

TOPOLOGIES ON SCHEMES

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1. Introduction

020L In this document we explain what the different topologies on the category of schemes are. Some references are [Gro71] and [BLR90]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

2. The general procedure

020M In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ . In order to get a site, as in Sites, Definition 6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say “consider all schemes with the Zariski topology” since that would give a “big” category. Instead, in each section of this chapter we will proceed as follows:

- (1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ .
- (2) We single out a notion of standard τ -covering within the category of affine schemes.

- (3) We define what is an “absolute” big τ -site Sch_τ . These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.
- (4) For any object S of Sch_τ we define the big τ -site $(Sch/S)_\tau$ and for suitable τ the small¹ τ -site S_τ .
- (5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ -covering of affines whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ -site of a scheme, or even the small τ -site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

3. The Zariski topology

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020O Definition 3.1. Let T be a scheme. A *Zariski covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

020P Lemma 3.2. *Let T be a scheme.*

- (1) *If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is a Zariski covering of T .*
- (2) *If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a Zariski covering.*
- (3) *If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

020Q Lemma 3.3. *Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be a Zariski covering of T . Then there exists a Zariski covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is a standard open of T , see Schemes, Definition 5.2. Moreover, we may choose each U_j to be an open of one of the T_i .*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 5.1. □

Thus we define the corresponding standard coverings of affines as follows.

020R Definition 3.4. Compare Schemes, Definition 5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a Zariski covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ with each $U_j \rightarrow T$ inducing an isomorphism with a standard affine open of T .

020S Definition 3.5. A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 6.2 constructed as follows:

- (1) Choose any set of schemes S_0 , and any set of Zariski coverings Cov_0 among these schemes.

¹The words big and small here do not relate to bigness/smallness of the corresponding categories.

- (2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .
- (3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 8.6 that, after having chosen the category Sch_α , the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α . It is shown in Sets, Lemma 9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 11.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S , let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

03WV **Lemma 3.6.** *Let Sch_{Zar} be a big Zariski site as in Definition 3.5. Let $T \in \text{Ob}(Sch_{Zar})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary Zariski covering of T . There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 8.2) to $\{T_i \rightarrow T\}_{i \in I}$.*

Proof. Since each $T_i \rightarrow T$ is an open immersion, we see by Sets, Lemma 9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar} . The covering $\{V_i \rightarrow T\}_{i \in I}$ is tautologically equivalent to $\{T_i \rightarrow T\}_{i \in I}$ (using the identity map on I both ways). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 11.1. \square

020T **Definition 3.7.** Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S .

- (1) The *big Zariski site of S* , denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 24.
- (2) The *small Zariski site of S* , which we denote S_{Zar} , is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \rightarrow S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{Zar}$ with $U \in \text{Ob}(S_{Zar})$.
- (3) The *big affine Zariski site of S* , denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are affine U/S . A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{Zar}$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site and the big affine Zariski site are sites. We check this now.

020U **Lemma 3.8.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S . Both S_{Zar} and $(Aff/S)_{Zar}$ are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{Zar}$ with $U \in \text{Ob}(S_{Zar})$ we also have $U_i \in \text{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. Let R be a ring. Let $f_1, \dots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \dots, n\}$ let $g_{i1}, \dots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i} . Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \text{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \text{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \rightarrow \text{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3). \square

020V **Lemma 3.9.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S . The underlying categories of the sites Sch_{Zar} , $(Sch/S)_{Zar}$, S_{Zar} , and $(Aff/S)_{Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S .*

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \rightarrow S$, $V \rightarrow U$, $W \rightarrow U$ morphisms of schemes with $U, V, W \in \text{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S , and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \rightarrow S$, $V \rightarrow U$ and $W \rightarrow U$ are open immersions then so is $V \times_U W \rightarrow S$ and hence we get the result for S_{Zar} . If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$. \square

Next, we check that the big affine site defines the same topos as the big site.

020W **Lemma 3.10.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S . The functor $(Aff/S)_{Zar} \rightarrow (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \rightarrow (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S , T affine, can be refined by a standard Zariski covering of T . This is the content of Lemma 3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. \square

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S .

020X **Lemma 3.11.** *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S .*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \rightarrow S$ is an isomorphism onto an open subscheme. Let \mathcal{F} be a sheaf on S . Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\text{Im}(U \rightarrow S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \text{Ob}(S_{Zar})$ with $\text{Im}(U' \rightarrow S) = U$ (here you have to use Sets, Lemma 9.9). Given a sheaf \mathcal{G} on S_{Zar} we define a sheaf on S by setting $\mathcal{G}(U) = \mathcal{G}(U'/S)$. To see that \mathcal{G}' is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \rightarrow U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_j \rightarrow U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 11.1, and we use Sites, Lemma 8.4. Details omitted. \square

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S . We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

020Y **Lemma 3.12.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \rightarrow S$ be a morphism in Sch_{Zar} . The functor $T_{Zar} \rightarrow (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf \mathcal{G} on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \rightarrow (Sch/S)_{Zar}$. In other words, given an open immersion $j : U \rightarrow T$ corresponding to an object of T_{Zar} we set $u(U \rightarrow T) = (f \circ j : U \rightarrow S)$. This functor commutes with fibre products, see Lemma 3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 20.5 and 20.6. \square

020Z **Lemma 3.13.** *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S . The inclusion functor $S_{Zar} \rightarrow (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 3.12. In particular the functor $i_S^{-1} = \pi_{S,}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.*

Proof. In this case the functor $u : S_{Zar} \rightarrow (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 3.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows. \square

04BS **Definition 3.14.** In the situation of Lemma 3.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf \mathcal{F} on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf \mathcal{F} on the big site and a sheaf \mathcal{G} on the big site that

$$\begin{aligned} \text{Mor}_{\text{Sh}(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) &= \text{Mor}_{\text{Sh}((\text{Sch}/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \text{Mor}_{\text{Sh}(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) &= \text{Mor}_{\text{Sh}((\text{Sch}/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F}) \end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

0210 **Lemma 3.15.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \rightarrow S$ be a morphism in Sch_{Zar} . The functor*

$$u : (\text{Sch}/T)_{Zar} \longrightarrow (\text{Sch}/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Sch}/S)_{Zar} \longrightarrow (\text{Sch}/T)_{Zar}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{big} : \text{Sh}((\text{Sch}/T)_{Zar}) \longrightarrow \text{Sh}((\text{Sch}/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big}!$ which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 3.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big}!$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. \square

0211 **Lemma 3.16.** *Let Sch_{Zar} be a big Zariski site. Let $f : T \rightarrow S$ be a morphism in Sch_{Zar} .*

- (1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 3.12 and i_T as in Lemma 3.13.*
- (2) *The functor $S_{Zar} \rightarrow T_{Zar}$, $(U \rightarrow S) \mapsto (U \times_S T \rightarrow T)$ is continuous and induces a morphism of topoi*

$$f_{small} : \text{Sh}(T_{Zar}) \longrightarrow \text{Sh}(S_{Zar}).$$

The functors f_{small}^{-1} and $f_{small,}$ agree with the usual notions f^{-1} and f_* if we identify sheaves on T_{Zar} , resp. S_{Zar} with sheaves on T , resp. S via Lemma 3.11.*

- (3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} T_{Zar} & \xleftarrow{\pi_T} & (\text{Sch}/T)_{Zar} \\ f_{small} \downarrow & & \downarrow f_{big} \\ S_{Zar} & \xleftarrow{\pi_S} & (\text{Sch}/S)_{Zar} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

- (4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 14.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T . \square

In the situation of the lemma, using the terminology of Definition 3.14 we have: for \mathcal{F} a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{Sch_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T , resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

0212 **Lemma 3.17.** *Given schemes X, Y, Z in $(Sch/S)_{Zar}$ and morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 3.15. For the functors on the small sites this is Sheaves, Lemma 21.2 via the identification of Lemma 3.11. \square

0DD9 **Lemma 3.18.** *Let Sch_{Zar} be a big Zariski site. Consider a cartesian diagram*

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

in Sch_{Zar} . Then $i_g^{-1} \circ f_{big,} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.*

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf \mathcal{F} on $(Sch/T)_{Zar}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on S'_{Zar} (use Lemmas 3.12 and 3.16). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 27.1. \square

We can think about a sheaf on the big Zariski site of S as a collection of “usual” sheaves on all schemes over S .

