

# SHEAVES ON ALGEBRAIC STACKS

06TF

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## 1. Introduction

06TG There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes  $\mathcal{O}$ -modules

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whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [Ols07]. We will return to this in Cohomology of Stacks, Section 1.

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site  $\mathcal{X}_{smooth}$  defined below.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 2 for an explanation.

## 2. Conventions

06TH The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 2. For convenience we repeat them here.

We work in a suitable big fppf site  $Sch_{fppf}$  as in Topologies, Definition 7.6. So, if not explicitly stated otherwise all schemes will be objects of  $Sch_{fppf}$ . We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base  $S$  contained in  $Sch_{fppf}$ . And we will then work with the big fppf site  $(Sch/S)_{fppf}$ , see Topologies, Definition 7.8. The absolute case can be recovered by taking  $S = \text{Spec}(\mathbf{Z})$ .

## 3. Presheaves

06TI In this section we define presheaves on categories fibred in groupoids over  $(Sch/S)_{fppf}$ , but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

06TJ **Definition 3.1.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids.

- (1) A *presheaf on  $\mathcal{X}$*  is a presheaf on the underlying category of  $\mathcal{X}$ .
- (2) A *morphism of presheaves on  $\mathcal{X}$*  is a morphism of presheaves on the underlying category of  $\mathcal{X}$ .

We denote  $PSh(\mathcal{X})$  the category of presheaves on  $\mathcal{X}$ .

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian presheaves*, i.e., presheaves of abelian groups, is denoted  $PAb(\mathcal{X})$ .

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Recall that this means just that  $f$  is a functor over  $(Sch/S)_{fppf}$ . The material in Sites, Section 18 provides us with a pair of adjoint functors<sup>1</sup>

06TK (3.1.1)  $f^p : PSh(\mathcal{Y}) \longrightarrow PSh(\mathcal{X}) \quad \text{and} \quad {}_p f : PSh(\mathcal{X}) \longrightarrow PSh(\mathcal{Y}).$

<sup>1</sup>These functors will be denoted  $f^{-1}$  and  $f_*$  after Lemma 4.4 has been proved.

The adjointness is

$$\text{Mor}_{\text{PSh}(\mathcal{X})}(f^p \mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{PSh}(\mathcal{Y})}(\mathcal{G}, {}_p f \mathcal{F})$$

where  $\mathcal{F} \in \text{Ob}(\text{PSh}(\mathcal{X}))$  and  $\mathcal{G} \in \text{Ob}(\text{PSh}(\mathcal{Y}))$ . We call  $f^p \mathcal{G}$  the *pullback* of  $\mathcal{G}$ . It follows from the definitions that

$$f^p \mathcal{G}(x) = \mathcal{G}(f(x))$$

for any  $x \in \text{Ob}(\mathcal{X})$ . The presheaf  ${}_p f \mathcal{F}$  is called the *pushforward* of  $\mathcal{F}$ . It is described by the formula

$$({}_p f \mathcal{F})(y) = \lim_{f(x) \rightarrow y} \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

06TL **Lemma 3.2.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  be 1-morphisms of categories fibred in groupoids over  $(\text{Sch}/S)_{fppf}$ . Then  $(g \circ f)^p = f^p \circ g^p$  and there is a canonical isomorphism  ${}_p(g \circ f) \rightarrow {}_p g \circ {}_p f$  compatible with adjointness of  $(f^p, {}_p f)$ ,  $(g^p, {}_p g)$ , and  $((g \circ f)^p, {}_p(g \circ f))$ .*

**Proof.** Let  $\mathcal{H}$  be a presheaf on  $\mathcal{Z}$ . Then  $(g \circ f)^p \mathcal{H} = f^p(g^p \mathcal{H})$  is given by the equalities

$$(g \circ f)^p \mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p \mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation  ${}_p(g \circ f) \rightarrow {}_p g \circ {}_p f$ . Let  $\mathcal{F}$  be a presheaf on  $\mathcal{X}$ . If  $z$  is an object of  $\mathcal{Z}$  then we get a category  $\mathcal{J}$  of quadruples  $(x, f(x) \rightarrow y, y, g(y) \rightarrow z)$  and a category  $\mathcal{I}$  of pairs  $(x, g(f(x)) \rightarrow z)$ . There is a canonical functor  $\mathcal{J} \rightarrow \mathcal{I}$  sending the object  $(x, \alpha : f(x) \rightarrow y, y, \beta : g(y) \rightarrow z)$  to  $(x, \beta \circ f(\alpha) : g(f(x)) \rightarrow z)$ . This gives the arrow in

$$\begin{aligned} ({}_p(g \circ f) \mathcal{F})(z) &= \lim_{g(f(x)) \rightarrow z} \mathcal{F}(x) \\ &= \lim_{\mathcal{I}} \mathcal{F} \\ &\rightarrow \lim_{\mathcal{J}} \mathcal{F} \\ &= \lim_{g(y) \rightarrow z} \left( \lim_{f(x) \rightarrow y} \mathcal{F}(x) \right) \\ &= ({}_p g \circ {}_p f \mathcal{F})(z) \end{aligned}$$

by Categories, Lemma 14.8. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define  ${}_p(g \circ f) \cong {}_p g \circ {}_p f$  as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of  $(f^p, {}_p f)$ ,  $(g^p, {}_p g)$ , and  $((g \circ f)^p, {}_p(g \circ f))$  means that given presheaves  $\mathcal{H}$  and  $\mathcal{F}$  as above we have a commutative diagram

$$\begin{array}{ccccc} \text{Mor}_{\text{PSh}(\mathcal{X})}(f^p g^p \mathcal{H}, \mathcal{F}) & \xlongequal{\quad} & \text{Mor}_{\text{PSh}(\mathcal{Y})}(g^p \mathcal{H}, {}_p f \mathcal{F}) & \xlongequal{\quad} & \text{Mor}_{\text{PSh}(\mathcal{Y})}(\mathcal{H}, {}_p g {}_p f \mathcal{F}) \\ \parallel & & & & \uparrow \\ \text{Mor}_{\text{PSh}(\mathcal{X})}((g \circ f)^p \mathcal{G}, \mathcal{F}) & \xlongequal{\quad} & & \xlongequal{\quad} & \text{Mor}_{\text{PSh}(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f) \mathcal{F}) \end{array}$$

Proof omitted. □

06TM **Lemma 3.3.** *Let  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  be 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $t : f \rightarrow g$  be a 2-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Assigned to  $t$  there are canonical isomorphisms of functors*

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_p t : {}_p f \longrightarrow {}_p g$$

which compatible with adjointness of  $(f^p, {}_p f)$  and  $(g^p, {}_p g)$  and with vertical and horizontal composition of 2-morphisms.

**Proof.** Let  $\mathcal{G}$  be a presheaf on  $\mathcal{Y}$ . Then  $t^p : g^p \mathcal{G} \rightarrow f^p \mathcal{G}$  is given by the family of maps

$$g^p \mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p \mathcal{G}(x)$$

parametrized by  $x \in \text{Ob}(\mathcal{X})$ . This makes sense as  $t_x : f(x) \rightarrow g(x)$  and  $\mathcal{G}$  is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation  ${}_p t$  for  $y \in \text{Ob}(\mathcal{Y})$  define  ${}_y^f \mathcal{I}$ , resp.  ${}_y^g \mathcal{I}$  to be the category of pairs  $(x, \psi : f(x) \rightarrow y)$ , resp.  $(x, \psi : g(x) \rightarrow y)$ , see Sites, Section 18. Note that  $t$  defines a functor  ${}_y t : {}_y^g \mathcal{I} \rightarrow {}_y^f \mathcal{I}$  given by the rule

$$(x, g(x) \rightarrow y) \mapsto (x, f(x) \xrightarrow{t_x} g(x) \rightarrow y).$$

Note that for  $\mathcal{F}$  a presheaf on  $\mathcal{X}$  the composition of  ${}_y t$  with  $\mathcal{F} : {}_y^f \mathcal{I}^{opp} \rightarrow \text{Sets}$ ,  $(x, f(x) \rightarrow y) \mapsto \mathcal{F}(x)$  is equal to  $\mathcal{F} : {}_y^g \mathcal{I}^{opp} \rightarrow \text{Sets}$ . Hence by Categories, Lemma 14.8 we get for every  $y \in \text{Ob}(\mathcal{Y})$  a canonical map

$$({}_p f \mathcal{F})(y) = \lim_{{}_y^f \mathcal{I}} \mathcal{F} \longrightarrow \lim_{{}_y^g \mathcal{I}} \mathcal{F} = ({}_p g \mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define  ${}_p t$  as the unique map compatible with the adjointness properties of the pairs  $(f^p, {}_p f)$  and  $(g^p, {}_p g)$  (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of  $(f^p, {}_p f)$  and  $(g^p, {}_p g)$  means that given presheaves  $\mathcal{G}$  and  $\mathcal{F}$  as above we have a commutative diagram

$$\begin{array}{ccc} \text{Mor}_{PSh(\mathcal{X})}(f^p \mathcal{G}, \mathcal{F}) & \xlongequal{\quad} & \text{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p f \mathcal{F}) \\ \downarrow - \circ t^p & & \downarrow {}_p t \circ - \\ \text{Mor}_{PSh(\mathcal{X})}(g^p \mathcal{G}, \mathcal{F}) & \xlongequal{\quad} & \text{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p g \mathcal{F}) \end{array}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 18.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for  $t^p$  is straightforward and one can conclude that this holds for the  ${}_p t$  maps using compatibility with adjointness.  $\square$

#### 4. Sheaves

06TN We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site  $Sch_{fppf}$  as in Topologies, Definition 7.6 and denote its underlying category  $Sch_\alpha$ . Besides being the underlying category of a fppf site, the category  $Sch_\alpha$  can also serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 10.1. We denote these sites  $Sch_{Zar}$ ,  $Sch_{étale}$ ,  $Sch_{smooth}$ , and  $Sch_{syntomic}$ . In this situation, since we have defined the big Zariski site  $(Sch/S)_{Zar}$  of  $S$ , the big étale site  $(Sch/S)_{étale}$  of  $S$ , the big smooth site  $(Sch/S)_{smooth}$  of  $S$ , the big syntomic site  $(Sch/S)_{syntomic}$  of  $S$ , and the big fppf site  $(Sch/S)_{fppf}$  of  $S$  as the localizations (see Sites, Section 24)  $Sch_{Zar}/S$ ,  $Sch_{étale}/S$ ,  $Sch_{smooth}/S$ ,  $Sch_{syntomic}/S$ , and  $Sch_{fppf}/S$  of these (absolute) big sites we see that all of these have the same underlying category, namely  $Sch_\alpha/S$ .

It follows that if we have a category  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  fibred in groupoids, then  $\mathcal{X}$  inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 10.2.

06TP **Definition 4.1.** Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ .

- (1) The *associated Zariski site*, denoted  $\mathcal{X}_{Zar}$ , is the structure of site on  $\mathcal{X}$  inherited from  $(Sch/S)_{Zar}$ .
- (2) The *associated étale site*, denoted  $\mathcal{X}_{étale}$ , is the structure of site on  $\mathcal{X}$  inherited from  $(Sch/S)_{étale}$ .
- (3) The *associated smooth site*, denoted  $\mathcal{X}_{smooth}$ , is the structure of site on  $\mathcal{X}$  inherited from  $(Sch/S)_{smooth}$ .
- (4) The *associated syntomic site*, denoted  $\mathcal{X}_{syntomic}$ , is the structure of site on  $\mathcal{X}$  inherited from  $(Sch/S)_{syntomic}$ .
- (5) The *associated fppf site*, denoted  $\mathcal{X}_{fppf}$ , is the structure of site on  $\mathcal{X}$  inherited from  $(Sch/S)_{fppf}$ .

This definition makes sense by the discussion above. If  $\mathcal{X}$  is an algebraic stack, the literature calls  $\mathcal{X}_{fppf}$  (or a site equivalent to it) the *big fppf site* of  $\mathcal{X}$  and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

06TQ **Remark 4.2.** We only use this notation when the symbol  $\mathcal{X}$  refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted  $X_{étale}$  (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on  $\mathcal{X}$ , i.e., define the corresponding topoi.

06TR **Definition 4.3.** Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\mathcal{F}$  be a presheaf on  $\mathcal{X}$ .

- (1) We say  $\mathcal{F}$  is a *Zariski sheaf*, or a *sheaf for the Zariski topology* if  $\mathcal{F}$  is a sheaf on the associated Zariski site  $\mathcal{X}_{Zar}$ .
- (2) We say  $\mathcal{F}$  is an *étale sheaf*, or a *sheaf for the étale topology* if  $\mathcal{F}$  is a sheaf on the associated étale site  $\mathcal{X}_{étale}$ .

- (3) We say  $\mathcal{F}$  is a *smooth sheaf*, or a *sheaf for the smooth topology* if  $\mathcal{F}$  is a sheaf on the associated smooth site  $\mathcal{X}_{smooth}$ .
- (4) We say  $\mathcal{F}$  is a *syntomic sheaf*, or a *sheaf for the syntomic topology* if  $\mathcal{F}$  is a sheaf on the associated syntomic site  $\mathcal{X}_{syntomic}$ .
- (5) We say  $\mathcal{F}$  is an *fppf sheaf*, or a *sheaf*, or a *sheaf for the fppf topology* if  $\mathcal{F}$  is a sheaf on the associated fppf site  $\mathcal{X}_{fppf}$ .

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves  $Sh(\mathcal{X}_{Zar})$ ,  $Sh(\mathcal{X}_{\acute{e}tale})$ ,  $Sh(\mathcal{X}_{smooth})$ ,  $Sh(\mathcal{X}_{syntomic})$ , and  $Sh(\mathcal{X}_{fppf})$ .

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian sheaves*, i.e., sheaves of abelian groups, is denoted  $Ab(\mathcal{X}_{fppf})$  and similarly for the other topologies. If  $\mathcal{X}$  is an algebraic stack, then  $Sh(\mathcal{X}_{fppf})$  is equivalent (modulo set theoretical problems) to what in the literature would be termed the *category of sheaves on the big fppf site of  $\mathcal{X}$* . Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{\acute{e}tale}) \subset Sh(\mathcal{X}_{Zar}) \subset PSh(\mathcal{X})$$

We sometimes write  $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$  and  $Ab(\mathcal{X}_{fppf}) = Ab(\mathcal{X})$  in accordance with our terminology that a sheaf on  $\mathcal{X}$  is an fppf sheaf on  $\mathcal{X}$ .

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

06TS **Lemma 4.4.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . The functors  $_p f$  and  $f^p$  of (3.1.1) transform  $\tau$  sheaves into  $\tau$  sheaves and define a morphism of topoi  $f : Sh(\mathcal{X}_\tau) \rightarrow Sh(\mathcal{Y}_\tau)$ .*

**Proof.** This follows immediately from Stacks, Lemma 10.3. □

In other words, pushforward and pullback of presheaves as defined in Section 3 also produces *pushforward* and *pullback* of  $\tau$ -sheaves. Having said all of the above we see that we can write  $f^p = f^{-1}$  and  $_p f = f_*$  without any possibility of confusion.

06TT **Definition 4.5.** Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the *associated morphism of fppf topoi* constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 43 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

### 5. Computing pushforward

06W5 Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\mathcal{F}$  be a presheaf on  $\mathcal{X}$ . Let  $y \in \text{Ob}(\mathcal{Y})$ . We can compute  $f_*\mathcal{F}(y)$  in the following way. Suppose that  $y$  lies over the scheme  $V$  and using the 2-Yoneda lemma think of  $y$  as a 1-morphism. Consider the projection

$$\text{pr} : (Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

06W6 (5.0.1) 
$$f_*\mathcal{F}(y) = \Gamma\left((Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{F}\right)$$

Namely, objects of the 2-fibre product are triples  $(h : U \rightarrow V, x, f(x) \rightarrow h^*y)$ . Dropping the  $h$  from the notation we see that this is equivalent to the data of an object  $x$  of  $\mathcal{X}$  and a morphism  $\alpha : f(x) \rightarrow y$  of  $\mathcal{Y}$ . Since  $f_*\mathcal{F}(y) = \lim_{f(x) \rightarrow y} \mathcal{F}(x)$  by definition the equality follows.

As a consequence we have the following “base change” result for pushforwards. This result is trivial and hinges on the fact that we are using “big” sites.

075B **Lemma 5.1.** *Let  $S$  be a scheme. Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of categories fibred in groupoids over  $S$ . Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf  $\mathcal{F}$  on  $\mathcal{X}$ .

**Proof.** Given an object  $y'$  of  $\mathcal{Y}'$  over  $V$  there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'), \mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y', \mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (5.0.1) a bijection  $g^{-1}f_*\mathcal{F}(y') \rightarrow f'_*(g')^{-1}\mathcal{F}(y')$ . We omit the verification that this is compatible with restriction mappings.  $\square$

In the case of a representable morphism of categories fibred in groupoids this formula (5.0.1) simplifies. We suggest the reader skip the rest of this section.

06W7 **Lemma 5.2.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . The following are equivalent*

- (1)  $f$  is representable, and
- (2) for every  $y \in \text{Ob}(\mathcal{Y})$  the functor  $\mathcal{X}^{opp} \rightarrow \text{Sets}, x \mapsto \text{Mor}_{\mathcal{Y}}(f(x), y)$  is representable.

