commutative diagrams

$$\begin{array}{ccccc} U_i = U \times_{U'} U_i' & \xrightarrow{a_i} & U_i' & \xrightarrow{x_i'} & \mathcal{X}_i \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_i} \\ U & \xrightarrow{a} & U' & \xrightarrow{x'} & \mathcal{X} \end{array}$$

Note that each a_i is a flat morphism of schemes as a base change of a. Denote $\psi_i : x_i \to x_i'$ the morphism of $\mathcal{X}_i$ lying over a_i with target x_i'. By assumption the comparison maps $c_{\psi_i} : (a_i)_{small}^*\big(f_i^*\mathcal{F}|_{(U_i')_{\acute{e}tale}}\big) \to f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ is an isomorphism. Because the vertical arrows $U_i' \to U'$ and $U_i \to U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U_i')_{\acute{e}tale}}$ and $f_i^*\mathcal{F}|_{(U_i)_{\acute{e}tale}}$ are the restrictions of $\mathcal{F}|_{U'_{\acute{e}tale}}$ and $\mathcal{F}|_{U_{\acute{e}tale}}$ and the map c_{ψ_i} is the restriction of c_φ to $(U_i)_{\acute{e}tale}$, see Sheaves on Stacks, Lemma 9.3. Since $\{U_i \to U\}$ is an étale covering, this implies that the comparison map c_φ is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \to Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each $\mathcal{F}_i$ has the flat base change property. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. Recall that $\operatorname{colim}_i \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(\operatorname{colim}_i \mathcal{F}_i)|_{U_{\acute{e}tale}} = \operatorname{colim}_i \mathcal{F}_i|_{U_{\acute{e}tale}}$$

and similarly for the restriction to $U'_{\acute{e}tale}$. Hence

$$\begin{aligned} f_{small}^*((\operatorname{colim}_i \mathcal{F}_i)|_{U'_{\acute{e}tale}}) &= f_{small}^*(\operatorname{colim}_i \mathcal{F}_i|_{U'_{\acute{e}tale}}) \\ &= \operatorname{colim}_i f_{small}^*(\mathcal{F}_i|_{U'_{\acute{e}tale}}) \\ &\xrightarrow{\operatorname{colim} c_\varphi} \operatorname{colim}_i \mathcal{F}_i|_{U_{\acute{e}tale}} \\ &= (\operatorname{colim}_i \mathcal{F}_i)|_{U_{\acute{e}tale}} \end{aligned}$$

For the second equality we used that f_{small}^* commutes with colimits (as a left adjoint). The arrow is an isomorphism as each $\mathcal{F}_i$ has the flat base change property. Thus the colimit has the flat base change property and (4) is true.

Part (5) holds because tensor products commute with pullbacks, see Modules on Sites, Lemma 26.2. Details omitted.

Let $\mathcal{F}$ and $\mathcal{G}$ be as in (6). Since $\mathcal{F}$ is quasi-coherent it has the flat base change property by Sheaves on Stacks, Lemma 12.2. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over the flat morphism of schemes $f : U \to U'$. As we are using the étale topology, we have

$$\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}})$$

and similarly for the restriction to $U'_{étale}$ (details omitted). Hence

$$\begin{aligned} f^*_{small}(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U'_{étale}}) &= f^*_{small}(\mathcal{H}om_{\mathcal{O}_{U'}}(\mathcal{F}|_{U'_{étale}}, \mathcal{G}|_{U'_{étale}})) \\ &= \mathcal{H}om_{\mathcal{O}_{U'}}(f^*_{small}(\mathcal{F}|_{U'_{étale}}), f^*_{small}(\mathcal{G}|_{U'_{étale}})) \\ &\xrightarrow{c_\varphi} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_{U_{étale}}, \mathcal{G}|_{U_{étale}}) \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} \end{aligned}$$

Here the second equality is Modules on Sites, Lemma 31.4 which uses that $f : U \to U'$ is flat and hence the morphism of ringed sites f_{small} is flat too. The arrow is an isomorphism as both $\mathcal{F}$ and $\mathcal{G}$ have the flat base change property. Thus our $\mathcal{H}om$ has the flat base change property too as desired. □

0765 **Lemma 7.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property. Then each $R^i f_* \mathcal{F}$ (computed in the étale topology) has the flat base change property.*

Proof. We will use Lemma 5.1 to prove this. For every algebraic stack $\mathcal{X}$ let $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ denote the full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) – (4) of Lemma 5.1 the lemma will follow. Properties (1), (2), and (3) follow from Sheaves on Stacks, Lemmas 12.3 and 12.4 and Lemmas 6.1 and 7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. In this case, suppose that $\psi : y \to y'$ is a morphism of $\mathcal{Y}$ lying over a flat morphism $b : V \to V'$ of schemes. For clarity denote $\mathcal{V} = (Sch/V)_{fppf}$ and $\mathcal{V}' = (Sch/V')_{fppf}$ the corresponding algebraic stacks. Consider the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{a} & \mathcal{Z}' & \xrightarrow{x'} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{b} & \mathcal{V}' & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated – even affine) we see that $\mathcal{Z}$ and $\mathcal{Z}'$ are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since $\mathcal{F}$ has the flat base change property we see that

$$a^*_{small}\big(\mathcal{F}|_{Z'_{étale}}\big) \longrightarrow \mathcal{F}|_{Z_{étale}}$$

is an isomorphism. Moreover,

$$R^i f_* \mathcal{F}|_{V'_{étale}} = R^i(f')_{small,*}\big(\mathcal{F}|_{Z'_{étale}}\big)$$

and

$$R^if_*\mathcal{F}|_{V_{étale}} = R^i(f'')_{small,*}\big(\mathcal{F}|_{Z_{étale}}\big)$$

by Sheaves on Stacks, Lemma 22.3. Hence we see that the comparison map

$$c_\psi : b^*_{small}(R^if_*\mathcal{F}|_{V'_{étale}}) \longrightarrow R^if_*\mathcal{F}|_{V_{étale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 11.2. Thus $R^if_*\mathcal{F}$ has the flat base change property. Since $R^if_*\mathcal{F}$ is locally quasi-coherent by Lemma 6.2 we win. □

8. Locally quasi-coherent modules with the flat base change property

0GQG Let $\mathcal{X}$ be an algebraic stack. We[2] will denote

$$LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

the full subcategory whose objects are étale $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ which are both locally quasi-coherent (Section 6) and have the flat base change property (Section 7). We have

$$QCoh(\mathcal{O}_\mathcal{X}) \subset LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$$

by Sheaves on Stacks, Lemma 12.2.

0771 **Proposition 8.1.** *Summary of results on locally quasi-coherent modules having the flat base change property.*

1. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, then $\mathcal{F}$ is a sheaf for the fppf topology, i.e., it is an object of $Mod(\mathcal{O}_\mathcal{X})$.*
2. *The category $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ is a weak Serre subcategory of both $Mod(\mathcal{O}_\mathcal{X})$ and $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
3. *Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^* : LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
4. *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and $\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, then*
   (a) *the total direct image $Rf_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and*
   (b) *each $R^if_*\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$.*
5. *The category $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ as well as in $Mod(\mathcal{O}_\mathcal{X})$.*
6. *Given $\mathcal{F}$ and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
7. *Given $\mathcal{F}$ of finite presentation and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) is Sheaves on Stacks, Lemma 23.1.

Part (2) for the embedding $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have seen in the proof of Lemma 7.3. Let us prove (2) for the embedding $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Since $\mathrm{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\mathrm{Ker}(\varphi)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in

[2] Apologies for the horrendous notation.

the étale topology. However, this étale cokernel is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ hence an fppf sheaf by (1) and we see that the cokernel is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ (i.e., using the fppf topology) with $\mathcal{F}_1$, $\mathcal{F}_2$ in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. In order to show that $\mathcal{F}_2$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H^1_{fppf}(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of $\mathcal{X}$). This is true because $H^1_{fppf}(x, \mathcal{F}_1) = H^1_{étale}(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 23.2 and because of locality of cohomology, see Cohomology on Sites, Lemma 7.3. This proves (2).

Part (3) follows from Lemma 7.2 and Sheaves on Stacks, Lemma 12.3.

Part (4)(b) for $R^i f_* \mathcal{F}$ computed in the étale cohomology follows from Lemma 7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 23.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 12.4 and Lemma 7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1).

Parts (6) and (7) follow from the corresponding parts of Lemma 7.2 and Sheaves on Stacks, Lemma 12.4. □

07AQ **Lemma 8.2.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is in $LQCoh^{fpc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}_i})$, then $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) follows from a combination of Lemmas 6.1 and 7.2. The proof of (2) is analogous to the proof of Lemma 6.3. Let $\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$.

First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_* a^* \mathcal{F} \to b_* b^* \mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 19.7 and Lemma 19.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 8.1. The modules $a_* a^* \mathcal{F}$ and $b_* b^* \mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 8.1. We conclude that $\mathcal{F}$ is locally quasi-coherent and has the flat base change property by Proposition 8.1.

Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change

property. Again by part (1) it suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 27.14 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. □

0GQH **Lemma 8.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is quasi-compact, quasi-separated, and representable by algebraic spaces. Let $\mathcal{F}$ be in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Then for an object $y : V \to \mathcal{Y}$ of $\mathcal{Y}$ we have*

$$(R^i f_*\mathcal{F})|_{V_{étale}} = R^i f'_{small,*}(\mathcal{F}|_{U_{étale}})$$

where $f' : U = V \times_\mathcal{Y} \mathcal{X} \to V$ is the base change of f.

