1. Introduction

In this document we explain what the different topologies on the category of schemes are. Some references are [Gro71] and [BLR90]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology $\tau$. In order to get a site, as in Sites, Definition 6.2 of schemes with that topology we have to do some work. Namely, we cannot simply say “consider all schemes with the Zariski topology” since that would give a “big” category. Instead, in each section of this chapter we will proceed as follows:

1. We define a class Cov$_{\tau}$ of coverings of schemes satisfying the axioms of Sites, Definition 6.2. It will always be the case that a Zariski open covering of a scheme is a covering for $\tau$.

2. We single out a notion of standard $\tau$-covering within the category of affine schemes.
We define what is an “absolute” big $\tau$-site $\text{Sch}_\tau$. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.

For any object $S$ of $\text{Sch}_\tau$ we define the big $\tau$-site $(\text{Sch}/S)_\tau$ and for suitable $\tau$ the small $\tau$-site $S_\tau$.

In addition there is a site $(\text{Aff}/S)_\tau$ using the notion of standard $\tau$-covering of affines whose category of sheaves is equivalent to the category of sheaves on $(\text{Sch}/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big $\tau$-site of a scheme, or even the small $\tau$-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

### 3. The Zariski topology

**Definition 3.1.** Let $T$ be a scheme. A Zariski covering of $T$ is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each $f_i$ is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

**Lemma 3.2.** Let $T$ be a scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and for each $i$ we have a Zariski covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a Zariski covering.
3. If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T\}_{i \in I}$ is a Zariski covering.

**Proof.** Omitted. \hfill \qed

**Lemma 3.3.** Let $T$ be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a Zariski covering of $T$. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each $U_j$ is a standard open of $T$, see Schemes, Definition 5.2. Moreover, we may choose each $U_j$ to be an open of one of the $T_i$.

**Proof.** Follows as $T$ is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 5.1. \hfill \qed

Thus we define the corresponding standard coverings of affines as follows.

**Definition 3.4.** Compare Schemes, Definition 5.2. Let $T$ be an affine scheme. A standard Zariski covering of $T$ is a a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of $T$.

**Definition 3.5.** A big Zariski site is any site $\text{Sch}_{Zar}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes $S_0$, and any set of Zariski coverings $\text{Cov}_0$ among these schemes.

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1The words big and small here do not relate to bigness/smallness of the corresponding categories.
As underlying category of $\mathbf{Sch}$ take any category $\mathbf{Sch}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set $S_0$.

As coverings of $\mathbf{Sch}$ choose any set of coverings as in Sets, Lemma 11.1 starting with the category $\mathbf{Sch}_\alpha$ and the class of Zariski coverings, and the set Cov$_0$ chosen above.

It is shown in Sites, Lemma 8.6 that, after having chosen the category $\mathbf{Sch}_\alpha$, the category of sheaves on $\mathbf{Sch}_\alpha$ does not depend on the choice of coverings chosen in (3) above. In other words, the topos $\mathcal{S}h(\mathbf{Sch}_{\text{Zar}})$ only depends on the choice of the category $\mathbf{Sch}_\alpha$. It is shown in Sets, Lemma 9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of $\mathbf{Sch}_\alpha$ does not matter too much, see Section 11.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define $\mathbf{Sch}_{\text{Zar}}$ to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme $S$, let us point out that the topology on a big Zariski site $\mathbf{Sch}_{\text{Zar}}$ is in some sense induced from the Zariski topology on the category of all schemes.

**Lemma 3.6.** Let $\mathbf{Sch}_{\text{Zar}}$ be a big Zariski site as in Definition 3.5. Let $T \in \text{Ob}(\mathbf{Sch}_{\text{Zar}})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary Zariski covering of $T$. There exists a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\mathbf{Sch}_{\text{Zar}}$ which is tautologically equivalent (see Sites, Definition 8.2) to $\{T_i \to T\}_{i \in I}$.

**Proof.** Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 9.9 that each $T_i$ is isomorphic to an object $V_i$ of $\mathbf{Sch}_{\text{Zar}}$. The covering $\{V_i \to T\}_{i \in I}$ is tautologically equivalent to $\{T_i \to T\}_{i \in I}$ (using the identity map on $I$ both ways). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\mathbf{Sch}_{\text{Zar}}$ by Sets, Lemma 11.1.

**Definition 3.7.** Let $S$ be a scheme. Let $\mathbf{Sch}_{\text{Zar}}$ be a big Zariski site containing $S$.

1. The **big Zariski site of $S$**, denoted $(\mathbf{Sch}/S)_{\text{Zar}}$, is the site $\mathbf{Sch}_{\text{Zar}}/S$ introduced in Sites, Section 24.
2. The **small Zariski site of $S$**, which we denote $S_{\text{Zar}}$, is the full subcategory of $(\mathbf{Sch}/S)_{\text{Zar}}$ whose objects are those $U/S$ such that $U \to S$ is an open immersion. A covering of $S_{\text{Zar}}$ is any covering $\{U_i \to U\}$ of $(\mathbf{Sch}/S)_{\text{Zar}}$ with $U \in \text{Ob}(S_{\text{Zar}})$.
3. The **big affine Zariski site of $S$**, denoted $(\mathbf{Aff}/S)_{\text{Zar}}$, is the full subcategory of $(\mathbf{Sch}/S)_{\text{Zar}}$ whose objects are affine $U/S$. A covering of $(\mathbf{Aff}/S)_{\text{Zar}}$ is any covering $\{U_i \to U\}$ of $(\mathbf{Sch}/S)_{\text{Zar}}$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site and the big affine Zariski site are sites. We check this now.

**Lemma 3.8.** Let $S$ be a scheme. Let $\mathbf{Sch}_{\text{Zar}}$ be a big Zariski site containing $S$. Both $S_{\text{Zar}}$ and $(\mathbf{Aff}/S)_{\text{Zar}}$ are sites.
Let us show that \( S_{\text{Zar}} \) is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since \((\text{Sch}/S)_{\text{Zar}}\) is a site, it suffices to prove that given any covering \( \{ U_i \to U \} \) of \((\text{Sch}/S)_{\text{Zar}}\) with \( U \in \text{Ob}(S_{\text{Zar}}) \) we also have \( U_i \in \text{Ob}(S_{\text{Zar}}) \). This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that \((\text{Aff}/S)_{\text{Zar}}\) is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. Let \( R \) be a ring. Let \( f_1, \ldots, f_n \in R \) generate the unit ideal. For each \( i \in \{1, \ldots, n\} \) let \( g_{i1}, \ldots, g_{in_i} \in R_{f_i} \) be elements generating the unit ideal of \( R_{f_i} \). Write \( g_{ij} = f_{ij}/f_{ij}^{*i} \) which is possible. After replacing \( f_{ij} \) by \( f_if_{ij} \) if necessary, we have that \( D(f_{ij}) \subset D(f_i) \cong \text{Spec}(R_{f_i}) \) is equal to \( D(g_{ij}) \subset \text{Spec}(R_{f_i}) \). Hence we see that the family of morphisms \( \{ D(g_{ij}) \to \text{Spec}(R) \} \) is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3).

\[ \square \]

\[ \textbf{Lemma 3.9.} \text{ Let } S \text{ be a scheme. Let } S_{\text{Zar}} \text{ be a big Zariski site containing } S. \text{ The underlying categories of the sites } S_{\text{Zar}}, (\text{Sch}/S)_{\text{Zar}}, S_{\text{Zar}}, \text{ and } (\text{Aff}/S)_{\text{Zar}} \text{ have fibre products. In each case the obvious functor into the category } S \text{ of all schemes commutes with taking fibre products. The categories } (\text{Sch}/S)_{\text{Zar}}, \text{ and } S_{\text{Zar}} \text{ both have a final object, namely } S/S. \]

\[ \text{Proof.} \text{ For } S_{\text{Zar}}, \text{ it is true by construction, see Sets, Lemma 9.9} \text{. Suppose we have } U \to S, V \to U, W \to U \text{ morphisms of schemes with } U, V, W \in \text{Ob}(S_{\text{Zar}}). \text{ The fibre product } V \times_U W \text{ in } S_{\text{Zar}} \text{ is a fibre product in } S \text{ and is the fibre product of } V/S \text{ with } W/S \text{ over } U/S \text{ in the category of all schemes over } S, \text{ and hence also a fibre product in } (\text{Sch}/S)_{\text{Zar}}. \text{ This proves the result for } (\text{Sch}/S)_{\text{Zar}}. \text{ If } U \to S, V \to U \text{ and } W \to U \text{ are open immersions then so is } V \times_U W \to S \text{ and hence we get the result for } S_{\text{Zar}}. \text{ If } U, V, W \text{ are affine, so is } V \times_U W \text{ and hence the result for } (\text{Aff}/S)_{\text{Zar}}. \text{ } \\]

Next, we check that the big affine site defines the same topos as the big site.

\[ \textbf{Lemma 3.10.} \text{ Let } S \text{ be a scheme. Let } S_{\text{Zar}} \text{ be a big Zariski site containing } S. \text{ The functor } (\text{Aff}/S)_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}} \text{ is a special cocontinuous functor. Hence it induces an equivalence of topoi from } Sh((\text{Aff}/S)_{\text{Zar}}) \text{ to } Sh((\text{Sch}/S)_{\text{Zar}}). \]

\[ \text{Proof.} \text{ The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2} \text{. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1} \text{. Denote the inclusion functor } u : (\text{Aff}/S)_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}}. \text{ Being cocontinuous just means that any Zariski covering of } T/S, T \text{ affine, can be refined by a standard Zariski covering of } T. \text{ This is the content of Lemma 3.3} \text{. Hence (1) holds. We see } u \text{ is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that } u \text{ is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. } \\]

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on \( S \).

\[ \textbf{Lemma 3.11.} \text{ The category of sheaves on } S_{\text{Zar}} \text{ is equivalent to the category of sheaves on the underlying topological space of } S. \]

\[ \square \]
Proof. We will use repeatedly that for any object $U/S$ of $S_{\text{Zar}}$ the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on $S$. Then we define a sheaf on $S_{\text{Zar}}$ by the rule $\mathcal{F}(U/S) = \mathcal{F}(\text{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \text{Ob}(S_{\text{Zar}})$ with $\text{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 0.9). Given a sheaf $\mathcal{G}$ on $S_{\text{Zar}}$ we define a sheaf on $S$ by setting $\mathcal{G}(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_i \to U'\}_{i \in I}$ in $S_{\text{Zar}}$ by Sets, Lemma 11.1 and we use Sites, Lemma 8.4. Details omitted. □

From now on we will not make any distinction between a sheaf on $S_{\text{Zar}}$ or a sheaf on $S$. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

020Y Lemma 3.12. Let $\text{Sch}_{\text{Zar}}$ be a big Zariski site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{Zar}}$. The functor $T_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}}$ is cocontinuous and induces a morphism of topoi

$$i_f : \text{Sh}(T_{\text{Zar}}) \to \text{Sh}((\text{Sch}/S)_{\text{Zar}})$$

For a sheaf $\mathcal{G}$ on $(\text{Sch}/S)_{\text{Zar}}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor $i_f^{-1}$ also has a left adjoint $i_{f!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of $T_{\text{Zar}}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 3.9. Moreover, $T_{\text{Zar}}$ has equalizers (as any two morphisms with the same source and target are the same) and $u$ commutes with them. It is clearly cocontinuous. It is also continuous as $u$ transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 20.5 and 20.6. □

020Z Lemma 3.13. Let $S$ be a scheme. Let $\text{Sch}_{\text{Zar}}$ be a big Zariski site containing $S$. The inclusion functor $S_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites

$$\pi_S : (\text{Sch}/S)_{\text{Zar}} \to S_{\text{Zar}}$$

and a morphism of topoi

$$i_S : \text{Sh}(S_{\text{Zar}}) \to \text{Sh}((\text{Sch}/S)_{\text{Zar}})$$

such that $\pi_S \circ i_S = \text{id}$. Moreover, $i_S = i_{\text{ids}}$ with $i_{\text{ids}}$ as in Lemma 3.12. In particular the functor $i_S^{-1} = \pi_{S*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{\text{Zar}} \to (\text{Sch}/S)_{\text{Zar}}$, in addition to the properties seen in the proof of Lemma 3.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows. □

04BS Definition 3.14. In the situation of Lemma 3.13 the functor $i_S^{-1} = \pi_{S*}$ is often called the restriction to the small Zariski site, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{\text{Zar}}}$ this restriction.
With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

\[
\text{Mor}_{\text{Sh}(\mathcal{S}_{\text{zar}})}(\mathcal{F}|_{\mathcal{S}_{\text{zar}}}, \mathcal{G}) = \text{Mor}_{\text{Sh}(\text{Sch}(S)_{\text{zar}})}(\mathcal{F}, i_{S,\ast} \mathcal{G})
\]

\[
\text{Mor}_{\text{Sh}(\mathcal{S}_{\text{zar}})}(\mathcal{G}, \mathcal{F}|_{\mathcal{S}_{\text{zar}}}) = \text{Mor}_{\text{Sh}(\text{Sch}(S)_{\text{zar}})}(\pi_{S,1}^{-1} \mathcal{G}, \mathcal{F})
\]

Moreover, we have $(i_{S,\ast} \mathcal{G})|_{\mathcal{S}_{\text{zar}}} = \mathcal{G}$ and we have $(\pi_{S,1}^{-1} \mathcal{G})|_{\mathcal{S}_{\text{zar}}} = \mathcal{G}$.