0213 **Lemma 3.19.** *Let S be a scheme contained in a big Zariski site Sch_{Zar} . A sheaf \mathcal{F} on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

- (1) *for every $T/S \in \text{Ob}((Sch/S)_{Zar})$ a sheaf \mathcal{F}_T on T ,*
- (2) *for every $f : T' \rightarrow T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \rightarrow \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

- (a) *given any $f : T' \rightarrow T$ and $g : T'' \rightarrow T'$ in $(Sch/S)_{Zar}$ the composition $c_g \circ g^{-1}c_f$ is equal to $c_{f \circ g}$, and*
- (b) *if $f : T' \rightarrow T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. Given a sheaf \mathcal{F} on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \rightarrow S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \rightarrow T$ an open immersion in $(Sch/T)_{Zar}$ with image U , see Lemmas 3.11 and 3.12. Hence given $f : T' \rightarrow T$ over S and $U, U' \rightarrow T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \rightarrow \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of \mathcal{F} with respect to the morphism $U' \times_T T' \rightarrow U'$ over S . The

collection of these maps are compatible with restrictions, and hence define an f -map c_f from \mathcal{F}_T to $\mathcal{F}_{T'}$, see Sheaves, Definition 21.7 and the discussion surrounding it. It is clear that $c_{f \circ g}$ is the composition of c_f and c_g , since composition of restriction maps of \mathcal{F} gives restriction maps.

Conversely, given a system (\mathcal{F}_T, c_f) as in the lemma we may define a presheaf \mathcal{F} on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \rightarrow T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f -map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. \square

4. The étale topology

0214 Let S be a scheme. We would like to define the étale-topology on the category of schemes over S . According to our general principle we first introduce the notion of an étale covering.

0215 **Definition 4.1.** Let T be a scheme. An *étale covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

0216 **Lemma 4.2.** Any Zariski covering is an étale covering.

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 34.9. \square

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0217 **Lemma 4.3.** Let T be a scheme.

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is an étale covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an étale covering.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is an étale covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is an étale covering.

Proof. Omitted. \square

0218 **Lemma 4.4.** Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be an étale covering of T . Then there exists an étale covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i .

Proof. Omitted. \square

Thus we define the corresponding standard coverings of affines as follows.

0219 **Definition 4.5.** Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \rightarrow T\}_{j=1, \dots, m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff/S , for example because of Algebra, Lemma 141.14 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra,

Lemma 141.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

021A **Definition 4.6.** A *big étale site* is any site $Sch_{\acute{e}tale}$ as in Sites, Definition 6.2 constructed as follows:

- (1) Choose any set of schemes S_0 , and any set of étale coverings Cov_0 among these schemes.
- (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S , let us point out that the topology on a big étale site $Sch_{\acute{e}tale}$ is in some sense induced from the étale topology on the category of all schemes.

03WW **Lemma 4.7.** Let $Sch_{\acute{e}tale}$ be a big étale site as in Definition 4.6. Let $T \in \text{Ob}(Sch_{\acute{e}tale})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary étale covering of T .

- (1) There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site $Sch_{\acute{e}tale}$ which refines $\{T_i \rightarrow T\}_{i \in I}$.
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{\acute{e}tale}$.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{\acute{e}tale}$.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T . By Lemma 4.3 the refinement $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T . Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar} . But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \rightarrow T\}_{i \in I}$ refines $\{T_i \rightarrow T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \rightarrow T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{\acute{e}tale}$ by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. \square

021B **Definition 4.8.** Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S .

- (1) The *big étale site of S* , denoted $(Sch/S)_{\acute{e}tale}$, is the site $Sch_{\acute{e}tale}/S$ introduced in Sites, Section 24.
- (2) The *small étale site of S* , which we denote $S_{\acute{e}tale}$, is the full subcategory of $(Sch/S)_{\acute{e}tale}$ whose objects are those U/S such that $U \rightarrow S$ is étale. A covering of $S_{\acute{e}tale}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{\acute{e}tale}$ with $U \in \text{Ob}(S_{\acute{e}tale})$.
- (3) The *big affine étale site of S* , denoted $(Aff/S)_{\acute{e}tale}$, is the full subcategory of $(Sch/S)_{\acute{e}tale}$ whose objects are affine U/S . A covering of $(Aff/S)_{\acute{e}tale}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{\acute{e}tale}$ which is a standard étale covering.

It is not completely clear that the big affine étale site or the small étale site are sites. We check this now.

021C **Lemma 4.9.** *Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S . Both $S_{\acute{e}tale}$ and $(Aff/S)_{\acute{e}tale}$ are sites.*

Proof. Let us show that $S_{\acute{e}tale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 6.2. Since $(Sch/S)_{\acute{e}tale}$ is a site, it suffices to prove that given any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{\acute{e}tale}$ with $U \in \text{Ob}(S_{\acute{e}tale})$ we also have $U_i \in \text{Ob}(S_{\acute{e}tale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(Aff/S)_{\acute{e}tale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard étale covering $\{T_i \rightarrow T\}_{i \in I}$ and for each i we have a standard étale covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a standard étale covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. \square

021D **Lemma 4.10.** *Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S . The underlying categories of the sites $Sch_{\acute{e}tale}$, $(Sch/S)_{\acute{e}tale}$, $S_{\acute{e}tale}$, and $(Aff/S)_{\acute{e}tale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{\acute{e}tale}$, and $S_{\acute{e}tale}$ both have a final object, namely S/S .*

Proof. For $Sch_{\acute{e}tale}$ it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \rightarrow S$, $V \rightarrow U$, $W \rightarrow U$ morphisms of schemes with $U, V, W \in \text{Ob}(Sch_{\acute{e}tale})$. The fibre product $V \times_U W$ in $Sch_{\acute{e}tale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S , and hence also a fibre product in $(Sch/S)_{\acute{e}tale}$. This proves the result for $(Sch/S)_{\acute{e}tale}$. If $U \rightarrow S$, $V \rightarrow U$ and $W \rightarrow U$ are étale then so is $V \times_U W \rightarrow S$ and hence we get the result for $S_{\acute{e}tale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{\acute{e}tale}$. \square

Next, we check that the big affine site defines the same topos as the big site.

021E **Lemma 4.11.** *Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S . The functor $(Aff/S)_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{\acute{e}tale})$ to $Sh((Sch/S)_{\acute{e}tale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$. Being cocontinuous just means that any étale covering of T/S , T affine, can be refined by a standard étale covering of T . This is the content of Lemma 4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. \square

Next, we establish some relationships between the topoi associated to these sites.