Proof. By Sheaves on Stacks, Lemma 21.3 we can reduce to the case where $\mathcal{X}$ is represented by U and $\mathcal{Y}$ is represented by V. Of course this also uses that the pullback of $\mathcal{F}$ to U is in $LQCoh^{fbc}(\mathcal{O}_U)$ by Proposition 8.1. Then the result follows from Sheaves on Stacks, Lemma 22.2 and the fact that $R^i f_*$ may be computed in the étale topology by Proposition 8.1. □

0GQI **Lemma 8.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functor $f_* : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ is exact and commutes with direct sums. The functors $R^i f_*$ for $i > 0$ vanish on $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*

Proof. The functors exist by Proposition 8.1. By Lemma 8.3 this reduces to the case of an affine morphism of algebraic spaces taking higher direct images in the setting of quasi-coherent modules on algebraic spaces. By the discussion in Cohomology of Spaces, Section 3 we reduce to the case of an affine morphism of schemes. For affine morphisms of schemes we have the vanishing of higher direct images on quasi-coherent modules by Cohomology of Schemes, Lemma 2.3. The vanishing for $R^1 f_*$ implies exactness of f_*. Commuting with direct sums follows from Morphisms, Lemma 11.7 for example. □

9. Parasitic modules

0772 The following definition is compatible with Descent, Definition 12.1.

0773 **Definition 9.1.** Let $\mathcal{X}$ be an algebraic stack. A presheaf of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ is *parasitic* if we have $\mathcal{F}(x) = 0$ for any object x of $\mathcal{X}$ which lies over a scheme U such that the corresponding morphism $x : U \to \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

0774 **Lemma 9.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules.*

(1) *If $\mathcal{F}$ is parasitic and $g : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.*
(2) *For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have*
   (a) *the τ sheafification of a parasitic presheaf of modules is parasitic, and*
   (b) *the full subcategory of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ consisting of parasitic modules is a Serre subcategory.*

(3) *Suppose $\mathcal{F}$ is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*
(4) *Suppose $\mathcal{F}$ is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*

Proof. To see part (1) let y be an object of $\mathcal{Y}$ which lies over a scheme V such that the corresponding morphism $y : V \to \mathcal{Y}$ is flat. Then $g(y) : V \to \mathcal{Y} \to \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 25.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \to x\}$ is a τ-covering of $\mathcal{X}$, then each of the morphisms $x_i \to x$ lies over a flat morphism of schemes. Hence if x lies over a scheme U such that $x : U \to \mathcal{X}$ is flat, so do all of the objects x_i. Hence the presheaf $\mathcal{F}^+$ (see Sites, Section 10) is parasitic if the presheaf $\mathcal{F}$ is parasitic. This proves (2)(a) as the sheafification of $\mathcal{F}$ is $(\mathcal{F}^+)^+$.

Let $\mathcal{F}$ be a parasitic τ-module. It is immediate from the definitions that any submodule of $\mathcal{F}$ is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient $\mathcal{F}/\mathcal{F}'$ is a parasitic module by (2)(a). Finally, we have to show that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ parasitic, then $\mathcal{F}_2$ is parasitic. This follows immediately on evaluating on x lying over a scheme flat over $\mathcal{X}$. This proves (2)(b), see Homology, Lemma 10.2.

Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of $\mathcal{X}$ which lies over a scheme U such that $x : U \to \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \to U \times_{x,\mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \to \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \to x\}$ is a covering for the smooth topology on $\mathcal{X}$. Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \to x\}_{i\in I}$ in $\mathcal{X}$ we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as the étale topology, see More on Morphisms, Lemma 38.7. Namely, $\{x_i \to x\}_{i\in I}$ lies over a smooth covering $\{U_i \to U\}_{i\in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \to U\}_{j\in J}$ which refines $\{U_i \to U\}_{i\in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \to x\}$ is an étale covering in $\mathcal{X}$ refining $\{x_i \to x\}_{i\in I}$. This means the map $\mathcal{F}(x) \to \prod_{j\in J} \mathcal{F}(x'_j)$, which is injective as $\mathcal{F}$ is a sheaf in the étale topology, factors through $\mathcal{F}(x) \to \prod_{i\in I} \mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). □

Parasitic modules are preserved under absolutely any pushforward.

0775 **Lemma 9.3.** *Let $\tau \in \{étale, fppf\}$. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a parasitic object of Mod$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

(1) *$H^i_\tau(\mathcal{X}, \mathcal{F}) = 0$ for all i.*
(2) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^i f_*\mathcal{F}$ (computed in τ-topology) is a parasitic object of Mod$(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$.*

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 21.2 we see that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V. By Lemma 9.2 it suffices to show that these cohomology groups are zero when $y : V \to \mathcal{Y}$ is flat. Note that $\mathrm{pr} : V \times_{y,\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is flat as a base change of y. Hence by Lemma 9.2 we see that $\mathrm{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 20.1 to reduce this to the case where $\mathcal{X}$ is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 9.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} \mathcal{U} \to \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of Lemma 5.1) we reduce to the case where the algebraic stack $\mathcal{X}$ is representable by a scheme X. Then $H^i_\tau(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 12.2. □

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ aren't exact in general.

0776 **Lemma 9.4.** *Let $\mathcal{X}$ be an algebraic stack. Let $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$ be maps in $QCoh(\mathcal{O}_\mathcal{X})$ with $\beta \circ \alpha = 0$. The following are equivalent:*

1. *in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$ the complex $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact at $\mathcal{G}$,*
2. *$\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ computed in either $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ is parasitic.*

Proof. We have $QCoh(\mathcal{O}_\mathcal{X}) \subset LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Section 8. Hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ agree, see Proposition 8.1. From now on we will use the étale topology on $\mathcal{X}$.

Let $\mathcal{E}$ be the cohomology of $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ computed in the abelian category $QCoh(\mathcal{O}_\mathcal{X})$. Let $x : U \to \mathcal{X}$ be a flat morphism where U is a scheme. As we are using the étale topology, the restriction functor $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to Mod(U_{étale}, \mathcal{O}_U)$ is exact. On the other hand, by Lemma 4.1 and Sheaves on Stacks, Lemma 14.2 the restriction functor

$$QCoh(\mathcal{O}_\mathcal{X}) \xrightarrow{x^*} QCoh((Sch/U)_{étale}, \mathcal{O}) \xrightarrow{-|_{U_{étale}}} QCoh(U_{étale}, \mathcal{O}_U)$$

is exact too. We conclude that $\mathcal{E}|_{U_{étale}} = (\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha))|_{U_{étale}}$.

If (1) holds, then $\mathcal{E} = 0$ hence $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$ and this is the definition of a parasitic module. If (2) holds, then $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$ hence $\mathcal{E}$ restricts to zero on $U_{étale}$ for all U flat over $\mathcal{X}$. This certainly implies that the quasi-coherent module $\mathcal{E}$ is zero, for example apply Lemma 4.2 to the map $0 \to \mathcal{E}$. □

10. Quasi-coherent modules

0777 We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 15. This fact is the basis for the following.

0778 **Lemma 10.1.** *Let $\mathcal{X}$ be an algebraic stack. Let $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ be the category of locally quasi-coherent modules with the flat base change property, see Section 8. The inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$Q : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Set $R = U \times_\mathcal{X} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 16.2. We may and do replace $\mathcal{X}$ by $[U/R]$. By Sheaves on Stacks, Proposition 14.3 there is an equivalence

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

Let us construct a functor

$$q_2 : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U, R, s, t, c)$$

by the following rule: if $\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ then we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{étale}}, \alpha)$$

where α is the isomorphism

$$t^*_{small}(f^*\mathcal{F}|_{U_{étale}}) \to t^*f^*\mathcal{F}|_{R_{étale}} \to s^*f^*\mathcal{F}|_{R_{étale}} \to s^*_{small}(f^*\mathcal{F}|_{U_{étale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because $\mathcal{F}$ is locally quasi-coherent and that we used (and needed) the flat base change property in the construction of the descent datum α. We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 12.1) holds. Looking at the proof of Sheaves on Stacks, Proposition 14.3 we see that $q_2 \circ i$ is the quasi-inverse to q_1. We define $Q = q_1 \circ q_2$. Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and let $\mathcal{G}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. We have

$$\begin{aligned}\mathrm{Mor}_{LQCoh^{fbc}(\mathcal{O}_\mathcal{X})}(i(\mathcal{G}), \mathcal{F}) &= \mathrm{Mor}_{QCoh(U,R,s,t,c)}(q_2(i(\mathcal{G})), q_2(\mathcal{F})) \\ &= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(\mathcal{G}, Q(\mathcal{F}))\end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 14.4 and the second equality holds because $q_1 \circ i$ and q_2 are quasi-inverse equivalences of categories. The assertion $Q \circ i \cong \mathrm{id}$ is a formal consequence of the fact that i is fully faithful. □

0779 **Lemma 10.2.** *Let $\mathcal{X}$ be an algebraic stack. Let $Q : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{X})$ be the functor constructed in Lemma 10.1.*

1. *The kernel of Q is exactly the collection of parasitic objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$.*
2. *For any object $\mathcal{F}$ of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic.*
3. *The functor Q is exact and commutes with all limits and colimits.*

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 10.1. Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. It is clear from the proof of Lemma 10.1 that $\mathcal{F}$ is in the kernel of Q if and only if $\mathcal{F}|_{U_{étale}} = 0$. In particular, if $\mathcal{F}$ is parasitic then $\mathcal{F}$ is in the kernel.

Next, let $x : V \to \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_{\mathcal{X}} U$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow[q]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q^*_{small} is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p^*_{small}\mathcal{F}|_{U_{étale}} \cong q^*_{small}\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of Q, then $\mathcal{F}|_{V_{étale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \to \mathcal{F}$ becomes an isomorphism after restricting to $U_{étale}$.