**Proof.** According to the discussion in Algebraic Stacks, Section 6 we see that  $f$  is representable if and only if for every  $y \in \text{Ob}(\mathcal{Y})$  lying over  $U$  the 2-fibre product  $(Sch/U)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$  is representable, i.e., of the form  $(Sch/V_y)_{fppf}$  for some scheme  $V_y$  over  $U$ . Objects in this 2-fibre products are triples  $(h : V \rightarrow U, x, \alpha : f(x) \rightarrow h^*y)$  where  $\alpha$  lies over  $\text{id}_V$ . Dropping the  $h$  from the notation we see that this is equivalent to the data of an object  $x$  of  $\mathcal{X}$  and a morphism  $f(x) \rightarrow y$ . Hence the 2-fibre product is representable by  $V_y$  and  $f(x_y) \rightarrow y$  where  $x_y$  is an object of  $\mathcal{X}$

over  $V_y$  if and only if the functor in (2) is representable by  $x_y$  with universal object a map  $f(x_y) \rightarrow y$ .  $\square$

Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & \searrow p \quad \swarrow q & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. Assume  $f$  is representable. For every  $y \in \text{Ob}(\mathcal{Y})$  we choose an object  $u(y) \in \text{Ob}(\mathcal{X})$  representing the functor  $x \mapsto \text{Mor}_{\mathcal{Y}}(f(x), y)$  of Lemma 5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms  $f(u(y)) \rightarrow y$  by construction. For every morphism  $\beta : y' \rightarrow y$  in  $\mathcal{Y}$  we obtain a unique morphism  $u(\beta) : u(y') \rightarrow u(y)$  in  $\mathcal{X}$  such that the diagram

$$\begin{array}{ccc} f(u(y')) & \xrightarrow{\quad f(u(\beta)) \quad} & f(u(y)) \\ \downarrow & & \downarrow \\ y' & \xrightarrow{\quad \quad \quad} & y \end{array}$$

commutes. In other words,  $u : \mathcal{Y} \rightarrow \mathcal{X}$  is a functor. In fact, we can say a little bit more. Namely, suppose that  $V' = q(y')$ ,  $V = q(y)$ ,  $U' = p(u(y'))$  and  $U = p(u(y))$ . Then

$$\begin{array}{ccc} U' & \xrightarrow{\quad p(u(\beta)) \quad} & U \\ \downarrow & & \downarrow \\ V' & \xrightarrow{\quad q(\beta) \quad} & V \end{array}$$

is a fibre product square. This is true because  $U' \rightarrow U$  represents the base change  $(Sch/V')_{fppf} \times_{y', \mathcal{Y}} \mathcal{X} \rightarrow (Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$  of  $V' \rightarrow V$ .

06W8 **Lemma 5.3.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a representable 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{\text{Zar}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . Then the functor  $u : \mathcal{Y}_\tau \rightarrow \mathcal{X}_\tau$  is continuous and defines a morphism of sites  $\mathcal{X}_\tau \rightarrow \mathcal{Y}_\tau$  which induces the same morphism of topoi  $\text{Sh}(\mathcal{X}_\tau) \rightarrow \text{Sh}(\mathcal{Y}_\tau)$  as the morphism  $f$  constructed in Lemma 4.4. Moreover,  $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$  for any presheaf  $\mathcal{F}$  on  $\mathcal{X}$ .*

**Proof.** Let  $\{y_i \rightarrow y\}$  be a  $\tau$ -covering in  $\mathcal{Y}$ . By definition this simply means that  $\{q(y_i) \rightarrow q(y)\}$  is a  $\tau$ -covering of schemes. By the final remark above the lemma we see that  $\{p(u(y_i)) \rightarrow p(u(y))\}$  is the base change of the  $\tau$ -covering  $\{q(y_i) \rightarrow q(y)\}$  by  $p(u(y)) \rightarrow q(y)$ , hence is itself a  $\tau$ -covering by the axioms of a site. Hence  $\{u(y_i) \rightarrow u(y)\}$  is a  $\tau$ -covering of  $\mathcal{X}$ . This proves that  $u$  is continuous.

Let's use the notation  $u_p, u_s, u^p, u^s$  of Sites, Sections 5 and 13. If we can show the final assertion of the lemma, then we see that  $f_* = u^p = u^s$  (by continuity of  $u$  seen above) and hence by adjointness  $f^{-1} = u_s$  which will prove  $u_s$  is exact, hence that  $u$  determines a morphism of sites, and the equality will be clear as well. To see that  $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$  note that by definition

$$f_*\mathcal{F}(y) = ({}_p f \mathcal{F})(y) = \lim_{f(x) \rightarrow y} \mathcal{F}(x).$$

Since  $u(y)$  is a final object in the category the limit is taken over we conclude.  $\square$



## 6. The structure sheaf

06TU Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. The 2-category of categories fibred in groupoids over  $(Sch/S)_{fppf}$  has a final object, namely,  $id : (Sch/S)_{fppf} \rightarrow (Sch/S)_{fppf}$  and  $p$  is a 1-morphism from  $\mathcal{X}$  to this final object. Hence any presheaf  $\mathcal{G}$  on  $(Sch/S)_{fppf}$  gives a presheaf  $p^{-1}\mathcal{G}$  on  $\mathcal{X}$  defined by the rule  $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$ . Moreover, the discussion in Section 4 shows that  $p^{-1}\mathcal{G}$  is a  $\tau$  sheaf whenever  $\mathcal{G}$  is a  $\tau$ -sheaf.

Recall that the site  $(Sch/S)_{fppf}$  is a ringed site with structure sheaf  $\mathcal{O}$  defined by the rule

$$(Sch/S)^{opp} \longrightarrow Rings, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 8.2.

06TV **Definition 6.1.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. The *structure sheaf of  $\mathcal{X}$*  is the sheaf of rings  $\mathcal{O}_{\mathcal{X}} = p^{-1}\mathcal{O}$ .

For an object  $x$  of  $\mathcal{X}$  lying over  $U$  we have  $\mathcal{O}_{\mathcal{X}}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$ . Needless to say  $\mathcal{O}_{\mathcal{X}}$  is also a Zariski,  $\acute{e}tale$ , smooth, and syntomic sheaf, and hence each of the sites  $\mathcal{X}_{Zar}$ ,  $\mathcal{X}_{\acute{e}tale}$ ,  $\mathcal{X}_{smooth}$ ,  $\mathcal{X}_{syntomic}$ , and  $\mathcal{X}_{fppf}$  is a ringed site. This construction is functorial as well.

06TW **Lemma 6.2.** Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . There is a canonical identification  $f^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$  which turns  $f : Sh(\mathcal{X}_{\tau}) \rightarrow Sh(\mathcal{Y}_{\tau})$  into a morphism of ringed topoi.

**Proof.** Denote  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  and  $q : \mathcal{Y} \rightarrow (Sch/S)_{fppf}$  the structural functors. Then  $q = p \circ f$ , hence  $q^{-1} = f^{-1} \circ p^{-1}$  by Lemma 3.2. The result follows.  $\square$

06TX **Remark 6.3.** In the situation of Lemma 6.2 the morphism of ringed topoi  $f : Sh(\mathcal{X}_{\tau}) \rightarrow Sh(\mathcal{Y}_{\tau})$  is flat as is clear from the equality  $f^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ . This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same thing happens when taking a closed immersion  $i : X \rightarrow Y$  of schemes: in this case the associated morphism of big  $\tau$ -sites  $i : (Sch/X)_{\tau} \rightarrow (Sch/Y)_{\tau}$  also is flat.

## 7. Sheaves of modules

06WA Since we have a structure sheaf we have modules.

06WB **Definition 7.1.** Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ .

- (1) A *presheaf of modules on  $\mathcal{X}$*  is a presheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules. The category of presheaves of modules is denoted  $PMod(\mathcal{O}_{\mathcal{X}})$ .
- (2) We say a presheaf of modules  $\mathcal{F}$  is an  $\mathcal{O}_{\mathcal{X}}$ -*module*, or more precisely a *sheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules* if  $\mathcal{F}$  is an fppf sheaf. The category of  $\mathcal{O}_{\mathcal{X}}$ -modules is denoted  $Mod(\mathcal{O}_{\mathcal{X}})$ .

These (pre)sheaves of modules occur in the literature as *(pre)sheaves of  $\mathcal{O}_{\mathcal{X}}$ -modules on the big fppf site of  $\mathcal{X}$* . We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski,  $\acute{e}tale$ , smooth, or syntomic topologies (without

necessarily being sheaves). If need be these will be denoted  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  and similarly for the other topologies.

Next, we address functoriality – first for presheaves of modules. Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & \searrow p \quad \swarrow q & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. The functors  $f^{-1}$ ,  $f_*$  on abelian presheaves extend to functors

$$06WD \quad (7.1.1) \quad f^{-1} : PMod(\mathcal{O}_{\mathcal{Y}}) \longrightarrow PMod(\mathcal{O}_{\mathcal{X}}) \quad \text{and} \quad f_* : PMod(\mathcal{O}_{\mathcal{Y}}) \longrightarrow PMod(\mathcal{O}_{\mathcal{X}})$$

This is immediate for  $f^{-1}$  because  $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$  which is a module over  $\mathcal{O}_{\mathcal{Y}}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_{\mathcal{X}}(x)$ . Alternatively it follows because  $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$  and because  $f^{-1}$  commutes with limits (on presheaves). Since  $f_*$  is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend  $f_*$  to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 12.1. We claim that the functors (7.1.1) form an adjoint pair of functors:

$$Mor_{PMod(\mathcal{O}_{\mathcal{X}})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{PMod(\mathcal{O}_{\mathcal{Y}})}(\mathcal{G}, f_*\mathcal{F}).$$

As  $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$  this follows from Modules on Sites, Lemma 12.3 by endowing  $\mathcal{X}$  and  $\mathcal{Y}$  with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by  $f$  also the induced morphism of ringed topoi, see Lemma 6.2 (for the fppf topologies right now). Note that the functors  $f^{-1}$  and  $f_*$  of (7.1.1) preserve the subcategories of sheaves of modules, see Lemma 4.4. Hence it follows immediately that

$$06WE \quad (7.1.2) \quad f^{-1} : Mod(\mathcal{O}_{\mathcal{Y}}) \longrightarrow Mod(\mathcal{O}_{\mathcal{X}}) \quad \text{and} \quad f_* : Mod(\mathcal{O}_{\mathcal{Y}}) \longrightarrow Mod(\mathcal{O}_{\mathcal{X}})$$

form an adjoint pair of functors:

$$Mor_{Mod(\mathcal{O}_{\mathcal{X}})}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{Mod(\mathcal{O}_{\mathcal{Y}})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that  $f^* = f^{-1}$  where  $f^*$  is as defined in Modules on Sites, Section 13 for the morphism of ringed topoi  $f$  above. Of course we could have seen this directly because  $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$  and because  $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$ .

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

## 8. Representable categories

076N In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

075I **Lemma 8.1.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)$ . Assume  $\mathcal{X}$  is representable by a scheme  $X$ . For  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$  there is a canonical equivalence*

$$(\mathcal{X}_{\tau}, \mathcal{O}_{\mathcal{X}}) = ((Sch/X)_{\tau}, \mathcal{O}_X)$$

*of ringed sites.*

**Proof.** This follows by choosing an equivalence  $(Sch/X)_\tau \rightarrow \mathcal{X}$  of categories fibred in groupoids over  $(Sch/S)_{fppf}$  and using the functoriality of the construction  $\mathcal{X} \rightsquigarrow \mathcal{X}_\tau$ .  $\square$

075J **Lemma 8.2.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of categories fibred in groupoids over  $S$ . Assume  $\mathcal{X}, \mathcal{Y}$  are representable by schemes  $X, Y$ . Let  $f : X \rightarrow Y$  be the morphism of schemes corresponding to  $f$ . For  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$  the morphism of ringed topoi  $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \rightarrow (Sh(\mathcal{Y}_\tau), \mathcal{O}_\mathcal{Y})$  agrees with the morphism of ringed topoi  $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \rightarrow (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$  via the identifications of Lemma 8.1.*

**Proof.** Follows by unwinding the definitions.  $\square$

## 9. Restriction

075C A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

06W0 **Lemma 9.1.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Let  $x \in \text{Ob}(\mathcal{X})$  lying over  $U = p(x)$ . The functor  $p$  induces an equivalence of sites  $\mathcal{X}_\tau/x \rightarrow (Sch/U)_\tau$ .*

**Proof.** Special case of Stacks, Lemma 10.4.  $\square$

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

06W1 **Definition 9.2.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $x \in \text{Ob}(\mathcal{X})$  lying over  $U = p(x)$ . Let  $\mathcal{F}$  be a presheaf on  $\mathcal{X}$ .

- (1) The *pullback*  $x^{-1}\mathcal{F}$  of  $\mathcal{F}$  is the restriction  $\mathcal{F}|_{(\mathcal{X}/x)}$  viewed as a presheaf on  $(Sch/U)_{fppf}$  via the equivalence  $\mathcal{X}/x \rightarrow (Sch/U)_{fppf}$  of Lemma 9.1.
- (2) The *restriction of  $\mathcal{F}$  to  $U_{\acute{e}tale}$*  is  $x^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$ , abusively written  $\mathcal{F}|_{U_{\acute{e}tale}}$ .

This notation makes sense because to the object  $x$  the 2-Yoneda lemma, see Algebraic Stacks, Section 5 associates a 1-morphism  $x : (Sch/U)_{fppf} \rightarrow \mathcal{X}/x$  which is quasi-inverse to  $p : \mathcal{X}/x \rightarrow (Sch/U)_{fppf}$ . Hence  $x^{-1}\mathcal{F}$  truly is the pullback of  $\mathcal{F}$  via this 1-morphism. In particular, by the material above, if  $\mathcal{F}$  is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then  $x^{-1}\mathcal{F}$  is a sheaf on  $(Sch/U)_{fppf}$  (or on  $(Sch/U)_{Zar}, (Sch/U)_{\acute{e}tale}, (Sch/U)_{smooth}, (Sch/U)_{syntomic}$ ).

Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\varphi : x \rightarrow y$  be a morphism of  $\mathcal{X}$  lying over the morphism of schemes  $a : U \rightarrow V$ . Recall that  $a$  induces a morphism of small étale sites  $a_{small} : U_{\acute{e}tale} \rightarrow V_{\acute{e}tale}$ , see Étale Cohomology, Section 34. Let  $\mathcal{F}$  be a presheaf on  $\mathcal{X}$ . Let  $\mathcal{F}|_{U_{\acute{e}tale}}$  and  $\mathcal{F}|_{V_{\acute{e}tale}}$  be the restrictions of  $\mathcal{F}$  via  $x$  and  $y$ . There is a natural *comparison map*

$$06W2 \quad (9.2.1) \quad c_\varphi : \mathcal{F}|_{V_{\acute{e}tale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{\acute{e}tale}})$$

of presheaves on  $U_{\acute{e}tale}$ . Namely, if  $V' \rightarrow V$  is étale, set  $U' = V' \times_V U$  and define  $c_\varphi$  on sections over  $V'$  via

$$\begin{array}{ccc} a_{small,*}(\mathcal{F}|_{U_{\acute{e}tale}})(V') & \xlongequal{\quad} & \mathcal{F}|_{U_{\acute{e}tale}}(U') \xlongequal{\quad} \mathcal{F}(x') \\ \uparrow c_\varphi & & \uparrow \mathcal{F}(\varphi') \\ \mathcal{F}|_{V_{\acute{e}tale}}(V') & \xlongequal{\quad} & \mathcal{F}(y') \end{array}$$

Here  $\varphi' : x' \rightarrow y'$  is a morphism of  $\mathcal{X}$  fitting into a commutative diagram

$$\begin{array}{ccc} x' & \longrightarrow & x \\ \varphi' \downarrow & & \downarrow \varphi \\ y' & \longrightarrow & y \end{array} \quad \text{lying over} \quad \begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow a \\ V' & \longrightarrow & V \end{array}$$

The existence and uniqueness of  $\varphi'$  follow from the axioms of a category fibred in groupoids. We omit the verification that  $c_\varphi$  so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in  $\mathcal{F}$ . In case  $\mathcal{F}$  is a sheaf for the étale topology we obtain a *comparison map*

$$06W3 \quad (9.2.2) \quad c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}}$$

which is also denoted  $c_\varphi$  as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

075D **Lemma 9.3.** *Let  $\mathcal{F}$  be an étale sheaf on  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$ .*

- (1) *If  $\varphi : x \rightarrow y$  and  $\psi : y \rightarrow z$  are morphisms of  $\mathcal{X}$  lying over  $a : U \rightarrow V$  and  $b : V \rightarrow W$ , then the composition*

$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{\acute{e}tale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{\acute{e}tale}}$$

*is equal to  $c_{\psi \circ \varphi}$  via the identification*

$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{\acute{e}tale}})).$$

- (2) *If  $\varphi : x \rightarrow y$  lies over an étale morphism of schemes  $a : U \rightarrow V$ , then (9.2.2) is an isomorphism.*
- (3) *Suppose  $f : \mathcal{Y} \rightarrow \mathcal{X}$  is a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$  and  $y$  is an object of  $\mathcal{Y}$  lying over the scheme  $U$  with image  $x = f(y)$ . Then there is a canonical identification  $f^{-1}\mathcal{F}|_{U_{\acute{e}tale}} = \mathcal{F}|_{U_{\acute{e}tale}}$ .*
- (4) *Moreover, given  $\psi : y' \rightarrow y$  in  $\mathcal{Y}$  lying over  $a : U' \rightarrow U$  the comparison map  $c_\psi : a_{small}^{-1}(F^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) \rightarrow F^{-1}\mathcal{F}|_{U'_{\acute{e}tale}}$  is equal to the comparison map  $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{\acute{e}tale}} \rightarrow \mathcal{F}|_{U'_{\acute{e}tale}}$  via the identifications in (3).*

**Proof.** The verification of these properties is omitted.  $\square$

Next, we turn to the restriction of (pre)sheaves of modules.