**Lemma 3.15.** Let $\text{Sch}_{\text{Zar}}$ be a big Zariski site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{Zar}}$. The functor

\[ u : (\text{Sch}/T)_{\text{Zar}} \longrightarrow (\text{Sch}/S)_{\text{Zar}}, \quad V/T \mapsto V/S \]

is cocontinuous, and has a continuous right adjoint

\[ v : (\text{Sch}/S)_{\text{Zar}} \longrightarrow (\text{Sch}/T)_{\text{Zar}}, \quad (U \to S) \mapsto (U \times_S T \to T). \]

They induce the same morphism of topoi

\[ f_{\text{big}} : \text{Sh}(\text{Sch}/T)_T \longrightarrow \text{Sh}(\text{Sch}/S)_T \]

We have $f_{\text{big}}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{\text{big}}_* (\mathcal{F})(U/T) = \mathcal{F}(U \times_S T/T)$. Also, $f_{\text{big}}^{-1}$ has a left adjoint $f_{\text{big}}^!$ which commutes with fibre products and equalizers.

**Proof.** The functor $u$ is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 3.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for $f_{\text{big}}$ and the existence of $f_{\text{big}}^!$. Moreover, the functor $v$ is a right adjoint because given $U/T$ and $V/S$ we have $\text{Mor}_{\text{Sh}}(u(U), V) = \text{Mor}_{T}(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{\text{big}}_*$. □

**Lemma 3.16.** Let $\text{Sch}_{\text{Zar}}$ be a big Zariski site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{Zar}}$.

\begin{enumerate}
  \item We have $i_{f} = f_{\text{big}} \circ i_T$ with $i_f$ as in Lemma 3.12 and $i_T$ as in Lemma 3.13.
  \item The functor $S_{\text{Zar}} \to T_{\text{Zar}}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi

  \[ f_{\text{small}} : \text{Sh}(T_{\text{Zar}}) \longrightarrow \text{Sh}(S_{\text{Zar}}). \]

  The functors $f_{\text{small}}^{-1}$ and $f_{\text{small}}_*$ agree with the usual notions $f^{-1}$ and $f_*$ is we identify sheaves on $T_{\text{Zar}}$, resp. $S_{\text{Zar}}$ with sheaves on $T$, resp. $S$ via Lemma 3.11.
  \item We have a commutative diagram of morphisms of sites

  \[
  \begin{array}{ccc}
  T_{\text{Zar}} & \xrightarrow{\pi_T} & (\text{Sch}/T)_{\text{Zar}} \\
  f_{\text{small}} \downarrow & & \downarrow f_{\text{big}} \\
  S_{\text{Zar}} & \xleftarrow{\pi_S} & (\text{Sch}/S)_{\text{Zar}}
  \end{array}
  \]

  so that $f_{\text{small}} \circ \pi_T = \pi_S \circ f_{\text{big}}$ as morphisms of topoi.
  \item We have $f_{\text{small}} = \pi_S \circ f_{\text{big}} \circ i_T = \pi_S \circ i_f$.
\end{enumerate}

**Proof.** The equality $i_f = f_{\text{big}} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{\text{big}}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Part (3) follows because $\pi_S$ and $\pi_T$ are given by the inclusion functors and $f_{\text{small}}$ and $f_{\text{big}}$ by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with $i_T$.  

In the situation of the lemma, using the terminology of Definition \ref{dfn:small-big-sites} we have: for $F$ a sheaf on the big Zariski site of $T$

$$\left(f_{\text{big},*}F\right)|_{S_{\text{zar}}} = f_{\text{small},*}(F|_{T_{\text{zar}}}).$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of $T$, resp. $S$ is given by $\pi_{T,*}$, resp. $\pi_{S,*}$.

A similar formula involving pullbacks and restrictions is false.

\begin{lemma}
Given schemes $X$, $Y$, in $(\text{Sch}/S)_{\text{Zar}}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{\text{big}} \circ f_{\text{big}} = (g \circ f)_{\text{big}}$ and $g_{\text{small}} \circ f_{\text{small}} = (g \circ f)_{\text{small}}$.

\begin{proof}
This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma \ref{lem:pullback-big-site}. For the functors on the small sites this is Sheaves, Lemma \ref{lem:pushforward-small-site} via the identification of Lemma \ref{lem:identification-sites}.
\end{proof}
\end{lemma}

\begin{lemma}
Let $S_{\text{Zar}}$ be a big Zariski site. Consider a cartesian diagram

\begin{equation}
\begin{array}{ccc}
T' & \longrightarrow & T \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}
\end{equation}

in $S_{\text{Zar}}$. Then $i_{g^{-1}} \circ f_{\text{big},*} = f'_{\text{small},*} \circ (i_{g'})^{-1}$ and $g_{\text{big}}^{-1} \circ f_{\text{big},*} = f'_{\text{big},*} \circ (g_{\text{big}})^{-1}$.

\begin{proof}
Since the diagram is cartesian, we have for $U'/S'$ that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_{g^{-1}} \circ f_{\text{big},*}$ and $f'_{\text{small},*} \circ (i_{g'})^{-1}$ send a sheaf $F$ on $(\text{Sch}/T)_{\text{Zar}}$ to the sheaf $U' \mapsto F(U' \times_{S'} T')$ on $S'_{\text{Zar}}$ (use Lemmas \ref{lem:pullback-big-site} and \ref{lem:pushforward-small-site}). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma \ref{lem:pullback-base-change}.
\end{proof}
\end{lemma}

We can think about a sheaf on the big Zariski site of $S$ as a collection of “usual” sheaves on all schemes over $S$.

\begin{lemma}
Let $S$ be a scheme contained in a big Zariski site $S_{\text{Zar}}$. A sheaf $F$ on the big Zariski site $(\text{Sch}/S)_{\text{Zar}}$ is given by the following data:

1. for every $T/S \in \text{Ob}((\text{Sch}/S)_{\text{Zar}})$ a sheaf $F_T$ on $T$,
2. for every $f : T' \to T$ in $(\text{Sch}/S)_{\text{Zar}}$ a map $c_f : f^{-1}F_T \to F_{T'}$.

These data are subject to the following conditions:

1. given any $f : T' \to T$ and $g : T'' \to T'$ in $(\text{Sch}/S)_{\text{Zar}}$ the composition $c_g \circ g^{-1}c_f$ is equal to $c_{fg}$, and
2. if $f : T' \to T$ in $(\text{Sch}/S)_{\text{Zar}}$ is an open immersion then $c_f$ is an isomorphism.

\begin{proof}
Given a sheaf $F$ on $\text{Sh}((\text{Sch}/S)_{\text{Zar}})$ we set $F_T = i_p^{-1}F$ where $p : T \to S$ is the structure morphism. Note that $F_T(U) = F(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(\text{Sch}/T)_{\text{Zar}}$ with image $U$, see Lemmas \ref{lem:pullback-big-site} and \ref{lem:pushforward-big-site}. Hence given $f : T' \to T$ over $S$ and $U,U' \to T$ we get a canonical map $F_T(U) = F(U'/S) \to F(U' \times_T T'/S) = F_T(f^{-1}(U))$ where the middle is the restriction map of $F$ with respect to the morphism $U' \times_T T' \to U'$ over $S$. The
collection of these maps are compatible with restrictions, and hence define an \( f \)-map \( c_f \) from \( \mathcal{F}_T \) to \( \mathcal{F}_{T'} \), see Sheaves, Definition 21.7 and the discussion surrounding it. It is clear that \( c_{f\circ g} \) is the composition of \( c_f \) and \( c_g \), since composition of restriction maps of \( \mathcal{F} \) gives restriction maps.

Conversely, given a system \( (\mathcal{F}_T, c_f) \) as in the lemma we may define a presheaf \( \mathcal{F} \) on \( Sh((Sch/S)_{zar}) \) by simply setting \( \mathcal{F}(T/S) = \mathcal{F}_T(T) \). As restriction mapping, given \( f : T' \to T \) we set for \( s \in \mathcal{F}(T) \) the pullback \( f^*(s) \) equal to \( c_f(s) \) (where we think of \( c_f \) as an \( f \)-map again). The condition on the \( c_f \) guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. \( \square \)

4. The étale topology

0214 Let \( S \) be a scheme. We would like to define the étale-topology on the category of schemes over \( S \). According to our general principle we first introduce the notion of an étale covering.

0215 \textbf{Definition 4.1.} Let \( T \) be a scheme. An étale covering of \( T \) is a family of morphisms \( \{ f_i : T_i \to T \}_{i \in I} \) of schemes such that each \( f_i \) is étale and such that \( T = \bigcup f_i(T_i) \).

0216 \textbf{Lemma 4.2.} Any Zariski covering is an étale covering.

\textbf{Proof.} This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 34.9. \( \square \)

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0217 \textbf{Lemma 4.3.} Let \( T \) be a scheme.

(1) If \( T' \to T \) is an isomorphism then \( \{ T' \to T \} \) is an étale covering of \( T \).

(2) If \( \{ T_i \to T \}_{i \in I} \) is an étale covering and for each \( i \) we have an étale covering \( \{ T_{ij} \to T_i \}_{j \in J_i} \), then \( \{ T_{ij} \to T \}_{i \in I, j \in J_i} \) is an étale covering.

(3) If \( \{ T_i \to T \}_{i \in I} \) is an étale covering and \( T' \to T \) is a morphism of schemes then \( \{ T' \times_T T_i \to T' \}_{i \in I} \) is an étale covering.

\textbf{Proof.} Omitted. \( \square \)

0218 \textbf{Lemma 4.4.} Let \( T \) be an affine scheme. Let \( \{ T_i \to T \}_{i \in I} \) be an étale covering of \( T \). Then there exists an étale covering \( \{ U_j \to T \}_{j=1,\ldots,m} \) which is a refinement of \( \{ T_i \to T \}_{i \in I} \) such that each \( U_j \) is an affine scheme. Moreover, we may choose each \( U_j \) to be open affine in one of the \( T_i \).

\textbf{Proof.} Omitted. \( \square \)

Thus we define the corresponding standard coverings of affines as follows.

0219 \textbf{Definition 4.5.} Let \( T \) be an affine scheme. A standard étale covering of \( T \) is a family \( \{ f_j : U_j \to T \}_{j=1,\ldots,m} \) with each \( U_j \) is affine and étale over \( T \) and \( T = \bigcup f_j(U_j) \).