To see part (3) note that Q is left exact as a right adjoint. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\mathcal{F}) & \longrightarrow & Q(\mathcal{G}) & \longrightarrow & Q(\mathcal{H}) & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \end{array}$$

Since the kernels and cokernels of a, b, and c are parasitic by part (2) and since the bottom row is a short exact sequence, we see that the top row as a complex of $\mathcal{O}_{\mathcal{X}}$-modules has parasitic cohomology sheaves (details omitted; this uses that the category of parasitic modules is a Serre subcategory of the category of all modules). By left exactness of Q we see that only exactness at $Q(\mathcal{H})$ is at issue. However, the cokernel $\mathcal{Q}$ of $Q(\mathcal{G}) \to Q(\mathcal{H}))$ may be computed either in $Mod(\mathcal{O}_{\mathcal{X}})$ or in $QCoh(\mathcal{O}_{\mathcal{X}})$ with the same result because the inclusion functor $QCoh(\mathcal{O}_{\mathcal{X}}) \to LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ is a left adjoint and hence right exact. Hence $\mathcal{Q} = Q(\mathcal{Q})$ is both quasi-coherent and parasitic, whence 0 by part (1) as desired.

As a right adjoint Q commutes with all limits. Since Q is exact, to show that Q commutes with all colimits it suffices to show that Q commutes with direct sums, see Categories, Lemma 14.12. Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$. To see that $Q(\bigoplus \mathcal{F}_i)$ is equal to $\bigoplus Q(\mathcal{F}_i)$ we look at the construction of Q in the proof of Lemma 10.1. This uses a presentation $\mathcal{X} = [U/R]$ where U is a scheme. Then $Q(\mathcal{F})$ is computed by first taking the pair $(\mathcal{F}|_{U_{étale}}, \alpha)$ in $QCoh(U, R, s, t, c)$ and then using the equivalence $QCoh(U, R, s, t, c) \cong QCoh(\mathcal{O}_{\mathcal{X}})$. Since the restriction functor $Mod(\mathcal{O}_{\mathcal{X}}) \to Mod(\mathcal{O}_{U_{étale}})$, $\mathcal{F} \mapsto \mathcal{F}|_{U_{étale}}$ commutes with direct sums, the desired equality is clear. □

0GQJ **Lemma 10.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Then $Q_{\mathcal{X}} \circ f^* = f^* \circ Q_{\mathcal{Y}}$ where $Q_{\mathcal{X}}$ and $Q_{\mathcal{Y}}$ are as in Lemma 10.1.*

Proof. Observe that f^* preserves both $QCoh$ and $LQCoh^{fbc}$, see Sheaves on Stacks, Lemma 11.2 and Proposition 8.1. If $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{Y}})$ then $Q_{\mathcal{Y}}(\mathcal{F}) \to \mathcal{F}$ has parasitic kernel and cokernel by Lemma 10.2. As f is flat we get that

$f^*Q_{\mathcal{Y}}(\mathcal{F}) \to f^*\mathcal{F}$ has parasitic kernel and cokernel by Lemma 9.2. Thus the induced map $f^*Q_{\mathcal{Y}}(\mathcal{F}) \to Q_{\mathcal{X}}(f^*\mathcal{F})$ has parasitic kernel and cokernel and hence is an isomorphism for example by Lemma 9.4. □

0GQK **Lemma 10.4.** *Let $\mathcal{X}$ be an algebraic stack. Let x be an object of $\mathcal{X}$ lying over the scheme U such that $x : U \to \mathcal{X}$ is flat. Then for $\mathcal{F}$ in $QCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ we have $Q(\mathcal{F})|_{U_{étale}} = \mathcal{F}|_{U_{étale}}$.*

Proof. True because the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic, see Lemma 10.2. □

0GQL **Remark 10.5.** Let $\mathcal{X}$ be an algebraic stack. The category $QCoh(\mathcal{O}_{\mathcal{X}})$ is abelian, the inclusion functor $QCoh(\mathcal{O}_{\mathcal{X}}) \to Mod(\mathcal{O}_{\mathcal{X}})$ is right exact, but not exact in general, see Sheaves on Stacks, Lemma 15.1. We can use the functor Q from Lemmas 10.1 and 10.2 to understand this. Namely, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. Then

(1) the cokernel $\mathrm{Coker}(\varphi)$ computed in $Mod(\mathcal{O}_{\mathcal{X}})$ is quasi-coherent and is the cokernel of φ in $QCoh(\mathcal{O}_{\mathcal{X}})$,
(2) the image $\mathrm{Im}(\varphi)$ computed in $Mod(\mathcal{O}_{\mathcal{X}})$ is quasi-coherent and is the image of φ in $QCoh(\mathcal{O}_{\mathcal{X}})$, and
(3) the kernel $\mathrm{Ker}(\varphi)$ computed in $Mod(\mathcal{O}_{\mathcal{X}})$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ by Proposition 8.1 and $Q(\mathrm{Ker}(\varphi))$ is the kernel in $QCoh(\mathcal{O}_{\mathcal{X}})$.

This follows from the references given.

0GQM **Remark 10.6.** Let $\mathcal{X}$ be an algebraic stack. Given two quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ and $\mathcal{G}$ the tensor product module $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ is quasi-coherent, see Sheaves on Stacks, Lemma 15.1 part (5). Similarly, given two locally quasi-coherent modules with the flat base change property, their tensor product has the same property, see Proposition 8.1. Thus the inclusion functors

$$QCoh(\mathcal{O}_{\mathcal{X}}) \to LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}}) \to Mod(\mathcal{O}_{\mathcal{X}})$$

are functors of symmetric monoidal categories. What is more interesting is that the functor

$$Q : LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{X}})$$

is a functor of symmetric monoidal categories as well. Namely, given $\mathcal{F}$ and $\mathcal{G}$ in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ we obtain

$$\begin{array}{ccc} Q(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{X}}} Q(\mathcal{G}) & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G} \\ & \searrow \quad \nearrow & \\ & Q(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}) & \end{array}$$

where the south-west arrow comes from the universal property of the north-west arrow (and the fact already mentioned that the object in the upper left corner is quasi-coherent). If we restrict this diagram to $U_{étale}$ for $U \to \mathcal{X}$ flat, then all three arrows become isomorphisms (see Lemmas 10.1 and 10.2 and Definition 9.1). Hence $Q(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{X}}} Q(\mathcal{G}) \to Q(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G})$ is an isomorphism, see for example Lemma 4.2.

07B2 **Remark 10.7.** Let $\mathcal{X}$ be an algebraic stack. Let $Parasitic(\mathcal{O}_{\mathcal{X}}) \subset Mod(\mathcal{O}_{\mathcal{X}})$ denote the full subcategory consisting of parasitic modules. The results of Lemmas

10.1 and 10.2 imply that

$$QCoh(\mathcal{O}_\mathcal{X}) = LQCoh^{fbc}(\mathcal{O}_\mathcal{X})/Parasitic(\mathcal{O}_\mathcal{X}) \cap LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 10.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ which is left adjoint to the quotient functor is a key feature of the situation. In Derived Categories of Stacks, Section 5 and especially Lemma 5.4 we prove that a similar result holds on the level of derived categories.

0GQN **Lemma 10.8.** *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. The internal homs $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ or $Mod(\mathcal{O}_\mathcal{X})$ agree and the common value is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. The quasi-coherent module $hom(\mathcal{F}, \mathcal{G}) = Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$ has the following universal property*

$$\mathrm{Hom}_\mathcal{X}(\mathcal{H}, hom(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_\mathcal{X}(\mathcal{H} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{F}, \mathcal{G})$$

for $\mathcal{H}$ in $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. The construction of $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules; the output $\mathcal{H}om$ is a sheaf for the fppf topology because $\mathcal{F}$ and $\mathcal{G}$ are assumed sheaves in the fppf topology, see Modules on Sites, Lemma 27.1. By Sheaves on Stacks, Lemma 12.4 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is locally quasi-coherent. By Lemma 7.2 we see that $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ has the flat base change property. Hence $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and it makes sense to apply the functor Q of Lemma 10.1 to it. By the universal property of Q we have

$$\mathrm{Hom}_\mathcal{X}(\mathcal{H}, Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) = \mathrm{Hom}_\mathcal{X}(\mathcal{H}, \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))$$

for $\mathcal{H}$ quasi-coherent, hence the displayed formula of the lemma follows from Modules on Sites, Lemma 27.6. □

0GQP **Lemma 10.9.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of algebraic stacks. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{Y}$-module of finite presentation and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then $f^*hom(\mathcal{F}, \mathcal{G}) = hom(f^*\mathcal{F}, f^*\mathcal{G})$ with notation as in Lemma 10.8.*

Proof. We have $f^*\,\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(f^*\mathcal{F}, f^*\mathcal{G})$ by Modules on Sites, Lemma 31.4. (Observe that this step is not where the flatness of f is used as the morphism of ringed topoi associated to f is always flat, see Sheaves on Stacks, Remark 6.3.) Then apply Lemma 10.3 (and here we do use flatness of f). □

11. Pushforward of quasi-coherent modules

070A Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}^2_k \setminus \{0\} \longrightarrow \mathbf{A}^2_k = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (Sch/X)_{fppf} \to (Sch/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_\mathcal{X}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_\mathcal{X}$ is quasi-coherent on $\mathcal{Y}$ then we would have $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. However, consider $T = \mathrm{Spec}(k) \to \mathbf{A}^2_k = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = 0$ because $X \times_Y T = \emptyset$ whereas $\Gamma(T, \mathcal{O}_\mathcal{Y}) = k$. On the positive side, for any flat morphism $T \to Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = \Gamma(T, \mathcal{O}_\mathcal{Y})$ as follows from Cohomology of Schemes, Lemma 5.2 using that j is quasi-compact and quasi-separated.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