06W9 **Lemma 9.4.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Let  $x \in \text{Ob}(\mathcal{X})$  lying over  $U = p(x)$ . The equivalence of Lemma 9.1 extends to an equivalence of ringed sites  $(\mathcal{X}_\tau/x, \mathcal{O}_{\mathcal{X}}|_x) \rightarrow ((Sch/U)_\tau, \mathcal{O})$ .*

**Proof.** This is immediate from the construction of the structure sheaves.  $\square$

Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\mathcal{F}$  be a (pre)sheaf of modules on  $\mathcal{X}$  as in Definition 7.1. Let  $x$  be an object of  $\mathcal{X}$  lying over  $U$ . Then Lemma 9.4 guarantees that the restriction  $x^{-1}\mathcal{F}$  is a (pre)sheaf of modules on  $(Sch/U)_{fppf}$ . We will sometimes write  $x^*\mathcal{F} = x^{-1}\mathcal{F}$  in this case. Similarly, if  $\mathcal{F}$  is a sheaf for the Zariski, étale, smooth, or syntomic topology, then  $x^{-1}\mathcal{F}$  is as well. Moreover, the restriction  $\mathcal{F}|_{U_{\acute{e}tale}} = x^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$  to  $U$  is a presheaf of  $\mathcal{O}_{U_{\acute{e}tale}}$ -modules. If  $\mathcal{F}$  is a sheaf for the étale topology, then  $\mathcal{F}|_{U_{\acute{e}tale}}$  is a sheaf of modules. Moreover, if  $\varphi : x \rightarrow y$  is a morphism of  $\mathcal{X}$  lying over  $a : U \rightarrow V$  then the

comparison map (9.2.2) is compatible with  $a_{small}^\#$  (see Descent, Remark 8.4) and induces a *comparison* map

$$06WC \quad (9.4.1) \quad c_\varphi : a_{small}^*(\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow \mathcal{F}|_{U_{\acute{e}tale}}$$

of  $\mathcal{O}_{U_{\acute{e}tale}}$ -modules. Note that the properties (1), (2), (3), and (4) of Lemma 9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

06W4 **Lemma 9.5.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . The site  $\mathcal{X}_\tau$  has enough points.*

**Proof.** By Sites, Lemma 37.5 we have to show that there exists a family of objects  $x$  of  $\mathcal{X}$  such that  $\mathcal{X}_\tau/x$  has enough points and such that the sheaves  $h_x^\#$  cover the final object of the category of sheaves. By Lemma 9.1 and Étale Cohomology, Lemma 30.1 we see that  $\mathcal{X}_\tau/x$  has enough points for every object  $x$  and we win.  $\square$

## 10. Restriction to algebraic spaces

076P In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 4.13 for algebraic spaces.

073M **Lemma 10.1.** *Let  $S$  be a scheme. Let  $\mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Assume  $\mathcal{X}$  is representable by an algebraic space  $F$ . Then there exists a continuous and cocontinuous functor  $F_{\acute{e}tale} \rightarrow \mathcal{X}_{\acute{e}tale}$  which induces a morphism of ringed sites*

$$\pi_F : (\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (F_{\acute{e}tale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{\acute{e}tale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_{\mathcal{X}})$$

such that  $\pi_F \circ i_F = id$ . Moreover  $\pi_{F,*} = i_F^{-1}$ .

**Proof.** Choose an equivalence  $j : \mathcal{S}_F \rightarrow \mathcal{X}$ , see Algebraic Stacks, Sections 7 and 8. An object of  $F_{\acute{e}tale}$  is a scheme  $U$  together with an étale morphism  $\varphi : U \rightarrow F$ . Then  $\varphi$  is an object of  $\mathcal{S}_F$  over  $U$ . Hence  $j(\varphi)$  is an object of  $\mathcal{X}$  over  $U$ . In this way  $j$  induces a functor  $u : F_{\acute{e}tale} \rightarrow \mathcal{X}$ . It is clear that  $u$  is continuous and cocontinuous for the étale topology on  $\mathcal{X}$ . Since  $j$  is an equivalence, the functor  $u$  is fully faithful. Also, fibre products and equalizers exist in  $F_{\acute{e}tale}$  and  $u$  commutes with them because these are computed on the level of underlying schemes in  $F_{\acute{e}tale}$ . Thus Sites, Lemmas 20.5, 20.6, and 20.7 apply. In particular  $u$  defines a morphism of topoi  $i_F : Sh(F_{\acute{e}tale}) \rightarrow Sh(\mathcal{X}_{\acute{e}tale})$  and there exists a left adjoint  $i_{F,!}$  of  $i_F^{-1}$  which commutes with fibre products and equalizers.

We claim that  $i_{F,!}$  is exact. If this is true, then we can define  $\pi_F$  by the rules  $\pi_F^{-1} = i_{F,!}$  and  $\pi_{F,*} = i_F^{-1}$  and everything is clear. To prove the claim, note that we already know that  $i_{F,!}$  is right exact and preserves fibre products. Hence it suffices to show that  $i_{F,!}^* = *$  where  $*$  indicates the final object in the category of sheaves of sets. Let  $U$  be a scheme and let  $\varphi : U \rightarrow F$  be surjective and étale. Set  $R = U \times_F U$ . Then

$$h_R \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} h_U \longrightarrow *$$

is a coequalizer diagram in  $Sh(F_{\acute{e}tale})$ . Using the right exactness of  $i_{F,!}$ , using  $i_{F,!} = (u_p)^\#$ , and using Sites, Lemma 5.6 we see that

$$h_{u(R)} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} h_{u(U)} \longrightarrow i_{F,!}^*$$

is a coequalizer diagram in  $Sh(F_{\acute{e}tale})$ . Using that  $j$  is an equivalence and that  $F = U/R$  it follows that the coequalizer in  $Sh(\mathcal{X}_{\acute{e}tale})$  of the two maps  $h_{u(R)} \rightarrow h_{u(U)}$  is  $*$ . We omit the proof that these morphisms are compatible with structure sheaves.  $\square$

Assume  $\mathcal{X}$  is an algebraic stack represented by the algebraic space  $F$ . Let  $j : \mathcal{S}_F \rightarrow \mathcal{X}$  be an equivalence and denote  $u : F_{\acute{e}tale} \rightarrow \mathcal{X}_{\acute{e}tale}$  the functor of the proof of Lemma 10.1 above. Given a sheaf  $\mathcal{F}$  on  $\mathcal{X}_{\acute{e}tale}$  we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of  $i_F^{-1}$  as a *restriction functor* similarly to Definition 9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

$$075K \quad (10.1.1) \quad \mathcal{F}|_{F_{\acute{e}tale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F}$$

in this situation.

073N **Lemma 10.2.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Assume  $\mathcal{X}, \mathcal{Y}$  are representable by algebraic spaces  $F, G$ . Denote  $f : F \rightarrow G$  the induced morphism of algebraic spaces, and  $f_{small} : F_{\acute{e}tale} \rightarrow G_{\acute{e}tale}$  the corresponding morphism of ringed topoi. Then*

$$\begin{array}{ccc} (Sh(\mathcal{X}_{\acute{e}tale}), \mathcal{O}_{\mathcal{X}}) & \xrightarrow{f} & (Sh(\mathcal{Y}_{\acute{e}tale}), \mathcal{O}_{\mathcal{Y}}) \\ \pi_F \downarrow & & \downarrow \pi_G \\ (Sh(F_{\acute{e}tale}), \mathcal{O}_F) & \xrightarrow{f_{small}} & (Sh(G_{\acute{e}tale}), \mathcal{O}_G) \end{array}$$

is a commutative diagram of ringed topoi.

**Proof.** This is similar to Topologies, Lemma 4.16 (3) but there is a small snag due to the fact that  $F \rightarrow G$  may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences  $j : \mathcal{S}_F \rightarrow \mathcal{X}$  and  $j' : \mathcal{S}_G \rightarrow \mathcal{Y}$  which induce functors  $u : F_{\acute{e}tale} \rightarrow \mathcal{X}$  and  $u' : G_{\acute{e}tale} \rightarrow \mathcal{Y}$  as in the proof of Lemma 10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over  $Sch_{fppf}$  (see discussion in Section 3) we may assume that  $\mathcal{X} = \mathcal{S}_F$  and  $\mathcal{Y} = \mathcal{S}_G$  and that  $f : \mathcal{S}_F \rightarrow \mathcal{S}_G$  is the functor associated to the morphism  $f : F \rightarrow G$ . Correspondingly we will omit  $u$  and  $u'$  from the notation, i.e., given an object  $U \rightarrow F$  of  $F_{\acute{e}tale}$  we denote  $U/F$  the corresponding object of  $\mathcal{X}$ . Similarly for  $G$ .

Let  $\mathcal{G}$  be a sheaf on  $\mathcal{X}_{\acute{e}tale}$ . To prove (2) we compute  $\pi_{G,*}f_*\mathcal{G}$  and  $f_{small,*}\pi_{F,*}\mathcal{G}$ . To do this let  $V \rightarrow G$  be an object of  $G_{\acute{e}tale}$ . Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\left((Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{G}\right)$$

see (5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_{\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space  $V \times_G F$ . And  $\text{pr}^{-1}\mathcal{G}$  is a sheaf on  $\mathcal{S}_{V \times_G F}$  for the étale topology.

In particular, if  $V \times_G F$  is representable, i.e., if it is a scheme, then  $\pi_{G,*}f_*\mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$  and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow V \times_G F$ . Set  $R = U \times_{V \times_G F} U$ . Then  $U/V \times_G F$  and  $R/V \times_G F$  are objects of the fibre product category above. Since  $\text{pr}^{-1}\mathcal{G}$  is a sheaf for the étale topology on  $\mathcal{S}_{V \times_G F}$  the diagram

$$\Gamma\left(\left(\mathcal{S}ch/V\right)_{fppf} \times_{\mathcal{Y}} \mathcal{X}, \text{pr}^{-1}\mathcal{G}\right) \longrightarrow \text{pr}^{-1}\mathcal{G}(U/V \times_G F) \rightrightarrows \text{pr}^{-1}\mathcal{G}(R/V \times_G F)$$

is an equalizer diagram. Note that  $\text{pr}^{-1}\mathcal{G}(U/V \times_G F) = \mathcal{G}(U/F)$  and  $\text{pr}^{-1}\mathcal{G}(R/V \times_G F) = \mathcal{G}(R/F)$  by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 18 (especially, Properties of Spaces, Remark 18.4 and Lemma 18.7) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) \longrightarrow \pi_{F,*}\mathcal{G}(U/F) \rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have  $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$  and  $\pi_{F,*}\mathcal{G}(R/F) = \mathcal{G}(R/F)$  we obtain a canonical identification  $f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{G,*}f_*\mathcal{G}(V)$ . We omit the proof that this is compatible with restriction mappings and that it is functorial in  $\mathcal{G}$ .  $\square$

Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  and  $f : F \rightarrow G$  be as in the second part of the lemma above. A consequence of the lemma, using (10.1.1), is that

$$075M \quad (10.2.1) \quad (f_*\mathcal{F})|_{G_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{F_{\acute{e}tale}})$$

for any sheaf  $\mathcal{F}$  on  $\mathcal{X}_{\acute{e}tale}$ . Moreover, if  $\mathcal{F}$  is a sheaf of  $\mathcal{O}$ -modules, then (10.2.1) is an isomorphism of  $\mathcal{O}_G$ -modules on  $G_{\acute{e}tale}$ .

Finally, suppose that we have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{a} & \mathcal{V} \\ & \searrow \varphi & \downarrow g \\ & & \mathcal{X} \end{array}$$

of 1-morphisms of categories fibred in groupoids over  $(\mathcal{S}ch/S)_{fppf}$ , that  $\mathcal{F}$  is a sheaf on  $\mathcal{X}_{\acute{e}tale}$ , and that  $\mathcal{U}, \mathcal{V}$  are representable by algebraic spaces  $U, V$ . Then we obtain a comparison map

$$076Q \quad (10.2.2) \quad c_{\varphi} : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}$$

where  $a : U \rightarrow V$  denotes the morphism of algebraic spaces corresponding to  $a$ . This is the analogue of (9.2.2). We define  $c_{\varphi}$  as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{\acute{e}tale}} \longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{\acute{e}tale}}) = (a_*f^{-1}\mathcal{F})|_{V_{\acute{e}tale}}$$

(equality by (10.2.1)) which is the restriction to  $V$  (10.1.1) of the map

$$g^{-1}\mathcal{F} \rightarrow a_*a^{-1}g^{-1}\mathcal{F} = a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules  $c_{\varphi}$  induces a *comparison map*

$$076R \quad (10.2.3) \quad c_{\varphi} : a_{small}^*(g^*\mathcal{F}|_{V_{\acute{e}tale}}) \longrightarrow f^*\mathcal{F}|_{U_{\acute{e}tale}}$$

of  $\mathcal{O}_{U_{\acute{e}tale}}$ -modules. Note that the properties (1), (2), (3), and (4) of Lemma 9.3 hold in this setting as well.

## 11. Quasi-coherent modules

06WF At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

06WG **Definition 11.1.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. A *quasi-coherent module on  $\mathcal{X}$* , or a *quasi-coherent  $\mathcal{O}_{\mathcal{X}}$ -module* is a quasi-coherent module on the ringed site  $(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$  as in Modules on Sites, Definition 23.1. The category of quasi-coherent sheaves on  $\mathcal{X}$  is denoted  $QCoh(\mathcal{O}_{\mathcal{X}})$ .

If  $\mathcal{X}$  is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that  $QCoh(\mathcal{O}_{\mathcal{X}})$  is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [Ols07, Definition 6.1] in Cohomology on Stacks, Lemma 11.6. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

06WH **Lemma 11.2.** Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . The pullback functor  $f^* = f^{-1} : Mod(\mathcal{O}_{\mathcal{Y}}) \rightarrow Mod(\mathcal{O}_{\mathcal{X}})$  preserves quasi-coherent sheaves.

**Proof.** This is a general fact, see Modules on Sites, Lemma 23.4.  $\square$

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 11.6 for a characterization in terms of restrictions.

06WI **Lemma 11.3.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules. Then  $\mathcal{F}$  is quasi-coherent if and only if  $x^*\mathcal{F}$  is a quasi-coherent sheaf on  $(Sch/U)_{fppf}$  for every object  $x$  of  $\mathcal{X}$  with  $U = p(x)$ .

**Proof.** By Lemma 11.2 the condition is necessary. Conversely, since  $x^*\mathcal{F}$  is just the restriction to  $\mathcal{X}_{fppf}/x$  we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18).  $\square$

0EM8 **Lemma 11.4.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\mathcal{F}$  be a presheaf of modules on  $\mathcal{X}$ . The following are equivalent

- (1)  $\mathcal{F}$  is an object of  $Mod(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$  and  $\mathcal{F}$  is a quasi-coherent module on  $(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$  in the sense of Modules on Sites, Definition 23.1,
- (2)  $\mathcal{F}$  is an object of  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  and  $\mathcal{F}$  is a quasi-coherent module on  $(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  in the sense of Modules on Sites, Definition 23.1, and
- (3)  $\mathcal{F}$  is a quasi-coherent module on  $\mathcal{X}$  in the sense of Definition 11.1.

**Proof.** Assume either (1), (2), or (3) holds. Let  $x$  be an object of  $\mathcal{X}$  lying over the scheme  $U$ . Recall that  $x^*\mathcal{F} = x^{-1}\mathcal{F}$  is just the restriction to  $\mathcal{X}/x = (Sch/U)_{\tau}$  where  $\tau = fppf$ ,  $\tau = \acute{e}tale$ , or  $\tau = Zar$ , see Section 9. By the definition of quasi-coherent modules on a ringed site this restriction is quasi-coherent provided



$\mathcal{F}$  is. By Descent, Proposition 8.11 we see that  $x^*\mathcal{F}$  is the sheaf associated to a quasi-coherent  $\mathcal{O}_U$ -module and is therefore a quasi-coherent module in the fppf, étale, and Zariski topology; here we also use Descent, Lemma 8.1 and Definition 8.2. Since this holds for every object  $x$  of  $\mathcal{X}$ , we see that  $\mathcal{F}$  is a sheaf in any of the three topologies. Moreover, we find that  $\mathcal{F}$  is quasi-coherent in any of the three topologies directly from the definition of being quasi-coherent and the fact that  $x$  is an arbitrary object of  $\mathcal{X}$ .  $\square$

Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

06WJ **Definition 11.5.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules. We say  $\mathcal{F}$  is *locally quasi-coherent*<sup>2</sup> if  $\mathcal{F}$  is a sheaf for the étale topology and for every object  $x$  of  $\mathcal{X}$  the restriction  $x^*\mathcal{F}|_{U_{\text{étale}}}$  is a quasi-coherent sheaf. Here  $U = p(x)$ .

We use  $LQCoh(\mathcal{O}_{\mathcal{X}})$  to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$\begin{array}{ccc} QCoh(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & Mod(\mathcal{O}_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ LQCoh(\mathcal{O}_{\mathcal{X}}) & \longrightarrow & Mod(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counterpart for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are “cartesian”, i.e., satisfy the second condition of the lemma below.

06WK **Lemma 11.6.** Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids. Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules. Then  $\mathcal{F}$  is quasi-coherent if and only if the following two conditions hold

- (1)  $\mathcal{F}$  is locally quasi-coherent, and
- (2) for any morphism  $\varphi : x \rightarrow y$  of  $\mathcal{X}$  lying over  $f : U \rightarrow V$  the comparison map  $c_{\varphi} : \int_{small}^* \mathcal{F}|_{V_{\text{étale}}} \rightarrow \mathcal{F}|_{U_{\text{étale}}}$  of (9.4.1) is an isomorphism.