021F **Lemma 4.12.** *Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \rightarrow S$ be a morphism in $Sch_{\acute{e}tale}$. The functor $T_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{\acute{e}tale}) \longrightarrow Sh((Sch/S)_{\acute{e}tale})$$

For a sheaf \mathcal{G} on $(Sch/S)_{\acute{e}tale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$. In other words, given an étale morphism $j : U \rightarrow T$ corresponding to an object of $T_{\acute{e}tale}$ we set $u(U \rightarrow T) = (f \circ j : U \rightarrow S)$. This functor commutes with fibre products, see Lemma 4.10. Let $a, b : U \rightarrow V$ be two morphisms in $T_{\acute{e}tale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T, V \times_T V, (a,b)}} U \times_T U$$

which is a fibre product of schemes étale over T , hence étale over T . Thus $T_{\acute{e}tale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 20.5 and 20.6. \square

021G **Lemma 4.13.** *Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big étale site containing S . The inclusion functor $S_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$ satisfies the hypotheses of Sites, Lemma 20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$

and a morphism of topoi

$$i_S : Sh(S_{\acute{e}tale}) \longrightarrow Sh((Sch/S)_{\acute{e}tale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 4.12. In particular the functor $i_S^{-1} = \pi_{S,}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.*

Proof. In this case the functor $u : S_{\acute{e}tale} \rightarrow (Sch/S)_{\acute{e}tale}$, in addition to the properties seen in the proof of Lemma 4.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 20.8. \square

04BT **Definition 4.14.** In the situation of Lemma 4.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf \mathcal{F} on the big étale site we denote $\mathcal{F}|_{S_{\acute{e}tale}}$ this restriction.

With this notation in place we have for a sheaf \mathcal{F} on the big site and a sheaf \mathcal{G} on the small site that

$$Mor_{Sh(S_{\acute{e}tale})}(\mathcal{F}|_{S_{\acute{e}tale}}, \mathcal{G}) = Mor_{Sh((Sch/S)_{\acute{e}tale})}(\mathcal{F}, i_{S,*}\mathcal{G})$$

$$Mor_{Sh(S_{\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{S_{\acute{e}tale}}) = Mor_{Sh((Sch/S)_{\acute{e}tale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$.

021H **Lemma 4.15.** *Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \rightarrow S$ be a morphism in $Sch_{\acute{e}tale}$. The functor*

$$u : (Sch/T)_{\acute{e}tale} \longrightarrow (Sch/S)_{\acute{e}tale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{\acute{e}tale} \longrightarrow (Sch/T)_{\acute{e}tale}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{\acute{e}tale}) \longrightarrow Sh((Sch/S)_{\acute{e}tale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big}!$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 4.12). Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big}!$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. \square

021I **Lemma 4.16.** *Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \rightarrow S$ be a morphism in $Sch_{\acute{e}tale}$.*

- (1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 4.12 and i_T as in Lemma 4.13.*
- (2) *The functor $S_{\acute{e}tale} \rightarrow T_{\acute{e}tale}$, $(U \rightarrow S) \mapsto (U \times_S T \rightarrow T)$ is continuous and induces a morphism of sites*

$$f_{small} : T_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$

We have $f_{small,}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.*

- (3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} T_{\acute{e}tale} & \xleftarrow{\pi_T} & (Sch/T)_{\acute{e}tale} \\ f_{small} \downarrow & & \downarrow f_{big} \\ S_{\acute{e}tale} & \xleftarrow{\pi_S} & (Sch/S)_{\acute{e}tale} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.

- (4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{\acute{e}tale} \rightarrow T_{\acute{e}tale}$, $u(U \rightarrow S) = (U \times_S T \rightarrow T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 4.3 (3) and 4.10. Moreover, both $S_{\acute{e}tale}$, $T_{\acute{e}tale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 14.6 the functor u corresponds to a morphism of sites $T_{\acute{e}tale} \rightarrow S_{\acute{e}tale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T . \square

In the situation of the lemma, using the terminology of Definition 4.14 we have: for \mathcal{F} a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{\acute{e}tale}} = f_{small,*}(\mathcal{F}|_{T_{\acute{e}tale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T , resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

021J **Lemma 4.17.** *Given schemes X, Y, Z in $Sch_{\acute{e}tale}$ and morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 4.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 4.16. \square

0DDA **Lemma 4.18.** *Let $Sch_{\acute{e}tale}$ be a big étale site. Consider a cartesian diagram*

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

in $Sch_{\acute{e}tale}$. Then $i_g^{-1} \circ f_{big,} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.*

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf \mathcal{F} on $(Sch/T)_{\acute{e}tale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{\acute{e}tale}$ (use Lemmas 4.12 and 4.15). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 27.1. \square

We can think about a sheaf on the big étale site of S as a collection of “usual” sheaves on all schemes over S .

021K **Lemma 4.19.** *Let S be a scheme contained in a big étale site $Sch_{\acute{e}tale}$. A sheaf \mathcal{F} on the big étale site $(Sch/S)_{\acute{e}tale}$ is given by the following data:*

- (1) *for every $T/S \in \text{Ob}((Sch/S)_{\acute{e}tale})$ a sheaf \mathcal{F}_T on $T_{\acute{e}tale}$,*
- (2) *for every $f : T' \rightarrow T$ in $(Sch/S)_{\acute{e}tale}$ a map $c_f : f_{small}^{-1} \mathcal{F}_T \rightarrow \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

- (a) *given any $f : T' \rightarrow T$ and $g : T'' \rightarrow T'$ in $(Sch/S)_{\acute{e}tale}$ the composition $c_g \circ g_{small}^{-1} c_f$ is equal to $c_{f \circ g}$, and*
- (b) *if $f : T' \rightarrow T$ in $(Sch/S)_{\acute{e}tale}$ is étale then c_f is an isomorphism.*

Proof. Given a sheaf \mathcal{F} on $Sh((Sch/S)_{\acute{e}tale})$ we set $\mathcal{F}_T = i_p^{-1} \mathcal{F}$ where $p : T \rightarrow S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \rightarrow T$ in $T_{\acute{e}tale}$ see Lemma 4.12. Hence given $f : T' \rightarrow T$ over S and $U \rightarrow T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \rightarrow \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of \mathcal{F} with respect to the morphism $U \times_T T' \rightarrow U$ over S . The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \rightarrow f_{small,*} \mathcal{F}_{T'}$ where $u : T_{\acute{e}tale} \rightarrow T'_{\acute{e}tale}$ is the base change functor associated to f . By adjunction of $f_{small,*}$ (see Sites, Section 13) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1} \mathcal{F}_T \rightarrow \mathcal{F}_{T'}$. It is clear that $c'_{f \circ g}$ is the composition of c'_f and $f_{small,*} c'_g$, since composition of restriction maps of \mathcal{F} gives restriction maps, and this gives the desired relationship among c_f , c_g and $c_{f \circ g}$.

Conversely, given a system (\mathcal{F}_T, c_f) as in the lemma we may define a presheaf \mathcal{F} on $Sh((Sch/S)_{\acute{e}tale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \rightarrow T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \rightarrow f_{small,*} \mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. \square

5. The smooth topology

021Y In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 34) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

021Z **Definition 5.1.** Let T be a scheme. An *smooth covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

0220 **Lemma 5.2.** Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 34.1 and Lemma 4.2. \square

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0221 **Lemma 5.3.** Let T be a scheme.

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is an smooth covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a smooth covering.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a smooth covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is a smooth covering.

Proof. Omitted. \square

0222 **Lemma 5.4.** Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be a smooth covering of T . Then there exists a smooth covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \rightarrow T$ is standard smooth, see Morphisms, Definition 32.1. Moreover, we may choose each U_j to be open affine in one of the T_i .

Proof. Omitted, but see Algebra, Lemma 135.10. \square

Thus we define the corresponding standard coverings of affines as follows.

0223 **Definition 5.5.** Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \rightarrow T\}_{j=1, \dots, m}$ with each U_j is affine, $U_j \rightarrow T$ standard smooth and $T = \bigcup f_j(U_j)$.