(1) f_* does preserve locally quasi-coherent modules (Lemma 6.2),
(2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 7.3), and
(3) locally quasi-coherent $\mathcal{O}_\mathcal{Y}$-modules with the flat base change property give rise to quasi-coherent modules on a presentation of $\mathcal{Y}$ and hence quasi-coherent modules on $\mathcal{Y}$, see Sheaves on Stacks, Section 15.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in \mathrm{Ob}(\mathcal{Y})$ such that the associated 1-morphism $y : V \to \mathcal{Y}$ is flat, see Lemma 11.2. Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$. However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

077A **Proposition 11.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{f_*} LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

where the functors f_* and Q are as in Proposition 8.1 and Lemma 10.1. Moreover, if we define $R^i f_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \xrightarrow{R^i f_*} LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

then the sequence of functors $\{R^i f_{QCoh,*}\}_{i\geq 0}$ forms a cohomological δ-functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then we have

$$\begin{aligned}
\mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}) \\
&= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, Q(f_*\mathcal{F})) \\
&= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_{QCoh,*}\mathcal{F})
\end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 8.1 we see that $f_*\mathcal{F}$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ (and can be

computed in either the fppf or étale topology) and we obtain the second equality by Lemma 10.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^i f_{QCoh,*}\}_{i\geq 0}$ is a cohomological δ-functor as defined in Homology, Definition 12.1 let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $QCoh(\mathcal{O}_\mathcal{X})$. This sequence may not be an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ but we know that it is up to parasitic modules, see Lemma 9.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned}
0 \to \mathcal{P}_1 \to \mathcal{F}_1 \to \mathcal{I}_2 \to 0 \\
0 \to \mathcal{I}_2 \to \mathcal{F}_2 \to \mathcal{Q}_2 \to 0 \\
0 \to \mathcal{P}_2 \to \mathcal{Q}_2 \to \mathcal{I}_3 \to 0 \\
0 \to \mathcal{I}_3 \to \mathcal{F}_3 \to \mathcal{P}_3 \to 0
\end{aligned}$$

of $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{P}_i$ parasitic. Note that each of the sheaves $\mathcal{P}_j$, $\mathcal{I}_j$, $\mathcal{Q}_j$ is an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 8.1. Applying $R^i f_*$ we obtain long exact sequences

$$\begin{aligned}
0 \to f_*\mathcal{P}_1 \to f_*\mathcal{F}_1 \to f_*\mathcal{I}_2 \to R^1 f_*\mathcal{P}_1 \to \ldots \\
0 \to f_*\mathcal{I}_2 \to f_*\mathcal{F}_2 \to f_*\mathcal{Q}_2 \to R^1 f_*\mathcal{I}_2 \to \ldots \\
0 \to f_*\mathcal{P}_2 \to f_*\mathcal{Q}_2 \to f_*\mathcal{I}_3 \to R^1 f_*\mathcal{P}_2 \to \ldots \\
0 \to f_*\mathcal{I}_3 \to f_*\mathcal{F}_3 \to f_*\mathcal{P}_3 \to R^1 f_*\mathcal{I}_3 \to \ldots
\end{aligned}$$

where are the terms are objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Y})$ by Proposition 8.1. By Lemma 9.3 the sheaves $R^i f_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q, see Lemma 10.2. Since Q is exact the maps

$$Q(R^i f_*\mathcal{F}_3) \cong Q(R^i f_*\mathcal{I}_3) \cong Q(R^i f_*\mathcal{Q}_2) \to Q(R^{i+1} f_*\mathcal{I}_2) \cong Q(R^{i+1} f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^i f_{QCoh,*}\}_{i\geq 0}$ into a cohomological δ-functor. □

0GQQ **Lemma 11.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $y : V \to \mathcal{Y}$ in $\mathrm{Ob}(\mathcal{Y})$ with y a flat morphism. Let $\mathcal{F}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then $(f_*\mathcal{F})(y) = (f_{QCoh,*}\mathcal{F})(y)$ and $(R^i f_*\mathcal{F})(y) = (R^i f_{QCoh,*}\mathcal{F})(y)$ for all $i \in \mathbf{Z}$.*

Proof. This follows from the construction of the functors $R^i f_{QCoh,*}$ in Proposition 11.1, the definition of parasitic modules in Definition 9.1, and Lemma 10.2 part (2). □

0GQR **Remark 11.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(\mathcal{O}_\mathcal{X})$. Then there is a canonical commutative diagram

$$\begin{array}{ccc}
f_{QCoh,*}\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_{QCoh,*}\mathcal{G} & \longrightarrow & f_*\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} f_*\mathcal{G} \\
\downarrow & & \downarrow c \\
f_{QCoh,*}(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}) & \longrightarrow & f_*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G})
\end{array}$$

The vertical arrow c on the right is the naive relative cup product (in degree 0), see Cohomology on Sites, Section 33. The source and target of c are in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, see Proposition 8.1. Applying Q to c we obtain the left vertical arrow as Q commutes with tensor products, see Remark 10.6. This construction is functorial in $\mathcal{F}$ and $\mathcal{G}$.

0782 **Lemma 11.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 14.5 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^q f_{QCoh,*}\mathcal{F} \longrightarrow R^q f_*\mathcal{F}$$

are parasitic modules on $\mathcal{Y}$ (Lemma 10.2) hence have vanishing cohomology (Lemma 9.3). It follows formally that $H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F}) = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$ and we win. □

0783 **Lemma 11.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = R^p g_{QCoh,*}(R^q f_{QCoh,*}\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 14.7 the Leray spectral sequence with

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converges to $R^{p+q}(g \circ f)_*\mathcal{F}$. By the results of Proposition 8.1 all the terms of this spectral sequence are objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{Z})$. Applying the exact functor $Q_\mathcal{Z} : LQCoh^{fbc}(\mathcal{O}_\mathcal{Z}) \to QCoh(\mathcal{O}_\mathcal{Z})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_\mathcal{Z})$ covering to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$. Hence the result follows if we can show that

$$Q_\mathcal{Z}(R^p g_*(R^q f_*\mathcal{F})) = Q_\mathcal{Z}(R^p g_*(Q_\mathcal{X}(R^q f_*\mathcal{F}))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_\mathcal{X}(R^q f_*\mathcal{F}) \longrightarrow R^q f_*\mathcal{F}$$

are parasitic (Lemma 10.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 9.3). □

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 20 and 21.

0784 **Proposition 11.6.** *Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Then there is a spectral sequence*

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ ($p+1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 20.1. □

0785 **Proposition 11.7.** *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that*

(1) *f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and*

(2) *g is quasi-compact and quasi-separated.*

If $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_{QCoh,*} f_p^* \mathcal{F} \Rightarrow R^{p+q} g_{QCoh,*} \mathcal{F}$$

in $QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q(g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

by Sheaves on Stacks, Proposition 21.1. Applying the exact functor $Q_\mathcal{Y} : LQCoh^{fbc}(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{Y})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_\mathcal{Y})$. □

12. Further remarks on quasi-coherent modules

0GQS In this section we collect some results that to help understand how to use quasi-coherent modules on algebraic stacks.

Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume $\mathcal{U}$ is represented by the algebraic space U. Consider the functor

$$a : Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow Mod(U_{étale}, \mathcal{O}_U), \quad \mathcal{F} \longmapsto f^*\mathcal{F}|_{U_{étale}}$$

given by pullback (Sheaves on Stacks, Section 7) followed by restriction (Sheaves on Stacks, Section 10). Applying this functor to locally quasi-coherent modules we obtain a functor

$$b : LQCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

See Sheaves on Stacks, Lemmas 12.3 and 14.1. We can further limit our functor to even smaller subcategories to obtain

$$c : LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

and

$$d : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(U_{étale}, \mathcal{O}_U)$$

About these functors we can say the following:[3]

(1) The functor a is exact. Namely, pullback $f^* = f^{-1}$ is exact (Sheaves on Stacks, Section 7) and restriction to $U_{étale}$ is exact, see Sheaves on Stacks, Equation (10.2.1).
(2) The functor b is exact. Namely, by Sheaves on Stacks, Lemma 12.4 the inclusion $LQCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is exact.
(3) The functor c is exact. Namely, by Proposition 8.1 the inclusion functor $LQCoh^{fbc}(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is exact.
(4) The functor d is right exact but not exact in general. Namely, by Sheaves on Stacks, Lemma 12.5 the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is right exact. We omit giving an example showing non-exactness.
(5) If f is flat, then d is exact. This follows on combining Lemma 4.1 and Sheaves on Stacks, Lemma 14.2.

[3]We suggest working out why these statements are true on a napkin instead of following the references given.

(6) If f is flat, then c kills parasitic objects. Namely, f^* preserves parasitic object by Lemma 9.2. Then for any scheme V étale over U and hence flat over $\mathcal{X}$ we see that $0 = f^*\mathcal{F}|_{V_{étale}} = c(\mathcal{F})|_{V_{étale}}$ by the compatibility of restriction with étale localization Sheaves on Stacks, Remark 10.2. Hence clearly $c(\mathcal{F}) = 0$.

(7) If f is flat, then $c = d \circ Q$. Namely, the kernel and cokernel of $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic by Lemma 10.2. Thus, since c is exact (3) and kills parasitic objects (6), we see that c applied to $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

(8) The functors a, b, c, d commute with colimits and arbitrary direct sums. This is true for f^* and restriction as left adjoints and hence it holds for a. Then it follows for b, c, d by the references given above.

(9) The functors a, b, c, d commute with tensor products.

(10) If f is flat and surjective, $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$, and $c(\mathcal{F}) = 0$, then $\mathcal{F}$ is parasitic. Namely, by (7) we get $d(Q(\mathcal{F})) = 0$. We may assume U is a scheme by the compatibility of restriction with étale localization (see reference above). Then Lemma 4.2 applied to $0 \to Q(\mathcal{F})$ and the morphism $f : U \to \mathcal{X}$ shows that $Q(\mathcal{F}) = 0$. Thus $\mathcal{F}$ is parasitic by Lemma 10.2.