**Proof.** Assume  $\mathcal{F}$  is quasi-coherent. Then  $\mathcal{F}$  is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of  $\mathcal{F}$  to a ringed topos is quasi-coherent, hence the restrictions  $x^*\mathcal{F}|_{U_{\text{étale}}}$  are quasi-coherent. This proves  $\mathcal{F}$  is locally quasi-coherent. Let  $y$  be an object of  $\mathcal{X}$  with  $V = p(y)$ . We have seen that  $\mathcal{X}/y = (Sch/V)_{fppf}$ . By Descent, Proposition 8.11 it follows that  $y^*\mathcal{F}$  is the quasi-coherent module associated to a (usual) quasi-coherent module  $\mathcal{F}_V$  on the scheme  $V$ . Hence certainly the comparison maps (9.4.1) are isomorphisms.

Conversely, suppose that  $\mathcal{F}$  satisfies (1) and (2). Let  $y$  be an object of  $\mathcal{X}$  with  $V = p(y)$ . Denote  $\mathcal{F}_V$  the quasi-coherent module on the scheme  $V$  corresponding to the restriction  $y^*\mathcal{F}|_{V_{\text{étale}}}$  which is quasi-coherent by assumption (1), see Descent, Proposition 8.11. Condition (2) now signifies that the restrictions  $x^*\mathcal{F}|_{U_{\text{étale}}}$  for  $x$  over  $y$  are each isomorphic to the (étale sheaf associated to the) pullback of  $\mathcal{F}_V$

<sup>2</sup>This is nonstandard notation.

via the corresponding morphism of schemes  $U \rightarrow V$ . Hence  $y^*\mathcal{F}$  is the sheaf on  $(Sch/V)_{fppf}$  associated to  $\mathcal{F}_V$ . Hence it is quasi-coherent (by Descent, Proposition 8.11 again) and we see that  $\mathcal{F}$  is quasi-coherent on  $\mathcal{X}$  by Lemma 11.3.  $\square$

06WL **Lemma 11.7.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . The pullback functor  $f^* = f^{-1} : Mod(\mathcal{Y}_{\acute{e}tale}, \mathcal{O}_{\mathcal{Y}}) \rightarrow Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  preserves locally quasi-coherent sheaves.*

**Proof.** Let  $\mathcal{G}$  be locally quasi-coherent on  $\mathcal{Y}$ . Choose an object  $x$  of  $\mathcal{X}$  lying over the scheme  $U$ . The restriction  $x^*f^*\mathcal{G}|_{U_{\acute{e}tale}}$  equals  $(f \circ x)^*\mathcal{G}|_{U_{\acute{e}tale}}$  hence is a quasi-coherent sheaf by assumption on  $\mathcal{G}$ .  $\square$

06WM **Lemma 11.8.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids.*

- (1) *The category  $LQCoh(\mathcal{O}_{\mathcal{X}})$  has colimits and they agree with colimits in the category  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ .*
- (2) *The category  $LQCoh(\mathcal{O}_{\mathcal{X}})$  is abelian with kernels and cokernels computed in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ , in other words the inclusion functor is exact.*
- (3) *Given a short exact sequence  $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$  of  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  if two out of three are locally quasi-coherent so is the third.*
- (4) *Given  $\mathcal{F}, \mathcal{G}$  in  $LQCoh(\mathcal{O}_{\mathcal{X}})$  the tensor product  $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$  in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  is an object of  $LQCoh(\mathcal{O}_{\mathcal{X}})$ .*
- (5) *Given  $\mathcal{F}, \mathcal{G}$  in  $LQCoh(\mathcal{O}_{\mathcal{X}})$  with  $\mathcal{F}$  locally of finite presentation on  $\mathcal{X}_{\acute{e}tale}$  the sheaf  $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})$  in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$  is an object of  $LQCoh(\mathcal{O}_{\mathcal{X}})$ .*

**Proof.** Each of these statements follows from the corresponding statement of Descent, Lemma 8.13. For example, suppose that  $\mathcal{I} \rightarrow LQCoh(\mathcal{O}_{\mathcal{X}})$ ,  $i \mapsto \mathcal{F}_i$  is a diagram. Consider the object  $\mathcal{F} = \text{colim}_i \mathcal{F}_i$  of  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ . For any object  $x$  of  $\mathcal{X}$  with  $U = p(x)$  the pullback functor  $x^*$  commutes with all colimits as it is a left adjoint. Hence  $x^*\mathcal{F} = \text{colim}_i x^*\mathcal{F}_i$ . Similarly we have  $x^*\mathcal{F}|_{U_{\acute{e}tale}} = \text{colim}_i x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$ . Now by assumption each  $x^*\mathcal{F}_i|_{U_{\acute{e}tale}}$  is quasi-coherent, hence the colimit is quasi-coherent by the aforementioned Descent, Lemma 8.13. This proves (1).

It follows from (1) that cokernels exist in  $LQCoh(\mathcal{O}_{\mathcal{X}})$  and agree with the cokernels computed in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ . Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of  $LQCoh(\mathcal{O}_{\mathcal{X}})$  and let  $\mathcal{K} = \text{Ker}(\varphi)$  computed in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ . If we can show that  $\mathcal{K}$  is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence  $\mathcal{K}|_{U_{\acute{e}tale}}$  is the kernel of the map  $\mathcal{F}|_{U_{\acute{e}tale}} \rightarrow \mathcal{G}|_{U_{\acute{e}tale}}$ , i.e., is the kernel of a map of quasi-coherent sheaves on  $U_{\acute{e}tale}$  whence quasi-coherent by Descent, Lemma 8.13. This proves (2).

Parts (3), (4), and (5) follow in exactly the same way. Details omitted.  $\square$

In the generality discussed here the category of quasi-coherent sheaves is not abelian. See Examples, Section 12. Here is what we can prove without any further work.

06WN **Lemma 11.9.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids.*

- (1) *The category  $QCoh(\mathcal{O}_{\mathcal{X}})$  has colimits and they agree with colimits in the categories  $Mod(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$ ,  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_{\mathcal{X}})$ ,  $Mod(\mathcal{O}_{\mathcal{X}})$ , and  $LQCoh(\mathcal{O}_{\mathcal{X}})$ .*
- (2) *Given  $\mathcal{F}, \mathcal{G}$  in  $QCoh(\mathcal{O}_{\mathcal{X}})$  the tensor product  $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$  in  $Mod(\mathcal{O}_{\mathcal{X}})$  is an object of  $QCoh(\mathcal{O}_{\mathcal{X}})$ .*
- (3) *Given  $\mathcal{F}, \mathcal{G}$  in  $QCoh(\mathcal{O}_{\mathcal{X}})$  with  $\mathcal{F}$  locally of finite presentation on  $\mathcal{X}_{fppf}$  the sheaf  $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})$  in  $Mod(\mathcal{O}_{\mathcal{X}})$  is an object of  $QCoh(\mathcal{O}_{\mathcal{X}})$ .*

**Proof.** Let  $\mathcal{I} \rightarrow QCoh(\mathcal{O}_{\mathcal{X}})$ ,  $i \mapsto \mathcal{F}_i$  be a diagram. Viewing  $\mathcal{F}_i$  as quasi-coherent modules in the Zariski topology (Lemma 11.4), we may consider the object  $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$  of  $Mod(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$ . For any object  $x$  of  $\mathcal{X}$  with  $U = p(x)$  the restriction functor  $x^*$  (Section 9) commutes with all colimits as it is a left adjoint. Hence  $x^*\mathcal{F} = \operatorname{colim}_i x^*\mathcal{F}_i$  in  $Mod((Sch/U)_{Zar}, \mathcal{O})$ . Observe that  $x_i^*\mathcal{F}_i$  is a quasi-coherent object (because restrictions of quasi-coherent modules are quasi-coherent). Thus by the equivalence in Descent, Proposition 8.11 and by the compatibility with colimits in Descent, Lemma 8.13 we conclude that  $x^*\mathcal{F}$  is quasi-coherent. Thus  $\mathcal{F}$  is quasi-coherent, see Lemma 11.4. Thus we see that  $QCoh(\mathcal{O}_{\mathcal{X}})$  has colimits and they agree with colimits in the category  $Mod(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$ . Since the other categories listed are full subcategories of  $Mod(\mathcal{X}_{Zar}, \mathcal{O}_{\mathcal{X}})$  we conclude part (1) holds.

Parts (2) and (3) are proved in the same way. Details omitted.  $\square$

## 12. Stackification and sheaves

06WP It turns out that the category of sheaves on a category fibred in groupoids only “knows about” the stackification.

06WQ **Lemma 12.1.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $f$  induces an equivalence of stackifications, then the morphism of topoi  $f : Sh(\mathcal{X}_{fppf}) \rightarrow Sh(\mathcal{Y}_{fppf})$  is an equivalence.*

**Proof.** We may assume  $\mathcal{Y}$  is the stackification of  $\mathcal{X}$ . We claim that  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a special cocontinuous functor, see Sites, Definition 28.2 which will prove the lemma. By Stacks, Lemma 10.3 the functor  $f$  is continuous and cocontinuous. By Stacks, Lemma 8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 28.1 hold.  $\square$

06WR **Lemma 12.2.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . If  $f$  induces an equivalence of stackifications, then  $f^*$  induces equivalences  $Mod(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{O}_{\mathcal{Y}})$  and  $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow QCoh(\mathcal{O}_{\mathcal{Y}})$ .*

**Proof.** We may assume  $\mathcal{Y}$  is the stackification of  $\mathcal{X}$ . The first assertion is clear from Lemma 12.1 and  $\mathcal{O}_{\mathcal{X}} = f^{-1}\mathcal{O}_{\mathcal{Y}}$ . Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 11.2. Hence it suffices to show that if  $f^*\mathcal{G}$  is quasi-coherent, then  $\mathcal{G}$  is. To see this, let  $y$  be an object of  $\mathcal{Y}$ . Translating the condition that  $\mathcal{Y}$  is the stackification of  $\mathcal{X}$  we see there exists an fppf covering  $\{y_i \rightarrow y\}$  in  $\mathcal{Y}$  such that  $y_i \cong f(x_i)$  for some  $x_i$  object of  $\mathcal{X}$ . Say  $x_i$  and  $y_i$  lie over the scheme  $U_i$ . Then  $f^*\mathcal{G}$  being quasi-coherent, means that  $x_i^*f^*\mathcal{G}$  is quasi-coherent. Since  $x_i^*f^*\mathcal{G}$  is isomorphic to  $y_i^*\mathcal{G}$  (as sheaves on  $(Sch/U_i)_{fppf}$  we see that  $y_i^*\mathcal{G}$  is quasi-coherent. It follows from Modules on Sites, Lemma 23.3 that the restriction of  $\mathcal{G}$  to  $\mathcal{Y}/y$  is quasi-coherent. Hence  $\mathcal{G}$  is quasi-coherent by Lemma 11.3.  $\square$

## 13. Quasi-coherent sheaves and presentations

06WS In Groupoids in Spaces, Definition 12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

06WT **Proposition 13.1.** *Let  $(U, R, s, t, c)$  be a groupoid in algebraic spaces over  $S$ . Let  $\mathcal{X} = [U/R]$  be the quotient stack. The category of quasi-coherent modules on  $\mathcal{X}$  is equivalent to the category of quasi-coherent modules on  $(U, R, s, t, c)$ .*

**Proof.** Denote  $QCoh(U, R, s, t, c)$  the category of quasi-coherent modules on the groupoid  $(U, R, s, t, c)$ . We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_{\mathcal{X}}) \longleftrightarrow QCoh(U, R, s, t, c).$$

According to Lemma 12.2 the stackification map  $[U/pR] \rightarrow [U/R]$  (see Groupoids in Spaces, Definition 19.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to prove the lemma with  $\mathcal{X} = [U/pR]$ .

Recall that an object  $x = (T, u)$  of  $\mathcal{X} = [U/pR]$  is given by a scheme  $T$  and a morphism  $u : T \rightarrow U$ . A morphism  $(T, u) \rightarrow (T', u')$  is given by a pair  $(f, r)$  where  $f : T \rightarrow T'$  and  $r : T \rightarrow R$  with  $s \circ r = u$  and  $t \circ r = u' \circ f$ . Let us call a *special morphism* any morphism of the form  $(f, e \circ u' \circ f) : (T, u' \circ f) \rightarrow (T', u')$ . The category of  $(T, u)$  with special morphisms is just the category of schemes over  $U$ .

Let  $\mathcal{F}$  be a quasi-coherent sheaf on  $\mathcal{X}$ . Then we obtain for every  $x = (T, u)$  a quasi-coherent sheaf  $\mathcal{F}_{(T, u)} = x^* \mathcal{F}|_{T_{\acute{e}t\acute{a}l\acute{e}}}$  on  $T$ . Moreover, for any morphism  $(f, r) : x = (T, u) \rightarrow (T', u') = x'$  we obtain a comparison isomorphism

$$c_{(f, r)} : f_{small}^* \mathcal{F}_{(T', u')} \longrightarrow \mathcal{F}_{(T, u)}$$

see Lemma 11.6. Moreover, these isomorphisms are compatible with compositions, see Lemma 9.3. If  $U, R$  are schemes, then we can construct the quasi-coherent sheaf on the groupoid as follows: First the object  $(U, \text{id})$  corresponds to a quasi-coherent sheaf  $\mathcal{F}_{(U, \text{id})}$  on  $U$ . Next, the isomorphism  $\alpha : t_{small}^* \mathcal{F}_{(U, \text{id})} \rightarrow s_{small}^* \mathcal{F}_{(U, \text{id})}$  comes from

- (1) the morphism  $(R, \text{id}_R) : (R, s) \rightarrow (R, t)$  in the category  $[U/pR]$  which produces an isomorphism  $\mathcal{F}_{(R, t)} \rightarrow \mathcal{F}_{(R, s)}$ ,
- (2) the special morphism  $(R, s) \rightarrow (U, \text{id})$  which produces an isomorphism  $s_{small}^* \mathcal{F}_{(U, \text{id})} \rightarrow \mathcal{F}_{(R, s)}$ , and
- (3) the special morphism  $(R, t) \rightarrow (U, \text{id})$  which produces an isomorphism  $t_{small}^* \mathcal{F}_{(U, \text{id})} \rightarrow \mathcal{F}_{(R, t)}$ .

The cocycle condition for  $\alpha$  follows from the condition that  $(U, R, s, t, c)$  is groupoid, i.e., that composition is associative (details omitted).

To do this in general, i.e., when  $U$  and  $R$  are algebraic spaces, it suffices to explain how to associate to an algebraic space  $(W, u)$  over  $U$  a quasi-coherent sheaf  $\mathcal{F}_{(W, u)}$  and to construct the comparison maps for morphisms between these. We set  $\mathcal{F}_{(W, u)} = x^* \mathcal{F}|_{W_{\acute{e}t\acute{a}l\acute{e}}}$  where  $x$  is the 1-morphism  $\mathcal{S}_W \rightarrow \mathcal{S}_U \rightarrow [U/pR]$  and the comparison maps are explained in (10.2.3).

Conversely, suppose that  $(\mathcal{G}, \alpha)$  is a quasi-coherent module on  $(U, R, s, t, c)$ . We are going to define a presheaf of modules  $\mathcal{F}$  on  $\mathcal{X}$  as follows. Given an object  $(T, u)$  of  $[U/pR]$  we set

$$\mathcal{F}(T, u) := \Gamma(T, u_{small}^* \mathcal{G}).$$

Given a morphism  $(f, r) : (T, u) \rightarrow (T', u')$  we get a map

$$\begin{aligned}
\mathcal{F}(T', u') &= \Gamma(T', (u')^*_{small} \mathcal{G}) \\
&\rightarrow \Gamma(T, f^*_{small} (u')^*_{small} \mathcal{G}) = \Gamma(T, (u' \circ f)^*_{small} \mathcal{G}) \\
&= \Gamma(T, (t \circ r)^*_{small} \mathcal{G}) = \Gamma(T, r^*_{small} t^*_{small} \mathcal{G}) \\
&\rightarrow \Gamma(T, r^*_{small} s^*_{small} \mathcal{G}) = \Gamma(T, (s \circ r)^*_{small} \mathcal{G}) \\
&= \Gamma(T, u^*_{small} \mathcal{G}) \\
&= \mathcal{F}(T, u)
\end{aligned}$$

where the first arrow is pullback along  $f$  and the second arrow is  $\alpha$ . Note that if  $(T, r)$  is a special morphism, then this map is just pullback along  $f$  as  $e^*_{small} \alpha = \text{id}$  by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that  $\mathcal{F}$  is a presheaf of modules (details omitted). It is immediate from the definition that  $\mathcal{F}$  is quasi-coherent when pulled back to  $(Sch/T)_{fppf}$  (by the simple description of the restriction maps of  $\mathcal{F}$  in case of a special morphism).

We omit the verification that the functors constructed above are quasi-inverse to each other.  $\square$

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 7.1. We will see later (Criteria for Representability, Theorem 17.2) that the assumptions on the groupoid imply that  $\mathcal{X}$  is an algebraic stack.