In the definition above we do not assume the morphisms \( f_j \) are standard étale. The reason is that if we did then the standard étale coverings would not define a site on \( Aff/S \), for example because of Algebra, Lemma 141.14 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra,
021A **Definition 4.6.** A *big étale site* is any site $\mathcal{S}_{\text{étale}}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes $S_0$, and any set of étale coverings $\text{Cov}_0$ among these schemes.
2. As underlying category take any category $\mathcal{S}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set $S_0$.
3. Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $\mathcal{S}_\alpha$ and the class of étale coverings, and the set $\text{Cov}_0$ chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme $S$, let us point out that the topology on a big étale site $\mathcal{S}_{\text{étale}}$ is in some sense induced from the étale topology on the category of all schemes.

021B **Definition 4.8.** Let $S$ be a scheme. Let $\mathcal{S}_{\text{étale}}$ be a big étale site containing $S$.

1. The *big étale site* of $S$, denoted $(\mathcal{S}/S)_{\text{étale}}$, is the site $\mathcal{S}_{\text{étale}}/S$ introduced in Sites, Section 24.
2. The *small étale site* of $S$, which we denote $\mathcal{S}_{\text{étale}}$, is the full subcategory of $(\mathcal{S}/S)_{\text{étale}}$ whose objects are those $U/S$ such that $U \to S$ is étale. A covering of $\mathcal{S}_{\text{étale}}$ is any covering $\{U_i \to U\}$ of $(\mathcal{S}/S)_{\text{étale}}$ with $U \in \text{Ob}(\mathcal{S}_{\text{étale}})$.
3. The *big affine étale site* of $S$, denoted $(\text{Aff}/S)_{\text{étale}}$, is the full subcategory of $(\mathcal{S}/S)_{\text{étale}}$ whose objects are affine $U/S$. A covering of $(\text{Aff}/S)_{\text{étale}}$ is any covering $\{U_i \to U\}$ of $(\mathcal{S}/S)_{\text{étale}}$ which is a standard étale covering.
It is not completely clear that the big affine étale site or the small étale site are sites. We check this now.

**Lemma 4.9.** Let $S$ be a scheme. Let $\mathbf{Sch}_{\text{étale}}$ be a big étale site containing $S$. Both $\mathbf{S}_{\text{étale}}$ and $(\text{Aff}/S)_{\text{étale}}$ are sites.

**Proof.** Let us show that $\mathbf{S}_{\text{étale}}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(\mathbf{Sch}/S)_{\text{étale}}$ is a site, it suffices to prove that given any covering \{$U_i \to U$\} of $(\mathbf{Sch}/S)_{\text{étale}}$ with $U \in \text{Ob}(\mathbf{S}_{\text{étale}})$ we also have $U_i \in \text{Ob}(\mathbf{S}_{\text{étale}})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(\text{Aff}/S)_{\text{étale}}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard étale covering \{$T_i \to T$\}_{i \in I} and for each $i$ we have a standard étale covering \{$T_{ij} \to T_i$\}_{j \in J_i}$, then \{$T_{ij} \to T$\}_{i \in I, j \in J_i} is a standard étale covering because $\bigcup_{i \in I} J_i$ is finite and each $T_{ij}$ is affine.

**Lemma 4.10.** Let $S$ be a scheme. Let $\mathbf{Sch}_{\text{étale}}$ be a big étale site containing $S$. The underlying categories of the sites $\mathbf{Sch}_{\text{étale}}$, $(\mathbf{Sch}/S)_{\text{étale}}$, $\mathbf{S}_{\text{étale}}$, and $(\text{Aff}/S)_{\text{étale}}$ have fibre products. In each case the obvious functor into the category $\mathbf{Sch}$ of all schemes commutes with taking fibre products. The categories $(\mathbf{Sch}/S)_{\text{étale}}$, and $\mathbf{S}_{\text{étale}}$ both have a final object, namely $S/S$.

**Proof.** For $\mathbf{Sch}_{\text{étale}}$ it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(\mathbf{Sch}_{\text{étale}})$. The fibre product $V \times_U W$ in $\mathbf{Sch}_{\text{étale}}$ is a fibre product in $\mathbf{Sch}$ and is the fibre product of $V/S$ with $W/S$ over $U/S$ in the category of all schemes over $S$, and hence also a fibre product in $(\mathbf{Sch}/S)_{\text{étale}}$. This proves the result for $(\mathbf{Sch}/S)_{\text{étale}}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $\mathbf{S}_{\text{étale}}$. If $U, V, W$ are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{\text{étale}}$.

Next, we check that the big affine site defines the same topos as the big site.

**Lemma 4.11.** Let $S$ be a scheme. Let $\mathbf{Sch}_{\text{étale}}$ be a big étale site containing $S$. The functor $(\text{Aff}/S)_{\text{étale}} \to (\mathbf{Sch}/S)_{\text{étale}}$ is special cocontinuous and induces an equivalence of topoi from $\text{Sh}((\text{Aff}/S)_{\text{étale}})$ to $\text{Sh}((\mathbf{Sch}/S)_{\text{étale}})$.

**Proof.** The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (\text{Aff}/S)_{\text{étale}} \to (\mathbf{Sch}/S)_{\text{étale}}$. Being cocontinuous just means that any étale covering of $T/S$, $T$ affine, can be refined by a standard étale covering of $T$. This is the content of Lemma 4.4. Hence (1) holds. We see $u$ is continuous simply because a standard étale covering is an étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that $u$ is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering.

Next, we establish some relationships between the topoi associated to these sites.
021F Lemma 4.12. Let $\text{Sch}_{\text{étale}}$ be a big étale site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{étale}}$. The functor $T_{\text{étale}} \to \left(\text{Sch}/S\right)_{\text{étale}}$ is cocontinuous and induces a morphism of topoi

$$i_f : \text{Sh}(T_{\text{étale}}) \to \text{Sh}(\left(\text{Sch}/S\right)_{\text{étale}})$$

For a sheaf $\mathcal{G}$ on $(\text{Sch}/S)_{\text{étale}}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor $i_f^{-1}$ also has a left adjoint $i_{f!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{\text{étale}}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 4.10. Let $a, b : U \to V$ be two morphisms in $T_{\text{étale}}$. In this case the equalizer of $a$ and $b$ (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over $T$, hence étale over $T$. Thus $T_{\text{étale}}$ has equalizers and $u$ commutes with them. It is clearly cocontinuous. It is also continuous as $u$ transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 20.5 and 20.6. □

021G Lemma 4.13. Let $S$ be a scheme. Let $\text{Sch}_{\text{étale}}$ be a big étale site containing $S$. The inclusion functor $S_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites

$$\pi_S : (\text{Sch}/S)_{\text{étale}} \to S_{\text{étale}}$$

and a morphism of topoi

$$i_S : \text{Sh}(S_{\text{étale}}) \to \text{Sh}(\left(\text{Sch}/S\right)_{\text{étale}})$$

such that $\pi_S \circ i_S = \text{id}$. Moreover, $i_S = i_{\text{id}_S}$ with $i_{\text{id}_S}$ as in Lemma 4.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}$, in addition to the properties seen in the proof of Lemma 4.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 20.8. □

04BT Definition 4.14. In the situation of Lemma 4.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the restriction to the small étale site, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{\text{étale}}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\text{Mor}_{\text{Sh}(S_{\text{étale}})}(\mathcal{F}|_{S_{\text{étale}}}, \mathcal{G}) = \text{Mor}_{\text{Sh}(\left(\text{Sch}/S\right)_{\text{étale}})}(\mathcal{F}, i_{S,*}\mathcal{G})$$

$$\text{Mor}_{\text{Sh}(S_{\text{étale}})}(\mathcal{G}, \mathcal{F}|_{S_{\text{étale}}}) = \text{Mor}_{\text{Sh}(\left(\text{Sch}/S\right)_{\text{étale}})}(\pi_{S,*}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{\text{étale}}} = \mathcal{G}$ and we have $(\pi_{S,*}\mathcal{G})|_{S_{\text{étale}}} = \mathcal{G}$.

021H Lemma 4.15. Let $\text{Sch}_{\text{étale}}$ be a big étale site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{étale}}$. The functor

$$u : (\text{Sch}/T)_{\text{étale}} \to (\text{Sch}/S)_{\text{étale}}, \quad V/T \mapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Sch}/S)_{\text{étale}} \to (\text{Sch}/T)_{\text{étale}}, \quad (U \to S) \mapsto (U \times_S T \to T).$$
They induce the same morphism of topoi

\[ f_{\text{big}} : \text{Sh}(\text{Sch}/T)_{\text{étale}} \rightarrow \text{Sh}(\text{Sch}/S)_{\text{étale}} \]

We have \( f_{\text{big}}^{-1}(G)(U/T) = G(U/S) \). We have \( f_{\text{big},*}(F)(U/S) = F(U \times_S T/T) \). Also, \( f_{\text{big}}^{-1} \) has a left adjoint \( f_{\text{big}!} \) which commutes with fibre products and equalizers.

**Proof.** The functor \( u \) is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for \( f_{\text{big}} \) and the existence of \( f_{\text{big}}^{-1} \). Moreover, the functor \( v \) is a right adjoint because given \( U/T \) and \( V/S \) we have \( \text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T) \) as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for \( f_{\text{big},*} \). \( \square \)

**Lemma 4.16.** Let \( \text{Sch}_{\text{étale}} \) be a big étale site. Let \( f : T \rightarrow S \) be a morphism in \( \text{Sch}_{\text{étale}} \).

1. We have \( i_f = f_{\text{big}} \circ i_T \) with \( i_f \) as in Lemma 4.12 and \( i_T \) as in Lemma 4.13.
2. The functor \( S_{\text{étale}} \rightarrow T_{\text{étale}} \), \( (U \rightarrow S) \mapsto (U \times_S T \rightarrow T) \) is continuous and induces a morphism of sites

\[ f_{\text{small}} : T_{\text{étale}} \rightarrow S_{\text{étale}} \]

We have \( f_{\text{small},*}(F)(U/S) = F(U \times_S T/T) \).
3. We have a commutative diagram of morphisms of sites

\[
\begin{array}{ccc}
S_{\text{étale}} & \xrightarrow{\pi_T} & (\text{Sch}/T)_{\text{étale}} \\
\downarrow f_{\text{small}} & & \downarrow f_{\text{big}} \\
T_{\text{étale}} & \xleftarrow{\pi}\ & (\text{Sch}/S)_{\text{étale}}
\end{array}
\]

so that \( f_{\text{small}} \circ \pi_T = \pi_S \circ f_{\text{big}} \) as morphisms of topoi.
4. We have \( f_{\text{small}} \circ \pi_T = \pi_S \circ f_{\text{big}} = \pi_S \circ i_f \).

**Proof.** The equality \( i_f = f_{\text{big}} \circ i_T \) follows from the equality \( i_f^{-1} = i_T^{-1} \circ f_{\text{big}}^{-1} \) which is clear from the descriptions of these functors above. Thus we see (1).

The functor \( u : S_{\text{étale}} \rightarrow T_{\text{étale}} \), \( u(U \rightarrow S) = (U \times_S T \rightarrow T) \) transforms coverings into coverings and commutes with fibre products, see Lemma 4.3 (3) and 4.10. Moreover, both \( S_{\text{étale}} \) and \( T_{\text{étale}} \) have final objects, namely \( S/S \) and \( T/T \) and \( u(S/S) = T/T \). Hence by Sites, Proposition 14.6 the functor \( u \) corresponds to a morphism of sites \( T_{\text{étale}} \rightarrow S_{\text{étale}} \). This in turn gives rise to the morphism of topoi, see Sites, Lemma 15.2. The description of the pushforward is clear from these references.

Part (3) follows because \( \pi_S \) and \( \pi_T \) are given by the inclusion functors and \( f_{\text{small}} \) and \( f_{\text{big}} \) by the base change functors \( U \rightarrow U \times_S T \).