03WY **Definition 5.6.** A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 6.2 constructed as follows:

- (1) Choose any set of schemes S_0 , and any set of smooth coverings Cov_0 among these schemes.
- (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S , let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

03WZ **Lemma 5.7.** *Let Sch_{smooth} be a big smooth site as in Definition 5.6. Let $T \in \text{Ob}(Sch_{smooth})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary smooth covering of T .*

- (1) *There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{smooth} which refines $\{T_i \rightarrow T\}_{i \in I}$.*
- (2) *If $\{T_i \rightarrow T\}_{i \in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth} .*
- (3) *If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth} .*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T . By Lemma 5.3 the refinement $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T . Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar} . But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \rightarrow T\}_{i \in I}$ refines $\{T_i \rightarrow T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \rightarrow T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. \square

03X0 **Definition 5.8.** Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S .

- (1) The *big smooth site of S* , denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 24.
- (2) The *big affine smooth site of S* , denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S . A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

06VC **Lemma 5.9.** *Let S be a scheme. Let $Sch_{\acute{e}tale}$ be a big smooth site containing S . The functor $(Aff/S)_{smooth} \rightarrow (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \rightarrow (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S , T affine, can be refined by a standard smooth covering of T . This is the content of Lemma 5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. \square

To be continued...

04HC **Lemma 5.10.** *Let Sch_{smooth} be a big smooth site. Let $f : T \rightarrow S$ be a morphism in Sch_{smooth} . The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (\text{Sch}/S)_{\text{smooth}} \longrightarrow (\text{Sch}/T)_{\text{smooth}}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{\text{big}} : \text{Sh}((\text{Sch}/T)_{\text{smooth}}) \longrightarrow \text{Sh}((\text{Sch}/S)_{\text{smooth}})$$

We have $f_{\text{big}}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{\text{big},*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{\text{big}!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{\text{big}!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\text{Mor}_S(u(U), V) = \text{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{\text{big},*}$. \square

6. The syntomic topology

0224 In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

0225 **Definition 6.1.** Let T be a scheme. An *syntomic covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

0226 **Lemma 6.2.** Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 32.7 and Lemma 5.2. \square

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0227 **Lemma 6.3.** Let T be a scheme.

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is an syntomic covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a syntomic covering.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a syntomic covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is a syntomic covering.

Proof. Omitted. \square

0228 **Lemma 6.4.** Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be a syntomic covering of T . Then there exists a syntomic covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \rightarrow T$ is standard syntomic, see Morphisms, Definition 29.1. Moreover, we may choose each U_j to be open affine in one of the T_i .

Proof. Omitted, but see Algebra, Lemma 134.15. \square

Thus we define the corresponding standard coverings of affines as follows.

0229 **Definition 6.5.** Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \rightarrow T\}_{j=1, \dots, m}$ with each U_j is affine, $U_j \rightarrow T$ standard syntomic and $T = \bigcup f_j(U_j)$.

03X1 **Definition 6.6.** A *big syntomic site* is any site Sch_{syntomic} as in Sites, Definition 6.2 constructed as follows:

- (1) Choose any set of schemes S_0 , and any set of syntomic coverings Cov_0 among these schemes.
- (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S , let us point out that the topology on a big syntomic site Sch_{syntomic} is in some sense induced from the syntomic topology on the category of all schemes.

03X2 **Lemma 6.7.** Let Sch_{syntomic} be a big syntomic site as in Definition 6.6. Let $T \in \text{Ob}(Sch_{\text{syntomic}})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary syntomic covering of T .

- (1) There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{syntomic} which refines $\{T_i \rightarrow T\}_{i \in I}$.
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in Sch_{syntomic} .
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in Sch_{syntomic} .

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T . By Lemma 6.3 the refinement $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T . Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar} . But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \rightarrow T\}_{i \in I}$ refines $\{T_i \rightarrow T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \rightarrow T\}_{j \in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{Zar} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. \square

03X3 **Definition 6.8.** Let S be a scheme. Let Sch_{syntomic} be a big syntomic site containing S .

- (1) The *big syntomic site of S* , denoted $(Sch/S)_{\text{syntomic}}$, is the site Sch_{syntomic}/S introduced in Sites, Section 24.
- (2) The *big affine syntomic site of S* , denoted $(\text{Aff}/S)_{\text{syntomic}}$, is the full subcategory of $(Sch/S)_{\text{syntomic}}$ whose objects are affine U/S . A covering of $(\text{Aff}/S)_{\text{syntomic}}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{\text{syntomic}}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

06VD **Lemma 6.9.** *Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S . The functor $(Aff/S)_{syntomic} \rightarrow (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \rightarrow (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S , T affine, can be refined by a standard syntomic covering of T . This is the content of Lemma 6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. \square

To be continued...

04HD **Lemma 6.10.** *Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \rightarrow S$ be a morphism in $Sch_{syntomic}$. The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big}!$ which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big}!$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. \square

7. The fppf topology

- 021L Let S be a scheme. We would like to define the fppf-topology² on the category of schemes over S . According to our general principle we first introduce the notion of an fppf-covering.
- 021M **Definition 7.1.** Let T be a scheme. An *fppf covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.
- 021N **Lemma 7.2.** *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

² The letters fppf stand for “fidèlement plat de présentation finie”.

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.6 and 29.7, and Lemma 6.2. \square

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

021O **Lemma 7.3.** *Let T be a scheme.*

- (1) *If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is an fppf covering of T .*
- (2) *If $\{T_i \rightarrow T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an fppf covering.*
- (3) *If $\{T_i \rightarrow T\}_{i \in I}$ is an fppf covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 24.5) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 20.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 24.7) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 20.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). \square

021P **Lemma 7.4.** *Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fppf covering of T . Then there exists an fppf covering $\{U_j \rightarrow T\}_{j=1, \dots, m}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i .*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 24.9. \square

Thus we define the corresponding standard coverings of affines as follows.

021Q **Definition 7.5.** *Let T be an affine scheme. A standard fppf covering of T is a family $\{f_j : U_j \rightarrow T\}_{j=1, \dots, m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.*

021R **Definition 7.6.** *A big fppf site is any site Sch_{fppf} as in Sites, Definition 6.2 constructed as follows:*

- (1) *Choose any set of schemes S_0 , and any set of fppf coverings Cov_0 among these schemes.*
- (2) *As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .*
- (3) *Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.*

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S , let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

03WX **Lemma 7.7.** *Let Sch_{fppf} be a big fppf site as in Definition 7.6. Let $T \in \text{Ob}(Sch_{fppf})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary fppf covering of T .*

- (1) *There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{fppf} which refines $\{T_i \rightarrow T\}_{i \in I}$.*
- (2) *If $\{T_i \rightarrow T\}_{i \in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf} .*
- (3) *If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf} .*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T . By Lemma 7.3 the refinement $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T . Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar} . But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \rightarrow T\}_{i \in I}$ refines $\{T_i \rightarrow T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \rightarrow T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. \square

021S **Definition 7.8.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S .

- (1) The *big fppf site of S* , denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 24.
- (2) The *big affine fppf site of S* , denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S . A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \rightarrow U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

021T **Lemma 7.9.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S . Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(Aff/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a standard fppf covering $\{T_i \rightarrow T\}_{i \in I}$ and for each i we have a standard fppf covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a standard fppf covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. \square

021U **Lemma 7.10.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S . The underlying categories of the sites Sch_{fppf} , $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S .*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \rightarrow S$, $V \rightarrow U$, $W \rightarrow U$ morphisms of schemes with $U, V, W \in \text{Ob}(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S , and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{fppf}$. \square

Next, we check that the big affine site defines the same topos as the big site.