076S **Lemma 13.2.** *Let  $(U, R, s, t, c)$  be a groupoid in algebraic spaces over  $S$ . Assume  $s, t$  are flat and locally of finite presentation. Let  $\mathcal{X} = [U/R]$  be the quotient stack. Denote  $\pi : \mathcal{S}_U \rightarrow \mathcal{X}$  the quotient map. Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_{\mathcal{X}}$ -module, and let  $\mathcal{H}$  be any object of  $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ . The map*

$$\text{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{H}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(x^* \mathcal{F}|_{U_{\acute{e}tale}}, x^* \mathcal{H}|_{U_{\acute{e}tale}}), \quad \phi \longmapsto x^* \phi|_{U_{\acute{e}tale}}$$

is injective and its image consists of exactly those  $\varphi : x^* \mathcal{F}|_{U_{\acute{e}tale}} \rightarrow x^* \mathcal{H}|_{U_{\acute{e}tale}}$  which give rise to a commutative diagram

$$\begin{array}{ccccc}
s^*_{small}(x^* \mathcal{F}|_{U_{\acute{e}tale}}) & \longrightarrow & (x \circ s)^* \mathcal{F}|_{R_{\acute{e}tale}} = (x \circ t)^* \mathcal{F}|_{R_{\acute{e}tale}} & \longleftarrow & t^*_{small}(x^* \mathcal{F}|_{U_{\acute{e}tale}}) \\
\downarrow s^*_{small} \varphi & & & & \downarrow t^*_{small} \varphi \\
s^*_{small}(x^* \mathcal{H}|_{U_{\acute{e}tale}}) & \longrightarrow & (x \circ s)^* \mathcal{H}|_{R_{\acute{e}tale}} = (x \circ t)^* \mathcal{H}|_{R_{\acute{e}tale}} & \longleftarrow & t^*_{small}(x^* \mathcal{H}|_{U_{\acute{e}tale}})
\end{array}$$

of modules on  $R_{\acute{e}tale}$  where the horizontal arrows are the comparison maps (10.2.3).

**Proof.** According to Lemma 12.2 the stackification map  $[U/pR] \rightarrow [U/R]$  (see Groupoids in Spaces, Definition 19.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf  $\mathcal{O}$ -modules. Thus it suffices to prove the lemma with  $\mathcal{X} = [U/pR]$ . By Proposition 13.1 and its proof there exists a quasi-coherent module  $(\mathcal{G}, \alpha)$  on  $(U, R, s, t, c)$  such that  $\mathcal{F}$  is given by the rule  $\mathcal{F}(T, u) = \Gamma(T, u^* \mathcal{G})$ . In particular  $x^* \mathcal{F}|_{U_{\acute{e}tale}} = \mathcal{G}$  and it is clear that the map of the statement of the lemma is injective. Moreover, given a map  $\varphi : \mathcal{G} \rightarrow x^* \mathcal{H}|_{U_{\acute{e}tale}}$  and given any object  $y = (T, u)$  of  $[U/pR]$  we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^* \mathcal{G}) \xrightarrow{u^*_{small} \varphi} \Gamma(T, u^*_{small} x^* \mathcal{H}|_{U_{\acute{e}tale}}) \rightarrow \Gamma(T, y^* \mathcal{H}|_{T_{\acute{e}tale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (9.4.1) for the sheaf  $\mathcal{H}$ . This assignment is compatible with the restriction mappings of the sheaves  $\mathcal{F}$  and  $\mathcal{G}$  for morphisms of  $[U/pR]$  if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of  $\mathcal{F}$  are made explicit in terms of  $(\mathcal{G}, \alpha)$  in the proof of Proposition 13.1.  $\square$

#### 14. Quasi-coherent sheaves on algebraic stacks

06WU Let  $\mathcal{X}$  be an algebraic stack over  $S$ . By Algebraic Stacks, Lemma 16.2 we can find an equivalence  $[U/R] \rightarrow \mathcal{X}$  where  $(U, R, s, t, c)$  is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_{\mathcal{X}}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 13.1. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 12.5 we see that  $QCoh(\mathcal{O}_{\mathcal{X}})$  is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_{\mathcal{X}}) \longrightarrow Mod(\mathcal{O}_{\mathcal{X}})$$

is in general **not** exact. However, exactly the same thing happens for schemes: for most schemes  $X$  the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow Mod((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 8.13. Parenthetically, the example in the proof of Descent, Lemma 8.13 shows that in general the strictly full embedding  $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow LQCoh(\mathcal{O}_{\mathcal{X}})$  isn't exact either.

We collect all the positive results obtained so far in a single statement.

06WV **Lemma 14.1.** *Let  $\mathcal{X}$  be an algebraic stack over  $S$ .*

- (1) *If  $[U/R] \rightarrow \mathcal{X}$  is a presentation of  $\mathcal{X}$  then there is a canonical equivalence  $QCoh(\mathcal{O}_{\mathcal{X}}) \cong QCoh(U, R, s, t, c)$ .*
- (2) *The category  $QCoh(\mathcal{O}_{\mathcal{X}})$  is abelian.*
- (3) *The category  $QCoh(\mathcal{O}_{\mathcal{X}})$  has colimits and they agree with colimits in the category  $Mod(\mathcal{O}_{\mathcal{X}})$ .*
- (4) *Given  $\mathcal{F}, \mathcal{G}$  in  $QCoh(\mathcal{O}_{\mathcal{X}})$  the tensor product  $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$  in  $Mod(\mathcal{O}_{\mathcal{X}})$  is an object of  $QCoh(\mathcal{O}_{\mathcal{X}})$ .*
- (5) *Given  $\mathcal{F}, \mathcal{G}$  in  $QCoh(\mathcal{O}_{\mathcal{X}})$  with  $\mathcal{F}$  locally of finite presentation on  $\mathcal{X}_{fppf}$  the sheaf  $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})$  in  $Mod(\mathcal{O}_{\mathcal{X}})$  is an object of  $QCoh(\mathcal{O}_{\mathcal{X}})$ .*

**Proof.** Properties (3), (4), and (5) were proven in Lemma 11.9. Part (1) is Proposition 13.1. Part (2) follows from Groupoids in Spaces, Lemma 12.5 as discussed above.  $\square$

0781 **Proposition 14.2.** *Let  $\mathcal{X}$  be an algebraic stack over  $S$ .*

- (1) *The category  $QCoh(\mathcal{O}_{\mathcal{X}})$  is a Grothendieck abelian category. Consequently,  $QCoh(\mathcal{O}_{\mathcal{X}})$  has enough injectives and all limits.*

(2) The inclusion functor  $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{O}_{\mathcal{X}})$  has a right adjoint<sup>3</sup>

$$Q : Mod(\mathcal{O}_{\mathcal{X}}) \longrightarrow QCoh(\mathcal{O}_{\mathcal{X}})$$

such that for every quasi-coherent sheaf  $\mathcal{F}$  the adjunction mapping  $Q(\mathcal{F}) \rightarrow \mathcal{F}$  is an isomorphism.

**Proof.** This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 23.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 32.2. We advise the reader to read either of those proofs first.

Part (1) means  $QCoh(\mathcal{O}_{\mathcal{X}})$  (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 10. By Lemma 14.1 colimits in  $QCoh(\mathcal{O}_{\mathcal{X}})$  exist and agree with colimits in  $Mod(\mathcal{O}_{\mathcal{X}})$ . By Modules on Sites, Lemma 14.2 filtered colimits are exact. Hence (a) and (b) hold.

Choose a presentation  $\mathcal{X} = [U/R]$  so that  $(U, R, s, t, c)$  is a smooth groupoid in algebraic spaces and in particular  $s$  and  $t$  are flat morphisms of algebraic spaces. By Lemma 14.1 above we have  $QCoh(\mathcal{O}_{\mathcal{X}}) = QCoh(U, R, s, t, c)$ . By Groupoids in Spaces, Lemma 13.2 there exists a set  $T$  and a family  $(\mathcal{F}_t)_{t \in T}$  of quasi-coherent sheaves on  $\mathcal{X}$  such that every quasi-coherent sheaf on  $\mathcal{X}$  is the directed colimit of its subsheaves which are isomorphic to one of the  $\mathcal{F}_t$ . Thus  $\bigoplus_t \mathcal{F}_t$  is a generator of  $QCoh(\mathcal{O}_{\mathcal{X}})$  and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 11.7 and Lemma 13.2.

Proof of (2). To construct  $Q$  we use the following general procedure. Given an object  $\mathcal{F}$  of  $Mod(\mathcal{O}_{\mathcal{X}})$  we consider the functor

$$QCoh(\mathcal{O}_{\mathcal{X}})^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \text{Hom}_{\mathcal{X}}(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 13.1. Thus there exists a quasi-coherent sheaf  $Q(\mathcal{F})$  and a functorial isomorphism  $\text{Hom}_{\mathcal{X}}(\mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{X}}(\mathcal{G}, Q(\mathcal{F}))$  for  $\mathcal{G}$  in  $QCoh(\mathcal{O}_{\mathcal{X}})$ . By the Yoneda lemma (Categories, Lemma 3.5) the construction  $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$  is functorial in  $\mathcal{F}$ . By construction  $Q$  is a right adjoint to the inclusion functor. The fact that  $Q(\mathcal{F}) \rightarrow \mathcal{F}$  is an isomorphism when  $\mathcal{F}$  is quasi-coherent is a formal consequence of the fact that the inclusion functor  $QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod(\mathcal{O}_{\mathcal{X}})$  is fully faithful.  $\square$

## 15. Cohomology

075E Let  $S$  be a scheme and let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . For any  $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$  the categories  $Ab(\mathcal{X}_{\tau})$  and  $Mod(\mathcal{X}_{\tau}, \mathcal{O}_{\mathcal{X}})$  have enough injectives, see Injectives, Theorems 7.4 and 8.4. Thus we can use the machinery of Cohomology on Sites, Section 3 to define the cohomology groups

$$H^p(\mathcal{X}_{\tau}, \mathcal{F}) = H^p_{\tau}(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_{\tau}(x, \mathcal{F})$$

for any  $x \in \text{Ob}(\mathcal{X})$  and any object  $\mathcal{F}$  of  $Ab(\mathcal{X}_{\tau})$  or  $Mod(\mathcal{X}_{\tau}, \mathcal{O}_{\mathcal{X}})$ . Moreover, if  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ , then we obtain the higher direct images  $R^i f_* \mathcal{F}$  in  $Ab(\mathcal{Y}_{\tau})$  or  $Mod(\mathcal{Y}_{\tau}, \mathcal{O}_{\mathcal{Y}})$ . Of course, as explained in Cohomology on Sites, Section 4 there are also derived versions of  $H^p(-)$  and  $R^i f_*$ .

<sup>3</sup>This functor is sometimes called the *coherator*.

075F **Lemma 15.1.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be a category fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$ . Let  $x \in \text{Ob}(\mathcal{X})$  be an object lying over the scheme  $U$ . Let  $\mathcal{F}$  be an object of  $\text{Ab}(\mathcal{X}_\tau)$  or  $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ . Then*

$$H_\tau^p(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if  $\tau = \text{étale}$ , then we also have

$$H_{\text{étale}}^p(x, \mathcal{F}) = H^p(U_{\text{étale}}, \mathcal{F}|_{U_{\text{étale}}}).$$

**Proof.** The first statement follows from Cohomology on Sites, Lemma 8.1 and the equivalence of Lemma 9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 20.5.  $\square$

## 16. Injective sheaves

06WW The pushforward of an injective abelian sheaf or module is injective.

06WX **Lemma 16.1.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{\text{Zar, étale, smooth, syntomic, fppf}\}$ .*

- (1)  $f_*\mathcal{I}$  is injective in  $\text{Ab}(\mathcal{Y}_\tau)$  for  $\mathcal{I}$  injective in  $\text{Ab}(\mathcal{X}_\tau)$ , and
- (2)  $f_*\mathcal{I}$  is injective in  $\text{Mod}(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$  for  $\mathcal{I}$  injective in  $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ .

**Proof.** This follows formally from the fact that  $f^{-1}$  is an exact left adjoint of  $f_*$ , see Homology, Lemma 26.1.  $\square$

In the rest of this section we prove that pullback  $f^{-1}$  has a left adjoint  $f_!$  on abelian sheaves and modules. If  $f$  is representable (by schemes or by algebraic spaces), then it will turn out that  $f_!$  is exact and  $f^{-1}$  will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

06WY **Lemma 16.2.** *Let  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  be a category fibred in groupoids.*

- (1) *The category  $\mathcal{X}$  has fibre products.*
- (2) *If the Isom-presheaves of  $\mathcal{X}$  are representable by algebraic spaces, then  $\mathcal{X}$  has equalizers.*
- (3) *If  $\mathcal{X}$  is an algebraic stack (or more generally a quotient stack), then  $\mathcal{X}$  has equalizers.*

**Proof.** Part (1) follows Categories, Lemma 34.14 as  $(Sch/S)_{fppf}$  has fibre products.

Let  $a, b : x \rightarrow y$  be morphisms of  $\mathcal{X}$ . Set  $U = p(x)$  and  $V = p(y)$ . The category of schemes has equalizers hence we can let  $W \rightarrow U$  be the equalizer of  $p(a)$  and  $p(b)$ . Denote  $c : z \rightarrow x$  a morphism of  $\mathcal{X}$  lying over  $W \rightarrow U$ . The equalizer of  $a$  and  $b$ , if it exists, is the equalizer of  $a \circ c$  and  $b \circ c$ . Thus we may assume that  $p(a) = p(b) = f : U \rightarrow V$ . As  $\mathcal{X}$  is fibred in groupoids, there exists a unique automorphism  $i : x \rightarrow x$  in the fibre category of  $\mathcal{X}$  over  $U$  such that  $a \circ i = b$ . Again the equalizer of  $a$  and  $b$  is the equalizer of  $\text{id}_x$  and  $i$ . Recall that the  $\text{Isom}_\mathcal{X}(x)$  is the presheaf on  $(Sch/U)_{fppf}$  which to  $T/U$  associates the set of automorphisms of  $x|_T$  in the fibre category of  $\mathcal{X}$  over  $T$ , see Stacks, Definition 2.2. If  $\text{Isom}_\mathcal{X}(x)$  is representable by an algebraic space  $G \rightarrow U$ , then we see that  $\text{id}_x$  and  $i$  define morphisms  $e, i : U \rightarrow G$  over  $U$ . Set  $M = U \times_{e, G, i} U$ , which by Morphisms of



Spaces, Lemma 4.7 is a scheme. Then it is clear that  $x|_M \rightarrow x$  is the equalizer of the maps  $\text{id}_x$  and  $i$  in  $\mathcal{X}$ . This proves (2).

If  $\mathcal{X} = [U/R]$  for some groupoid in algebraic spaces  $(U, R, s, t, c)$  over  $S$ , then the hypothesis of (2) holds by Bootstrap, Lemma 11.5. If  $\mathcal{X}$  is an algebraic stack, then we can choose a presentation  $[U/R] \cong \mathcal{X}$  by Algebraic Stacks, Lemma 16.2.  $\square$

06WZ **Lemma 16.3.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(\text{Sch}/S)_{\text{fppf}}$ .*

- (1) *The functor  $f$  transforms fibre products into fibre products.*
- (2) *If  $f$  is faithful, then  $f$  transforms equalizers into equalizers.*

**Proof.** By Categories, Lemma 34.14 we see that a fibre product in  $\mathcal{X}$  is any commutative square lying over a fibre product diagram in  $(\text{Sch}/S)_{\text{fppf}}$ . Similarly for  $\mathcal{Y}$ . Hence (1) is clear.

Let  $x \rightarrow x'$  be the equalizer of two morphisms  $a, b : x' \rightarrow x''$  in  $\mathcal{X}$ . We will show that  $f(x) \rightarrow f(x')$  is the equalizer of  $f(a)$  and  $f(b)$ . Let  $y \rightarrow f(x)$  be a morphism of  $\mathcal{Y}$  equalizing  $f(a)$  and  $f(b)$ . Say  $x, x', x''$  lie over the schemes  $U, U', U''$  and  $y$  lies over  $V$ . Denote  $h : V \rightarrow U'$  the image of  $y \rightarrow f(x)$  in the category of schemes. The morphism  $y \rightarrow f(x)$  is isomorphic to  $f(h^*x') \rightarrow f(x')$  by the axioms of fibred categories. Hence, as  $f$  is faithful, we see that  $h^*x' \rightarrow x'$  equalizes  $a$  and  $b$ . Thus we obtain a unique morphism  $h^*x' \rightarrow x$  whose image  $y = f(h^*x') \rightarrow f(x)$  is the desired morphism in  $\mathcal{Y}$ .  $\square$

06X0 **Lemma 16.4.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $g : \mathcal{Z} \rightarrow \mathcal{Y}$  be faithful 1-morphisms of categories fibred in groupoids over  $(\text{Sch}/S)_{\text{fppf}}$ .*

- (1) *the functor  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Y}$  is faithful, and*
- (2) *if  $\mathcal{X}, \mathcal{Z}$  have equalizers, so does  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ .*

**Proof.** We think of objects in  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$  as quadruples  $(U, x, z, \alpha)$  where  $\alpha : f(x) \rightarrow g(z)$  is an isomorphism over  $U$ , see Categories, Lemma 31.3. A morphism  $(U, x, z, \alpha) \rightarrow (U', x', z', \alpha')$  is a pair of morphisms  $a : x \rightarrow x'$  and  $b : z \rightarrow z'$  compatible with  $\alpha$  and  $\alpha'$ . Thus it is clear that if  $f$  and  $g$  are faithful, so is the functor  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Y}$ . Now, suppose that  $(a, b), (a', b') : (U, x, z, \alpha) \rightarrow (U', x', z', \alpha')$  are two morphisms of the 2-fibre product. Then consider the equalizer  $x'' \rightarrow x$  of  $a$  and  $a'$  and the equalizer  $z'' \rightarrow z$  of  $b$  and  $b'$ . Since  $f$  commutes with equalizers (by Lemma 16.3) we see that  $f(x'') \rightarrow f(x)$  is the equalizer of  $f(a)$  and  $f(a')$ . Similarly,  $g(z'') \rightarrow g(z)$  is the equalizer of  $g(b)$  and  $g(b')$ . Picture

$$\begin{array}{ccccc}
 f(x'') & \longrightarrow & f(x) & \begin{array}{c} \xrightarrow{f(a)} \\ \xrightarrow{f(a')} \end{array} & f(x') \\
 \alpha'' \downarrow \text{dotted} & & \alpha \downarrow & & \downarrow \alpha' \\
 g(z'') & \longrightarrow & g(z) & \begin{array}{c} \xrightarrow{g(b)} \\ \xrightarrow{g(b')} \end{array} & g(z')
 \end{array}$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of  $\alpha''$  in the category of schemes is the identity of its source. On the other hand, the existence of  $\alpha''$  means that we can assume that  $x''$  and  $z''$  are defined over the same scheme and that the morphisms  $x'' \rightarrow x$  and  $z'' \rightarrow z$  have the same image in the category of schemes. Redoing the diagram

above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted.  $\square$

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that  $f$  is faithful.