Statement (4) follows from (3) by precomposing with \( i_T \). \( \square \)

In the situation of the lemma, using the terminology of Definition 4.14 we have: for \( F \) a sheaf on the big étale site of \( T \)

\[ (f_{\text{big},*}F)|_{S_{\text{étale}}} = f_{\text{small},*}(F|_{T_{\text{étale}}}). \]

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of \( T \), resp. \( S \) is given by \( \pi_{T,*} \), resp. \( \pi_{S,*} \). A similar formula involving pullbacks and restrictions is false.
021J Lemma 4.17. Given schemes $X, Y, Z$ in $\text{Sch}_{\text{étale}}$ and morphisms $f : X \to Y$, $g : Y \to Z$, we have $g_{\text{big}} \circ f_{\text{big}} = (g \circ f)_{\text{big}}$ and $g_{\text{small}} \circ f_{\text{small}} = (g \circ f)_{\text{small}}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.16. □

0DDA Lemma 4.18. Let $\text{Sch}_{\text{étale}}$ be a big étale site. Consider a cartesian diagram

$$
\begin{array}{ccc}
T' & \xrightarrow{g'} & T \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{\alpha} & S
\end{array}
$$

in $\text{Sch}_{\text{étale}}$. Then $i_g^{-1} \circ f_{\text{big}*} = f'_{\text{small}*} \circ (i_{g'})^{-1}$ and $g_{\text{big}*} \circ f_{\text{big}*} = f'_{\text{big}*} \circ (g_{\text{big}})^{-1}$.

Proof. Since the diagram is cartesian, we have for $U'/S'$ that $U' \times_{S'} T' = U' \times_{S} T$. Hence both $i_g^{-1} \circ f_{\text{big}*}$ and $f'_{\text{small}*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(\text{Sch}/T)_{\text{étale}}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{\text{étale}}$ (use Lemmas 4.12 and 4.15). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 27.1. □

We can think about a sheaf on the big étale site of $S$ as a collection of “usual” sheaves on all schemes over $S$.

021K Lemma 4.19. Let $S$ be a scheme contained in a big étale site $\text{Sch}_{\text{étale}}$. A sheaf $\mathcal{F}$ on the big étale site $(\text{Sch}/S)_{\text{étale}}$ is given by the following data:

1. for every $T/S \in \text{Ob}((\text{Sch}/S)_{\text{étale}})$ a sheaf $\mathcal{F}_T$ on $T_{\text{étale}}$,
2. for every $f : T' \to T$ in $(\text{Sch}/S)_{\text{étale}}$ a map $c_f : f_{\text{small}}^{-1} \mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(a) given any $f : T' \to T$ and $g : T'' \to T'$ in $(\text{Sch}/S)_{\text{étale}}$ the composition $c_g \circ g_{\text{small}}^{-1} c_f$ is equal to $c_{f \circ g}$, and
(b) if $f : T' \to T$ in $(\text{Sch}/S)_{\text{étale}}$ is étale then $c_f$ is an isomorphism.

Proof. Given a sheaf $\mathcal{F}$ on $\text{Sh}((\text{Sch}/S)_{\text{étale}})$ we set $\mathcal{F}_T = i_p^{-1} \mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{\text{étale}}$ see Lemma 4.12. Hence given $f : T' \to T$ over $S$ and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over $S$.

The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{\text{small}*} \mathcal{F}_{T'}$ where $u : T_{\text{étale}} \to T'_{\text{étale}}$ is the base change functor associated to $f$. By adjunction of $f_{\text{small}*}$ (see Sites, Section 13) with $f_{\text{small}}^{-1}$ this is the same as a map $c_{f \circ g}' : f_{\text{small}}^{-1} \mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c_{f \circ g}'$ is the composition of $c_f'$ and $f_{\text{small}*} c_g'$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among $c_f$, $c_g$ and $c_{f \circ g}'$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $\text{Sh}((\text{Sch}/S)_{\text{étale}})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of $c_f$ as a map $\mathcal{F}_T \to f_{\text{small}*} \mathcal{F}_{T'}$ again. The condition on the $c_f$ guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □
5. The smooth topology

In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 34) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

Definition 5.1. Let $T$ be a scheme. An smooth covering of $T$ is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each $f_i$ is smooth and such that $T = \bigcup f_i(T_i)$.

Lemma 5.2. Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 34.1 and Lemma 4.2.

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

Lemma 5.3. Let $T$ be a scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an smooth covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is a smooth covering and for each $i$ we have a smooth covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a smooth covering.
3. If $\{T_i \to T\}_{i \in I}$ is a smooth covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a smooth covering.

Proof. Omitted.

Lemma 5.4. Let $T$ be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a smooth covering of $T$. Then there exists a smooth covering $\{U_j \to T\}_{j=1,...,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each $U_j$ is an affine scheme, and such that each morphism $U_j \to T$ is standard smooth, see Morphisms, Definition 32.1. Moreover, we may choose each $U_j$ to be open affine in one of the $T_i$.

Proof. Omitted, but see Algebra, Lemma 135.10.

Thus we define the corresponding standard coverings of affines as follows.

Definition 5.5. Let $T$ be an affine scheme. A standard smooth covering of $T$ is a family $\{f_j : U_j \to T\}_{j=1,...,m}$ with each $U_j$ is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

Definition 5.6. A big smooth site is any site $\text{Sch}_{\text{smooth}}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes $S_0$, and any set of smooth coverings $\text{Cov}_0$ among these schemes.
2. As underlying category take any category $\text{Sch}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set $S_0$.
3. Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $\text{Sch}_\alpha$ and the class of smooth coverings, and the set $\text{Cov}_0$ chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme $S$, let us point out that the topology on a big smooth site $\text{Sch}_{\text{smooth}}$ is in some sense induced from the smooth topology on the category of all schemes.
Let $\mathcal{S}$ be a big smooth site as in Definition \ref{def:big-smooth-site}. Let $T \in \text{Ob}(\mathcal{S})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of $T$.

1. There exists a covering $\{U_j \to T\}_{j\in J}$ of $T$ in the site $\mathcal{S}$ which refines $\{T_i \to T\}_{i\in I}$.
2. If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of $\mathcal{S}$.
3. If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of $\mathcal{S}$.

**Proof.** For each $i$ choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each $T_{ij}$ maps into an affine open subscheme of $T$. By Lemma \ref{lem:standard-smooth-covering} the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an smooth covering of $T$ as well. Hence we may assume each $T_i$ is affine, and maps into an affine open $W_i$ of $T$. Applying Sets, Lemma \ref{lem:zariski-covering} we see that $W_i$ is isomorphic to an object of $\mathcal{S}_{zar}$. But then $T_i$ as a finite type scheme over $W_i$ is isomorphic to an object of $\mathcal{S}_{zar}$ by a second application of Sets, Lemma \ref{lem:zariski-covering}. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic).

Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of $T$ in the site $\mathcal{S}_{zar}$ by Sets, Lemma \ref{lem:zariski-covering}. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes $T_i$ is isomorphic to an object of $\mathcal{S}$ by Sets, Lemma \ref{lem:zariski-covering} and another application of Sets, Lemma \ref{lem:zariski-covering}.

Let $S$ be a scheme. Let $\mathcal{S}$ be a big smooth site containing $S$.

1. The big smooth site of $S$, denoted $(\mathcal{S}/S)_{smooth}$, is the site $\mathcal{S}/S$ introduced in Sites, Section \ref{sec:smooth-site}.
2. The big affine smooth site of $S$, denoted $(\text{Aff}/S)_{smooth}$, is the full subcategory of $(\mathcal{S}/S)_{smooth}$ whose objects are affine $U/S$. A covering of $(\text{Aff}/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(\mathcal{S}/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

Let $S$ be a scheme. Let $\mathcal{S}_{etale}$ be a big smooth site containing $S$. The functor $(\text{Aff}/S)_{smooth} \to (\mathcal{S}/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $\text{Sh}(\text{Aff}/S)_{smooth}$ to $\text{Sh}(\mathcal{S}/S)_{smooth}$.

**Proof.** The notion of a special cocontinuous functor is introduced in Sites, Definition \ref{def:cocontinuous-functor}. Thus we have to verify assumptions (1) – (5) of Sites, Lemma \ref{lem:cocontinuous-functor}.

Denote the inclusion functor $u : (\text{Aff}/S)_{smooth} \to (\mathcal{S}/S)_{smooth}$. Being cocontinuous just means that any smooth covering of $T/S$, $T$ affine, can be refined by a standard smooth covering of $T$. This is the content of Lemma \ref{lem:standard-smooth-covering}. Hence (1) holds.

We see $u$ is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that $u$ is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering.

To be continued...

Let $\mathcal{S}$ be a big smooth site. Let $f : T \to S$ be a morphism in $\mathcal{S}$. The functor $u : (\mathcal{S}/T)_{smooth} \to (\mathcal{S}/S)_{smooth}$, $V/T \mapsto V/S$
is cocontinuous, and has a continuous right adjoint
\[ v : (\text{Sch}/S)_{\text{smooth}} \longrightarrow (\text{Sch}/T)_{\text{smooth}}, \quad (U \to S) \mapsto (U \times_S T \to T). \]
They induce the same morphism of topoi
\[ f_{\text{big}} : \text{Sh}((\text{Sch}/T)_{\text{smooth}}) \longrightarrow \text{Sh}((\text{Sch}/S)_{\text{smooth}}). \]
We have \( f_{\text{big}}^{-1}(G)(U/T) = G(U/S) \). We have \( f_{\text{big},*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T) \). Also, \( f_{\text{big}}^{-1} \) has a left adjoint \( f_{\text{big}}! \) which commutes with fibre products and equalizers.

**Proof.** The functor \( v \) is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.3 and 20.6 apply and we deduce the formula for \( f_{\text{big}}^{-1} \) and the existence of \( f_{\text{big}}! \). Moreover, the functor \( v \) is a right adjoint because given \( U/T \) and \( V/S \) we have \( \text{Mor}_S(v(U), V) = \text{Mor}_T(U, V \times_S T) \) as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for \( f_{\text{big},*} \). \( \square \)

### 6. The syntomic topology

0224 In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

0225 **Definition 6.1.** Let \( T \) be a scheme. An **syntomic covering of** \( T \) is a family of morphisms \( \{f_i : T_i \to T\}_{i \in I} \) of schemes such that each \( f_i \) is syntomic and such that \( T = \bigcup f_i(T_i) \).

0226 **Lemma 6.2.** Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.

**Proof.** This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 32.7 and Lemma 5.2. \( \square \)

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0227 **Lemma 6.3.** Let \( T \) be a scheme.

1. If \( T' \to T \) is an isomorphism then \( \{T' \to T\} \) is a syntomic covering of \( T \).
2. If \( \{T_i \to T\}_{i \in I} \) is a syntomic covering and for each \( i \) we have a syntomic covering \( \{T_{ij} \to T_i\}_{j \in J_i} \), then \( \{T_{ij} \to T\}_{i \in I, j \in J_i} \) is a syntomic covering.
3. If \( \{T_i \to T\}_{i \in I} \) is a syntomic covering and \( T' \to T \) is a morphism of schemes then \( \{T' \times_T T_i \to T'\}_{i \in I} \) is a syntomic covering.

**Proof.** Omitted. \( \square \)

0228 **Lemma 6.4.** Let \( T \) be an affine scheme. Let \( \{T_i \to T\}_{i \in I} \) be a syntomic covering of \( T \). Then there exists a syntomic covering \( \{U_j \to T\}_{j = 1, \ldots, m} \) which is a refinement of \( \{T_i \to T\}_{i \in I} \) such that each \( U_j \) is an affine scheme, and such that each morphism \( U_j \to T \) is standard syntomic, see Morphisms, Definition 29.7. Moreover, we may choose each \( U_j \) to be open affine in one of the \( T_i \).

**Proof.** Omitted, but see Algebra, Lemma 134.15. \( \square \)

Thus we define the corresponding standard coverings of affines as follows.