(11) If f is flat and surjective, then the functor d reflects exactness. More precisely, let $\mathcal{F}^\bullet$ be a complex in $QCoh(\mathcal{O}_\mathcal{X})$. Then $\mathcal{F}^\bullet$ is exact in $QCoh(\mathcal{O}_\mathcal{X})$ if and only if $d(\mathcal{F}^\bullet)$ is exact. Namely, we have seen one implication in (5). For the other, suppose that $H^i(d(\mathcal{F}^\bullet)) = 0$. Then $\mathcal{G} = H^i(\mathcal{F}^\bullet)$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$ with $d(\mathcal{G}) = 0$. Hence $\mathcal{G}$ is both quasi-coherent and parasitic by (10), whence 0 for example by Remark 10.7.

0GQT (12) If f is flat, $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(QCoh(\mathcal{O}_\mathcal{X}))$, and $\mathcal{F}$ of finite presentation and let then we have

$$d(hom(\mathcal{F}, \mathcal{G})) = \mathcal{H}om_{\mathcal{O}_U}(d(\mathcal{F}), d(\mathcal{G}))$$

with notation as in Lemma 10.8. Perhaps the easiest way to see this is as follows

$$\begin{aligned} d(hom(\mathcal{F}, \mathcal{G})) &= d(Q(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G}))) \\ &= c(\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})) \\ &= f^* \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})|_{U_{étale}} \\ &= \mathcal{H}om_{\mathcal{O}_\mathcal{U}}(f^*\mathcal{F}, f^*\mathcal{G})|_{U_{étale}} \\ &= \mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{F}|_{U_{étale}}, f^*\mathcal{G}|_{U_{étale}}) \end{aligned}$$

The first equality by construction of hom. The second equality by (7). The third equality by definition of c. The fourth equality by Modules on Sites, Lemma 31.4. The final equality by the same reference applied to the flat morphism of ringed topoi $i_U(U_{étale}, \mathcal{O}_U) \to (\mathcal{U}_{étale}, \mathcal{O}_\mathcal{U})$ of Sheaves on Stacks, Lemma 10.1.

(13) add more here.

13. Colimits and cohomology

0GQU The following lemma in particular applies to diagrams of quasi-coherent sheaves.

0GQV **Lemma 13.1.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Then*

$$\mathrm{colim}_i\, H^p(\mathcal{X}, \mathcal{F}_i) \longrightarrow H^p(\mathcal{X}, \mathrm{colim}_i\, \mathcal{F}_i)$$

is an isomorphism for every filtered diagram of abelian sheaves on $\mathcal{X}$. The same is true for abelian sheaves on $\mathcal{X}_{étale}$ taking cohomology in the étale topology.

Proof. Let $\tau = fppf$, resp. $\tau = étale$. The lemma follows from Cohomology on Sites, Lemma 16.2 applied to the site $\mathcal{X}_\tau$. In order to check the assumptions we use Cohomology on Sites, Remark 16.3. Namely, let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{X}_\tau)$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (1) – (4) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjective and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_\tau$.
(2) Since coverings in $\mathcal{X}_\tau$ are fppf, resp. étale coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 7.4, resp. Lemma 4.4.
(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_\tau)$ is equal to the sheaf on $\mathcal{X}_\tau$ determined by the algebraic space $W = U \times_{x,\mathcal{X},x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_\tau$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 7.8 for example. Hence we can choose an affine scheme U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^\# \to h_x^\# \times h_{x'}^\#$ is surjective as desired.
(4) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. Let $a, b : U \to U'$ be a morphism over $\mathcal{X}$, i.e., $a, b : x \to x'$ is a morphism in $\mathcal{X}_\tau$. Then the equalizer of h_a and h_b is represented by the equalizer of $a, b : U \to U'$ which is affine scheme over $\mathcal{X}$ and hence in $\mathcal{B}$.

This finished the proof. □

0GQW **Lemma 13.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $\mathcal{X}$. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \mathrm{colim}\, R^p f_* \mathcal{F}_i.$$

The same is true for abelian sheaves on $\mathcal{X}_{étale}$ taking higher direct images in the étale topology.

Proof. We will prove this for the fppf topology; the proof for the étale topology is the same. Recall that $R^i f_* \mathcal{F}$ is the sheaf on $\mathcal{Y}_{fppf}$ associated to the presheaf

$$(y : V \to \mathcal{Y}) \longmapsto H^i(V \times_{y,\mathcal{Y}} \mathcal{X}, \mathrm{pr}^{-1}\mathcal{F})$$

See Sheaves on Stacks, Lemma 21.2. Recall that the colimit is the sheaf associated to the presheaf colimit. When V is affine, the fibre product $V \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact and quasi-separated. Hence we can apply Lemma 13.1 to $H^p(V \times_{\mathcal{Y}} \mathcal{X}, -)$ where V is affine. Since every V has an fppf covering by affine objects this proves the lemma. Some details omitted. □

0GQX **Lemma 13.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f_{QCoh,*}$ and the functors $R^i f_{QCoh,*}$ commute with direct sums and filtered colimits.*

Proof. The functors f_* and R^if_* commute with direct sums and filtered colimits on all modules by Lemma 13.2. The lemma follows as $f_{QCoh,*} = Q \circ f_*$ and $R^if_{QCoh,*} = Q \circ R^if_*$ and Q commutes with all colimits, see Lemma 10.2. □

0GQY **Lemma 13.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks. The functors $R^if_{QCoh,*}$, $i > 0$ vanish and the functor $f_{QCoh,*}$ is exact and commutes with direct sums and all colimits.*

Proof. Since we have $R^if_{QCoh,*} = Q \circ R^if_*$ we obtain the vanishing from Lemma 8.4. The vanishing implies that $f_{QCoh,*}$ is exact as $\{R^if_{QCoh,*}\}_{i\geq 0}$ form a δ-functor, see Proposition 11.1. Then $f_{QCoh,*}$ commutes with direct sums for example by Lemma 13.3. An exact functor which commutes with direct sums commutes with all colimits. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic stacks.

0GQZ **Lemma 13.5.** *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack. Let I be a directed set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of $\mathcal{O}_\mathcal{X}$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_\mathcal{X}$-module of finite presentation. Then we have*

$$\operatorname{colim}_i \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_\mathcal{X}(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

In particular, $\operatorname{Hom}_\mathcal{X}(\mathcal{G}, -)$ commutes with filtered colimits in $QCoh(\mathcal{O}_\mathcal{X})$.

Proof. The displayed equality is a special case of Modules on Sites, Lemma 27.12. In order to apply it, we need to check the hypotheses of Sites, Lemma 17.8 part (4) for the site $\mathcal{X}_{fppf}$. In order to do this, we will check hypotheses (2)(a), (2)(b), (2)(c) of Sites, Remark 17.9. Namely, let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{X}_{fppf})$ be the set of objects lying over affine schemes. In other words, an element of $\mathcal{B}$ is a morphism $x : U \to \mathcal{X}$ with U affine. We check each of the conditions (2)(a), (2)(b), and (2)(c) of the remark in turn:

(1) Since $\mathcal{X}$ is quasi-compact, there exists a surjective and smooth morphism $x : U \to \mathcal{X}$ with U affine (Properties of Stacks, Lemma 6.2). Then $h_x^\# \to *$ is a surjective map of sheaves on $\mathcal{X}_{fppf}$.
(2) Since coverings in $\mathcal{X}_{fppf}$ are fppf coverings, we see that every covering of $U \in \mathcal{B}$ is refined by a finite affine fppf covering, see Topologies, Lemma 7.4.
(3) Let $x : U \to \mathcal{X}$ and $x' : U' \to \mathcal{X}$ be in $\mathcal{B}$. The product $h_x^\# \times h_{x'}^\#$ in $Sh(\mathcal{X}_{fppf})$ is equal to the sheaf on $\mathcal{X}_{fppf}$ determined by the algebraic space $W = U \times_{x,\mathcal{X},x'} U'$ over $\mathcal{X}$: for an object $y : V \to \mathcal{X}$ of $\mathcal{X}_{fppf}$ we have $(h_x^\# \times h_{x'}^\#)(y) = \{f : V \to W \mid y = x \circ \mathrm{pr}_1 \circ f = x' \circ \mathrm{pr}_2 \circ f\}$. The algebraic space W is quasi-compact because $\mathcal{X}$ is quasi-separated, see Morphisms of Stacks, Lemma 7.8 for example. Hence we can choose an affine scheme U'' and a surjective étale morphism $U'' \to W$. Denote $x'' : U'' \to \mathcal{X}$ the composition of $U'' \to W$ and $W \to \mathcal{X}$. Then $h_{x''}^\# \to h_x^\# \times h_{x'}^\#$ is surjective as desired.

For the final statement, observe that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ commutes with colimits and that finitely presented modules are quasi-coherent. See Sheaves on Stacks, Lemma 15.1. □

14. The lisse-étale and the flat-fppf sites

0786 In the book [LMB00] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 60 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the Stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

0787 **Definition 14.1.** Let $\mathcal{X}$ be an algebraic stack.

(1) The *lisse-étale site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{lisse,étale}$[4] of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{lisse,étale}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{lisse,étale}$ which forms a covering of $\mathcal{X}_{étale}$.

(2) The *flat-fppf site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{flat,fppf}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is flat. A covering of $\mathcal{X}_{flat,fppf}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{flat,fppf}$ which forms a covering of $\mathcal{X}_{fppf}$.

We denote $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ the restriction of $\mathcal{O}_{\mathcal{X}}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to "topological" properties in this lemma).