06X1 **Lemma 16.5.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . The functor  $f^{-1} : Ab(\mathcal{Y}_\tau) \rightarrow Ab(\mathcal{X}_\tau)$  has a left adjoint  $f_! : Ab(\mathcal{X}_\tau) \rightarrow Ab(\mathcal{Y}_\tau)$ . If  $f$  is faithful and  $\mathcal{X}$  has equalizers, then*

- (1)  $f_!$  is exact, and
- (2)  $f^{-1}\mathcal{I}$  is injective in  $Ab(\mathcal{X}_\tau)$  for  $\mathcal{I}$  injective in  $Ab(\mathcal{Y}_\tau)$ .

**Proof.** By Stacks, Lemma 10.3 the functor  $f$  is continuous and cocontinuous. Hence by Modules on Sites, Lemma 16.2 the functor  $f^{-1} : Ab(\mathcal{Y}_\tau) \rightarrow Ab(\mathcal{X}_\tau)$  has a left adjoint  $f_! : Ab(\mathcal{X}_\tau) \rightarrow Ab(\mathcal{Y}_\tau)$ . To see (1) we apply Modules on Sites, Lemma 16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 16.2 and 16.3 above. Part (2) follows from this formally, see Homology, Lemma 26.1.  $\square$

06X2 **Lemma 16.6.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . The functor  $f^* : Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \rightarrow Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$  has a left adjoint  $f_! : Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \rightarrow Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$  which agrees with the functor  $f_!$  of Lemma 16.5 on underlying abelian sheaves. If  $f$  is faithful and  $\mathcal{X}$  has equalizers, then*

- (1)  $f_!$  is exact, and
- (2)  $f^{-1}\mathcal{I}$  is injective in  $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$  for  $\mathcal{I}$  injective in  $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ .

**Proof.** Recall that  $f$  is a continuous and cocontinuous functor of sites and that  $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ . Hence Modules on Sites, Lemma 40.1 implies  $f^*$  has a left adjoint  $f_!^{Mod}$ . Let  $x$  be an object of  $\mathcal{X}$  lying over the scheme  $U$ . Then  $f$  induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to  $(Sch/U)_\tau$ , see Lemma 9.4. Modules on Sites, Remark 40.2 shows that  $f_!$  agrees with the functor on abelian sheaves.

Assume now that  $\mathcal{X}$  has equalizers and that  $f$  is faithful. Lemma 16.5 tells us that  $f_!$  is exact. Finally, Homology, Lemma 26.1 implies the statement on pullbacks of injective modules.  $\square$

## 17. The Čech complex

06X3 To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad f \quad} & \mathcal{X} \\ & \searrow q & \swarrow p \\ & (Sch/S)_{fppf} & \end{array}$$

06X4 (17.0.1)

We are going to think about  $\mathcal{U}$  as a “covering” of  $\mathcal{X}$ . Hence we want to consider the simplicial object

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightrightarrows \mathcal{U}$$

in the category of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . However, since this is a  $(2, 1)$ -category and not a category, we should say explicitly what we mean. Namely, we let  $\mathcal{U}_n$  be the category with objects  $(u_0, \dots, u_n, x, \alpha_0, \dots, \alpha_n)$  where  $\alpha_i : f(u_i) \rightarrow x$  is an isomorphism in  $\mathcal{X}$ . We denote  $f_n : \mathcal{U}_n \rightarrow \mathcal{X}$  the 1-morphism which assigns to  $(u_0, \dots, u_n, x, \alpha_0, \dots, \alpha_n)$  the object  $x$ . Note that  $\mathcal{U}_0 = \mathcal{U}$  and  $f_0 = f$ . Given a map  $\varphi : [m] \rightarrow [n]$  we consider the 1-morphism  $\mathcal{U}_\varphi : \mathcal{U}_m \rightarrow \mathcal{U}_n$  given by

$$(u_0, \dots, u_m, x, \alpha_0, \dots, \alpha_m) \mapsto (u_{\varphi(0)}, \dots, u_{\varphi(m)}, x, \alpha_{\varphi(0)}, \dots, \alpha_{\varphi(m)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over  $\mathcal{X}$ . We denote  $\mathcal{U}_\bullet$  this simplicial object. If  $\mathcal{F}$  is a presheaf of sets on  $\mathcal{X}$ , then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \rightrightarrows \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \rightrightarrows \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If  $\mathcal{F}$  is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let  $\mathcal{U} \rightarrow \mathcal{X}$  be as above and let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{X}$ . The Čech complex associated to the situation is denoted  $\check{C}^\bullet(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{F})$ . It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 25. It has terms

$$\check{C}^n(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1} : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \rightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where  $\delta_i^{n+1}$  corresponds to the map  $[n] \rightarrow [n+1]$  omitting the index  $i$ . Note that the map  $\Gamma(\mathcal{X}, \mathcal{F}) \rightarrow \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$  is in the kernel of the differential  $d^0$ . Hence we define the *extended Čech complex* to be the complex

$$\dots \rightarrow 0 \rightarrow \Gamma(\mathcal{X}, \mathcal{F}) \rightarrow \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \rightarrow \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \rightarrow \dots$$

with  $\Gamma(\mathcal{X}, \mathcal{F})$  placed in degree  $-1$ . The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \rightarrow \check{C}^\bullet(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

06X5 **Lemma 17.1.** *Generalities on Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ g \downarrow & & \downarrow f \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

*is 2-commutative diagram of categories fibred in groupoids over  $(Sch/S)_{fppf}$ , then there is a morphism of Čech complexes*

$$\check{C}^\bullet(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{F}) \rightarrow \check{C}^\bullet(\mathcal{V} \rightarrow \mathcal{Y}, e^{-1}\mathcal{F})$$

- (2) if  $h$  and  $e$  are equivalences, then the map of (1) is an isomorphism,  
(3) if  $f, f' : \mathcal{U} \rightarrow \mathcal{X}$  are 2-isomorphic, then the associated Čech complexes are isomorphic,

**Proof.** In the situation of (1) let  $t : f \circ h \rightarrow e \circ g$  be a 2-morphism. The map on complexes is given in degree  $n$  by pullback along the 1-morphisms  $\mathcal{V}_n \rightarrow \mathcal{U}_n$  given by the rule

$$(v_0, \dots, v_n, y, \beta_0, \dots, \beta_n) \mapsto (h(v_0), \dots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \dots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2).  $\square$

06X6 **Lemma 17.2.** *If there exists a 1-morphism  $s : \mathcal{X} \rightarrow \mathcal{U}$  such that  $f \circ s$  is 2-isomorphic to  $\text{id}_{\mathcal{X}}$  then the extended Čech complex is homotopic to zero.*

**Proof.** Set  $\mathcal{U}' = \mathcal{U} \times_{\mathcal{X}} \mathcal{X}$  equal to the fibre product as described in Categories, Lemma 31.3. Set  $f' : \mathcal{U}' \rightarrow \mathcal{X}$  equal to the second projection. Then  $\mathcal{U} \rightarrow \mathcal{U}'$ ,  $u \mapsto (u, f(x), 1)$  is an equivalence over  $\mathcal{X}$ , hence we may replace  $(\mathcal{U}, f)$  by  $(\mathcal{U}', f')$  by Lemma 17.1. The advantage of this is that now  $f'$  has a section  $s'$  such that  $f' \circ s' = \text{id}_{\mathcal{X}}$  on the nose. Namely, if  $t : s \circ f \rightarrow \text{id}_{\mathcal{X}}$  is a 2-isomorphism then we can set  $s'(x) = (s(x), x, t_x)$ . Thus we may assume that  $f \circ s = \text{id}_{\mathcal{X}}$ .

In the case that  $f \circ s = \text{id}_{\mathcal{X}}$  the result follows from general principles. We give the homotopy explicitly. Namely, for  $n \geq 0$  define  $s_n : \mathcal{U}_n \rightarrow \mathcal{U}_{n+1}$  to be the 1-morphism defined by the rule on objects

$$(u_0, \dots, u_n, x, \alpha_0, \dots, \alpha_n) \mapsto (u_0, \dots, u_n, s(x), x, \alpha_0, \dots, \alpha_n, \text{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along  $s_n$ . We also set  $s_{-1} = s$  and  $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \rightarrow \Gamma(\mathcal{X}, \mathcal{F})$  equal to pullback along  $s_{-1}$ . Then the family of maps  $\{h^n\}_{n \geq 0}$  is a homotopy between 1 and 0 on the extended Čech complex.  $\square$

## 18. The relative Čech complex

06X7 Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$  as in (17.0.1). Consider the associated simplicial object  $\mathcal{U}_\bullet$  and the maps  $f_n : \mathcal{U}_n \rightarrow \mathcal{X}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Finally, suppose that  $\mathcal{F}$  is a sheaf (of sets) on  $\mathcal{X}_\tau$ . Then

$$f_{0,*}f_0^{-1}\mathcal{F} \rightrightarrows f_{1,*}f_1^{-1}\mathcal{F} \rightrightarrows f_{2,*}f_2^{-1}\mathcal{F}$$

is a cosimplicial sheaf on  $\mathcal{X}_\tau$  where we use the pullback maps introduced in Sites, Section 44. If  $\mathcal{F}$  is an abelian sheaf, then  $f_{n,*}f_n^{-1}\mathcal{F}$  form a cosimplicial abelian sheaf on  $\mathcal{X}_\tau$ . The associated complex (see Simplicial, Section 25)

$$\dots \rightarrow 0 \rightarrow f_{0,*}f_0^{-1}\mathcal{F} \rightarrow f_{1,*}f_1^{-1}\mathcal{F} \rightarrow f_{2,*}f_2^{-1}\mathcal{F} \rightarrow \dots$$

is called the *relative Čech complex* associated to the situation. We will denote this complex  $\mathcal{K}^\bullet(f, \mathcal{F})$ . The *extended relative Čech complex* is the complex

$$\dots \rightarrow 0 \rightarrow \mathcal{F} \rightarrow f_{0,*}f_0^{-1}\mathcal{F} \rightarrow f_{1,*}f_1^{-1}\mathcal{F} \rightarrow f_{2,*}f_2^{-1}\mathcal{F} \rightarrow \dots$$

with  $\mathcal{F}$  in degree  $-1$ . The extended relative Čech complex is acyclic if and only if the map  $\mathcal{F}[0] \rightarrow \mathcal{K}^\bullet(f, \mathcal{F})$  is a quasi-isomorphism of complexes of sheaves.

06X8 **Remark 18.1.** We can define the complex  $\mathcal{K}^\bullet(f, \mathcal{F})$  also if  $\mathcal{F}$  is a presheaf, only we cannot use the reference to Sites, Section 44 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ & \searrow g & \swarrow f \\ & & \mathcal{X} \end{array}$$

of categories fibred in groupoids over  $(Sch/S)_{fppf}$  and a presheaf  $\mathcal{G}$  on  $\mathcal{U}$  we can define the pullback map  $f_*\mathcal{G} \rightarrow g_*h^{-1}\mathcal{G}$  as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map  $\mathcal{G} \rightarrow h_*h^{-1}\mathcal{G}$ . This works because in our situation the functors  $h_*$  and  $h^{-1}$  are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 3 and 4.

06X9 **Lemma 18.2.** *Generalities on relative Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ g \downarrow & & \downarrow f \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

*is 2-commutative diagram of categories fibred in groupoids over  $(Sch/S)_{fppf}$ , then there is a morphism  $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \rightarrow \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ .*

(2) *if  $h$  and  $e$  are equivalences, then the map of (1) is an isomorphism,*

(3) *if  $f, f' : \mathcal{U} \rightarrow \mathcal{X}$  are 2-isomorphic, then the associated relative Čech complexes are isomorphic,*

**Proof.** Literally the same as the proof of Lemma 17.1 using the pullback maps of Remark 18.1.  $\square$

06XA **Lemma 18.3.** *If there exists a 1-morphism  $s : \mathcal{X} \rightarrow \mathcal{U}$  such that  $f \circ s$  is 2-isomorphic to  $id_{\mathcal{X}}$  then the extended relative Čech complex is homotopic to zero.*

**Proof.** Literally the same as the proof of Lemma 17.2.  $\square$

06XB **Remark 18.4.** Let us “compute” the value of the relative Čech complex on an object  $x$  of  $\mathcal{X}$ . Say  $p(x) = U$ . Consider the 2-fibre product diagram (which serves to introduce the notation  $g : \mathcal{V} \rightarrow \mathcal{Y}$ )

$$\begin{array}{ccccc} \mathcal{V} & \xlongequal{\quad} & (Sch/U)_{fppf} \times_{x, \mathcal{X}} \mathcal{U} & \xrightarrow{\quad} & \mathcal{U} \\ g \downarrow & & \downarrow & & \downarrow f \\ \mathcal{Y} & \xlongequal{\quad} & (Sch/U)_{fppf} & \xrightarrow{x} & \mathcal{X} \end{array}$$

Note that the morphism  $\mathcal{V}_n \rightarrow \mathcal{U}_n$  of the proof of Lemma 17.1 induces an equivalence  $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x, \mathcal{X}} \mathcal{U}_n$ . Hence we see from (5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \rightarrow \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object  $x$  of  $\mathcal{X}$  is the Čech complex of the base change of  $f$  to  $\mathcal{X}/x \cong (Sch/U)_{fppf}$ . This implies for example that Lemma 17.2 implies Lemma 18.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

06XC **Lemma 18.5.** *Let*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ g \downarrow & & \downarrow f \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over  $(Sch/S)_{fppf}$  and let  $\mathcal{F}$  be an abelian presheaf on  $\mathcal{X}$ . Then the map  $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \rightarrow \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$  of Lemma 18.2 is an isomorphism of complexes of abelian presheaves.

**Proof.** Let  $y$  be an object of  $\mathcal{Y}$  lying over the scheme  $T$ . Set  $x = e(y)$ . We are going to show that the map induces an isomorphism on sections over  $y$ . Note that  $\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{C}^\bullet((Sch/T)_{fppf} \times_{x, \mathcal{X}} \mathcal{U} \rightarrow (Sch/T)_{fppf}, x^{-1}\mathcal{F})$  by Remark 18.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{C}^\bullet((Sch/T)_{fppf} \times_{y, \mathcal{Y}} \mathcal{V} \rightarrow (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 18.4. Note that  $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$  and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y, \mathcal{Y}} \mathcal{V} \rightarrow (Sch/T)_{fppf} \times_{x, \mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over  $y$  is an isomorphism by Lemma 17.1.  $\square$

Exactness can be checked on a “covering”.

06XD **Lemma 18.6.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Let*

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$$

be a complex in  $Ab(\mathcal{X}_\tau)$ . Assume that

- (1) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ , and
- (2)  $f^{-1}\mathcal{F} \rightarrow f^{-1}\mathcal{G} \rightarrow f^{-1}\mathcal{H}$  is exact.

Then the sequence  $\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$  is exact.

**Proof.** Let  $x$  be an object of  $\mathcal{X}$  lying over the scheme  $T$ . Consider the sequence  $x^{-1}\mathcal{F} \rightarrow x^{-1}\mathcal{G} \rightarrow x^{-1}\mathcal{H}$  of abelian sheaves on  $(Sch/T)_\tau$ . It suffices to show this sequence is exact. By assumption there exists a  $\tau$ -covering  $\{T_i \rightarrow T\}$  such that  $x|_{T_i}$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$  over  $T_i$  and moreover the sequence  $u_i^{-1}f^{-1}\mathcal{F} \rightarrow u_i^{-1}f^{-1}\mathcal{G} \rightarrow u_i^{-1}f^{-1}\mathcal{H}$  of abelian sheaves on  $(Sch/T_i)_\tau$  is exact. Since  $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$  we conclude that the sequence  $x^{-1}\mathcal{F} \rightarrow x^{-1}\mathcal{G} \rightarrow x^{-1}\mathcal{H}$  become exact after localizing at each of the members of a covering, hence the sequence is exact.  $\square$

06XE **Proposition 18.7.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . If*

- (1)  $\mathcal{F}$  is an abelian sheaf on  $\mathcal{X}_\tau$ , and

- (2) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ ,

then the extended relative Čech complex

$$\dots \rightarrow 0 \rightarrow \mathcal{F} \rightarrow f_{0,*}f_0^{-1}\mathcal{F} \rightarrow f_{1,*}f_1^{-1}\mathcal{F} \rightarrow f_{2,*}f_2^{-1}\mathcal{F} \rightarrow \dots$$

is exact in  $Ab(\mathcal{X}_\tau)$ .