0229 **Definition 6.5.** Let \( T \) be an affine scheme. A **standard syntomic covering of** \( T \) is a family \( \{f_j : U_j \to T\}_{j = 1, \ldots, m} \) with each \( U_j \) is affine, \( U_j \to T \) standard syntomic and \( T = \bigcup f_j(U_j) \).
Definition 6.6. A **big syntomic site** is any site $\text{Sch}_{\text{syntomic}}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes $S_0$, and any set of syntomic coverings $\text{Cov}_0$ among these schemes.
2. As underlying category take any category $\text{Sch}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set $S_0$.
3. Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $\text{Sch}_\alpha$ and the class of syntomic coverings, and the set $\text{Cov}_0$ chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme $S$, let us point out that the topology on a big syntomic site $\text{Sch}_{\text{syntomic}}$ is in some sense induced from the syntomic topology on the category of all schemes.

Lemma 6.7. Let $\text{Sch}_{\text{syntomic}}$ be a big syntomic site as in Definition 6.6. Let $T \in \text{Ob}(\text{Sch}_{\text{syntomic}})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary syntomic covering of $T$.

1. There exists a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\text{Sch}_{\text{syntomic}}$ which refines $\{T_i \to T\}_{i \in I}$.
2. If $\{T_i \to T\}_{i \in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $\text{Sch}_{\text{syntomic}}$.
3. If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $\text{Sch}_{\text{syntomic}}$.

Proof. For each $i$ choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each $T_{ij}$ maps into an affine open subscheme of $T$. By Lemma 6.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a syntomic covering of $T$ as well. Hence we may assume each $T_i$ is affine, and maps into an affine open $W_i$ of $T$. Applying Sets, Lemma 9.9 we see that $W_i$ is isomorphic to an object of $\text{Sch}_{\text{Zar}}$. But then $T_i$ as a finite type scheme over $W_i$ is isomorphic to an object of $\text{Sch}_{\text{Zar}}$ by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\text{Sch}_{\text{Zar}}$ by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes $T_i$ is isomorphic to an object of $\text{Sch}_{\text{Zar}}$ by Sets, Lemma 9.9 and another application of Sets, Lemma 11.1 gives what we want. □

Definition 6.8. Let $S$ be a scheme. Let $\text{Sch}_{\text{syntomic}}$ be a big syntomic site containing $S$.

1. The **big syntomic site** of $S$, denoted $(\text{Sch}/S)_{\text{syntomic}}$, is the site $\text{Sch}_{\text{syntomic}}/S$ introduced in Sites, Section 24.
2. The **big affine syntomic site** $\text{Aff}_{\text{syntomic}}$ of $S$, denoted $(\text{Aff}/S)_{\text{syntomic}}$, is the full subcategory of $(\text{Sch}/S)_{\text{syntomic}}$ whose objects are affine $U/S$. A covering of $(\text{Aff}/S)_{\text{syntomic}}$ is any covering $\{U_i \to U\}$ of $(\text{Sch}/S)_{\text{syntomic}}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.
Lemma 6.9. Let $S$ be a scheme. Let $\text{Sch}_{\text{syntomic}}$ be a big syntomic site containing $S$. The functor $(\text{Aff}/S)_{\text{syntomic}} \to (\text{Sch}/S)_{\text{syntomic}}$ is special cocontinuous and induces an equivalence of topoi from $\text{Sh}((\text{Aff}/S)_{\text{syntomic}})$ to $\text{Sh}((\text{Sch}/S)_{\text{syntomic}})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (\text{Aff}/S)_{\text{syntomic}} \to (\text{Sch}/S)_{\text{syntomic}}$. Being cocontinuous just means that any syntomic covering of $T/S$, $T$ affine, can be refined by a standard syntomic covering of $T$. This is the content of Lemma 6.4. Hence (1) holds. We see $u$ is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that $u$ is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 6.10. Let $\text{Sch}_{\text{syntomic}}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $\text{Sch}_{\text{syntomic}}$. The functor $u : (\text{Sch}/T)_{\text{syntomic}} \to (\text{Sch}/S)_{\text{syntomic}}$, $V/T \mapsto V/S$ is cocontinuous, and has a continuous right adjoint $v : (\text{Sch}/S)_{\text{syntomic}} \to (\text{Sch}/T)_{\text{syntomic}}$, $(U \to S) \mapsto (U \times_S T \to T)$. They induce the same morphism of topoi $f_{\text{big}} : \text{Sh}((\text{Sch}/T)_{\text{syntomic}}) \to \text{Sh}((\text{Sch}/S)_{\text{syntomic}})$.

We have $f_{\text{big}}^{-1}(G)(U/T) = G(U/S)$. We have $f_{\text{big},*}(F)(U/S) = F(U \times_S T/T)$. Also, $f_{\text{big}}^{-1}$ has a left adjoint $f_{\text{big}}!$ which commutes with fibre products and equalizers.

Proof. The functor $u$ is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for $f_{\text{big}}^{-1}$ and the existence of $f_{\text{big}}!$. Moreover, the functor $v$ is a right adjoint because given $U/T$ and $V/S$ we have $\text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{\text{big},*}$. □

7. The fppf topology

Let $S$ be a scheme. We would like to define the fppf-topology on the category of schemes over $S$. According to our general principle we first introduce the notion of an fppf-covering.

Definition 7.1. Let $T$ be a scheme. An fppf covering of $T$ is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each $f_i$ is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Lemma 7.2. Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.6 and 29.7, and Lemma 6.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2. 

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2 The letters fppf stand for “fidèlement plat de présentation finie”.

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Lemma 7.3. Let $T$ be a scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each $i$ we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.
3. If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 24.5) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 20.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 24.7) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 20.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). □

Lemma 7.4. Let $T$ be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fppf covering of $T$. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each $U_j$ is an affine scheme. Moreover, we may choose each $U_j$ to be open affine in one of the $T_i$.

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 24.9 □

Thus we define the corresponding standard coverings of affines as follows.

Definition 7.5. Let $T$ be an affine scheme. A standard fppf covering of $T$ is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each $U_j$ is affine, flat and of finite presentation over $T$ and $T = \bigcup f_j(U_j)$.

Definition 7.6. A big fppf site is any site $\text{Sch}_{fppf}$ as in Sites, Definition 6.2 constructed as follows:

1. Choose any set of schemes $S_0$, and any set of fppf coverings Cov$_0$ among these schemes.
2. As underlying category take any category $\text{Sch}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set $S_0$.
3. Choose any set of coverings as in Sets, Lemma 11.1 starting with the category $\text{Sch}_\alpha$ and the class of fppf coverings, and the set Cov$_0$ chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme $S$, let us point out that the topology on a big fppf site $\text{Sch}_{fppf}$ is in some sense induced from the fppf topology on the category of all schemes.

Lemma 7.7. Let $\text{Sch}_{fppf}$ be a big fppf site as in Definition 7.6. Let $T \in \text{Ob}(\text{Sch}_{fppf})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary fppf covering of $T$.

1. There exists a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\text{Sch}_{fppf}$ which refines $\{T_i \to T\}_{i \in I}$.
2. If $\{T_i \to T\}_{i \in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of $\text{Sch}_{fppf}$.
3. If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of $\text{Sch}_{fppf}$.
Proof. For each $i$ choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each $T_{ij}$ maps into an affine open subscheme of $T$. By Lemma 7.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering of $T$ as well. Hence we may assume each $T_i$ is affine, and maps into an affine open $W_i$ of $T$. Applying Sets, Lemma 9.9 we see that $W_i$ is isomorphic to an object of $\mathcal{S}_{\text{Zar}}$. But then $T_i$ as a finite type scheme over $W_i$ is isomorphic to an object $V_i$ of $\mathcal{S}_{\text{Zar}}$ by a second application of Sets, Lemma 9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of $T$ in the site $\mathcal{S}_{\text{Zar}}$ by Sets, Lemma 9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes $T_i$ is isomorphic to an object of $\mathcal{S}_{\text{fppf}}$ by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. 

Definition 7.8. Let $S$ be a scheme. Let $\mathcal{S}_{\text{fppf}}$ be a big fppf site containing $S$.

(1) The big fppf site of $S$, denoted $(\mathcal{S}/S)_{\text{fppf}}$, is the site $\mathcal{S}_{\text{fppf}}/S$ introduced in Sites, Section 23.

(2) The big affine fppf site of $S$, denoted $(\text{Aff}/S)_{\text{fppf}}$, is the full subcategory of $(\mathcal{S}/S)_{\text{fppf}}$ whose objects are affine $U/S$. A covering of $(\text{Aff}/S)_{\text{fppf}}$ is any covering $\{U_i \to U\}$ of $(\mathcal{S}/S)_{\text{fppf}}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

Lemma 7.9. Let $S$ be a scheme. Let $\mathcal{S}_{\text{fppf}}$ be a big fppf site containing $S$. Then $(\text{Aff}/S)_{\text{fppf}}$ is a site.

Proof. Let us show that $(\text{Aff}/S)_{\text{fppf}}$ is a site. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i \in I}$ and for each $i$ we have a standard fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fppf covering because $\bigcup_{i \in I} J_i$ is finite and each $T_{ij}$ is affine.

Lemma 7.10. Let $S$ be a scheme. Let $\mathcal{S}_{\text{fppf}}$ be a big fppf site containing $S$. The underlying categories of the sites $\mathcal{S}_{\text{fppf}}$, $(\mathcal{S}/S)_{\text{fppf}}$, and $(\text{Aff}/S)_{\text{fppf}}$ have fibre products. In each case the obvious functor into the category $\mathcal{S}$ of all schemes commutes with taking fibre products. The category $(\mathcal{S}/S)_{\text{fppf}}$ has a final object, namely $S/S$.

Proof. For $\mathcal{S}_{\text{fppf}}$ it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(\mathcal{S}_{\text{fppf}})$. The fibre product $V \times_U W$ in $\mathcal{S}_{\text{fppf}}$ is a fibre product in $\mathcal{S}$ and is the fibre product of $V/S$ with $W/S$ over $U/S$ in the category of all schemes over $S$, and hence also a fibre product in $(\mathcal{S}/S)_{\text{fppf}}$. This proves the result for $(\mathcal{S}/S)_{\text{fppf}}$. If $U, V, W$ are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{\text{fppf}}$. 

Next, we check that the big affine site defines the same topos as the big site.

Lemma 7.11. Let $S$ be a scheme. Let $\mathcal{S}_{\text{fppf}}$ be a big fppf site containing $S$. The functor $(\text{Aff}/S)_{\text{fppf}} \to (\mathcal{S}/S)_{\text{fppf}}$ is cocontinuous and induces an equivalence of topoi from $\text{Sh}((\text{Aff}/S)_{\text{fppf}})$ to $\text{Sh}((\mathcal{S}/S)_{\text{fppf}})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1.
Denote the inclusion functor \( u : (\text{Aff}/S)_{\text{fppf}} \rightarrow (\text{Sch}/S)_{\text{fppf}} \). Being cocontinuous just means that any fppf covering of \( T/S \), \( T \) affine, can be refined by a standard fppf covering of \( T \). This is the content of Lemma 7.4. Hence (1) holds. Parts (3) and (4) follow immediately from the fact that \( u \) is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we establish some relationships between the topoi associated to these sites.

Lemma 7.12. Let \( \text{Sch}_{\text{fppf}} \) be a big fppf site. Let \( f : T \rightarrow S \) be a morphism in \( \text{Sch}_{\text{fppf}} \). The functor \( u : (\text{Sch}/T)_{\text{fppf}} \rightarrow (\text{Sch}/S)_{\text{fppf}} \), \( V/T \mapsto V/S \) is cocontinuous, and has a continuous right adjoint \( v : (\text{Sch}/S)_{\text{fppf}} \rightarrow (\text{Sch}/T)_{\text{fppf}} \), \( (U \rightarrow S) \mapsto (U \times_S T \rightarrow T) \).