0788 **Lemma 14.2.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *The inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{lisse,étale}) \longrightarrow Sh(\mathcal{X}_{étale})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

(2) *The inclusion functor $\mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

[4] In the literature the site is denoted Lis-ét($\mathcal{X}$) or Lis-Et($\mathcal{X}$) and the associated topos is denoted $\mathcal{X}_{\text{lis-ét}}$ or $\mathcal{X}_{\text{lis-et}}$. In the Stacks project our convention is to name the site and denote the corresponding topos by $Sh(\mathcal{C})$.

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 13.1 and 20.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 21.5 and 21.7. Parts (d), (e) follow from Modules on Sites, Lemmas 16.2 and 16.4. Part (f) is immediate. □

0GR0 **Lemma 14.3.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 14.2.*

(1) *For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have*
   (a) *$H^p(\mathcal{X}_{étale}, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}, g^{-1}\mathcal{F})$, and*
   (b) *$H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{lisse,étale}/x, g^{-1}\mathcal{F})$ for any object x of $\mathcal{X}_{lisse,étale}$.*
   The same holds for sheaves of modules.
(2) *For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{fppf}$ we have*
   (a) *$H^p(\mathcal{X}_{fppf}, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}, g^{-1}\mathcal{F})$, and*
   (b) *$H^p(x, \mathcal{F}) = H^p(\mathcal{X}_{flat,fppf}/x, g^{-1}\mathcal{F})$ for any object x of $\mathcal{X}_{flat,fppf}$.*
   The same holds for sheaves of modules.

Proof. Part (1)(a) follows from Sheaves on Stacks, Lemma 23.3 applied to the inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$. Part (1)(b) follows from part (1)(a). Namely, if x lies over the scheme U, then the site $\mathcal{X}_{étale}/x$ is equivalent to $(Sch/U)_{étale}$ and $\mathcal{X}_{lisse,étale}$ is equivalent to $U_{lisse,étale}$. Part (2) is proved in the same manner. □

0789 **Lemma 14.4.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 14.2.*

(1) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$$
   which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.
(2) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$$
   which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 41.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps Modules on Sites, Equation (41.2.1) are isomorphisms.

Lisse-étale case. Let $x \in \mathrm{Ob}(\mathcal{X}_{lisse,étale})$ lying over a scheme U with $x : U \to \mathcal{X}$ smooth. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{lisse,étale}/x \longrightarrow \mathcal{X}_{étale}/x$$
The right hand side is identified with $(Sch/U)_{étale}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 49.2 applies.

Flat-fppf case. Let $x \in \mathrm{Ob}(\mathcal{X}_{flat,fppf})$ lying over a scheme U with $x : U \to \mathcal{X}$ flat. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{flat,fppf}/x \longrightarrow \mathcal{X}_{fppf}/x$$
The right hand side is identified with $(Sch/U)_{fppf}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 49.2 applies.

In both cases the equality $g^*g_! = \mathrm{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 14.2. □

078A **Lemma 14.5.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemmas 14.2 and 14.4.*

(1) *We have $g_!\mathcal{O}_{\mathcal{X}_{lisse,étale}} = \mathcal{O}_{\mathcal{X}}$.*
(2) *We have $g_!\mathcal{O}_{\mathcal{X}_{flat,fppf}} = \mathcal{O}_{\mathcal{X}}$.*

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{étale}$ (resp. $\mathcal{C} = \mathcal{X}_{fppf}$) and we denote $\mathcal{C}' = \mathcal{X}_{lisse,étale}$ (resp. $\mathcal{C}' = \mathcal{X}_{flat,fppf}$). Then $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. In this proof we will think of objects V of $\mathcal{C}$ as schemes over $\mathcal{X}$ and objects U of $\mathcal{C}'$ as schemes smooth (resp. flat) over $\mathcal{X}$. Finally, we write $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the $\mathcal{O}$-module homomorphism $g_!\mathcal{O}' \to \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

Recall that $g_!\mathcal{O}'$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}'$ given by the rule

$$V \longmapsto \mathrm{colim}_{V \to U} \mathcal{O}'(U)$$

where the colimit is taken in the category of abelian groups (Modules on Sites, Definition 16.1). Below we will use frequently that if

$$V \to U \to U'$$

are morphisms and if $f' \in \mathcal{O}'(U')$ restricts to $f \in \mathcal{O}'(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}' \to \mathcal{O}$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Let us prove that $g_!\mathcal{O}' \to \mathcal{O}$ is surjective. Let $h \in \mathcal{O}(V)$ for some object V of $\mathcal{C}$. It suffices to show that h is locally in the image. Choose an object U of $\mathcal{C}'$ corresponding to a surjective smooth morphism $U \to \mathcal{X}$. Since $U \times_{\mathcal{X}} V \to V$ is surjective smooth, after replacing V by the members of an étale covering of V we may assume there exists a morphism $V \to U$, see Topologies on Spaces, Lemma 4.4. Using h we obtain a morphism $V \to U \times \mathbf{A}^1$ such that writing $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[t])$ the element $t \in \mathcal{O}(U \times \mathbf{A}^1)$ pulls back to h. Since $U \times \mathbf{A}^1$ is an object of $\mathcal{C}'$ we see that $(V \to U \times \mathbf{A}^1, t)$ is an element of the colimit above which maps to $h \in \mathcal{O}(V)$ as desired.

Suppose that $s \in g_!\mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$\mathrm{colim}_{V \to U} \mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \to U_i$, U_i smooth (resp. flat) over $\mathcal{X}$, and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U_n$. Note that W is still smooth (resp. flat) over $\mathcal{X}$, i.e., defines an object of $\mathcal{C}'$. The fibre product

$$V' = V \times_{(\varphi_1, \ldots, \varphi_n), U} W$$

is surjective étale over V, hence it suffices to show that s maps to zero in $g_!\mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W. In other words, we have reduced to the case of (φ, s) where

$\varphi : V \to U$ is a morphism with U in $\mathcal{C}'$ and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{(\varphi,0)} & U \times \mathbf{A}^1 \\ & \searrow^{\varphi} & \uparrow{\scriptstyle (\mathrm{id},0)} \\ & & U \end{array}$$

we see that $((\varphi, 0) : V \to U \times \mathbf{A}^1, \mathrm{pr}_2^* x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\mathrm{pr}_1^* s + \mathrm{pr}_2^* x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over $\mathcal{X}$. Then we see that $(V \to Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. □

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

07AR **Lemma 14.6.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module with the flat base change property on $\mathcal{X}_{étale}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*
   (b) *$g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{lisse,étale}) \to Sh(\mathcal{X}_{étale})$ is as in Lemma 14.2.*

(2) *Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{fppf}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*
   (b) *$g^*\mathcal{F} = 0$ where $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ is as in Lemma 14.2.*

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) $\Rightarrow$ (1)(b) is immediate as well. To see (1)(b) $\Rightarrow$ (1)(a) let U be a scheme and let $x : U \to \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of $\mathcal{X}$. Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{étale}} = 0$. Let $V \to \mathcal{X}$ be an flat morphism where V is a scheme. Set $W = U \times_\mathcal{X} V$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow[q]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q_{small}^* is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p_{small}^*\mathcal{F}|_{U_{étale}} \cong q_{small}^*\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of g^*, then $\mathcal{F}|_{V_{étale}} = 0$ as desired. □

15. Functoriality of the lisse-étale and flat-fppf sites

0GR1 The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks. We warn the reader that the lisse-étale and flat-fppf topoi are not functorial with respect to all morphisms of algebraic stacks, see Examples, Section 60.

07AT **Lemma 15.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is smooth, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{lisse,étale}) & \xrightarrow{g'} & Sh(\mathcal{X}_{étale}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{lisse,étale}) & \xrightarrow{g} & Sh(\mathcal{Y}_{étale}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

(2) *If f is flat, then f restricts to a continuous and cocontinuous functor $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$ which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{flat,fppf}) & \xrightarrow{g'} & Sh(\mathcal{X}_{fppf}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{flat,fppf}) & \xrightarrow{g} & Sh(\mathcal{Y}_{fppf}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ and $g'_!(f')^{-1} = f^{-1}g_!$.

Proof. The initial statement comes from the fact that if $x \in \mathrm{Ob}(\mathcal{X})$ lies over a scheme U such that $x : U \to \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \to \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 33.2 and 25.2. The induced functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 14.2) and Sites, Lemma 21.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let $\mathcal{F}$ be a sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 45. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}f_*\mathcal{F}\right)(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \to V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \to \mathcal{X}$ is smooth (resp. flat). Denote

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{lisse,étale} \text{ (resp. } \mathcal{X}_{flat,fppf})$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (5.0.1) shows that for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) we have

07AU (15.1.1)
$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

Since $(g')^{-1}$ is restriction we see that

$$\left(f'_*(g')^{-1}\mathcal{F}\right)(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V \times_{y,\mathcal{Y}} \mathcal{X})'})$$

By Sheaves on Stacks, Lemma 23.3 we see that

$$\Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'}) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_*, the adjointness of $g_!$ and g^{-1}, and their "primed" versions. □

0GR2 **Lemma 15.2.** *With assumptions and notation as in Lemma 15.1. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Then*

07AW (15.2.1) $$R^p f'_* \mathcal{H} = \text{sheaf associated to } y \longmapsto H^p((V \times_{y,\mathcal{Y}} \mathcal{X})', (pr')^{-1}\mathcal{H})$$

Here y is an object of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) lying over the scheme V and the notation $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ and pr' are explained in the proof.