021V **Lemma 7.11.** *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S . The functor $(Aff/S)_{fppf} \rightarrow (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (Aff/S)_{fppf} \rightarrow (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S , T affine, can be refined by a standard fppf covering of T . This is the content of Lemma 7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. \square

Next, we establish some relationships between the topoi associated to these sites.

021W **Lemma 7.12.** *Let Sch_{fppf} be a big fppf site. Let $f : T \rightarrow S$ be a morphism in Sch_{fppf} . The functor*

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big}!$ which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big}!$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. \square

021X **Lemma 7.13.** *Given schemes X, Y, Z in $(Sch/S)_{fppf}$ and morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 7.12. \square

8. The ph topology

0DBC In this section we define the ph topology. This is the topology generated by Zariski coverings and proper surjective morphisms, see Lemma 8.14.

We borrow our notation/terminology from the paper [GL01] by Goodwillie and Lichtenbaum. These authors show that if we restrict to the subcategory of Noetherian schemes, then the ph topology is the same as the “h topology” as originally defined by Voevodsky: this is the topology generated by Zariski open coverings and

finite type morphisms which are universally submersive. They also show that the two topologies do not agree on non-Noetherian schemes, see [GL01, Example 4.5].

It may be worth pointing out that the definition of the h topology on the category of schemes depends on the reference. For example, in the paper [Ryd07] David Rydh (re)defines the h topology as the topology generated by Zariski open coverings and morphisms of finite presentation which are universally “subtrusive”.

Before we can define the coverings in our topology we need to do a bit of work.

0DBD **Definition 8.1.** Let T be an affine scheme. A *standard ph covering* is a family $\{f_j : U_j \rightarrow T\}_{j=1,\dots,m}$ constructed from a proper surjective morphism $f : U \rightarrow T$ and an affine open covering $U = \bigcup_{j=1,\dots,m} U_j$ by setting $f_j = f|_{U_j}$.

It follows immediately from Chow’s lemma that we can refine a standard ph covering by a standard ph covering corresponding to a surjective projective morphism.

0DBE **Lemma 8.2.** Let $\{f_j : U_j \rightarrow T\}_{j=1,\dots,m}$ be a standard ph covering. Let $T' \rightarrow T$ be a morphism of affine schemes. Then $\{U_j \times_T T' \rightarrow T'\}_{j=1,\dots,m}$ is a standard ph covering.

Proof. Let $f : U \rightarrow T$ be proper surjective and let an affine open covering $U = \bigcup_{j=1,\dots,m} U_j$ be given as in Definition 8.1. Then $U \times_T T' \rightarrow T'$ is proper surjective (Morphisms, Lemmas 9.4 and 39.5). Also, $U \times_T T' = \bigcup_{j=1,\dots,m} U_j \times_T T'$ is an affine open covering. This concludes the proof. \square

0DBF **Lemma 8.3.** Let T be an affine scheme. Each of the following types of families of maps with target T has a refinement by a standard ph covering:

- (1) any Zariski open covering of T ,
- (2) $\{W_{ji} \rightarrow T\}_{j=1,\dots,m, i=1,\dots,n_j}$ where $\{W_{ji} \rightarrow U_j\}_{i=1,\dots,n_j}$ and $\{U_j \rightarrow T\}_{j=1,\dots,m}$ are standard ph coverings.

Proof. Part (1) follows from the fact that any Zariski open covering of T can be refined by a finite affine open covering.

Proof of (3). Choose $U \rightarrow T$ proper surjective and $U = \bigcup_{j=1,\dots,m} U_j$ as in Definition 8.1. Choose $W_j \rightarrow U_j$ proper surjective and $W_j = \bigcup W_{ji}$ as in Definition 8.1. By Chow’s lemma (Limits, Lemma 12.1) we can find $W'_j \rightarrow W_j$ proper surjective and closed immersions $W'_j \rightarrow \mathbf{P}_{U_j}^{e_j}$. Thus, after replacing W_j by W'_j and $W_j = \bigcup W_{ji}$ by a suitable affine open covering of W'_j , we may assume there is a closed immersion $W_j \subset \mathbf{P}_{U_j}^{e_j}$ for all $j = 1, \dots, m$.

Let $\overline{W}_j \subset \mathbf{P}_U^{e_j}$ be the scheme theoretic closure of W_j . Then $W_j \subset \overline{W}_j$ is an open subscheme; in fact W_j is the inverse image of $U_j \subset U$ under the morphism $\overline{W}_j \rightarrow U$. (To see this use that $W_j \rightarrow \mathbf{P}_U^{e_j}$ is quasi-compact and hence formation of the scheme theoretic image commutes with restriction to opens, see Morphisms, Section 6.) Let $Z_j = U \setminus U_j$ with reduced induced closed subscheme structure. Then

$$V_j = \overline{W}_j \amalg Z_j \rightarrow U$$

is proper surjective and the open subscheme $W_j \subset V_j$ is the inverse image of U_j . Hence for $v \in V_j$, $v \notin W_j$ we can pick an affine open neighbourhood $v \in V_{j,v} \subset V_j$ which maps into $U_{j'}$ for some $1 \leq j' \leq m$.

To finish the proof we consider the proper surjective morphism

$$V = V_1 \times_U V_2 \times_U \dots \times_U V_m \longrightarrow U \longrightarrow T$$

and the covering of V by the affine opens

$$V_{1,v_1} \times_U \dots \times_U V_{j-1,v_{j-1}} \times_U W_{ji} \times_U V_{j+1,v_{j+1}} \times_U \dots \times_U V_{m,v_m}$$

These do indeed form a covering, because each point of U is in some U_j and the inverse image of U_j in V is equal to $V_1 \times \dots \times V_{j-1} \times W_j \times V_{j+1} \times \dots \times V_m$. Observe that the morphism from the affine open displayed above to T factors through W_{ji} ; thus we obtain a refinement. Finally, we only need a finite number of these affine opens as V is quasi-compact (as a scheme proper over the affine scheme T). \square

0DBG **Definition 8.4.** Let T be a scheme. A *ph covering* of T is a family of morphisms $\{T_i \rightarrow T\}_{i \in I}$ of schemes such that f_i is locally of finite type and such that for every affine open $U \subset T$ there exists a standard ph covering $\{U_j \rightarrow U\}_{j=1,\dots,m}$ refining the family $\{T_i \times_T U \rightarrow U\}_{i \in I}$.

0DBH **Lemma 8.5.** A Zariski covering is a ph covering³.

Proof. This is true because a Zariski covering of an affine scheme can be refined by a standard ph covering by Lemma 8.3. \square

0DES **Lemma 8.6.** Let $f : Y \rightarrow X$ be a surjective proper morphism of schemes. Then $\{Y \rightarrow X\}$ is a ph covering.

Proof. Omitted. \square

Next, we show that this notion satisfies the conditions of Sites, Definition 6.2.

0DBI **Lemma 8.7.** Let T be a scheme.

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is a ph covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a ph covering.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a ph covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is a ph covering.

Proof. Assertion (1) is clear.

Proof of (3). The base change $T_i \times_T T' \rightarrow T'$ is locally of finite type by Morphisms, Lemma 14.4. hence we only need to check the condition on affine opens. Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \dots \cup U'_n$ such that $U'_j \rightarrow T$ maps into an affine open $U_j \subset T$. Choose a standard ph covering $\{U_{jl} \rightarrow U_j\}_{l=1,\dots,n_j}$ refining $\{T_i \times_T U_j \rightarrow U_j\}$. By Lemma 8.2 the base change $\{U_{jl} \times_{U_j} U'_j \rightarrow U'_j\}$ is a standard ph covering. Note that $\{U'_j \rightarrow U'\}$ is a standard ph covering as well. By Lemma 8.3 the family $\{U_{jl} \times_{U_j} U'_j \rightarrow U'\}$ can be refined by a standard ph covering. Since $\{U_{jl} \rightarrow U_j\}$ refines $\{T_i \times_T U_j \rightarrow U_j\}$ we conclude.