They induce the same morphism of topoi

\[
\tilde{f}_{\text{big}} : \text{Sh}(\text{Sch}/T)_{\text{fppf}} \rightarrow \text{Sh}(\text{Sch}/S)_{\text{fppf}}
\]

We have \( f_{\text{big}}^{-1}(G)(U/T) = G(U/S) \). We have \( f_{\text{big}*}(F)(U/S) = F(U \times_S T/T) \). Also, \( f_{\text{big}}^{-1} \) has a left adjoint \( f_{\text{big}}! \) which commutes with fibre products and equalizers.

Proof. The functor \( u \) is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for \( f_{\text{big}}^{-1} \) and the existence of \( f_{\text{big}}! \). Moreover, the functor \( v \) is a right adjoint because given \( U/T \) and \( V/S \) we have \( \text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T/T) \) as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for \( f_{\text{big}}* \). □

Lemma 7.13. Given schemes \( X, Y, Z \) in \( (\text{Sch}/S)_{\text{fppf}} \) and morphisms \( f : X \rightarrow Y \), \( g : Y \rightarrow Z \) we have \( g_{\text{big}} \circ f_{\text{big}} = (g \circ f)_{\text{big}} \).

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.12. □

8. The ph topology

In this section we define the ph topology. This is the topology generated by Zariski coverings and proper surjective morphisms, see Lemma 8.14.

We borrow our notation/terminology from the paper [GL01] by Goodwillie and Lichtenbaum. These authors show that if we restrict to the subcategory of Noetherian schemes, then the ph topology is the same as the “h topology” as originally defined by Voevodsky: this is the topology generated by Zariski open coverings and finite type morphisms which are universally submersive. They also show that the two topologies do not agree on non-Noetherian schemes, see [GL01, Example 4.5].

It may be worth pointing out that the definition of the h topology on the category of schemes depends on the reference. For example, in the paper [Ryd07] David Rydh (re)defines the h topology as the topology generated by Zariski open coverings and morphisms of finite presentation which are universally “subtrusive”.

Before we can define the coverings in our topology we need to do a bit of work.
Let $T$ be an affine scheme. A standard ph covering is a family 
$f_j : U_j \to T$ for $j = 1, \ldots, m$ constructed from a proper surjective morphism $f : U \to T$
and an affine open covering $U = \bigcup_{j=1, \ldots, m} U_j$ by setting $f_j = f|_{U_j}$.

It follows immediately from Chow’s lemma that we can refine a standard ph covering
by a standard ph covering corresponding to a surjective projective morphism.

Let $\{f_j : U_j \to T\}_{j=1, \ldots, m}$ be a standard ph covering. Let $T' \to T$
be a morphism of affine schemes. Then $\{U_j \times_T T' \to T'\}_{j=1, \ldots, m}$ is a standard ph covering.

Proof. Let $f : U \to T$ be proper surjective and let an affine open covering $U = \bigcup_{j=1, \ldots, m} U_j$ be given as in Definition 8.1. Then $U \times_T T' \to T'$ is proper surjective
(Morphisms, Lemmas 9.4 and 39.5). Also, $U \times_T T' = \bigcup_{j=1, \ldots, m} U_j \times_T T'$ is an affine open covering. This concludes the proof.

Proof of (3). Choose $U \to T$ proper surjective and $U = \bigcup_{j=1, \ldots, m} U_j$ as in Definition 8.1. Choose $W_j \to U_j$ proper surjective and $W_j = \bigcup W_{ji}$ as in Definition 8.1. By Chow’s lemma (Limits, Lemma 12.1) we can find $W'_j \to W_j$ proper surjective and closed immersions $W'_j \to \mathbf{P}^n_{U_j}$. Thus, after replacing $W_j$ by $W'_j$ and $W_j = \bigcup W_{ji}$ by a suitable affine open covering of $W'_j$, we may assume there is a closed immersion $W_j \subset \mathbf{P}^n_{U_j}$ for all $j = 1, \ldots, m$.

Let $\overline{W}_j \subset \mathbf{P}^n_{U_j}$ be the scheme theoretic closure of $W_j$. Then $W_j \subset \overline{W}_j$ is an open subscheme; in fact $W_j$ is the inverse image of $U_j \subset U$ under the morphism $\overline{W}_j \to U$.

(To see this use that $W_j \to \mathbf{P}^n_{U_j}$ is quasi-compact and hence formation of the scheme theoretic image commutes with restriction to opens, see Morphisms, Section 12.) Let $Z_j = U \setminus U_j$ with reduced induced closed subscheme structure. Then

$V_j = \overline{W}_j \cup Z_j \to U$

is proper surjective and the open subscheme $W_j \subset V_j$ is the inverse image of $U_j$.

Hence for $v \in V_j$, $v \notin W_j$ we can pick an affine open neighbourhood $v \in V_{j',v} \subset V_j$
which maps into $U_{j'}$ for some $1 \leq j' \leq m$.

To finish the proof we consider the proper surjective morphism

$V = V_1 \times_U V_2 \times_U \ldots \times_U V_m \to U \to T$

and the covering of $V$ by the affine opens

$V_1 \times_U \ldots \times_U V_{j-1, v_{j-1}} \times_U W_{j_i} \times_U V_{j+1, v_{j+1}} \times_U \ldots \times_U V_{m, v_m}$

These do indeed form a covering, because each point of $U$ is in some $U_j$ and the
inverse image of $U_j$ in $V$ is equal to $V_1 \times \ldots \times V_{j-1} \times W_j \times V_{j+1} \times \ldots \times V_m$. Observe
that the morphism from the affine open displayed above to $T$ factors through $W_{j_i}$
Thus we obtain a refinement. Finally, we only need a finite number of these affine opens as $V$ is quasi-compact (as a scheme proper over the affine scheme $T$).

**Definition 8.4.** Let $T$ be a scheme. A **ph covering of $T$** is a family of morphisms $\{T_i \to T\}_{i \in I}$ of schemes such that $f_i$ is locally of finite type and such that for every affine open $U \subset T$ there exists a standard ph covering $\{U_j \to U\}_{j=1, \ldots, m}$ refining the family $\{T_i \times_T U \to U\}_{i \in I}$.

**Lemma 8.5.** A Zariski covering is a ph covering.

**Proof.** This is true because a Zariski covering of an affine scheme can be refined by a standard ph covering by Lemma 8.3.

**Lemma 8.6.** Let $f : Y \to X$ be a surjective proper morphism of schemes. Then $\{Y \to X\}$ is a ph covering.

**Proof.** Omitted.

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

**Lemma 8.7.** Let $T$ be a scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a ph covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is a ph covering and for each $i$ we have a ph covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a ph covering.
3. If $\{T_i \to T\}_{i \in I}$ is a ph covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T\}_{i \in I}$ is a ph covering.

**Proof.** Assertion (1) is clear.

Proof of (3). The base change $T_i \times_T T' \to T'$ is locally of finite type by Morphisms, Lemma 14.3 hence we only need to check the condition on affine opens. Let $U' \subset T'$ be an affine open subscheme. Since $U'$ is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_i \to T'$ maps into an affine open $U_j \subset T$. Choose a standard ph covering $\{U_{ij} \to U_j\}_{i=1, \ldots, n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 8.2 the base change $\{U_{ij} \times_{U_j} U'_j \to U'_j\}$ is a standard ph covering. Note that $U'_j \to U'$ is a standard ph covering as well. By Lemma 8.3 the family $\{U_{ij} \times_{U_j} U'_j \to U'\}$ can be refined by a standard ph covering. Since $\{U_{ij} \to U_j\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Composition preserves being locally of finite type, see Morphisms, Lemma 14.3 hence we only need to check the condition on affine opens. Let $U \subset T$ be affine open. First we pick a standard ph covering $\{U_k \to U\}_{k=1, \ldots, m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over $T$. Then

$$\{T_{i_k} \times_{T_k} U_k \to U_k\}_{j \in J_k}$$

is a ph covering by part (3). As $U_k$ is affine, we can find a standard ph covering $\{U_{ka} \to U_k\}_{a=1, \ldots, b_k}$ refining this family. Then we apply Lemma 8.3 to see that $\{U_{ka} \to U\}$ can be refined by a standard ph covering. Since $\{U_{ka} \to U\}$ refines $T_{i_k} \times_T U \to U$ this finishes the proof.

**Definition 8.8.** A **big ph site** is any site $\text{Sch}_{ph}$ as in Sites, Definition 6.2 constructed as follows:

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3We will see in More on Morphisms, Lemma 41.7 that fppf coverings (and hence syntomic, smooth, or étale coverings) are ph coverings as well.
Let Sch be a big ph site as in Definition \[8.8\]. Let \( T \in \text{Ob}(\text{Sch}_{ph}) \).

Let \( \{T_i \to T\}_{i \in I} \) be an arbitrary ph covering of \( T \).

1. There exists a covering \( \{U_j \to T\}_{j \in J} \) of \( T \) in the site \( \text{Sch}_{ph} \) which refines \( \{T_i \to T\}_{i \in I} \).
2. If \( \{T_i \to T\}_{i \in I} \) is a standard ph covering, then it is tautologically equivalent to a covering of \( \text{Sch}_{ph} \).
3. If \( \{T_i \to T\}_{i \in I} \) is a Zariski covering, then it is tautologically equivalent to a covering of \( \text{Sch}_{ph} \).

**Proof.** For each \( i \) choose an affine open covering \( T_i = \bigcup_{j \in J} T_{ij} \) such that each \( T_{ij} \) maps into an affine open subscheme of \( T \). By Lemma \[7.3\] the refinement \( \{T_{ij} \to T\}_{i \in I, j \in J} \) is an fpqc covering of \( T \) as well. Hence we may assume each \( T_i \) is affine, and maps into an affine open \( W_i \) of \( T \). Applying Sets, Lemma \[9.9\] we see that \( W_i \) is isomorphic to an object of \( \text{Sch}_{Zar} \). But then \( T_i \) as a finite type scheme over \( W_i \) is isomorphic to an object \( V_i \) of \( \text{Sch}_{Zar} \) by a second application of Sets, Lemma \[9.9\]. The covering \( \{V_i \to T\}_{i \in I} \) refines \( \{T_i \to T\}_{i \in I} \) (because they are isomorphic).

Moreover, \( \{V_i \to T\}_{i \in I} \) is combinatorially equivalent to a covering \( \{U_j \to T\}_{j \in J} \) of \( T \) in the site \( \text{Sch}_{Zar} \) by Sets, Lemma \[9.9\]. The covering \( \{U_j \to T\}_{j \in J} \) is a refinement as in (1). In the situation of (2), (3) each of the schemes \( T_i \) is isomorphic to an object of \( \text{Sch}_{fpqc} \) by Sets, Lemma \[9.9\] and another application of Sets, Lemma \[11.1\] gives what we want.

**Definition 8.10.** Let \( S \) be a scheme. Let \( \text{Sch}_{ph} \) be a big ph site containing \( S \).

1. The big ph site of \( S \), denoted \( (\text{Sch}/S)_{ph} \), is the site \( \text{Sch}_{ph}/S \) introduced in Sites, Section \[24\].
2. The big affine ph site of \( S \), denoted \( (\text{Aff}/S)_{ph} \), is the full subcategory of \( (\text{Sch}/S)_{ph} \) whose objects are affine \( U/S \). A covering of \( (\text{Aff}/S)_{ph} \) is any finite covering \( \{U_i \to U\} \) of \( (\text{Sch}/S)_{ph} \) with \( U_i \) and \( U \) affine.

We explicitly state that the big affine ph site is a site.

**Lemma 8.11.** Let \( S \) be a scheme. Let \( \text{Sch}_{ph} \) be a big ph site containing \( S \). Then \( (\text{Aff}/S)_{ph} \) is a site.