Proof. As in the proof of Lemma 15.1 let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects (x, φ) where x is an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) and $\varphi : f(x) \to y$ is a morphism in $\mathcal{Y}$. By Equation (15.1.1) we have

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

where pr' is the projection. For an object (x, φ) of $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ we can think of φ as a section of $(f')^{-1}h_y$ over x. Thus $(V \times_{\mathcal{Y}} \mathcal{X})'$ is the localization of the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) at the sheaf of sets $(f')^{-1}h_y$, see Sites, Lemma 30.3. The morphism

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{lisse,étale} \text{ (resp. } \mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{flat,fppf})$$

is the localization morphism. In particular, the pullback $(\mathrm{pr}')^{-1}$ preserves injective abelian sheaves, see Cohomology on Sites, Lemma 13.3.

Choose an injective resolution $\mathcal{H} \to \mathcal{I}^\bullet$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). By the formula for pushforward we see that $R^i f'_* \mathcal{H}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{array}{c} \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((V \times_{y,\mathcal{Y}} \mathcal{X})', (\mathrm{pr}')^{-1}\mathcal{I}^{i+1}\Big) \end{array}$$

Since $(\mathrm{pr}')^{-1}$ is exact and preserves injectives the complex $(\mathrm{pr}')^{-1}\mathcal{I}^\bullet$ is an injective resolution of $(\mathrm{pr}')^{-1}\mathcal{H}$ and the proof is complete. □

0GR3 **Lemma 15.3.** *With assumptions and notation as in Lemma 15.1 the canonical (base change) map*

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$).

Proof. Comparing the formula for $g^{-1}R^pf_*\mathcal{F}$ and $R^pf'_*(g')^{-1}\mathcal{F}$ given in Sheaves on Stacks, Lemma 21.2 and Lemma 15.2 we see that it suffices to show

$$H^p((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'}) = H^p_\tau(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

where $\tau = étale$ (resp. $\tau = fppf$). Here y is an object of $\mathcal{Y}$ lying over a scheme V such that the morphism $y : V \to \mathcal{Y}$ is smooth (resp. flat). This equality follows from Sheaves on Stacks, Lemma 23.3. Although we omit the verification of the assumptions of the lemma, we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition. □

16. Quasi-coherent modules and the lisse-étale and flat-fppf sites

07AY In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

07AZ **Lemma 16.1.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*

(2) *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*

Proof. Proof of (1). We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 33.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still quasi-coherent, see Modules on Sites, Lemma 23.4. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a smooth surjective morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 19.10 there exists an étale covering $\{x_i \to x\}_{i\in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^*f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 23.3. By Sheaves on Stacks, Lemma 11.4 we see that $x^*\mathcal{F}$ is an fppf sheaf and since x was arbitrary we see that $\mathcal{F}$ is a sheaf in the fppf topology. Applying Sheaves on Stacks, Lemma 11.3 we see that $\mathcal{F}$ is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 25.2 and 27.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 19.10 there exists an fppf covering $\{x_i \to x\}_{i\in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^*f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 23.3. By Sheaves on Stacks, Lemma 11.3 we see that $\mathcal{F}$ is quasi-coherent. □

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 23.

07B0 **Lemma 16.2.** *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 14.2.*

(1) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*
(2) *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 15.1 we see that $f_i^{-1}g_!\mathcal{H} = g_{i,!}(f_i')^{-1}\mathcal{H}$. Using Lemma 16.1 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation. Say we have

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 14.4 and Categories, Lemma 24.5) we conclude that there exists an exact sequence

$$\bigoplus\nolimits_{j\in J} g_!\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} g_!\mathcal{O} \longrightarrow g_!\mathcal{H} \longrightarrow 0$$

Lemma 14.5 shows that $g_!\mathcal{O} = \mathcal{O}_{\mathcal{X}}$. In case (2) we are done. In case (1) we apply Sheaves on Stacks, Lemma 11.4 to conclude. □

07B1 **Lemma 16.3.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *With g as in Lemma 14.2 for the lisse-étale site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

   (b) *if $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 10.1.*
(2) *With g as in Lemma 14.2 for the flat-fppf site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_{\mathcal{X}}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

   (b) *if $\mathcal{F}$ is in $LQCoh^{fbc}(\mathcal{O}_{\mathcal{X}})$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 10.1.*

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 23.4. Hence g^{-1} preserves the categories of quasi-coherent modules; here we use that $QCoh(\mathcal{O}_{\mathcal{X}}) = QCoh(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ by Sheaves on Stacks, Lemma 11.4. The same is true for $g_!$ by Lemma 16.2. We

know that $\mathcal{H} \to g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 14.2. Conversely, if $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then the map $g_!g^{-1}\mathcal{F} \to \mathcal{F}$ is a map of quasi-coherent modules on $\mathcal{X}$ whose restriction to any scheme smooth over $\mathcal{X}$ is an isomorphism. Then the discussion in Sheaves on Stacks, Sections 14 and 15 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let $\mathcal{F}$ be an object of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. By Lemma 10.2 the kernel and cokernel of the map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic. Hence by Lemma 14.6 and since $g^* = g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F}) \to g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F}) \to Q(\mathcal{F})$ is an isomorphism by our arguments above. □

07B4 **Lemma 16.4.** *Let $\mathcal{X}$ be an algebraic stack.*

(1) *$QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$.*
(2) *$QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$.*

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 10.3.

Since 0 is a quasi-coherent module on any ringed site we see that (1) holds.

By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $Mod(\mathcal{O})$, so (2) holds.

Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{lisse,étale}$ or $\mathcal{X}_{flat,fppf}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for $\mathcal{G}$ and φ, see Lemma 14.4. By Lemma 16.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. By Sheaves on Stacks, Lemma 15.1 we have that $\mathrm{Coker}(g_!\varphi)$ is a quasi-coherent module on $\mathcal{X}$ (and the cokernel in the category of quasi-coherent modules on $\mathcal{X}$). Since g^* is exact (see Lemma 14.2) $g^*\,\mathrm{Coker}(g_!\varphi) = \mathrm{Coker}(g^*g_!\varphi) = \mathrm{Coker}(\varphi)$ is quasi-coherent too (see Lemma 16.3). By Proposition 8.1 the kernel $\mathrm{Ker}(g_!\varphi)$ is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$. Since g^* is exact, we have $g^*\,\mathrm{Ker}(g_!\varphi) = \mathrm{Ker}(g^*g_!\varphi) = \mathrm{Ker}(\varphi)$. Since g^* maps objects of $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ to quasi-coherent modules by Lemma 16.3 we conclude that $\mathrm{Ker}(\varphi)$ is quasi-coherent as well. This proves (3).

Finally, suppose that

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is an extension of $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-modules (resp. $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-modules) with $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent. To prove (4) and finish the proof we have to show that $\mathcal{E}$ is quasi-coherent on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). Let U be an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$; we think of U as a scheme smooth (resp. flat) over $\mathcal{X}$. We have to show that the restriction of $\mathcal{E}$ to $U_{lisse,étale}$ (resp. $= U_{flat,fppf}$) is quasi-coherent. Thus we may assume that $\mathcal{X} = U$ is a scheme. Because $\mathcal{G}$ is quasi-coherent on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$), we may assume, after replacing U by the members of an étale (resp. fppf) covering, that $\mathcal{G}$ has a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

on $U_{lisse,étale}$ (resp. $U_{flat,fppf}$) where $\mathcal{O}$ is the structure sheaf on the site. We may also assume U is affine. Since $\mathcal{F}$ is quasi-coherent, we have

$$H^1(U_{lisse,étale}, \mathcal{F}) = 0, \quad \text{resp.} \quad H^1(U_{flat,fppf}, \mathcal{F}) = 0$$

Namely, $\mathcal{F}$ is the pullback of a quasi-coherent module $\mathcal{F}'$ on the big site of U (by Lemma 16.3), cohomology of $\mathcal{F}$ and $\mathcal{F}'$ agree (by Lemma 14.3), and we know that

the cohomology of $\mathcal{F}'$ on the big site of the affine scheme U is zero (to get this in the current situation you have to combine Descent, Propositions 8.9 and 9.3 with Cohomology of Schemes, Lemma 2.2). Thus we can lift the map $\bigoplus_{i\in I}\mathcal{O}\to\mathcal{G}$ to $\mathcal{E}$. A diagram chase shows that we obtain an exact sequence

$$\bigoplus\nolimits_{j\in J}\mathcal{O}\to\mathcal{F}\oplus\bigoplus\nolimits_{i\in I}\mathcal{O}\to\mathcal{E}\to 0$$

By (3) proved above, we conclude that $\mathcal{E}$ is quasi-coherent as desired. □

17. Coherent sheaves on locally Noetherian stacks

0GR4 This section is the analogue of Cohomology of Spaces, Section 12. We have defined the notion of a coherent module on any ringed topos in Modules on Sites, Section 23. However, for any algebraic stack $\mathcal{X}$ the category of coherent $\mathcal{O}_\mathcal{X}$-modules is zero, essentially because the site $\mathcal{X}$ contains too many non-Noetherian objects (even if $\mathcal{X}$ is itself locally Noetherian). Instead, we will define coherent modules using the following lemma.