Proof of (2). Composition preserves being locally of finite type, see Morphisms, Lemma 14.3. Hence we only need to check the condition on affine opens. Let $U \subset T$ be affine open. First we pick a standard ph covering $\{U_k \rightarrow U\}_{k=1,\dots,m}$

³We will see in More on Morphisms, Lemma 41.7 that fppf coverings (and hence syntomic, smooth, or étale coverings) are ph coverings as well.

refining $\{T_i \times_T U \rightarrow U\}$. Say the refinement is given by morphisms $U_k \rightarrow T_{i_k}$ over T . Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \rightarrow U_k\}_{j \in J_{i_k}}$$

is a ph covering by part (3). As U_k is affine, we can find a standard ph covering $\{U_{ka} \rightarrow U_k\}_{a=1, \dots, b_k}$ refining this family. Then we apply Lemma 8.3 to see that $\{U_{ka} \rightarrow U\}$ can be refined by a standard ph covering. Since $\{U_{ka} \rightarrow U\}$ refines $\{T_{i_j} \times_T U \rightarrow U\}$ this finishes the proof. \square

0DBJ **Definition 8.8.** A *big ph site* is any site Sch_{ph} as in Sites, Definition 6.2 constructed as follows:

- (1) Choose any set of schemes S_0 , and any set of ph coverings Cov_0 among these schemes.
- (2) As underlying category take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set S_0 .
- (3) Choose any set of coverings as in Sets, Lemma 11.1 starting with the category Sch_α and the class of ph coverings, and the set Cov_0 chosen above.

See the remarks following Definition 3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big ph site of a scheme S , let us point out that the topology on a big ph site Sch_{ph} is in some sense induced from the ph topology on the category of all schemes.

0DBK **Lemma 8.9.** Let Sch_{ph} be a big ph site as in Definition 8.8. Let $T \in \text{Ob}(Sch_{ph})$. Let $\{T_i \rightarrow T\}_{i \in I}$ be an arbitrary ph covering of T .

- (1) There exists a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{ph} which refines $\{T_i \rightarrow T\}_{i \in I}$.
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a standard ph covering, then it is tautologically equivalent to a covering of Sch_{ph} .
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{ph} .

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T . By Lemma 7.3 the refinement $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T . Applying Sets, Lemma 9.9 we see that W_i is isomorphic to an object of Sch_{Zar} . But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 9.9. The covering $\{V_i \rightarrow T\}_{i \in I}$ refines $\{T_i \rightarrow T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \rightarrow T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \rightarrow T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 9.9. The covering $\{U_j \rightarrow T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 9.9, and another application of Sets, Lemma 11.1 gives what we want. \square

0DBL **Definition 8.10.** Let S be a scheme. Let Sch_{ph} be a big ph site containing S .

- (1) The *big ph site of S* , denoted $(Sch/S)_{ph}$, is the site Sch_{ph}/S introduced in Sites, Section 24.

- (2) The *big affine ph site* of S , denoted $(\text{Aff}/S)_{ph}$, is the full subcategory of $(\text{Sch}/S)_{ph}$ whose objects are affine U/S . A covering of $(\text{Aff}/S)_{ph}$ is any finite covering $\{U_i \rightarrow U\}$ of $(\text{Sch}/S)_{ph}$ with U_i and U affine.

We explicitly state that the big affine ph site is a site.

0DBM **Lemma 8.11.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S . Then $(\text{Aff}/S)_{ph}$ is a site.*

Proof. Reasoning as in the proof of Lemma 4.9 it suffices to show that the collection of finite ph coverings $\{U_i \rightarrow U\}$ with U, U_i affine satisfies properties (1), (2) and (3) of Sites, Definition 6.2. This is clear since for example, given a finite ph covering $\{T_i \rightarrow T\}_{i \in I}$ with T_i, T affine, and for each i a finite ph covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$ with T_{ij} affine, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a ph covering (Lemma 8.7), $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. \square

0DBN **Lemma 8.12.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S . The underlying categories of the sites Sch_{ph} , $(\text{Sch}/S)_{ph}$, and $(\text{Aff}/S)_{ph}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(\text{Sch}/S)_{ph}$ has a final object, namely S/S .*

Proof. For Sch_{ph} it is true by construction, see Sets, Lemma 9.9. Suppose we have $U \rightarrow S, V \rightarrow U, W \rightarrow U$ morphisms of schemes with $U, V, W \in \text{Ob}(\text{Sch}_{ph})$. The fibre product $V \times_U W$ in Sch_{ph} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S , and hence also a fibre product in $(\text{Sch}/S)_{ph}$. This proves the result for $(\text{Sch}/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{ph}$. \square

Next, we check that the big affine site defines the same topos as the big site.

0DBP **Lemma 8.13.** *Let S be a scheme. Let Sch_{ph} be a big ph site containing S . The functor $(\text{Aff}/S)_{ph} \rightarrow (\text{Sch}/S)_{ph}$ is cocontinuous and induces an equivalence of topoi from $\text{Sh}((\text{Aff}/S)_{ph})$ to $\text{Sh}((\text{Sch}/S)_{ph})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 28.1. Denote the inclusion functor $u : (\text{Aff}/S)_{ph} \rightarrow (\text{Sch}/S)_{ph}$. Being cocontinuous follows because any ph covering of $T/S, T$ affine, can be refined by a standard ph covering of T by definition. Hence (1) holds. We see u is continuous simply because a finite ph covering of an affine by affines is a ph covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a ph covering). \square

0DBQ **Lemma 8.14.** *Let \mathcal{F} be a presheaf on $(\text{Sch}/S)_{ph}$. Then \mathcal{F} is a sheaf if and only if*

- (1) \mathcal{F} satisfies the sheaf condition for Zariski coverings, and
- (2) if $f : V \rightarrow U$ is proper surjective, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \rightarrow \mathcal{F}(V \times_U V)$.

Proof. We will show that if (1) and (2) hold, then \mathcal{F} is sheaf. Let $\{T_i \rightarrow T\}$ be a ph covering, i.e., a covering in $(\text{Sch}/S)_{ph}$. We will verify the sheaf condition for

this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}$ restricting to s_i over T_i . Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s . Consider the ph coverings $\{T_i \times_T U_j \rightarrow U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \rightarrow U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \rightarrow U_j\}$ refines $\{T_i \times_T U_j \rightarrow U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \rightarrow \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties. \square

Next, we establish some relationships between the topoi associated to these sites.

0DBR **Lemma 8.15.** *Let Sch_{ph} be a big ph site. Let $f : T \rightarrow S$ be a morphism in Sch_{ph} . The functor*

$$u : (Sch/T)_{ph} \longrightarrow (Sch/S)_{ph}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{ph} \longrightarrow (Sch/T)_{ph}, \quad (U \rightarrow S) \longmapsto (U \times_S T \rightarrow T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{ph}) \longrightarrow Sh((Sch/S)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big}!$ which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 20.5 and 20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big}!$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $Mor_S(u(U), V) = Mor_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 21.1 and 21.2 to get the formula for $f_{big,*}$. \square

0DBS **Lemma 8.16.** *Given schemes X, Y, Z in $(Sch/S)_{ph}$ and morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 8.15. \square

9. The fpqc topology

022A

022B **Definition 9.1.** Let T be a scheme. An *fpqc covering* of T is a family of morphisms $\{f_i : T_i \rightarrow T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \dots, n\} \rightarrow I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \dots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

03L7 **Lemma 9.2.** *Let T be a scheme. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with target T . The following are equivalent*

- (1) $\{f_i : T_i \rightarrow T\}_{i \in I}$ is an fpqc covering,
- (2) each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,
- (3) each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \dots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1, \dots, n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.