**Proof.** By Lemma 18.6 it suffices to check exactness after pulling back to  $\mathcal{U}$ . By Lemma 18.5 the pullback of the extended relative Čech complex is isomorphic to the extended relative Čech complex for the morphism  $\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \rightarrow \mathcal{U}$  and an abelian sheaf on  $\mathcal{U}_\tau$ . Since there is a section  $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \rightarrow \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$  exactness follows from Lemma 18.3.  $\square$

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

06XF **Lemma 18.8.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Assume*

- (1)  $\mathcal{F}$  is an abelian sheaf on  $\mathcal{X}_\tau$ ,
- (2) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ ,
- (3) the category  $\mathcal{U}$  has equalizers, and
- (4) the functor  $f$  is faithful.

Then there is a first quadrant spectral sequence of abelian groups

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of  $\mathcal{F}$  in the  $\tau$ -topology.

**Proof.** Before we start the proof we make some remarks. By Lemma 16.4 (and induction) all of the categories fibred in groupoids  $\mathcal{U}_p$  have equalizers and all of the morphisms  $f_p : \mathcal{U}_p \rightarrow \mathcal{X}$  are faithful. Let  $\mathcal{I}$  be an injective object of  $Ab(\mathcal{X}_\tau)$ . By Lemma 16.5 we see  $f_p^{-1}\mathcal{I}$  is an injective object of  $Ab((\mathcal{U}_p)_\tau)$ . Hence  $f_{p,*}f_p^{-1}\mathcal{I}$  is an injective object of  $Ab(\mathcal{X}_\tau)$  by Lemma 16.1. Hence Proposition 18.7 shows that the extended relative Čech complex

$$\dots \rightarrow 0 \rightarrow \mathcal{I} \rightarrow f_{0,*}f_0^{-1}\mathcal{I} \rightarrow f_{1,*}f_1^{-1}\mathcal{I} \rightarrow f_{2,*}f_2^{-1}\mathcal{I} \rightarrow \dots$$

is an exact complex in  $Ab(\mathcal{X}_\tau)$  all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex  $\check{C}^\bullet(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{I})$  is quasi-isomorphic to  $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$ .

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 22)

$$\check{C}^\bullet(\mathcal{U} \rightarrow \mathcal{X}, \mathcal{I}^\bullet)$$

where  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  is an injective resolution in  $Ab(\mathcal{X}_\tau)$ . The discussion above shows that Homology, Lemma 22.7 applies which shows that  $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$  is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex  $f_p^{-1}\mathcal{I}^\bullet$  is an injective resolution of  $f_p^{-1}\mathcal{F}$ . Hence the other spectral sequence is as indicated in the lemma.  $\square$

To be sure there is a version for modules as well.

06XG **Lemma 18.9.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Assume*

- (1)  $\mathcal{F}$  is an object of  $Mod(\mathcal{X}_\tau, \mathcal{O}_{\mathcal{X}})$ ,
- (2) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ ,
- (3) the category  $\mathcal{U}$  has equalizers, and
- (4) the functor  $f$  is faithful.

Then there is a first quadrant spectral sequence of  $\Gamma(\mathcal{O}_{\mathcal{X}})$ -modules

$$E_1^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^* \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of  $\mathcal{F}$  in the  $\tau$ -topology.

**Proof.** The proof of this lemma is identical to the proof of Lemma 18.8 except that it uses an injective resolution in  $Mod(\mathcal{X}_\tau, \mathcal{O}_{\mathcal{X}})$  and it uses Lemma 16.6 instead of Lemma 16.5.  $\square$

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

06XH **Lemma 18.10.** *Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ .*

- (1) Assume that  $f$  is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object  $y$  of  $\mathcal{Y}$  there exists an fppf covering  $\{y_i \rightarrow y\}$  and objects  $x_i$  of  $\mathcal{X}$  such that  $f(x_i) \cong y_i$  in  $\mathcal{Y}$ .
- (2) Assume that  $f$  is representable by algebraic spaces, surjective, and smooth. Then for any object  $y$  of  $\mathcal{Y}$  there exists an étale covering  $\{y_i \rightarrow y\}$  and objects  $x_i$  of  $\mathcal{X}$  such that  $f(x_i) \cong y_i$  in  $\mathcal{Y}$ .

**Proof.** Proof of (1). Suppose that  $y$  lies over the scheme  $V$ . We may think of  $y$  as a morphism  $(Sch/V)_{fppf} \rightarrow \mathcal{Y}$ . By definition the 2-fibre product  $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$  is representable by an algebraic space  $W$  and the morphism  $W \rightarrow V$  is surjective, flat, and locally of finite presentation. Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow W$ . Then  $U \rightarrow V$  is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 39.7, 39.8, 5.4, 28.2, and 30.3). Hence  $\{U \rightarrow V\}$  is an fppf covering. Denote  $x$  the object of  $\mathcal{X}$  over  $U$  corresponding to the 1-morphism  $(Sch/U)_{fppf} \rightarrow \mathcal{X}$ . Then  $\{f(x) \rightarrow y\}$  is the desired fppf covering of  $\mathcal{Y}$ .

Proof of (1). Suppose that  $y$  lies over the scheme  $V$ . We may think of  $y$  as a morphism  $(Sch/V)_{fppf} \rightarrow \mathcal{Y}$ . By definition the 2-fibre product  $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$  is representable by an algebraic space  $W$  and the morphism  $W \rightarrow V$  is surjective and smooth. Choose a scheme  $U$  and a surjective étale morphism  $U \rightarrow W$ . Then  $U \rightarrow V$  is also surjective and smooth (see Morphisms of Spaces, Lemmas 39.6, 5.4, and 37.2). Hence  $\{U \rightarrow V\}$  is a smooth covering. By More on Morphisms, Lemma 34.7 there exists an étale covering  $\{V_i \rightarrow V\}$  such that each  $V_i \rightarrow V$  factors through  $U$ . Denote  $x_i$  the object of  $\mathcal{X}$  over  $V_i$  corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \rightarrow (Sch/U)_{fppf} \rightarrow \mathcal{X}.$$

Then  $\{f(x_i) \rightarrow y\}$  is the desired étale covering of  $\mathcal{Y}$ .  $\square$



072D **Lemma 18.11.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  and  $g : \mathcal{X} \rightarrow \mathcal{Y}$  be composable 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Assume*

- (1)  $\mathcal{F}$  is an abelian sheaf on  $\mathcal{X}_\tau$ ,
- (2) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ ,
- (3) the category  $\mathcal{U}$  has equalizers, and
- (4) the functor  $f$  is faithful.

Then there is a first quadrant spectral sequence of abelian sheaves on  $\mathcal{Y}_\tau$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

where all higher direct images are computed in the  $\tau$ -topology.

**Proof.** Note that the assumptions on  $f : \mathcal{U} \rightarrow \mathcal{X}$  and  $\mathcal{F}$  are identical to those in Lemma 18.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \rightarrow g_* \mathcal{I} \rightarrow (g \circ f_0)_* f_0^{-1} \mathcal{I} \rightarrow (g \circ f_1)_* f_1^{-1} \mathcal{I} \rightarrow \dots$$

is exact if  $\mathcal{I}$  is an injective object of  $Ab(\mathcal{X}_\tau)$ . Having said this, consider the two spectral sequences of Homology, Section 22 associated to the double complex  $\mathcal{C}^{\bullet,\bullet}$  with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_* \mathcal{I}^q$$

where  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  is an injective resolution in  $Ab(\mathcal{X}_\tau)$ . The first spectral sequence implies, via Homology, Lemma 22.7, that  $g_* \mathcal{I}^\bullet$  is quasi-isomorphic to the total complex associated to  $\mathcal{C}^{\bullet,\bullet}$ . Since  $f_p^{-1} \mathcal{I}^\bullet$  is an injective resolution of  $f_p^{-1} \mathcal{F}$  (see Lemma 16.5) the second spectral sequence has terms  $E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F}$  as in the statement of the lemma.  $\square$

072E **Lemma 18.12.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  and  $g : \mathcal{X} \rightarrow \mathcal{Y}$  be composable 1-morphisms of categories fibred in groupoids over  $(Sch/S)_{fppf}$ . Let  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . Assume*

- (1)  $\mathcal{F}$  is an object of  $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ ,
- (2) for every object  $x$  of  $\mathcal{X}$  there exists a covering  $\{x_i \rightarrow x\}$  in  $\mathcal{X}_\tau$  such that each  $x_i$  is isomorphic to  $f(u_i)$  for some object  $u_i$  of  $\mathcal{U}$ ,
- (3) the category  $\mathcal{U}$  has equalizers, and
- (4) the functor  $f$  is faithful.

Then there is a first quadrant spectral sequence in  $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

where all higher direct images are computed in the  $\tau$ -topology.

**Proof.** The proof is identical to the proof of Lemma 18.11 except that it uses an injective resolution in  $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$  and it uses Lemma 16.6 instead of Lemma 16.5.  $\square$

### 19. Cohomology on algebraic stacks

06XI Let  $\mathcal{X}$  be an algebraic stack over  $S$ . In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on  $\mathcal{X}$ . In fact, we have constructed a site  $\mathcal{X}_\tau$  for each  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . There is a notion of an abelian sheaf  $\mathcal{F}$  on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the *derived global sections*

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{\acute{e}tale}(\mathcal{X}, \mathcal{F}), \dots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as  $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$  where  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  is an injective resolution in  $Ab(\mathcal{X}_\tau)$ . The  $i$ th cohomology group is the  $i$ th cohomology of the total derived cohomology. We will denote this

$$H_{Zar}^i(\mathcal{X}, \mathcal{F}), H_{\acute{e}tale}^i(\mathcal{X}, \mathcal{F}), \dots, H_{fppf}^i(\mathcal{X}, \mathcal{F}).$$

It will turn out that  $H_{\acute{e}tale}^i = H_{smooth}^i$  because of More on Morphisms, Lemma 34.7. If  $\mathcal{F}$  is a presheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules which is a sheaf in the  $\tau$ -topology, then we use injective resolutions in  $Mod(\mathcal{X}_\tau, \mathcal{O}_{\mathcal{X}})$  to compute total derived global sections and cohomology groups; of course the end result is quasi-isomorphic resp. isomorphic by the general fact Cohomology on Sites, Lemma 13.4.

So far our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are  $(p+1)$ -factors. Also, recall that a sheaf on  $\mathcal{X}$  is a sheaf for the fppf topology. Note that if  $\mathcal{U}$  is an algebraic space, then  $f : \mathcal{U} \rightarrow \mathcal{X}$  is representable by algebraic spaces, see Algebraic Stacks, Lemma 10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack  $\mathcal{X}$  by a scheme.

06XJ **Proposition 19.1.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  be a 1-morphism of algebraic stacks.*

- (1) *Let  $\mathcal{F}$  be an abelian étale sheaf on  $\mathcal{X}$ . Assume that  $f$  is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence*

$$E_1^{p,q} = H_{\acute{e}tale}^q(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H_{\acute{e}tale}^{p+q}(\mathcal{X}, \mathcal{F})$$

- (2) *Let  $\mathcal{F}$  be an abelian sheaf on  $\mathcal{X}$ . Assume that  $f$  is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence*

$$E_1^{p,q} = H_{fppf}^q(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H_{fppf}^{p+q}(\mathcal{X}, \mathcal{F})$$

**Proof.** To see this we will check the hypotheses (1) – (4) of Lemma 18.8. The 1-morphism  $f$  is faithful by Algebraic Stacks, Lemma 15.2. This proves (4). Hypothesis (3) follows from the fact that  $\mathcal{U}$  is an algebraic stack, see Lemma 16.2. To see (2) apply Lemma 18.10. Condition (1) is satisfied by fiat.  $\square$

### 20. Higher direct images and algebraic stacks

072F Let  $g : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of algebraic stacks over  $S$ . In the sections above we have constructed a morphism of ringed topoi  $g : Sh(\mathcal{X}_\tau) \rightarrow Sh(\mathcal{Y}_\tau)$  for each  $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$ . In the chapter on cohomology of sites

we have explained how to define higher direct images. Hence the *derived direct image*  $Rg_*\mathcal{F}$  is defined as  $g_*\mathcal{I}^\bullet$  where  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  is an injective resolution in  $Ab(\mathcal{X}_\tau)$ . The  $i$ th higher direct image  $R^i g_*\mathcal{F}$  is the  $i$ th cohomology of the derived direct image. Important: it matters which topology  $\tau$  is used here!

If  $\mathcal{F}$  is a presheaf of  $\mathcal{O}_\mathcal{X}$ -modules which is a sheaf in the  $\tau$ -topology, then we use injective resolutions in  $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$  to compute derived direct image and higher direct images.

So far our only tool to compute the higher direct images of  $g_*$  is the result on Čech complexes proved above. This requires the choice of a “covering”  $f : \mathcal{U} \rightarrow \mathcal{X}$ . If  $\mathcal{U}$  is an algebraic space, then  $f : \mathcal{U} \rightarrow \mathcal{X}$  is representable by algebraic spaces, see Algebraic Stacks, Lemma 10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack  $\mathcal{X}$  by a scheme.

072G **Proposition 20.1.** *Let  $f : \mathcal{U} \rightarrow \mathcal{X}$  and  $g : \mathcal{X} \rightarrow \mathcal{Y}$  be composable 1-morphisms of algebraic stacks.*

- (1) *Assume that  $f$  is representable by algebraic spaces, surjective and smooth.*  
 (a) *If  $\mathcal{F}$  is in  $Ab(\mathcal{X}_{\acute{e}tale})$  then there is a spectral sequence*

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

*in  $Ab(\mathcal{Y}_{\acute{e}tale})$  with higher direct images computed in the étale topology.*

- (b) *If  $\mathcal{F}$  is in  $Mod(\mathcal{X}_{\acute{e}tale}, \mathcal{O}_\mathcal{X})$  then there is a spectral sequence*

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

*in  $Mod(\mathcal{Y}_{\acute{e}tale}, \mathcal{O}_\mathcal{Y})$ .*

- (2) *Assume that  $f$  is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*

- (a) *If  $\mathcal{F}$  is in  $Ab(\mathcal{X})$  then there is a spectral sequence*

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

*in  $Ab(\mathcal{Y})$  with higher direct images computed in the fppf topology.*

- (b) *If  $\mathcal{F}$  is in  $Mod(\mathcal{O}_\mathcal{X})$  then there is a spectral sequence*

$$E_1^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

*in  $Mod(\mathcal{O}_\mathcal{Y})$ .*

**Proof.** To see this we will check the hypotheses (1) – (4) of Lemma 18.11 and Lemma 18.12. The 1-morphism  $f$  is faithful by Algebraic Stacks, Lemma 15.2. This proves (4). Hypothesis (3) follows from the fact that  $\mathcal{U}$  is an algebraic stack, see Lemma 16.2. To see (2) apply Lemma 18.10. Condition (1) is satisfied by fiat in all four cases.  $\square$

Here is a description of higher direct images for a morphism of algebraic stacks.

075G **Lemma 20.2.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism of algebraic stacks<sup>4</sup> over  $S$ . Let  $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$ . Let  $\mathcal{F}$  be an object of  $Ab(\mathcal{X}_\tau)$  or  $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ . Then the sheaf  $R^i f_* \mathcal{F}$  is the sheaf associated to the presheaf*

$$y \longmapsto H_\tau^i \left( (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}, pr^{-1} \mathcal{F} \right)$$

*Here  $y$  is a typical object of  $\mathcal{Y}$  lying over the scheme  $V$ .*

<sup>4</sup>This result should hold for any 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$ .

**Proof.** Choose an injective resolution  $\mathcal{F}[0] \rightarrow \mathcal{I}^\bullet$ . By the formula for pushforward (5.0.1) we see that  $R^i f_* \mathcal{F}$  is the sheaf associated to the presheaf which associates to  $y$  the cohomology of the complex

$$\begin{array}{c} \Gamma\left((Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1} \mathcal{I}^{i-1}\right) \\ \downarrow \\ \Gamma\left((Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1} \mathcal{I}^i\right) \\ \downarrow \\ \Gamma\left((Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}, \text{pr}^{-1} \mathcal{I}^{i+1}\right) \end{array}$$

Since  $\text{pr}^{-1}$  is exact, it suffices to show that  $\text{pr}^{-1}$  preserves injectives. This follows from Lemmas 16.5 and 16.6 as well as the fact that  $\text{pr}$  is a representable morphism of algebraic stacks (so that  $\text{pr}$  is faithful by Algebraic Stacks, Lemma 15.2 and that  $(Sch/V)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$  has equalizers by Lemma 16.2).  $\square$

Here is a trivial base change result.