0GR5 **Lemma 17.1.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module. The following are equivalent*

1. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_\mathcal{X}$-module,*
2. *$\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module of finite presentation,*
3. *$\mathcal{F}$ is quasi-coherent and for any morphism $f:U\to\mathcal{X}$ where U is a locally Noetherian algebraic space, the pullback $f^*\mathcal{F}|_{U_{étale}}$ is coherent, and*
4. *$\mathcal{F}$ is quasi-coherent and there exists an algebraic space U and a morphism $f:U\to\mathcal{X}$ which is locally of finite type, flat, and surjective, such that the pullback $f^*\mathcal{F}|_{U_{étale}}$ is coherent.*

Proof. Let $f:U\to\mathcal{X}$ be as in (4). Then U is locally Noetherian (Morphisms of Stacks, Lemma 17.5) and we see that the statement of the lemma makes sense. Additionally, f is locally of finite presentation by Morphisms of Stacks, Lemma 27.5. Let x be an object of $\mathcal{X}$ lying over the scheme V. In order to prove (2) we have to show that, after replacing V by the members of an fppf covering of V, the restriction $x^*\mathcal{F}$ has a global finite presentation on $\mathcal{X}/x\cong(Sch/V)_{fppf}$. The projection $W=U\times_\mathcal{X}V\to V$ is locally of finite presentation, flat, and surjective. Hence we may replace V by the members of an étale covering of W by schemes and assume we have a morphism $h:V\to U$ with $f\circ h=x$. Since $\mathcal{F}$ is quasi-coherent, we see that the restriction $x^*\mathcal{F}$ is the pullback of $h^*_{small}(f^*\mathcal{F})|_{U_{étale}}$ by π_V, see Sheaves on Stacks, Lemma 14.2. Since $f^*\mathcal{F}|_{U_{étale}}$ locally in the étale topology has a finite presentation by assumption, we conclude (4) ⇒ (2).

Part (2) implies (1) for any ringed topos (immediate from the definition). The properties "finite type" and "quasi-coherent" are preserved under pullback by any morphism of ringed topoi, see Modules on Sites, Lemma 23.4. Hence (1) implies (3), see Cohomology of Spaces, Lemma 12.2. Finally, (3) trivially implies (4). □

0GR6 **Definition 17.2.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack. An $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ is called *coherent* if $\mathcal{F}$ satisfies one (and hence all) of the equivalent conditions of Lemma 17.1. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is denote $Coh(\mathcal{O}_\mathcal{X})$.

0GR7 **Lemma 17.3.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The module $\mathcal{O}_\mathcal{X}$ is coherent, any invertible $\mathcal{O}_\mathcal{X}$-module is coherent, and more generally any finite locally free $\mathcal{O}_\mathcal{X}$-module is coherent.*

Proof. Follows from the definition and Cohomology of Spaces, Lemma 12.2. □

0GR8 **Lemma 17.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally Noetherian algebraic stacks. Then f^* sends coherent modules on $\mathcal{Y}$ to coherent modules on $\mathcal{X}$.*

Proof. Immediate from the definition and the fact that pullback for any morphism of ringed topoi preserves finitely presented modules, see Modules on Sites, Lemma 23.4. □

0GR9 **Lemma 17.5.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. The category of coherent $\mathcal{O}_\mathcal{X}$-modules is abelian. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of coherent $\mathcal{O}_\mathcal{X}$-modules, then*

(1) *the cokernel* $\mathrm{Coker}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module,*
(2) *the image* $\mathrm{Im}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ is a coherent $\mathcal{O}_\mathcal{X}$-module, and*
(3) *the kernel* $\mathrm{Ker}(\varphi)$ *computed in Mod$(\mathcal{O}_\mathcal{X})$ may not be coherent, but it is in $LQCoh^{fbc}(\mathcal{O}_\mathcal{X})$ and $Q(\mathrm{Ker}(\varphi))$ is coherent and is the kernel of φ in Coh$(\mathcal{O}_\mathcal{X})$.*

The inclusion functor Coh$(\mathcal{O}_\mathcal{X}) \to$ QCoh$(\mathcal{O}_\mathcal{X})$ is exact.

Proof. The rules given for taking kernels, images, and cokernels in Coh$(\mathcal{O}_\mathcal{X})$ agree with the prescription for quasi-coherent modules in Remark 10.5. Hence the lemma will follow if we can show that the quasi-coherent modules $\mathrm{Coker}(\varphi)$, $\mathrm{Im}(\varphi)$, and $Q(\mathrm{Ker}(\varphi))$ are coherent. By Lemma 17.1 it suffices to prove this after restricting to $U_{étale}$ for some surjective smooth morphism $f : U \to \mathcal{X}$. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ is exact. Hence $f^* \mathrm{Coker}(\varphi)$ and $f^* \mathrm{Im}(\varphi)$ are the cokernel and image of a map between coherent $\mathcal{O}_U$-modules hence coherent as desired. The functor $\mathcal{F} \mapsto f^*\mathcal{F}|_{U_{étale}}$ kills parasitic modules by Lemma 9.2. Hence $f^*Q(\mathrm{Ker}(\varphi))|_{U_{étale}} = f^* \mathrm{Ker}(\varphi)|_{U_{étale}}$ by part (2) of Lemma 10.2. Thus we conclude that $Q(\mathrm{Ker}(\varphi))$ is coherent in the same way. □

0GRA **Lemma 17.6.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in Mod$(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ coherent, then $\mathcal{F}_2$ is coherent.*

Proof. By Sheaves on Stacks, Lemma 15.1 part (7) we see that $\mathcal{F}_2$ is quasi-coherent. Then we can check that $\mathcal{F}_2$ is coherent by restricting to $U_{étale}$ for some $U \to \mathcal{X}$ surjective and smooth. This follows from Cohomology of Spaces, Lemma 12.3. Some details omitted. □

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. This does not hold for modules on a general ringed topos.

0GRB **Lemma 17.7.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Then Coh$(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of QCoh$(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. We have*

(1) *if $\mathcal{F}$ is coherent and φ surjective, then $\mathcal{G}$ is coherent,*
(2) *if $\mathcal{F}$ is coherent, then* $\mathrm{Im}(\varphi)$ *is coherent, and*
(3) *if $\mathcal{G}$ coherent and* $\mathrm{Ker}(\varphi)$ *parasitic, then $\mathcal{F}$ is coherent.*

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Then the functor $f^* : QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_U)$ is exact (Lemma 4.1) and moreover by definition Coh$(\mathcal{O}_\mathcal{X})$ is the full subcategory of QCoh$(\mathcal{O}_\mathcal{X})$ consisting of objects $\mathcal{F}$

such that $f^*\mathcal{F}$ is in $Coh(\mathcal{O}_U)$. The statement that $Coh(\mathcal{O}_\mathcal{X})$ is a Serre subcategory of $QCoh(\mathcal{O}_\mathcal{X})$ follows immediately from this and the corresponding fact for U, see Cohomology of Spaces, Lemmas 12.3 and 12.4. We omit the proof of (1), (2), and (3). Hint: compare with the proof of Lemma 17.5. □

Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be surjective, locally of finite presentation, and flat. Observe that U is locally Noetherian (Morphisms of Stacks, Lemma 17.5). Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ the isomorphism constructed in Algebraic Stacks, Lemma 16.1 and Remark 16.3. As in Sheaves on Stacks, Section 15 we obtain equivalences

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Sheaves on Stacks, Proposition 14.3. Recall that in Groupoids in Spaces, Section 13 we have defined the full subcategory

$$Coh(U, R, s, t, c) \subset QCoh(U, R, s, t, c)$$

of *coherent modules* as those $(\mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a coherent $\mathcal{O}_U$-module.

0GRC **Lemma 17.8.** *In the situation discussed above, the equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$ sends coherent sheaves to coherent sheaves and vice versa, i.e., induces an equivalence $Coh(\mathcal{O}_\mathcal{X}) \cong Coh(U, R, s, t, c)$.*

Proof. This is immediate from the definition of coherent $\mathcal{O}_\mathcal{X}$-modules. For bookkeeping purposes: the material above uses Morphisms of Stacks, Lemma 17.5, Algebraic Stacks, Lemma 16.1 and Remark 16.3, Sheaves on Stacks, Section 15, Sheaves on Stacks, Proposition 14.3, and Groupoids in Spaces, Section 13. □

0GRD **Lemma 17.9.** *Let $\mathcal{X}$ be a locally Noetherian algebraic stack. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent be $\mathcal{O}_\mathcal{X}$-modules. Then the internal hom $hom(\mathcal{F}, \mathcal{G})$ constructed in Lemma 10.8 is a coherent $\mathcal{O}_\mathcal{X}$-module.*

Proof. Let $U \to \mathcal{X}$ be a smooth surjective morphism from a scheme. By item (12) in Section 12 we see that the restriction of $hom(\mathcal{F}, \mathcal{G})$ to U is the Hom sheaf of the restrictions. Hence this lemma follows from the case of algebraic spaces, see Cohomology of Spaces, Lemma 12.5. □

18. Coherent sheaves on Noetherian stacks

0GRE This section is the analogue of Cohomology of Spaces, Section 13.

0GRF **Lemma 18.1.** *Let $\mathcal{X}$ be a Noetherian algebraic stack. Every quasi-coherent $\mathcal{O}_\mathcal{X}$-module is the filtered colimit of its coherent submodules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_\mathcal{X}$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_\mathcal{X}$-submodule which contains both of them, see Lemma 17.7. In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to \mathcal{X}$ a surjective smooth morphism (Properties of Stacks, Lemma 6.2). Set $R = U \times_\mathcal{X} U$ so that $\mathcal{X} = [U/R]$ as in Algebraic Stacks, Lemma 16.2. By Lemma 17.8 we have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$

and $Coh(\mathcal{O}_X) = Coh(U, R, s, t, c)$. In this way we reduce to the problem of proving the corresponding thing for $QCoh(U, R, s, t, c)$. This is Groupoids in Spaces, Lemma 13.4; we check its assumptions in the next paragraph.

We urge the reader to skip the rest of the proof. The affine scheme U is Noetherian; this follows from our definition of $\mathcal{X}$ being locally Noetherian, see Properties of Stacks, Definition 7.2 and Remark 7.3. The projection morphisms $s, t : R \to U$ are smooth (see reference given above) and quasi-separated and quasi-compact (Morphisms of Stacks, Lemma 7.8). In particular, R is a quasi-compact and quasi-separated algebraic space smooth over U and hence Noetherian (Morphisms of Spaces, Lemma 28.6). □

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