If T is quasi-separated, these are also equivalent to

- (4) each f_i is flat, and for every $t \in T$ there exist $i_1, \dots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1, \dots, n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T .

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U . By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T . Since T is quasi-separated we see that $U_j = U'_j \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished. \square

040I **Lemma 9.3.** *Let T be a scheme. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with target T . The following are equivalent*

- (1) $\{f_i : T_i \rightarrow T\}_{i \in I}$ is an fpqc covering, and
- (2) setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$ the family $\{f : T' \rightarrow T\}$ is an fpqc covering.

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \dots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1, \dots, n} f_{i_j}(U_j)$. Then $U_1 \amalg \dots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U . Thus $\{f : T' \rightarrow T\}$ is an fpqc covering by Lemma 9.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j . By Lemma 9.2 we see that (1) holds. \square

03L8 **Lemma 9.4.** *Let T be a scheme. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with target T . Assume that*

- (1) each f_i is flat, and
- (2) the family $\{f_i : T_i \rightarrow T\}_{i \in I}$ can be refined by a fpqc covering of T .

Then $\{f_i : T_i \rightarrow T\}_{i \in I}$ is a fpqc covering of T .

Proof. Let $\{g_j : X_j \rightarrow T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \rightarrow T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \dots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \rightarrow T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f_{i_{j_k}}^{-1}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \rightarrow T\}_{i \in I}$ is an fpqc covering by Lemma 9.2. \square

03L9 **Lemma 9.5.** *Let T be a scheme. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with target T . Assume that*

- (1) each f_i is flat, and
- (2) there exists an fpqc covering $\{g_j : S_j \rightarrow T\}_{j \in J}$ such that each $\{S_j \times_T T_i \rightarrow S_j\}_{i \in I}$ is an fpqc covering.

Then $\{f_i : T_i \rightarrow T\}_{i \in I}$ is a fpqc covering of T .

Proof. We will use Lemma 9.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \text{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \rightarrow T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \rightarrow T\}$ we conclude by Lemma 9.4. \square

022C **Lemma 9.6.** *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 7.2. Let $\{f_i : U_i \rightarrow U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 9.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 24.9), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V . Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. \square

The fpqc⁴ topology cannot be treated in the same way as the fppf topology⁵. Namely, suppose that R is a nonzero ring. We will see in Lemma 9.14 that there does not exist a set A of fpqc-coverings of $\text{Spec}(R)$ such that every fpqc-covering can be refined by an element of A . If $R = k$ is a field, then the reason for this unboundedness is that there does not exist a field extension of k such that every field extension of k is contained in it.

If you ignore set theoretic difficulties, then you run into presheaves which do not have a sheafification, see [Wat75, Theorem 5.5]. A mildly interesting option is to consider only those faithfully flat ring extensions $R \rightarrow R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : \text{Sch}^{opp} \rightarrow \text{Sets}$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

022D **Lemma 9.7.** *Let T be a scheme.*

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is an fpqc covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is an fpqc covering.

⁴The letters fpqc stand for “fidèlement plat quasi-compacte”.

⁵A more precise statement would be that the analogue of Lemma 7.7 for the fpqc topology does not hold.

- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is an fpqc covering and $T' \rightarrow T$ is a morphism of schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is an fpqc covering.

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 24.7 and 24.5). Thus we can apply Lemma 9.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \rightarrow T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \rightarrow T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(W_{ij})$. Thus $\{T_{ij} \rightarrow T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \rightarrow T\}_{i \in I}$ is an fpqc covering and $T' \rightarrow T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \text{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is an fpqc covering by Lemma 9.2. \square

- 022E **Lemma 9.8.** Let T be an affine scheme. Let $\{T_i \rightarrow T\}_{i \in I}$ be an fpqc covering of T . Then there exists an fpqc covering $\{U_j \rightarrow T\}_{j=1, \dots, n}$ which is a refinement of $\{T_i \rightarrow T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i .

Proof. This follows directly from the definition. \square

- 022F **Definition 9.9.** Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \rightarrow T\}_{j=1, \dots, n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

- 03LA **Lemma 9.10.** Let T be an affine scheme.

- (1) If $T' \rightarrow T$ is an isomorphism then $\{T' \rightarrow T\}$ is a standard fpqc covering of T .
- (2) If $\{T_i \rightarrow T\}_{i \in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \rightarrow T_i\}_{j \in J_i}$, then $\{T_{ij} \rightarrow T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.
- (3) If $\{T_i \rightarrow T\}_{i \in I}$ is a standard fpqc covering and $T' \rightarrow T$ is a morphism of affine schemes then $\{T' \times_T T_i \rightarrow T'\}_{i \in I}$ is a standard fpqc covering.

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 24.7 and 24.5) and that fibre products of affine schemes are affine (Schemes, Lemma 17.2). \square

- 03LB **Lemma 9.11.** Let T be a scheme. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with target T . Assume that

- (1) each f_i is flat, and
- (2) every affine scheme Z and morphism $h : Z \rightarrow T$ there exists a standard fpqc covering $\{Z_j \rightarrow Z\}_{j=1, \dots, n}$ which refines the family $\{T_i \times_T Z \rightarrow Z\}_{i \in I}$.

Then $\{f_i : T_i \rightarrow T\}_{i \in I}$ is a fpqc covering of T .

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \rightarrow U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 9.4. As $\{U_\alpha \rightarrow T\}$ is an fpqc covering we conclude that $\{T_i \rightarrow T\}$ is an fpqc covering by Lemma 9.5. \square

022G **Definition 9.12.** Let F be a contravariant functor on the category of schemes with values in sets.

- (1) Let $\{U_i \rightarrow T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. We say that F satisfies the sheaf property for the given family if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \times_T U_j} = \xi_j|_{U_i \times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
- (2) We say that F satisfies the sheaf property for the fpqc topology if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology “ F is a sheaf” in this situation since we are not defining a category of fpqc sheaves as we explained above.

022H **Lemma 9.13.** Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies

- (1) the sheaf property for every Zariski covering, and
- (2) the sheaf property for any standard fpqc covering.

Moreover, in the presence of (1) property (2) is equivalent to property

- (2') the sheaf property for $\{V \rightarrow U\}$ with V, U affine and $V \rightarrow U$ faithfully flat.

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \rightarrow T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i . Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i .

By Lemma 9.8 we can find a standard fpqc covering $\{U_j \rightarrow U\}_{j=1, \dots, n}$ refining $\{U \times_T T_i \rightarrow U\}$, say by morphisms $h_j : U_j \rightarrow T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \rightarrow U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \text{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \dots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \rightarrow V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V . In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \rightarrow T\}$ is a standard fpqc covering, then $\coprod T_i \rightarrow T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_i)$. Thus the sheaf condition for $\{T_i \rightarrow T\}$ and $\{\coprod T_i \rightarrow T\}$ is the same. \square

The following lemma is here just to point out set theoretical difficulties do indeed arise and should be ignored by most readers.