**Proof.** Reasoning as in the proof of Lemma \[4.9\] it suffices to show that the collection of finite ph coverings \( \{U_i \to U\} \) with \( U, U_i \) affine satisfies properties (1), (2) and (3) of Sites, Definition \[6.2\]. This is clear since for example, given a finite ph covering \( \{T_i \to T\}_{i \in I} \) with \( T_i, T \) affine, and for each \( i \) a finite ph covering...
The underlying categories of the sites $\text{Sch}_{\text{ph}}$, $\text{Sch}_{\text{ph}}$, $(\text{Sch}/S)_{\text{ph}}$, and $(\text{Aff}/S)_{\text{ph}}$ have fibre products. In each case the obvious functor into the category $\text{Sch}$ of all schemes commutes with taking fibre products. The category $(\text{Sch}/S)_{\text{ph}}$ has a final object, namely $S/S$.

**Proof.** For $\text{Sch}_{\text{ph}}$ it is true by construction, see Sets, Lemma [8.9]. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(\text{Sch}_{\text{ph}})$. The fibre product $V \times_U W$ in $\text{Sch}_{\text{ph}}$ is a fibre product in $\text{Sch}$ and is the fibre product of $V/S$ with $W/S$ over $U/S$ in the category of all schemes over $S$, and hence also a fibre product in $(\text{Sch}/S)_{\text{ph}}$. This proves the result for $(\text{Sch}/S)_{\text{fppf}}$. If $U, V, W$ are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{\text{ph}}$. □

Next, we check that the big affine site defines the same topos as the big site.

**Lemma 8.12.** Let $S$ be a scheme. Let $\text{Sch}_{\text{ph}}$ be a big ph site containing $S$. The underlying categories of the sites $\text{Sch}_{\text{ph}}$, $(\text{Sch}/S)_{\text{ph}}$, and $(\text{Aff}/S)_{\text{ph}}$ have fibre products. In each case the obvious functor into the category $\text{Sch}$ of all schemes commutes with taking fibre products. The category $(\text{Sch}/S)_{\text{ph}}$ has a final object, namely $S/S$.

**Proof.** For $\text{Sch}_{\text{ph}}$ it is true by construction, see Sets, Lemma [8.9]. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(\text{Sch}_{\text{ph}})$. The fibre product $V \times_U W$ in $\text{Sch}_{\text{ph}}$ is a fibre product in $\text{Sch}$ and is the fibre product of $V/S$ with $W/S$ over $U/S$ in the category of all schemes over $S$, and hence also a fibre product in $(\text{Sch}/S)_{\text{ph}}$. This proves the result for $(\text{Sch}/S)_{\text{fppf}}$. If $U, V, W$ are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{\text{ph}}$. □

**Lemma 8.13.** Let $S$ be a scheme. Let $\text{Sch}_{\text{ph}}$ be a big ph site containing $S$. The functor $(\text{Aff}/S)_{\text{ph}} \to (\text{Sch}/S)_{\text{ph}}$ is cocontinuous and induces an equivalence of topoi from $\text{Sh}(\text{Aff}/S)_{\text{ph}}$ to $\text{Sh}(\text{Sch}/S)_{\text{ph}}$.

**Proof.** The notion of a special cocontinuous functor is introduced in Sites, Definition [28.2]. Thus we have to verify assumptions (1) - (5) of Sites, Lemma [28.1]. Denote the inclusion functor $u : (\text{Aff}/S)_{\text{ph}} \to (\text{Sch}/S)_{\text{ph}}$. Being cocontinuous follows because any ph covering of $T/S$, $T$ affine, can be refined by a standard ph covering of $T$ by definition. Hence (1) holds. We see $u$ is continuous simply because a finite ph covering of an affine by affines is a ph covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that $u$ is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a ph covering). □

**Lemma 8.14.** Let $\mathcal{F}$ be a presheaf on $(\text{Sch}/S)_{\text{ph}}$. Then $\mathcal{F}$ is a sheaf if and only if

1. $\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and
2. if $f : V \to U$ is proper surjective, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.

**Proof.** We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(\text{Sch}/S)_{\text{ph}}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_j$. We will show that there exists a unique section $s \in \mathcal{F}$ restricting to $s_i$ over $T_i$. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_j'$ in order to produce $s$. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_j = s_i|_{T_i \times_T U_j}$ are sections agreeing over $T_i \times_T U_j \times_U U_j$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of $s_j$ to $V_{jk}$ by the implied morphisms, then we find that $s_{jk}$ glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that $s'_j$ is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_U V_j)$. Hence by (2) we find that $s'_j$ comes from a
Lemma 8.15. Let $\text{Sch}_{ph}$ be a big ph site. Let $f : T \to S$ be a morphism in $\text{Sch}_{ph}$. The functor

$$u : (\text{Sch}/T)_{ph} \to (\text{Sch}/S)_{ph}, \quad V/T \mapsto V/S$$

is cocontinuous, and has a continuous right adjoint $v : (\text{Sch}/S)_{ph} \to (\text{Sch}/T)_{ph}$, $(U \to S) \mapsto (U \times_ST \to T)$.

They induce the same morphism of topos $f_{big} : \text{Sh}((\text{Sch}/T)_{ph}) \to \text{Sh}((\text{Sch}/S)_{ph})$

We have $f_{big}^{-1}(G)(U/T) = G(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_ST/T)$. Also, $f_{big}^{-1}$ has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor $u$ is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for $f_{big}^{-1}$ and the existence of $f_{big!}$. Moreover, the functor $v$ is a right adjoint because given $U/T$ and $V/S$ we have $\text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_ST)$. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. □

Lemma 8.16. Given schemes $X, Y, Z$ in $(\text{Sch}/S)_{ph}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 8.15. □

9. The fpqc topology

Definition 9.1. Let $T$ be a scheme. An fpqc covering of $T$ is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each $f_i$ is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_1 \subset T_{a(1)}, \ldots, V_n \subset T_{a(n)}$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

Lemma 9.2. Let $T$ be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target $T$. The following are equivalent

1. $\{f_i : T_i \to T\}$ is an fpqc covering,
2. each $f_i$ is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,
3. each $f_i$ is flat and there exists an affine covering $T = \bigcup_{a \in A} U_a$ and for each $a \in A$ there exist $i_1, \ldots, i_n(a) \in I$ and quasi-compact opens $U_{a,j} \subset T_{a,j}$ such that $U_a = \bigcup_{j=1, \ldots, n(a)} f_{a,j}(U_{a,j})$.

If $T$ is quasi-separated, these are also equivalent to

4. each $f_i$ is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1, \ldots, n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of $t$ in $T$.  

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume $T$ is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_j$ such that $f_{ij}(U_j) \subset U$ and such that $\bigcup f_{ij}(U_j)$ is a neighbourhood of $t$ in $U$. By assumption there do exist finitely many quasi-compact opens $U_j' \subset T_j$ such that such that $\bigcup f_{ij}(U_j')$ is a neighbourhood of $t$ in $T$. Since $T$ is quasi-separated we see that $U_j = U_j' \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished.

\[\square\]

**Lemma 9.3.** Let $T$ be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target $T$. The following are equivalent

1. $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering, and
2. setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$, the family $\{f : T' \to T\}$ is an fpqc covering.

**Proof.** Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_j$ such that $U = \bigcup_{j=1}^{n} f_{ij}(U_j)$. Then $U \cap \coprod_{j=1}^{n} U_j = U'$ is a quasi-compact open surjecting onto $U$. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 9.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in $T_j$ and empty for almost all $j$. By Lemma 9.2 we see that (1) holds.

\[\square\]

**Lemma 9.4.** Let $T$ be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target $T$. Assume that

1. each $f_i$ is flat, and
2. the family $\{f_i : T_i \to T\}_{i \in I}$ can be refined by an fpqc covering of $T$.

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of $T$.

**Proof.** Let $\{g_j : X_j \to T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and affine open such that $U = \bigcup_{j \in J} g_j(V_k)$. For each $j$ pick $i_j \in I$ and a morphism $h_j : X_j \to T_i$ such that $g_j = f_{ij} \circ h_j$. Since $h_j(U_k)$ is quasi-compact we can find a quasi-compact open $h_{ik}(V_k) \subset U_k \subset f_{ik}^{-1}(U)$. Then $U = \bigcup f_{ik}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering by Lemma 9.2.

\[\square\]

**Lemma 9.5.** Let $T$ be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target $T$. Assume that

1. each $f_i$ is flat, and
2. there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of $T$.

**Proof.** We will use Lemma 9.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each $j$ we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \text{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 9.4.

\[\square\]

**Lemma 9.6.** Any fpqc covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.
Let consider only those faithfully flat ring extensions $R$ have a sheafification, see [Wat75, Theorem 5.5]. A mildly interesting option is to if you ignore set theoretic difficulties, then you run into presheaves which do not unboundedness is that there does not exist a field extension of $k$ such that every fpqc-covering can be refined by an element of $A$. If $R = k$ is a field, then the reason for this unboundedness is that there does not exist a set theoretic difficulties, then you run into presheaves which do not.

The fpqc4 topology cannot be treated in the same way as the fppf topology. Namely, suppose that $R$ is a nonzero ring. We will see in Lemma 9.14 that there does not exist a set $A$ of fpqc-coverings of $\text{Spec}(R)$ such that every fpqc-covering can be refined by an element of $A$. However, it is not so clear what happens if you change the cardinality to a bigger one. For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : \text{Sch}^{\text{fppf}} \to \text{Sets}$ it does make sense to ask whether $F$ satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

022D

**Lemma 9.7.** Let $T$ be a scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each $i$ we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.
3. If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.

**Proof.** Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 24.7 and 24.5). Thus we can apply Lemma 9.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each $i$ we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each $i$ we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$.

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4The letters fpqc stand for “fid`ement plat quasi-compapecte”.

5A more precise statement would be that the analogue of Lemma 7.7 for the fpqc topology does not hold.
Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_T U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \text{pr}_1(U' \times_T U_i)$. Since $T'$ can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering by Lemma 9.2.

022E Lemma 9.8. Let $T$ be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of $T$. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each $U_j$ is an affine scheme. Moreover, we may choose each $U_j$ to be open affine in one of the $T_i$.

Proof. This follows directly from the definition.

022F Definition 9.9. Let $T$ be an affine scheme. A standard fpqc covering of $T$ is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each $U_j$ is affine, flat over $T$ and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

03LA Lemma 9.10. Let $T$ be an affine scheme.

1. If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of $T$.
2. If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and for each $i$ we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.
3. If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a standard fpqc covering.

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 24.7 and 24.5) and that fibre products of affine schemes are affine (Schemes, Lemma 17.2).

03LB Lemma 9.11. Let $T$ be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target $T$. Assume that

1. each $f_i$ is flat, and
2. every affine scheme $Z$ and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i \in I}$.

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of $T$.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each $\alpha$ the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 9.4. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 9.5.

022G Definition 9.12. Let $F$ be a contravariant functor on the category of schemes with values in sets.

1. Let $\{U_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. We say that $F$ satisfies the sheaf property for the given family if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \times_T U_j} = \xi_j|_{U_i \times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
2. We say that $F$ satisfies the sheaf property for the fpqc topology if it satisfies the sheaf property for any fpqc covering.
We try to avoid using the terminology “$F$ is a sheaf” in this situation since we are not defining a category of fpqc sheaves as we explained above.

**Lemma 9.13.** Let $F$ be a contravariant functor on the category of schemes with values in sets. Then $F$ satisfies the sheaf property for the fpqc topology if and only if it satisfies

1. the sheaf property for every Zariski covering, and
2. the sheaf property for any standard fpqc covering.

Moreover, in the presence of (1) property (2) is equivalent to property

(2') the sheaf property for $\{V \to U\}$ with $V$, $U$ affine and $V \to U$ faithfully flat.

**Proof.** Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that $s_i$ and $s_j$ map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all $i$. Such a maximal open exists because $F$ satisfies the sheaf property for Zariski coverings; in fact $W$ is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all $i$.

By Lemma 9.8 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,...,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{ij}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{ij})$. Note that for any scheme $V \to U$ over $U$ there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ pr_2)(s_{ij})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if $V$ is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of $V$. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{ij}|_{T_{ij} \times_T T_i} = s_i|_{T_{ij} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$, which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\prod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\prod T_i) = \prod F(T_i)$ and similarly $F((\prod T_i) \times_T (\prod T_i)) = \prod F(T_i \times_T T_i)$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\prod T_i \to T\}$ is the same. $\square$

The following lemma is here just to point out set theoretical difficulties do indeed arise and should be ignored by most readers.