075H **Lemma 20.3.** *Let  $S$  be a scheme. Let  $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$ . Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

*be a 2-cartesian diagram of algebraic stacks over  $S$ . Then the base change map is an isomorphism*

$$g^{-1} Rf_* \mathcal{F} \longrightarrow Rf'_*(g')^{-1} \mathcal{F}$$

*functorial for  $\mathcal{F}$  in  $\text{Ab}(\mathcal{X}_\tau)$  or  $\mathcal{F}$  in  $\text{Mod}(\mathcal{X}_\tau, \mathcal{O}_{\mathcal{X}})$ .*

**Proof.** The isomorphism  $g^{-1} f_* \mathcal{F} = f'_*(g')^{-1} \mathcal{F}$  is Lemma 5.1 (and it holds for arbitrary presheaves). For the derived direct images, there is a base change map because the morphisms  $g$  and  $g'$  are flat, see Cohomology on Sites, Section 16. To see that this map is a quasi-isomorphism we can use that for an object  $y'$  of  $\mathcal{Y}'$  over a scheme  $V$  there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'), \mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y', \mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

We conclude that the induced map  $g^{-1} R^i f_* \mathcal{F} \rightarrow R^i f'_*(g')^{-1} \mathcal{F}$  is an isomorphism by Lemma 20.2.  $\square$

## 21. Comparison

073L In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

075L **Lemma 21.1.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$  representable by the algebraic space  $F$ .*

- (1) *If  $\mathcal{I}$  injective in  $\text{Ab}(\mathcal{X}_{\text{étale}})$ , then  $\mathcal{I}|_{F_{\text{étale}}}$  is injective in  $\text{Ab}(F_{\text{étale}})$ ,*
- (2) *If  $\mathcal{I}^\bullet$  is a  $K$ -injective complex in  $\text{Ab}(\mathcal{X}_{\text{étale}})$ , then  $\mathcal{I}^\bullet|_{F_{\text{étale}}}$  is a  $K$ -injective complex in  $\text{Ab}(F_{\text{étale}})$ .*

*The same does not hold for modules.*

**Proof.** This follows formally from the fact that the restriction functor  $\pi_{F,*} = i_F^{-1}$  (see Lemma 10.1) is right adjoint to the exact functor  $\pi_F^{-1}$ , see Homology, Lemma 26.1 and Derived Categories, Lemma 29.9. To see that the lemma does not hold for modules, we refer the reader to Étale Cohomology, Lemma 81.1.  $\square$

075N **Lemma 21.2.** *Let  $S$  be a scheme. Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of algebraic stacks over  $S$ . Assume  $\mathcal{X}, \mathcal{Y}$  are representable by algebraic spaces  $F, G$ . Denote  $f : F \rightarrow G$  the induced morphism of algebraic spaces.*

(1) *For any  $\mathcal{F} \in \text{Ab}(\mathcal{X}_{\text{étale}})$  we have*

$$(Rf_*\mathcal{F})|_{G_{\text{étale}}} = Rf_{small,*}(\mathcal{F}|_{F_{\text{étale}}})$$

*in  $D(G_{\text{étale}})$ .*

(2) *For any object  $\mathcal{F}$  of  $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$  we have*

$$(Rf_*\mathcal{F})|_{G_{\text{étale}}} = Rf_{small,*}(\mathcal{F}|_{F_{\text{étale}}})$$

*in  $D(\mathcal{O}_G)$ .*

**Proof.** Part (1) follows immediately from Lemma 21.1 and (10.2.1) on choosing an injective resolution of  $\mathcal{F}$ .

Part (2) can be proved as follows. In Lemma 10.2 we have seen that  $\pi_G \circ f = f_{small} \circ \pi_F$  as morphisms of ringed sites. Hence we obtain  $R\pi_{G,*} \circ Rf_* = Rf_{small,*} \circ R\pi_{F,*}$  by Cohomology on Sites, Lemma 20.2. Since the restriction functors  $\pi_{F,*}$  and  $\pi_{G,*}$  are exact, we conclude.  $\square$

075P **Lemma 21.3.** *Let  $S$  be a scheme. Consider a 2-fibre product square*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

*of algebraic stacks over  $S$ . Assume that  $f$  is representable by algebraic spaces and that  $\mathcal{Y}'$  is representable by an algebraic space  $G'$ . Then  $\mathcal{X}'$  is representable by an algebraic space  $F'$  and denoting  $f' : F' \rightarrow G'$  the induced morphism of algebraic spaces we have*

$$g^{-1}(Rf_*\mathcal{F})|_{G'_{\text{étale}}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{\text{étale}}})$$

*for any  $\mathcal{F}$  in  $\text{Ab}(\mathcal{X}_{\text{étale}})$  or in  $\text{Mod}(\mathcal{X}_{\text{étale}}, \mathcal{O}_{\mathcal{X}})$*

**Proof.** Follows formally on combining Lemmas 20.3 and 21.2.  $\square$

## 22. Change of topology

075Q Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of  $\mathcal{X}$  lying over flat morphisms.

076T **Lemma 22.1.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . Let  $\mathcal{F}$  be a presheaf of  $\mathcal{O}_{\mathcal{X}}$ -modules. Assume*

(a)  *$\mathcal{F}$  is locally quasi-coherent, and*

(b) *for any morphism  $\varphi : x \rightarrow y$  of  $\mathcal{X}$  which lies over a morphism of schemes  $f : U \rightarrow V$  which is flat and locally of finite presentation the comparison map  $c_\varphi : f_{small}^*\mathcal{F}|_{V_{\text{étale}}} \rightarrow \mathcal{F}|_{U_{\text{étale}}}$  of (9.4.1) is an isomorphism.*

Then  $\mathcal{F}$  is a sheaf for the fppf topology.

**Proof.** Let  $\{x_i \rightarrow x\}$  be an fppf covering of  $\mathcal{X}$  lying over the fppf covering  $\{f_i : U_i \rightarrow U\}$  of schemes over  $S$ . By assumption the restriction  $\mathcal{G} = \mathcal{F}|_{U_{\acute{e}tale}}$  is quasi-coherent and the comparison maps  $f_{i,small}^* \mathcal{G} \rightarrow \mathcal{F}|_{U_i, \acute{e}tale}$  are isomorphisms. Hence the sheaf condition for  $\mathcal{F}$  and the covering  $\{x_i \rightarrow x\}$  is equivalent to the sheaf condition for  $\mathcal{G}^a$  on  $(Sch/U)_{fppf}$  and the covering  $\{U_i \rightarrow U\}$  which holds by Descent, Lemma 8.1.  $\square$

075R **Lemma 22.2.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . Let  $\mathcal{F}$  be a presheaf  $\mathcal{O}_{\mathcal{X}}$ -module such that*

- (a)  $\mathcal{F}$  is locally quasi-coherent, and
- (b) for any morphism  $\varphi : x \rightarrow y$  of  $\mathcal{X}$  which lies over a morphism of schemes  $f : U \rightarrow V$  which is flat and locally of finite presentation, the comparison map  $c_\varphi : f_{small}^* \mathcal{F}|_{V_{\acute{e}tale}} \rightarrow \mathcal{F}|_{U_{\acute{e}tale}}$  of (9.4.1) is an isomorphism.

Then  $\mathcal{F}$  is an  $\mathcal{O}_{\mathcal{X}}$ -module and we have the following

- (1) If  $\epsilon : \mathcal{X}_{fppf} \rightarrow \mathcal{X}_{\acute{e}tale}$  is the comparison morphism, then  $R\epsilon_* \mathcal{F} = \epsilon_* \mathcal{F}$ .
- (2) The cohomology groups  $H_{fppf}^p(\mathcal{X}, \mathcal{F})$  are equal to the cohomology groups computed in the étale topology on  $\mathcal{X}$ . Similarly for the cohomology groups  $H_{fppf}^p(x, \mathcal{F})$  and the derived versions  $R\Gamma(\mathcal{X}, \mathcal{F})$  and  $R\Gamma(x, \mathcal{F})$ .
- (3) If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a 1-morphism of categories fibred in groupoids over  $(Sch/S)_{fppf}$  then  $R^i f_* \mathcal{F}$  is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived pull-back.

**Proof.** The assertion that  $\mathcal{F}$  is an  $\mathcal{O}_{\mathcal{X}}$ -module follows from Lemma 22.1. Note that  $\epsilon$  is a morphism of sites given by the identity functor on  $\mathcal{X}$ . The sheaf  $R^p \epsilon_* \mathcal{F}$  is therefore the sheaf associated to the presheaf  $x \mapsto H_{fppf}^p(x, \mathcal{F})$ , see Cohomology on Sites, Lemma 8.4. To prove (1) it suffices to show that  $H_{fppf}^p(x, \mathcal{F}) = 0$  for  $p > 0$  whenever  $x$  lies over an affine scheme  $U$ . By Lemma 15.1 we have  $H_{fppf}^p(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1} \mathcal{F})$ . Combining Descent, Lemma 9.4 with Cohomology of Schemes, Lemma 2.2 we see that these cohomology groups are zero.

We have seen above that  $\epsilon_* \mathcal{F}$  and  $\mathcal{F}$  are the sheaves on  $\mathcal{X}_{\acute{e}tale}$  and  $\mathcal{X}_{fppf}$  corresponding to the same presheaf on  $\mathcal{X}$  (and this is true more generally for any sheaf in the fppf topology on  $\mathcal{X}$ ). We often abusively identify  $\mathcal{F}$  and  $\epsilon_* \mathcal{F}$  and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 15.6.

Finally we prove (3). The sheaf  $R^i f_* \mathcal{F}$  (resp.  $Rf_{\acute{e}tale,*} \mathcal{F}$ ) is the sheaf associated to the presheaf

$$y \longmapsto H_\tau^i \left( (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}, \text{pr}^{-1} \mathcal{F} \right)$$

where  $\tau$  is fppf (resp. étale), see Lemma 20.2. Note that  $\text{pr}^{-1} \mathcal{F}$  satisfies properties (a) and (b) also (by Lemmas 11.7 and 9.3), hence these two presheaves are equal by (2). This immediately implies (3).  $\square$

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

07AK **Lemma 22.3.** *Let  $S$  be a scheme. Let  $\mathcal{X}$  be an algebraic stack over  $S$ . Let  $\tau = \text{étale}$  (resp.  $\tau = \text{fppf}$ ). Let  $\mathcal{X}' \subset \mathcal{X}$  be a full subcategory with the following properties*

- (1) *if  $x \rightarrow x'$  is a morphism of  $\mathcal{X}$  which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and  $x' \in \text{Ob}(\mathcal{X}')$ , then  $x \in \text{Ob}(\mathcal{X}')$ , and*
- (2) *there exists an object  $x \in \text{Ob}(\mathcal{X}')$  lying over a scheme  $U$  such that the associated 1-morphism  $x : (\text{Sch}/U)_{\text{fppf}} \rightarrow \mathcal{X}$  is smooth and surjective.*

*We get a site  $\mathcal{X}'_\tau$  by declaring a covering of  $\mathcal{X}'$  to be any family of morphisms  $\{x_i \rightarrow x\}$  in  $\mathcal{X}'$  which is a covering in  $\mathcal{X}_\tau$ . Then the inclusion functor  $\mathcal{X}' \rightarrow \mathcal{X}_\tau$  is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi*

$$g : \text{Sh}(\mathcal{X}'_\tau) \longrightarrow \text{Sh}(\mathcal{X}_\tau)$$

*and  $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$  for all  $p \geq 0$  and all  $\mathcal{F} \in \text{Ab}(\mathcal{X}_\tau)$ .*

**Proof.** Note that assumption (1) implies that if  $\{x_i \rightarrow x\}$  is a covering of  $\mathcal{X}_\tau$  and  $x \in \text{Ob}(\mathcal{X}')$ , then we have  $x_i \in \text{Ob}(\mathcal{X}')$ . Hence we see that  $\mathcal{X}' \rightarrow \mathcal{X}$  is continuous and cocontinuous as the coverings of objects of  $\mathcal{X}'_\tau$  agree with their coverings seen as objects of  $\mathcal{X}_\tau$ . We obtain the morphism  $g$  and the functor  $g^{-1}$  is identified with the restriction functor, see Sites, Lemma 20.5.

In particular, if  $\{x_i \rightarrow x\}$  is a covering in  $\mathcal{X}'_\tau$ , then for any abelian sheaf  $\mathcal{F}$  on  $\mathcal{X}$  then

$$\check{H}^p(\{x_i \rightarrow x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \rightarrow x\}, \mathcal{F})$$

Thus if  $\mathcal{I}$  is an injective abelian sheaf on  $\mathcal{X}_\tau$  then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 11.2). Hence  $H^p(x, g^{-1}\mathcal{I}) = 0$  for all objects  $x$  of  $\mathcal{X}'$  (Cohomology on Sites, Lemma 11.9). In other words injective abelian sheaves on  $\mathcal{X}_\tau$  are right acyclic for the functor  $H^0(x, g^{-1}-)$ . It follows that  $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$  for all  $\mathcal{F} \in \text{Ab}(\mathcal{X})$  and all  $x \in \text{Ob}(\mathcal{X}')$ .

Choose an object  $x \in \mathcal{X}'$  lying over a scheme  $U$  as in assumption (2). In particular  $\mathcal{X}/x \rightarrow \mathcal{X}$  is a morphism of algebraic stacks which representable by algebraic spaces, surjective, and smooth. (Note that  $\mathcal{X}/x$  is equivalent to  $(\text{Sch}/U)_{\text{fppf}}$ , see Lemma 9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in  $\text{Sh}(\mathcal{X}_\tau)$  is surjective. Namely, for any object  $x'$  of  $\mathcal{X}$  there exists a  $\tau$ -covering  $\{x'_i \rightarrow x'\}$  such that there exist morphisms  $x'_i \rightarrow x$ , see Lemma 18.10. Since  $g$  is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in  $\text{Sh}(\mathcal{X}'_\tau)$  is surjective also. Let  $h_{x,n}$  be the  $(n+1)$ -fold product  $h_x \times \dots \times h_x$ . Then we have spectral sequences

$$07AL \quad (22.3.1) \quad E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

and

$$07AM \quad (22.3.2) \quad E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F})$$

see Cohomology on Sites, Lemma 14.2.

Case I:  $\mathcal{X}$  has a final object  $x$  which is also an object of  $\mathcal{X}'$ . This case follows immediately from the discussion in the second paragraph above.

Case II:  $\mathcal{X}$  is representable by an algebraic space  $F$ . In this case the sheaves  $h_{x,n}$  are representable by an object  $x_n$  in  $\mathcal{X}$ . (Namely, if  $\mathcal{S}_F = \mathcal{X}$  and  $x : U \rightarrow F$  is the given object, then  $h_{x,n}$  is representable by the object  $U \times_F \dots \times_F U \rightarrow F$  of  $\mathcal{S}_F$ .) It follows that  $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$ . The morphisms  $x_n \rightarrow x$  lie over smooth morphisms of schemes, hence  $x_n \in \mathcal{X}'$  for all  $n$ . Hence  $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$ . Thus in the two spectral sequences (22.3.1) and (22.3.2) above the  $E_1^{p,q}$  terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III:  $\mathcal{X}$  is an algebraic stack. We claim that in this case the cohomology groups  $H^q(h_{x,p}, \mathcal{F})$  and  $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$  agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category  $\mathcal{X}/h_{x,n}$ , see Sites, Lemma 29.3. Since  $h_{x,n}$  is the  $(n+1)$ -fold product of  $h_x$  an object of this category is an  $(n+2)$ -tuple  $(y, s_0, \dots, s_n)$  where  $y$  is an object of  $\mathcal{X}$  and each  $s_i : y \rightarrow x$  is a morphism of  $\mathcal{X}$ . This is a category over  $(Sch/S)_{fppf}$ . There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_{\mathcal{X}} \dots \times_{\mathcal{X}} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over  $(Sch/S)_{fppf}$ . Namely, if  $x : (Sch/U)_{fppf} \rightarrow \mathcal{X}$  also denotes the 1-morphism associated with  $x$  and  $p : \mathcal{X} \rightarrow (Sch/S)_{fppf}$  the structure functor, then we can think of  $(y, s_0, \dots, s_n)$  as  $(y, f_0, \dots, f_n, \alpha_0, \dots, \alpha_n)$  where  $y$  is an object of  $\mathcal{X}$ ,  $f_i : p(y) \rightarrow p(x)$  is a morphism of schemes, and  $\alpha_i : y \rightarrow x(f_i)$  an isomorphism. The category of  $2n+3$ -tuples  $(y, f_0, \dots, f_n, \alpha_0, \dots, \alpha_n)$  is an incarnation of the  $(n+1)$ -fold fibred product  $\mathcal{U}_n$  of algebraic stacks displayed above, as we discussed in Section 17. By Cohomology on Sites, Lemma 14.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the ‘‘primed’’ analogue of this. Namely,  $\mathcal{X}'/h_{x,n}$  corresponds, via the equivalence above to the full subcategory  $\mathcal{U}'_n \subset \mathcal{U}_n$  consisting of those tuples  $(y, f_0, \dots, f_n, \alpha_0, \dots, \alpha_n)$  with  $y \in \mathcal{X}'$ . Hence certainly property (1) of the statement of the lemma holds for the inclusion  $\mathcal{U}'_n \subset \mathcal{U}_n$ . To see property (2) choose an object  $\xi = (y, s_0, \dots, s_n)$  which lies over a scheme  $W$  such that  $(Sch/W)_{fppf} \rightarrow \mathcal{U}_n$  is smooth and surjective (this is possible as  $\mathcal{U}_n$  is an algebraic stack). Then  $(Sch/W)_{fppf} \rightarrow \mathcal{U}_n \rightarrow (Sch/U)_{fppf}$  is smooth as a composition of base changes of the morphism  $x : (Sch/U)_{fppf} \rightarrow \mathcal{X}$ , see Algebraic Stacks, Lemmas 10.6 and 10.5. Thus axiom (1) for  $\mathcal{X}$  implies that  $y$  is an object of  $\mathcal{X}'$  whence  $\xi$  is an object of  $\mathcal{U}'_n$ . Using again

$$H^p(\mathcal{U}'_n, \mathcal{F}|_{\mathcal{U}'_n}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for  $\mathcal{U}'_n \subset \mathcal{U}_n$  to conclude.  $\square$

### 23. Other chapters

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- |                       |                          |
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| (6) Sheaves on Spaces | (13) Derived Categories  |
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- (16) Smoothing Ring Maps
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### References

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