0BBK **Lemma 9.14.** *Let R be a nonzero ring. There does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A .*

Proof. Let us first explain this when $R = k$ is a field. For any set I consider the purely transcendental field extension $k \subset k_I = k(\{t_i\}_{i \in I})$. Since $k \rightarrow k_I$ is faithfully flat we see that $\{\mathrm{Spec}(k_I) \rightarrow \mathrm{Spec}(k)\}$ is an fpqc covering. Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \rightarrow \mathrm{Spec}(k)\}_{j \in J_\alpha}$ be an fpqc covering. If \mathcal{U}_α refines $\{\mathrm{Spec}(k_I) \rightarrow \mathrm{Spec}(k)\}$ then the morphisms $S_{\alpha,j} \rightarrow \mathrm{Spec}(k)$ factor through $\mathrm{Spec}(k_I)$. Since \mathcal{U}_α is a covering, at least some $S_{\alpha,j}$ is nonempty. Pick a point $s \in S_{\alpha,j}$. Since we have the factorization $S_{\alpha,j} \rightarrow \mathrm{Spec}(k_I) \rightarrow \mathrm{Spec}(k)$ we obtain a homomorphism of fields $k_I \rightarrow \kappa(s)$. In particular, we see that the cardinality of $\kappa(s)$ is at least the cardinality of I . Thus if we take I to be a set of cardinality bigger than the cardinalities of the residue fields of all the schemes $S_{\alpha,j}$, then such a factorization does not exist and the lemma holds for $R = k$.

General case. Since R is nonzero it has a maximal prime ideal \mathfrak{m} with residue field κ . Let I be a set and consider $R_I = S_I^{-1}R[\{t_i\}_{i \in I}]$ where $S_I \subset R[\{t_i\}_{i \in I}]$ is the multiplicative subset of $f \in R[\{t_i\}_{i \in I}]$ such that f maps to a nonzero element of $R/\mathfrak{p}[\{t_i\}_{i \in I}]$ for all primes \mathfrak{p} of R . Then R_I is a faithfully flat R -algebra and $\{\mathrm{Spec}(R_I) \rightarrow \mathrm{Spec}(R)\}$ is an fpqc covering. We leave it as an exercise to the reader to show that $R_I \otimes_R \kappa \cong \kappa(\{t_i\}_{i \in I}) = \kappa_I$ with notation as above (hint: use that $R \rightarrow \kappa$ is surjective and that any $f \in R[\{t_i\}_{i \in I}]$ one of whose monomials occurs with coefficient 1 is an element of S_I). Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \rightarrow \mathrm{Spec}(R)\}_{j \in J_\alpha}$ be an fpqc covering. If \mathcal{U}_α refines $\{\mathrm{Spec}(R_I) \rightarrow \mathrm{Spec}(R)\}$, then by base change we conclude that $\{S_{\alpha,j} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \rightarrow \mathrm{Spec}(\kappa)\}$ refines $\{\mathrm{Spec}(\kappa_I) \rightarrow \mathrm{Spec}(\kappa)\}$. Hence by the result of the previous paragraph, there exists an I such that this is not the case and the lemma is proved. \square

10. Change of topologies

03FE Let $f : X \rightarrow Y$ be a morphism of schemes over a base scheme S . In this case we have the following morphisms of sites⁶ (with suitable choices of sites as in Remark 10.1 below):

- (1) $(\mathrm{Sch}/X)_{fppf} \rightarrow (\mathrm{Sch}/Y)_{fppf}$,
- (2) $(\mathrm{Sch}/X)_{fppf} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{syntomic}}$,
- (3) $(\mathrm{Sch}/X)_{fppf} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{smooth}}$,
- (4) $(\mathrm{Sch}/X)_{fppf} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{étale}}$,
- (5) $(\mathrm{Sch}/X)_{fppf} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{Zar}}$,
- (6) $(\mathrm{Sch}/X)_{\mathrm{syntomic}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{syntomic}}$,
- (7) $(\mathrm{Sch}/X)_{\mathrm{syntomic}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{smooth}}$,
- (8) $(\mathrm{Sch}/X)_{\mathrm{syntomic}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{étale}}$,
- (9) $(\mathrm{Sch}/X)_{\mathrm{syntomic}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{Zar}}$,
- (10) $(\mathrm{Sch}/X)_{\mathrm{smooth}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{smooth}}$,
- (11) $(\mathrm{Sch}/X)_{\mathrm{smooth}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{étale}}$,
- (12) $(\mathrm{Sch}/X)_{\mathrm{smooth}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{Zar}}$,
- (13) $(\mathrm{Sch}/X)_{\mathrm{étale}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{étale}}$,
- (14) $(\mathrm{Sch}/X)_{\mathrm{étale}} \rightarrow (\mathrm{Sch}/Y)_{\mathrm{Zar}}$,

⁶We have not included the comparison between the ph topology and the others; for this see More on Morphisms, Remark 41.8.

- (15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar},$
- (16) $(Sch/X)_{fppf} \longrightarrow Y_{\acute{e}tale},$
- (17) $(Sch/X)_{syntomic} \longrightarrow Y_{\acute{e}tale},$
- (18) $(Sch/X)_{smooth} \longrightarrow Y_{\acute{e}tale},$
- (19) $(Sch/X)_{\acute{e}tale} \longrightarrow Y_{\acute{e}tale},$
- (20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar},$
- (21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar},$
- (22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar},$
- (23) $(Sch/X)_{\acute{e}tale} \longrightarrow Y_{Zar},$
- (24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar},$
- (25) $X_{\acute{e}tale} \longrightarrow Y_{\acute{e}tale},$
- (26) $X_{\acute{e}tale} \longrightarrow Y_{Zar},$
- (27) $X_{Zar} \longrightarrow Y_{Zar},$

In each case the underlying continuous functor $Sch/Y \rightarrow Sch/X$, or $Y_\tau \rightarrow Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \rightarrow (Sch/Y)_\tau$ and $f_{small} : X_\tau \rightarrow Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \rightarrow Y_\tau$ for $\tau \in \{\acute{e}tale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \rightarrow (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ' . Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \rightarrow Y \rightarrow Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \rightarrow Z$.

03FF **Remark 10.1.** Take any category Sch_α constructed as in Sets, Lemma 9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $Cov_\tau \subset Cov_{fppf}$ simply the subset of τ -coverings. It is straightforward to check that this gives rise to a big site Sch_τ .

11. Change of big sites

022I In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$, we say that Sch_τ is contained in $Sch'_{\tau'}$, if $Ob(Sch_\tau) \subset Ob(Sch'_{\tau'})$ and $Cov(Sch_\tau) \subset Cov(Sch'_{\tau'})$. In this case τ is stronger than τ' , for example, no fppf site can be contained in an étale site.

022J **Lemma 11.1.** *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 9.2 and 11.1 allow one to start with any initially given set of schemes and coverings. \square

022K **Lemma 11.2.** *Let $\tau \in \{\text{Zariski}, \text{étale}, \text{smooth}, \text{syntomic}, \text{fppf}\}$. Suppose given big sites Sch_τ and Sch'_τ . Assume that Sch_τ is contained in Sch'_τ . The inclusion functor $Sch_\tau \rightarrow Sch'_\tau$ satisfies the assumptions of Sites, Lemma 20.8. There are morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \rightarrow (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 20.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 20.8 are immediate for the functors $Sch_\tau \rightarrow Sch'_\tau$ and $(Sch/S)_\tau \rightarrow (Sch'/S)_\tau$. Property (a) holds by Lemma 3.6, 4.7, 5.7, 6.7, or 7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. \square

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf \mathcal{G} on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets \mathcal{F} on Sch_τ there exists a canonical sheaf \mathcal{F}' on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf \mathcal{F}' has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto \text{colim}_{V \rightarrow U} \mathcal{F}(U)$$

where U is an object of Sch_τ . This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 20.5 and 20.8.

12. Other chapters

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