**Lemma 9.14.** Let $R$ be a nonzero ring. There does not exist a set $A$ of fpqc-coverings of $\text{Spec}(R)$ such that every fpqc-covering can be refined by an element of $A$.

**Proof.** Let us first explain this when $R = k$ is a field. For any set $I$ consider the purely transcendental field extension $k \subset k_I = k(\{t_i\}_{i \in I})$. Since $k \subset k_I$ is faithfully flat we see that $\{\text{Spec}(k_I) \to \text{Spec}(k)\}$ is an fpqc covering. Let $A$ be a set and for each $\alpha \in A$ let $U_\alpha = \{\text{Spec}(k) \to \text{Spec}(k)\}_{j \in J_\alpha}$ be an fpqc covering. If $U_\alpha$ refines $\{\text{Spec}(k_I) \to \text{Spec}(k)\}$ then the morphisms $S_{\alpha,j} : \text{Spec}(k_I) \to \text{Spec}(k)$ factor through $\text{Spec}(k_I)$. Since $U_\alpha$ is a covering, at least some $S_{\alpha,j}$ is nonempty. Pick a point point $s \in S_{\alpha,j}$. Since we have the factorization $S_{\alpha,j} : \text{Spec}(k_I) \to \text{Spec}(k)$ we obtain a homomorphism of fields $k_I \to k(s)$. In particular, we see that the cardinality of
\(\kappa(s)\) is at least the cardinality of \(I\). Thus if we take \(I\) to be a set of cardinality bigger than the cardinalities of the residue fields of all the schemes \(S_{\alpha,j}\), then such a factorization does not exist and the lemma holds for \(R = k\).

General case. Since \(R\) is nonzero it has a maximal prime ideal \(\mathfrak{m}\) with residue field \(\kappa\). Let \(I\) be a set and consider \(R_I = S_I^{-1}R[\{t_i\}_{i \in I}]\) where \(S_I \subset R[\{t_i\}_{i \in I}]\) is the multiplicative subset of \(f \in R[\{t_i\}_{i \in I}]\) such that \(f\) maps to a nonzero element of \(R/p[\{t_i\}_{i \in I}]\) for all primes \(p\) of \(R\). Then \(R_I\) is a faithfully flat \(R\)-algebra and \(\{\text{Spec}(R_I) \to \text{Spec}(R)\}\) is an fpqc covering. We leave it as an exercise to the reader to show that \(R_I \otimes_R \kappa \cong \kappa[\{t_i\}_{i \in I}] = \kappa_I\) with notation as above (hint: use that \(R \to \kappa\) is surjective and that any \(f \in R[\{t_i\}_{i \in I}]\) one of whose monomials occurs with coefficient 1 is an element of \(S_I\)). Let \(A\) be a set and for each \(\alpha \in A\) let \(U_\alpha = \{S_{\alpha,j} \to \text{Spec}(R)\}_{j \in I_\alpha}\) be an fpqc covering. If \(U_\alpha\) refines \(\{\text{Spec}(R_I) \to \text{Spec}(R)\}\), then by base change we conclude that \(\{S_{\alpha,j} \times_{\text{Spec}(R)} \text{Spec}(\kappa) \to \text{Spec}(\kappa)\}\) refines \(\{\text{Spec}(\kappa_I) \to \text{Spec}(\kappa)\}\). Hence by the result of the previous paragraph, there exists an \(I\) such that this is not the case and the lemma is proved. \(\square\)

10. Change of topologies

Let \(f : X \to Y\) be a morphism of schemes over a base scheme \(S\). In this case we have the following morphisms of sites\(^6\) (with suitable choices of sites as in Remark 10.1 below):

\begin{align*}
(1) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow (\text{Sch}/Y)_{\text{fppf}}, \\
(2) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow (\text{Sch}/Y)_{\text{syntomic}}, \\
(3) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow (\text{Sch}/Y)_{\text{smooth}}, \\
(4) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow (\text{Sch}/Y)_{\text{étale}}, \\
(5) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow (\text{Sch}/Y)_{\text{Zar}}, \\
(6) \quad & (\text{Sch}/X)_{\text{syntomic}} \longrightarrow (\text{Sch}/Y)_{\text{syntomic}}, \\
(7) \quad & (\text{Sch}/X)_{\text{syntomic}} \longrightarrow (\text{Sch}/Y)_{\text{smooth}}, \\
(8) \quad & (\text{Sch}/X)_{\text{syntomic}} \longrightarrow (\text{Sch}/Y)_{\text{étale}}, \\
(9) \quad & (\text{Sch}/X)_{\text{syntomic}} \longrightarrow (\text{Sch}/Y)_{\text{Zar}}, \\
(10) \quad & (\text{Sch}/X)_{\text{smooth}} \longrightarrow (\text{Sch}/Y)_{\text{smooth}}, \\
(11) \quad & (\text{Sch}/X)_{\text{smooth}} \longrightarrow (\text{Sch}/Y)_{\text{étale}}, \\
(12) \quad & (\text{Sch}/X)_{\text{smooth}} \longrightarrow (\text{Sch}/Y)_{\text{Zar}}, \\
(13) \quad & (\text{Sch}/X)_{\text{étale}} \longrightarrow (\text{Sch}/Y)_{\text{étale}}, \\
(14) \quad & (\text{Sch}/X)_{\text{étale}} \longrightarrow (\text{Sch}/Y)_{\text{Zar}}, \\
(15) \quad & (\text{Sch}/X)_{\text{Zar}} \longrightarrow (\text{Sch}/Y)_{\text{Zar}}, \\
(16) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow Y_{\text{étale}}, \\
(17) \quad & (\text{Sch}/X)_{\text{smooth}} \longrightarrow Y_{\text{étale}}, \\
(18) \quad & (\text{Sch}/X)_{\text{étale}} \longrightarrow Y_{\text{étale}}, \\
(19) \quad & (\text{Sch}/X)_{\text{fppf}} \longrightarrow Y_{\text{Zar}}, \\
(20) \quad & (\text{Sch}/X)_{\text{syntomic}} \longrightarrow Y_{\text{Zar}}, \\
(21) \quad & (\text{Sch}/X)_{\text{smooth}} \longrightarrow Y_{\text{Zar}}, \\
(22) \quad & (\text{Sch}/X)_{\text{étale}} \longrightarrow Y_{\text{Zar}}, \\
(23) \quad & (\text{Sch}/X)_{\text{Zar}} \longrightarrow Y_{\text{Zar}}, \\
(24) \quad & X_{\text{étale}} \longrightarrow Y_{\text{étale}}, \\
(25) \quad & X_{\text{étale}} \longrightarrow Y_{\text{étale}},
\end{align*}

\(^6\)We have not included the comparison between the ph topology and the others; for this see More on Morphisms, Remark \[41.8\]
In each case the underlying continuous functor $\text{Sch}/Y \to \text{Sch}/X$, or $Y \to \text{Sch}/X$ is the functor $Y'/Y \to X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{\text{big}} : (\text{Sch}/X)_\tau \to (\text{Sch}/Y)_\tau$ and $f_{\text{small}} : X \to Y$ for $\tau$ as above. We also have seen the morphisms of sites $\pi_Y : (\text{Sch}/Y)_\tau \to Y_\tau$ for $\tau \in \{\text{étale, Zariski}\}$. On the other hand, it is clear that the identity functor $(\text{Sch}/X)_\tau \to (\text{Sch}/X)_{\tau'}$ defines a morphism of sites when $\tau$ is a stronger topology than $\tau'$. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(\text{Sch}/X)_{\tau_0} \to (\text{Sch}/Y)_{\tau_1} \to (\text{Sch}/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

**Remark 10.1.** Take any category $\text{Sch}_\alpha$ constructed as in Sets, Lemma 9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings $\text{Cov}_{\text{fppf}}$ on $\text{Sch}_\alpha$ as in Sets, Lemma 11.1 starting with the category $\text{Sch}_\alpha$ and the class of fppf coverings. Let $\text{Sch}_{\text{fppf}}$ denote the big fppf site so obtained. Next, for $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$ let $\text{Sch}_\tau$ have the same underlying category as $\text{Sch}_{\text{fppf}}$ with coverings $\text{Cov}_\tau \subset \text{Cov}_{\text{fppf}}$ simply the subset of $\tau$-coverings. It is straightforward to check that this gives rise to a big site $\text{Sch}_\tau$.

### 11. Change of big sites

In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Given two big sites $\text{Sch}_\tau$ and $\text{Sch}'_{\tau'}$, we say that $\text{Sch}_\tau$ is contained in $\text{Sch}'_{\tau'}$ if $\text{Ob}(\text{Sch}_\tau) \subset \text{Ob}(\text{Sch}'_{\tau'})$ and $\text{Cov}(\text{Sch}_\tau) \subset \text{Cov}(\text{Sch}'_{\tau'})$. In this case $\tau$ is stronger than $\tau'$, for example, no fppf site can be contained in an étale site.

**Lemma 11.1.** Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.

**Proof.** This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 9.2 and 11.1 allow one to start with any initially given set of schemes and coverings.

**Lemma 11.2.** Let $\tau \in \{\text{Zariski, étale, smooth, syntomic, fppf}\}$. Suppose given big sites $\text{Sch}_\tau$ and $\text{Sch}'_{\tau'}$. Assume that $\text{Sch}_\tau$ is contained in $\text{Sch}'_{\tau'}$. The inclusion functor $\text{Sch}_\tau \to \text{Sch}'_{\tau'}$ satisfies the assumptions of Sites, Lemma 20.8. There are morphisms of topoi

$$g : \text{Sh}(\text{Sch}_\tau) \to \text{Sh}(\text{Sch}'_{\tau'})$$

$$f : \text{Sh}(\text{Sch}'_{\tau'}) \to \text{Sh}(\text{Sch}_\tau)$$

such that $f \circ g \cong \text{id}$. For any object $S$ of $\text{Sch}_\tau$ the inclusion functor $(\text{Sch}/S)_\tau \to (\text{Sch}'/S)_{\tau'}$ satisfies the assumptions of Sites, Lemma 20.8 also. Hence similarly we
obtain morphisms
\[ g : \text{Sh}((\text{Sch}/S)_\tau) \to \text{Sh}((\text{Sch}'/S)_\tau) \]
\[ f : \text{Sh}((\text{Sch}'/S)_\tau) \to \text{Sh}((\text{Sch}/S)_\tau) \]

with \( f \circ g \cong \text{id} \).

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 20.8 are immediate for the functors \( \text{Sch}_\tau \to \text{Sch}'_\tau \) and \( (\text{Sch}/S)_\tau \to (\text{Sch}'/S)_\tau \). Property (a) holds by Lemma 3.6, 4.7, 5.7, 6.7, or 7.7. Property (d) holds because fibre products in the categories \( \text{Sch}_\tau \), \( \text{Sch}'_\tau \) exist and are compatible with fibre products in the category of schemes. \( \square \)

Discussion: The functor \( g^{-1} = f_* \) is simply the restriction functor which associates to a sheaf \( G \) on \( \text{Sch}'_\tau \) the restriction \( G|_{\text{Sch}_\tau} \). Hence this lemma simply says that given any sheaf of sets \( F \) on \( \text{Sch}_\tau \) there exists a canonical sheaf \( F' \) on \( \text{Sch}'_\tau \) such that \( F'|_{\text{Sch}_\tau} = F \). In fact the sheaf \( F' \) has the following description: it is the sheafification of the presheaf
\[ \text{Sch}'_\tau \to \text{Sets}, \ V \mapsto \text{colim}_{V \to U} F(U) \]
where \( U \) is an object of \( \text{Sch}_\tau \). This is true because \( F' = f^{-1}F = (u_p,F)' \) according to Sites, Lemmas 20.5 and 20.8.
References


